Short Course LID, Prague, 19-23 September 2005

Modeling of Localized Inelastic Deformation

Milan Jirásek

General outline:

A. Introduction
B. Elastoplasticity
C. Damage mechanics
D. Strain localization
E. Regularized continuum models
F. Strong discontinuity models

F. Strong discontinuity models

F.1

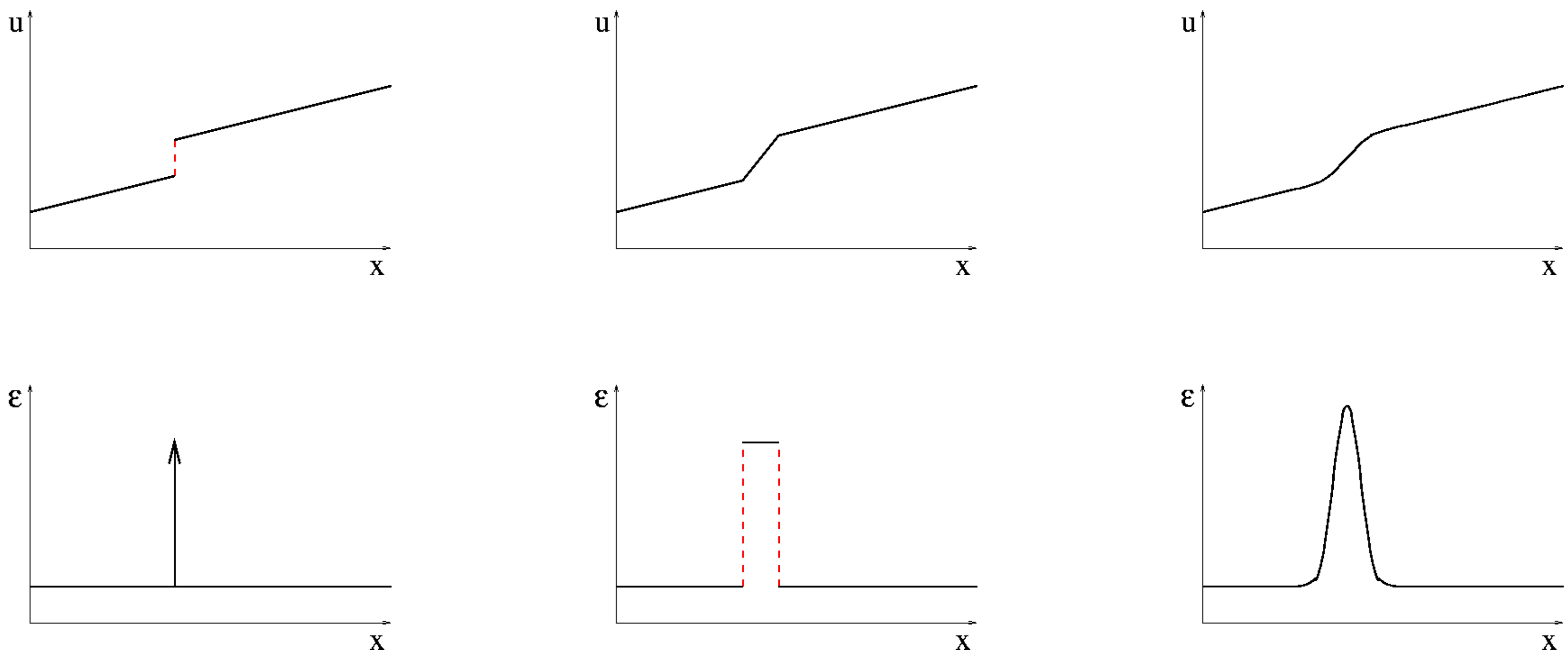
Introduction

Classification of models: kinematic aspects

Strong discontinuity

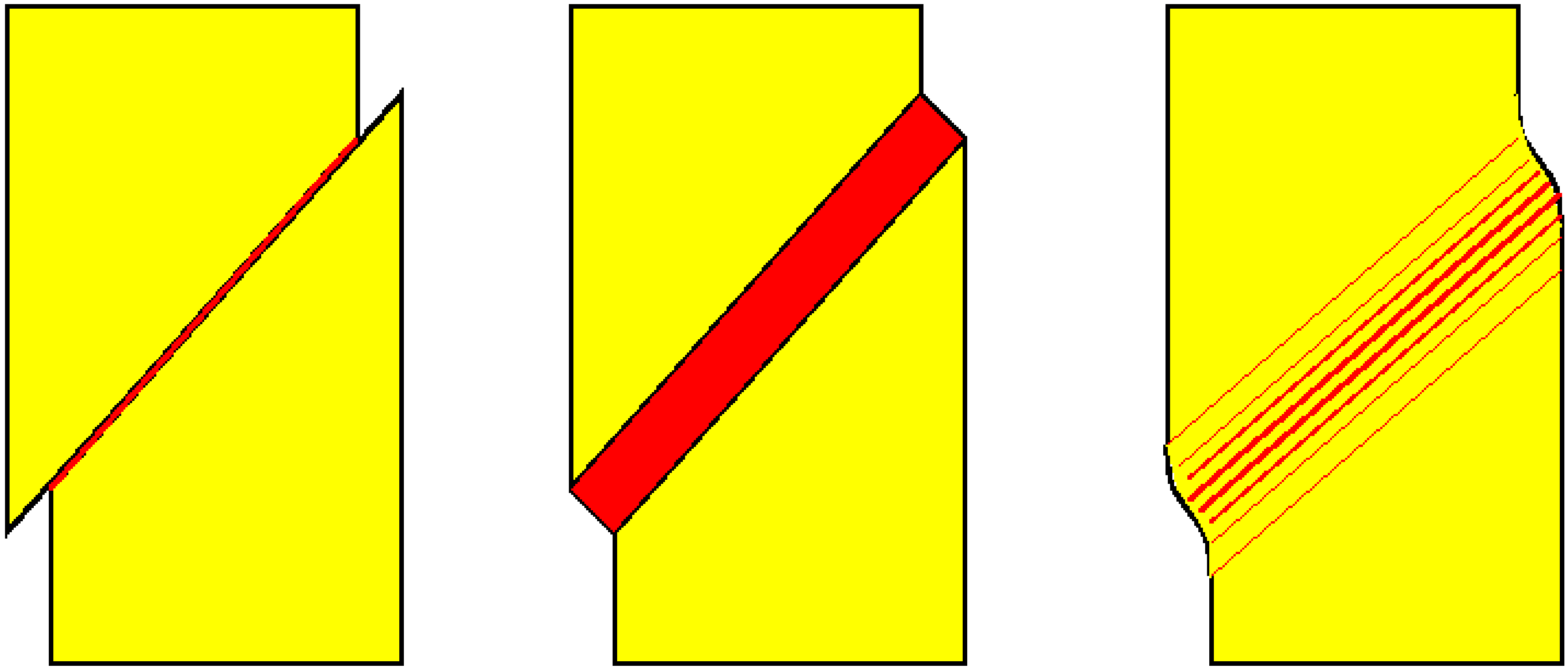
Weak discontinuity

Regularized localization zone



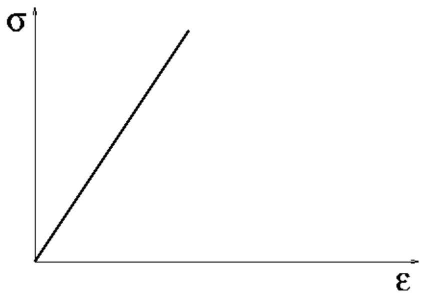
Classification of models: kinematic aspects

Strong discontinuity | Weak discontinuity | Regularized localization zone

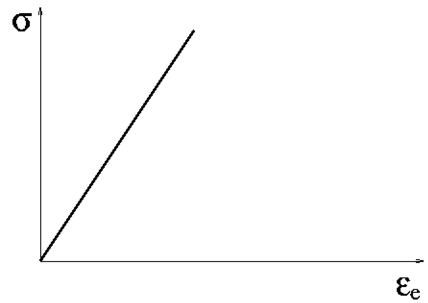


Classification of models: material laws

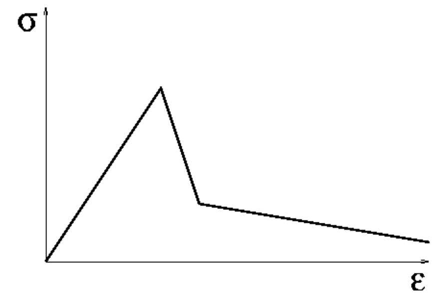
Stress-strain law



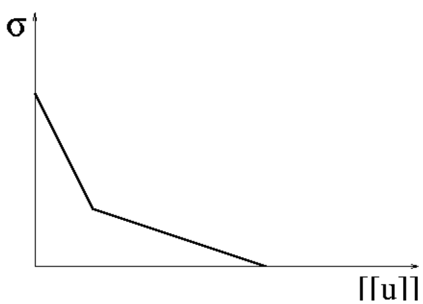
Stress-strain law
(pre-localization part)



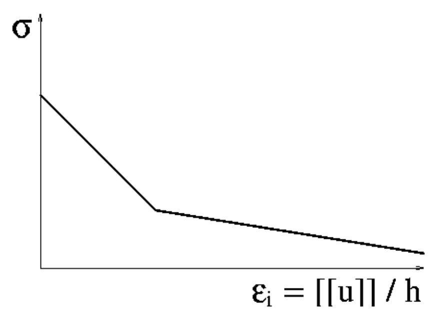
Stress-strain law



Traction-separation law



Stress-strain law
(post-localization part)

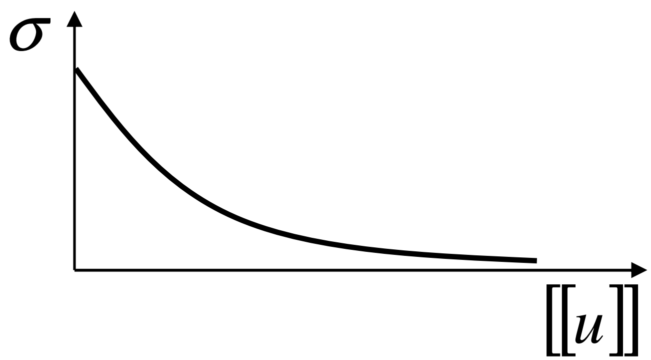


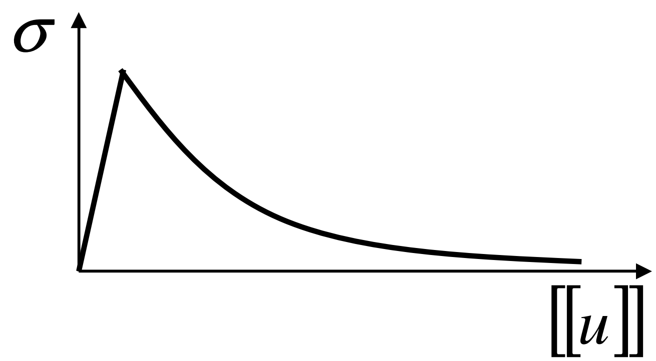
Enrichment acting
as localization limiter:

- nonlocal
- gradient
- Cosserat
- viscosity

Traction-separation laws

1) Formulated directly in the traction-separation space
   a) with nonzero elastic compliance (elasto-plastic, …)
   b) with zero elastic compliance (rigid-plastic, …)





For general applications, we need a link between the separation **vector** (displacement jump vector) and the traction **vector**:

$$[[\mathbf{u}]] \longrightarrow \mathbf{t}$$

Traction-separation laws

2) “Derived“ from a stress-strain law (softening continuum) using the strong discontinuity approach

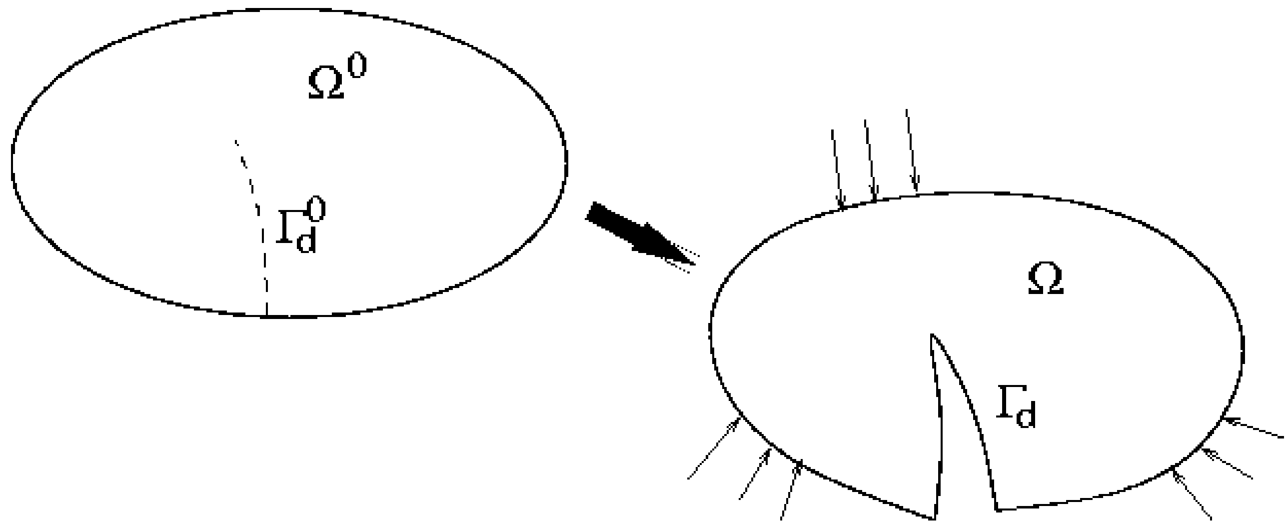
$$[\![\mathbf{u}]\!]$$

$$\mathbf{t} = \mathbf{n} \cdot \boldsymbol{\sigma}$$

$$\boldsymbol{\varepsilon} = \left([\![\mathbf{u}]\!] \cdot \mathbf{n}\right)_{sym}$$

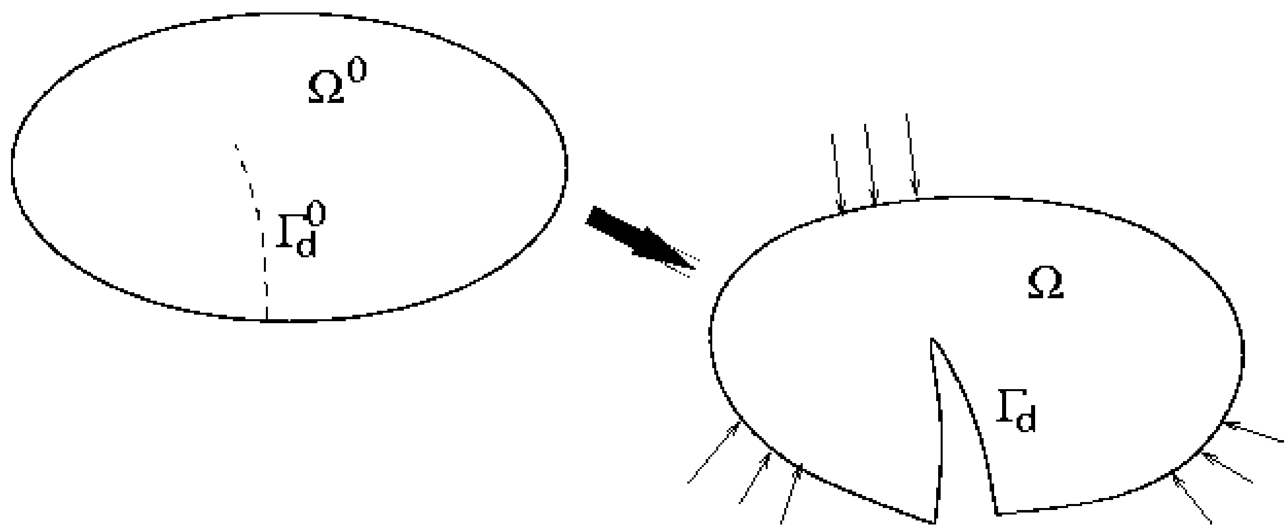
$$\boldsymbol{\sigma} = \tilde{\boldsymbol{\sigma}}\left(\boldsymbol{\varepsilon}, \ldots\right)$$

Finite element representation of strong discontinuities



1) Discontinuities at element interfaces:
   a) Remeshing
   b) Interspersed potential discontinuities

Finite element representation of strong discontinuities

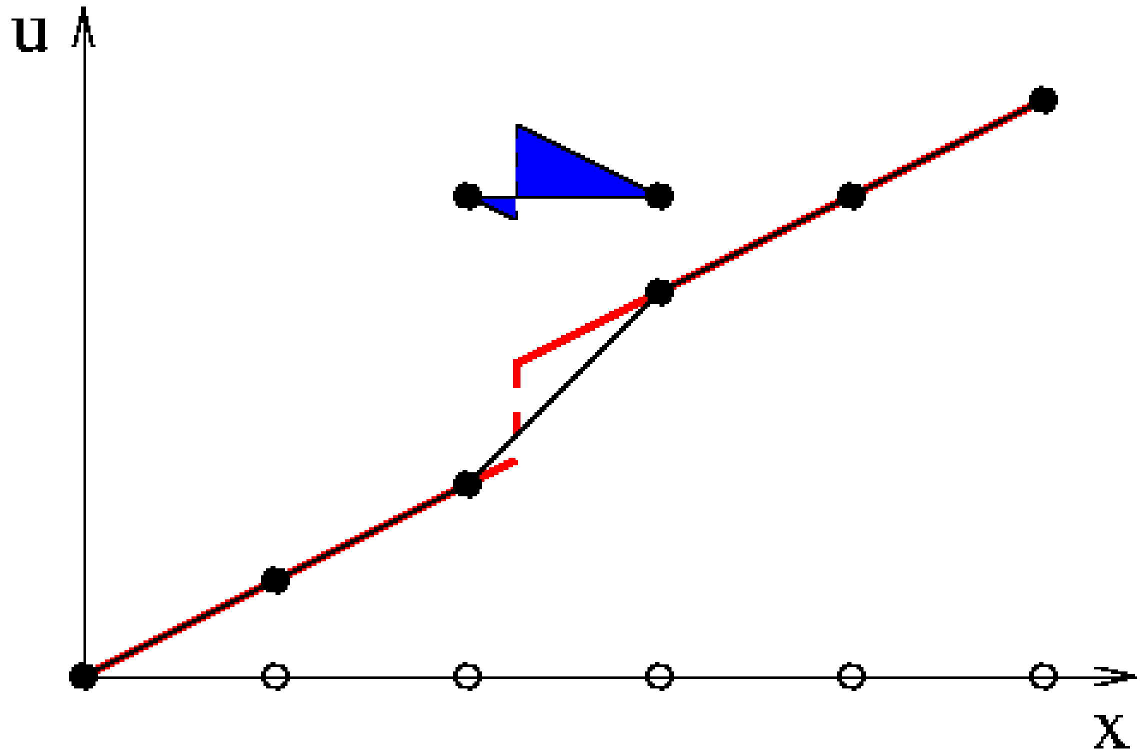


2) Arbitrary discontinuities across elements:

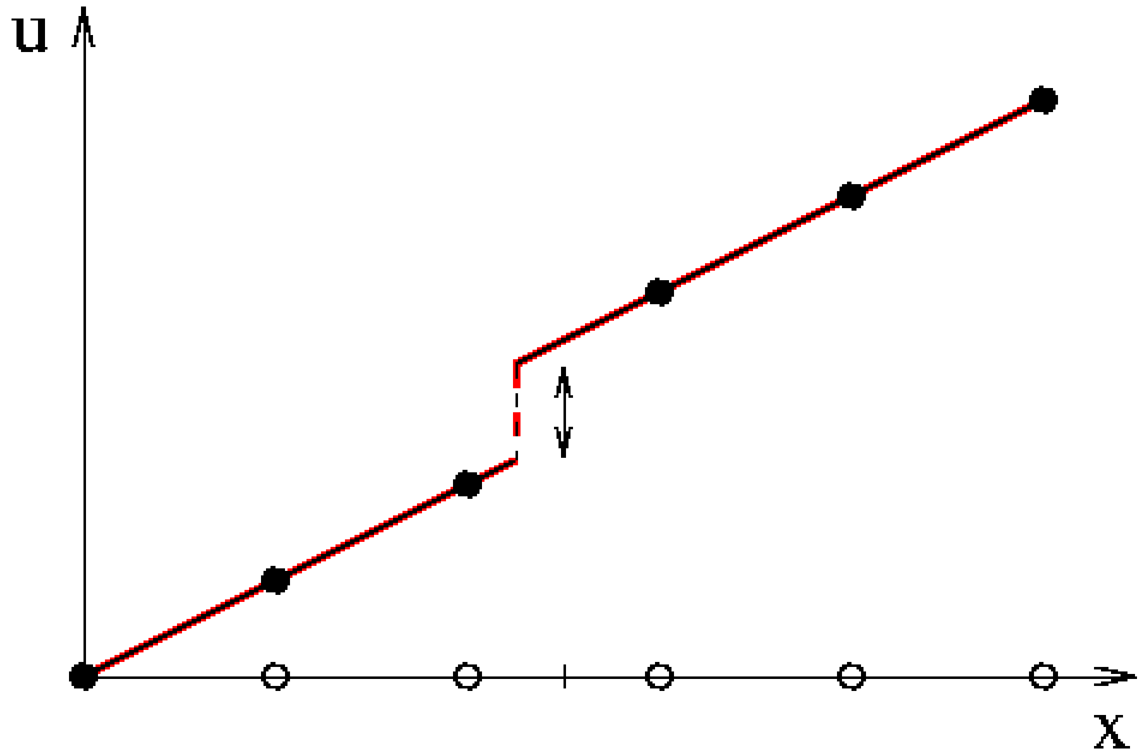
   a) Elements with embedded discontinuities using the enhanced assumed strain formulation (EED-EAS)

   b) Extended finite elements based on the partition-of-unity concept (XFEM-PUM)

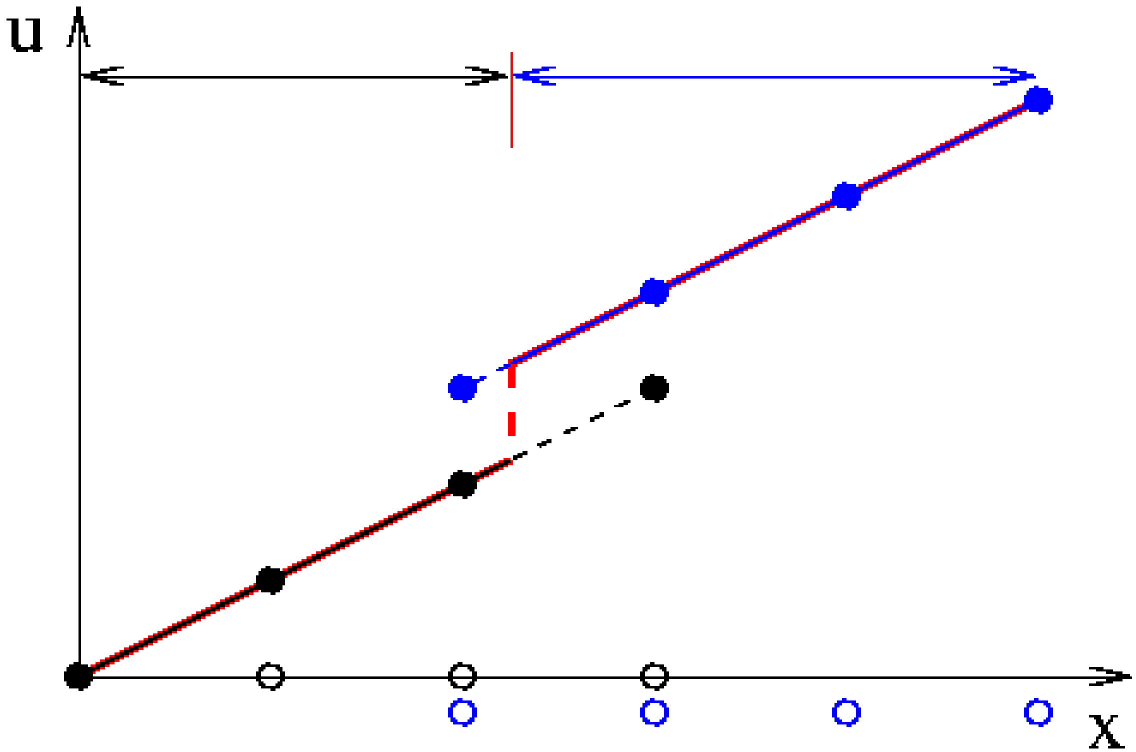
Embedded discontinuity (enhanced assumed strain)



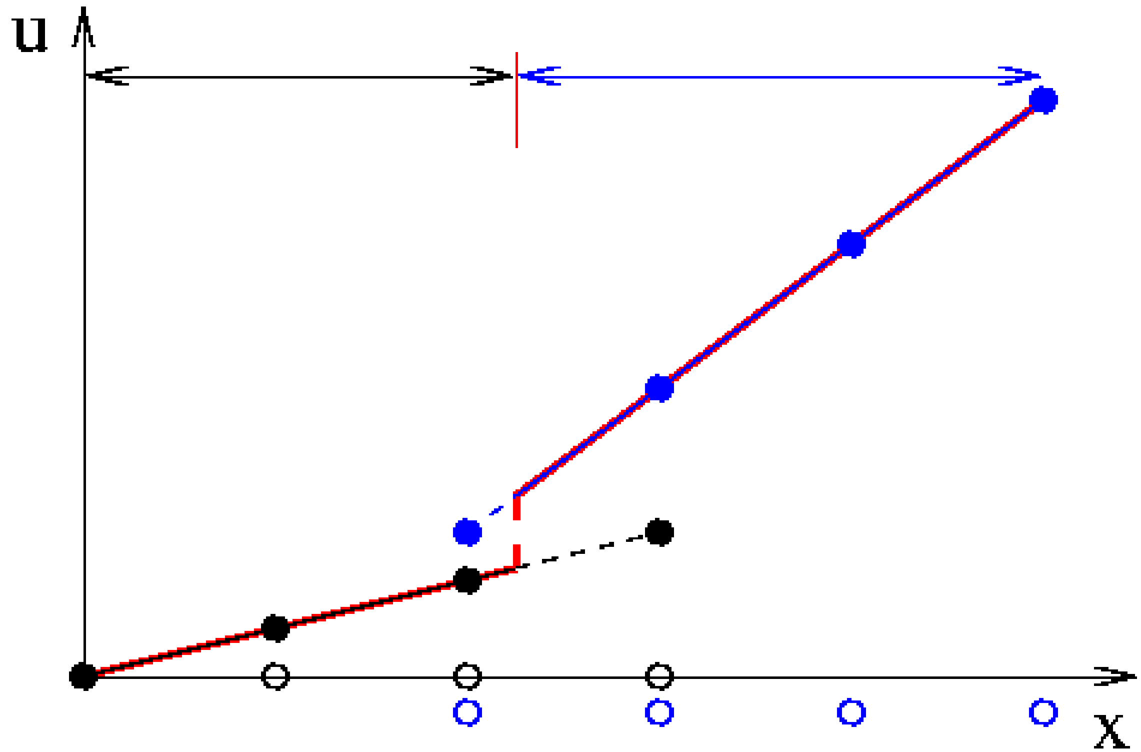
Embedded discontinuity (enhanced assumed strain)



Approximation on two overlapping meshes (XFEM)

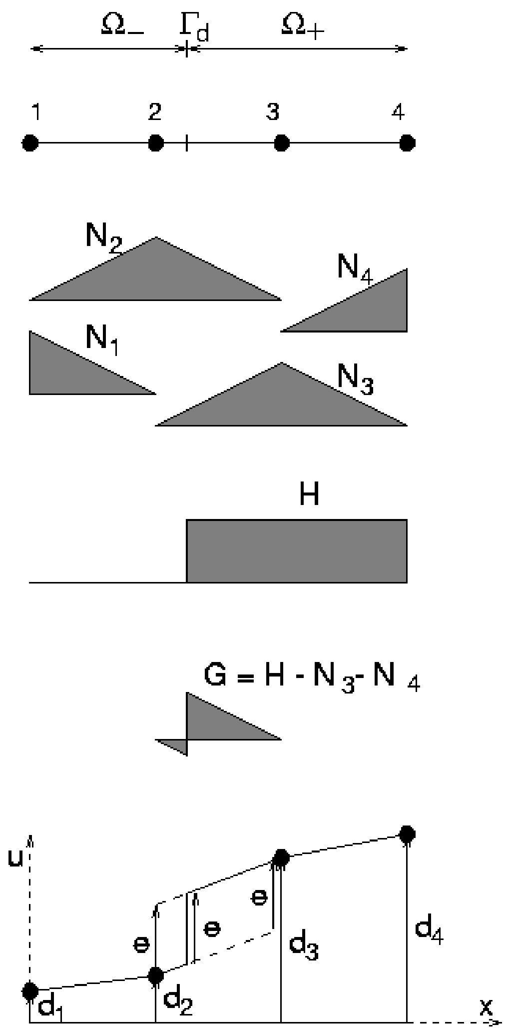


Approximation on two overlapping meshes (XFEM)



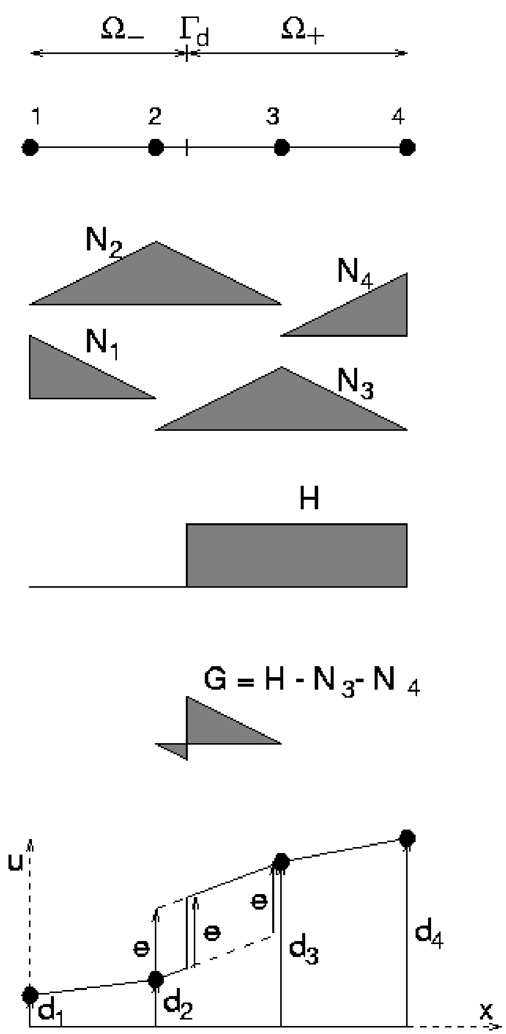
Enrichment of interpolation functions in one dimension

EED-EAS

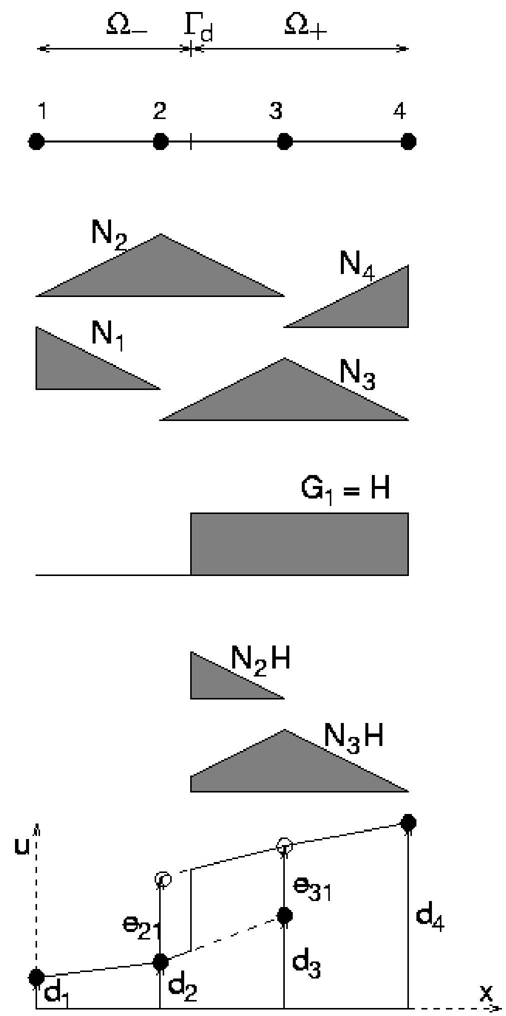


Enrichment of interpolation functions in one dimension

EED-EAS

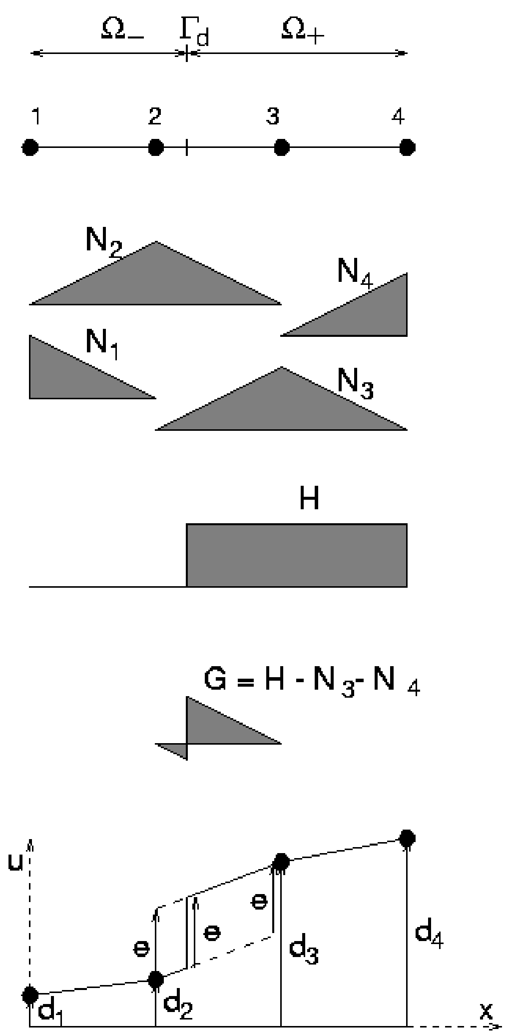


XFEM-PUM

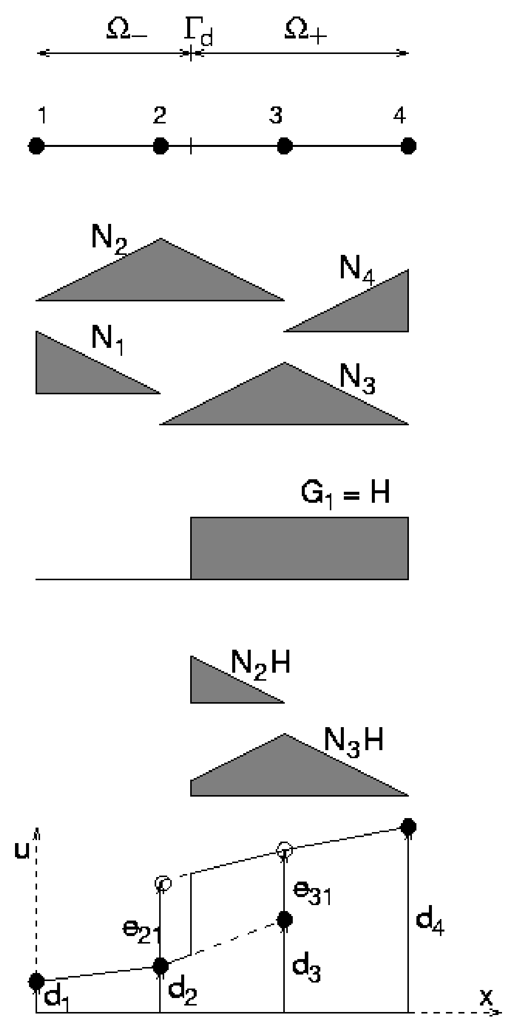


Enrichment of interpolation functions in one dimension

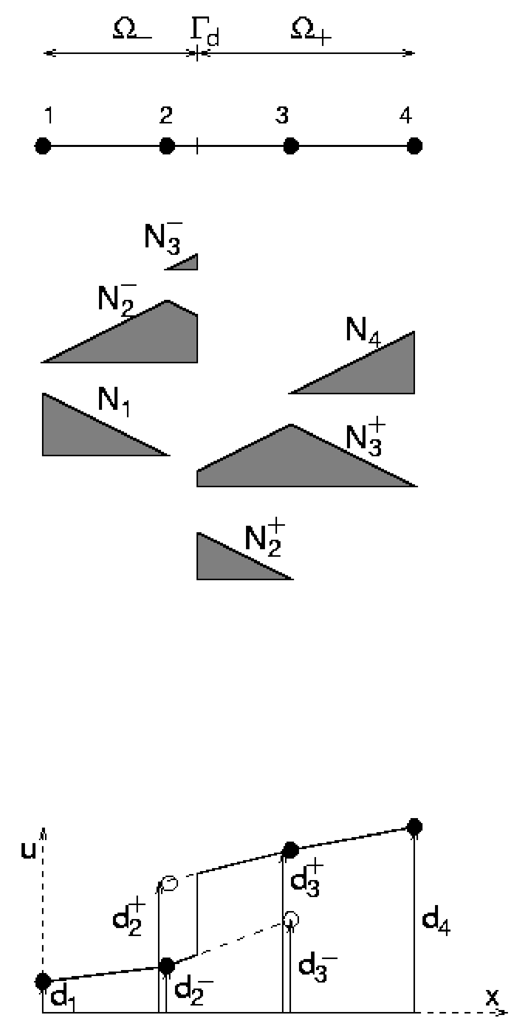
EED-EAS



XFEM-PUM



XFEM-PUM



F.2

Elements with Embedded Discontinuities (EAS)

Elements with embedded discontinuities

$$\mathbf{d}$$

$$\downarrow \quad \boldsymbol{\varepsilon} = \mathbf{B}\mathbf{d}$$

$$\boldsymbol{\varepsilon}$$

$$\downarrow \quad \boldsymbol{\sigma} = \tilde{\boldsymbol{\sigma}}(\boldsymbol{\varepsilon}, \ldots)$$

$$\boldsymbol{\sigma}$$

$$\downarrow \quad \mathbf{f}_{\text{int}} = \int_V \mathbf{B}^T \boldsymbol{\sigma} \, \mathrm{d}V$$

$$\mathbf{f}_{\text{int}}$$

Elements with embedded discontinuities

$\mathbf{d}$

$\boldsymbol{\varepsilon}$ $\mathbf{e}$ … new degrees of freedom
characterizing separation (displacement jump)

$\boldsymbol{\sigma}$ $\mathbf{t}$ … traction

$\mathbf{f}_{\text{int}}$

Elements with embedded discontinuities

$\mathbf{d}$

? kinematics ?

$\boldsymbol{\varepsilon}$ $\qquad$ $\mathbf{e}$

↓ material ↓

$\boldsymbol{\sigma}$ $\qquad$ $\mathbf{t}$

? equilibrium ?

$\mathbf{f}_{\text{int}}$

Elements with embedded discontinuities

$\mathbf{d}$

kinematics

$\boldsymbol{\varepsilon}$ $\mathbf{e}$

↓ material ↓

$\boldsymbol{\sigma}$ $\mathbf{t}$

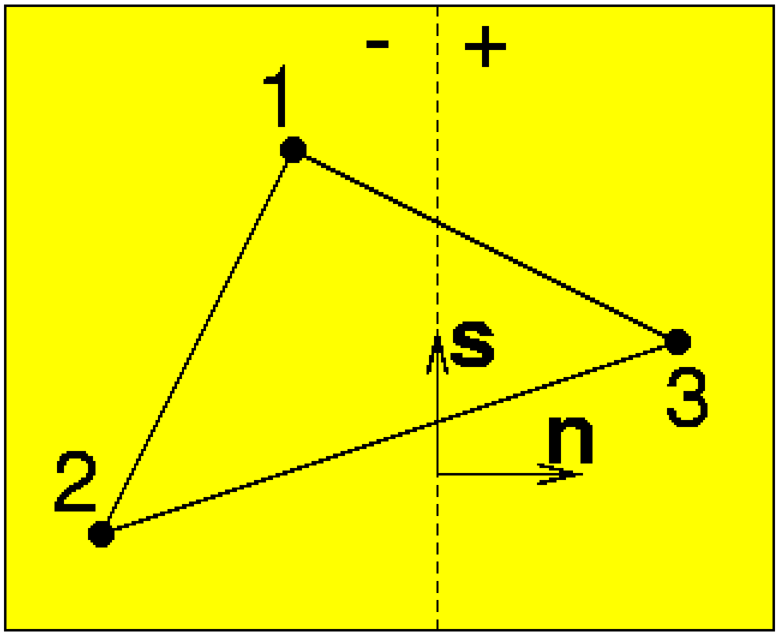
equilibrium

$\mathbf{f}_{\text{int}}$

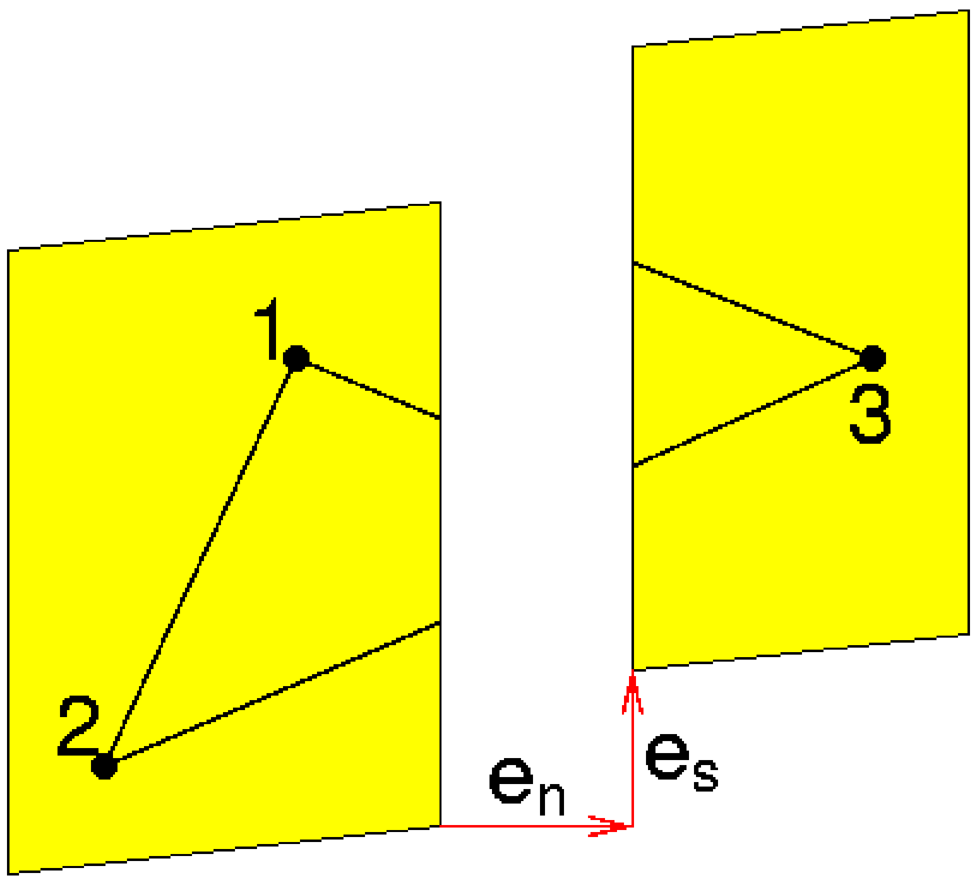
Three types of formulations:

- KOS … kinematically optimal symmetric
- SOS … statically optimal symmetric
- **SKON … kinematically and statically optimal nonsymmetric**

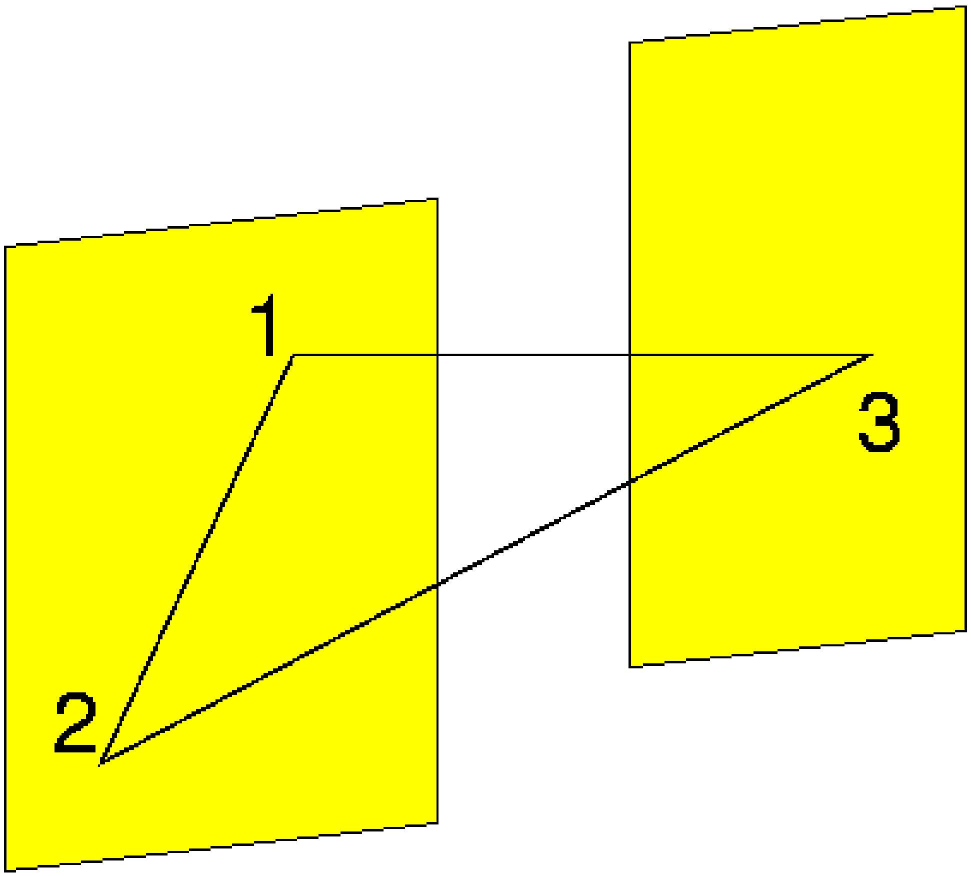
Elements with embedded discontinuities



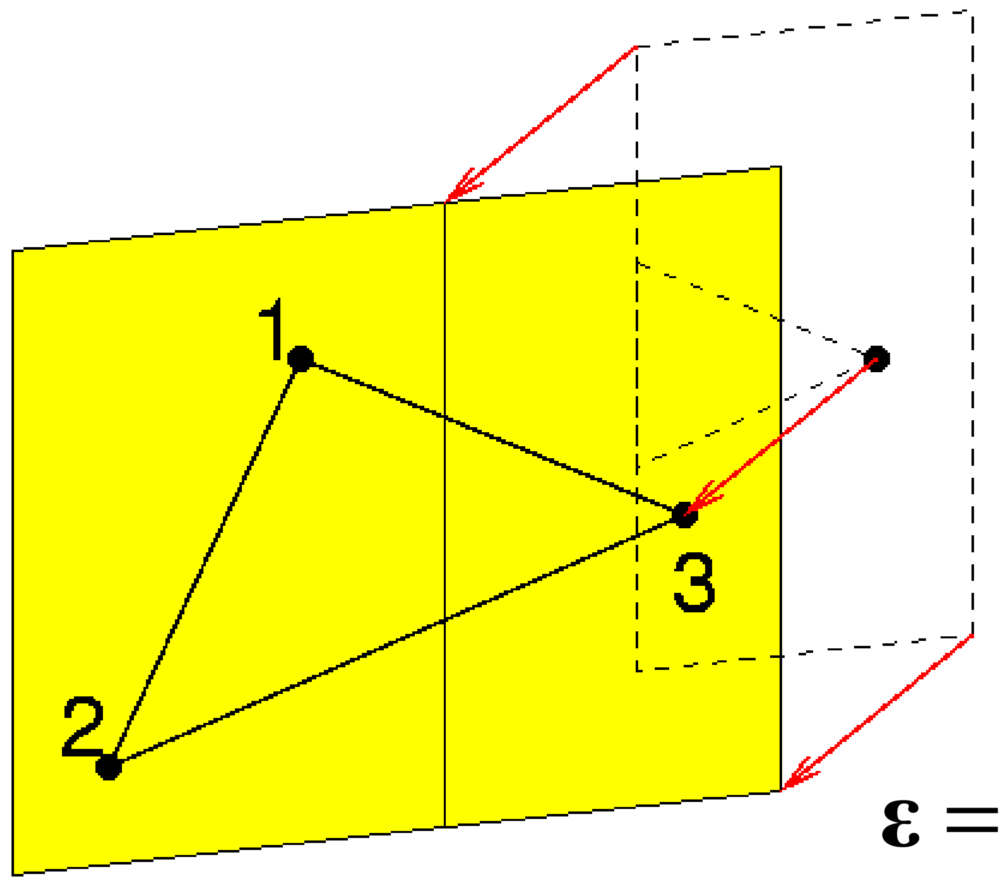
Elements with embedded discontinuities



Elements with embedded discontinuities



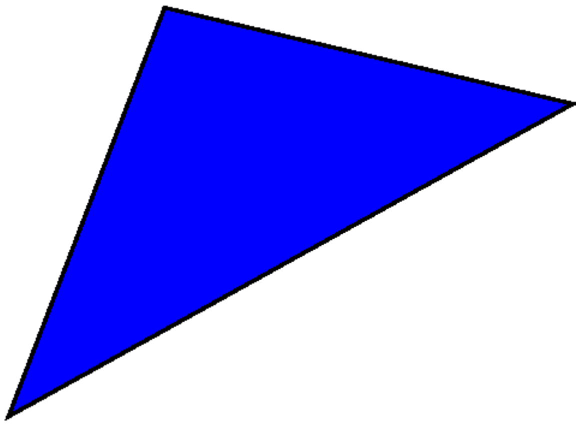
Elements with embedded discontinuities



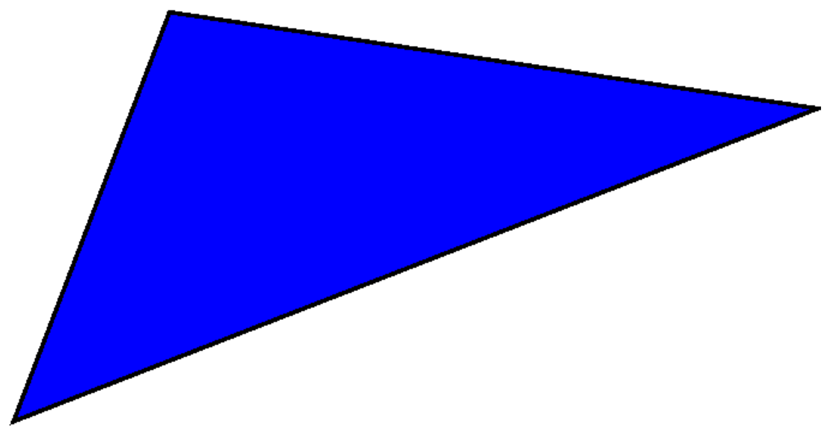
$$\boldsymbol{\varepsilon} = \mathbf{B}\left(\mathbf{d} - \mathbf{H}\mathbf{e}\right)$$

$$\mathbf{t} = \mathbf{P}^T \boldsymbol{\sigma}$$

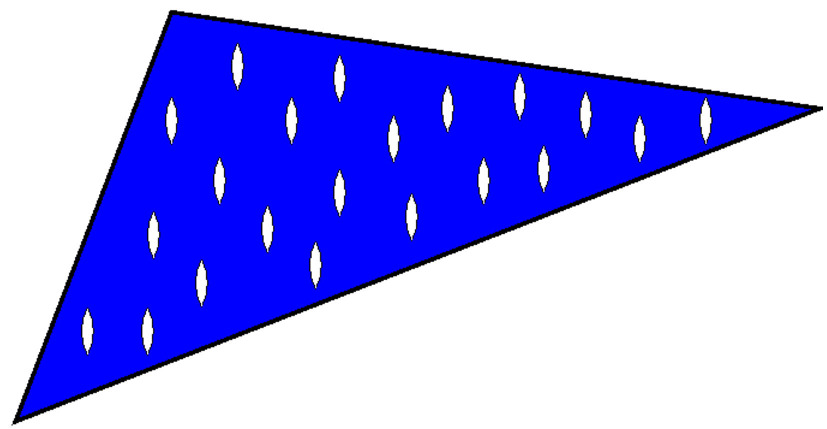
Smeared crack



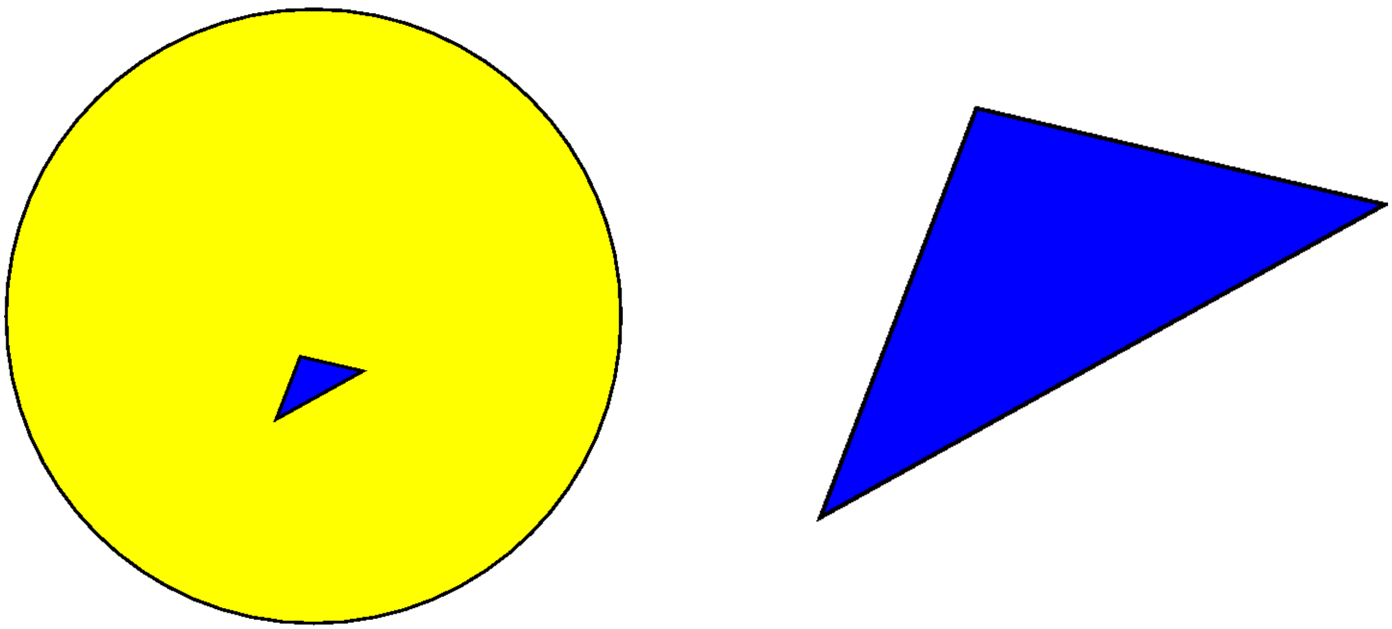
Smeared crack



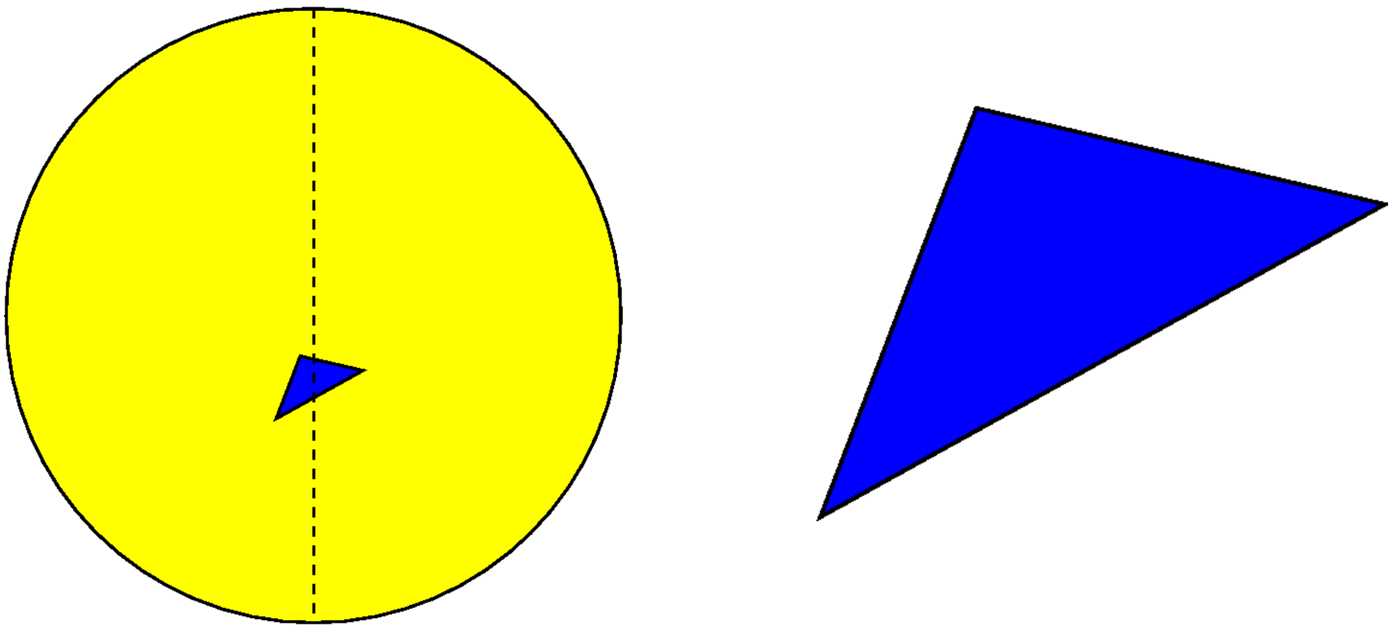
Smeared crack



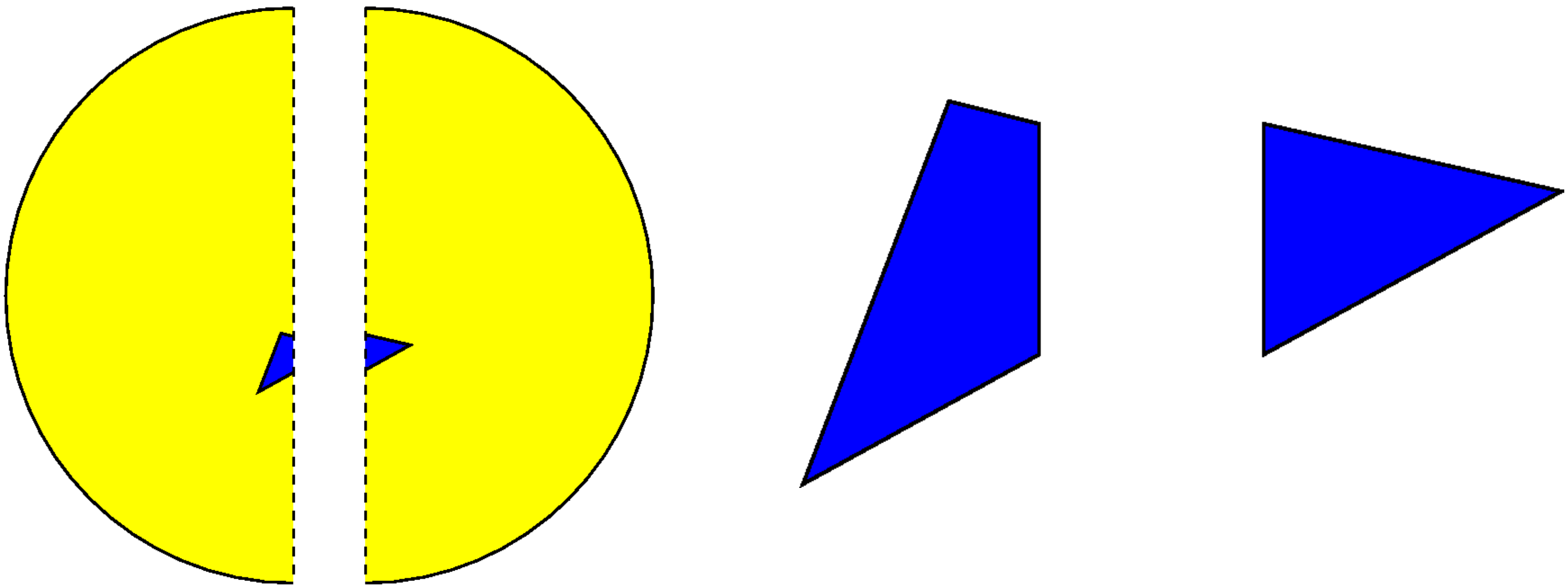
Smeared crack



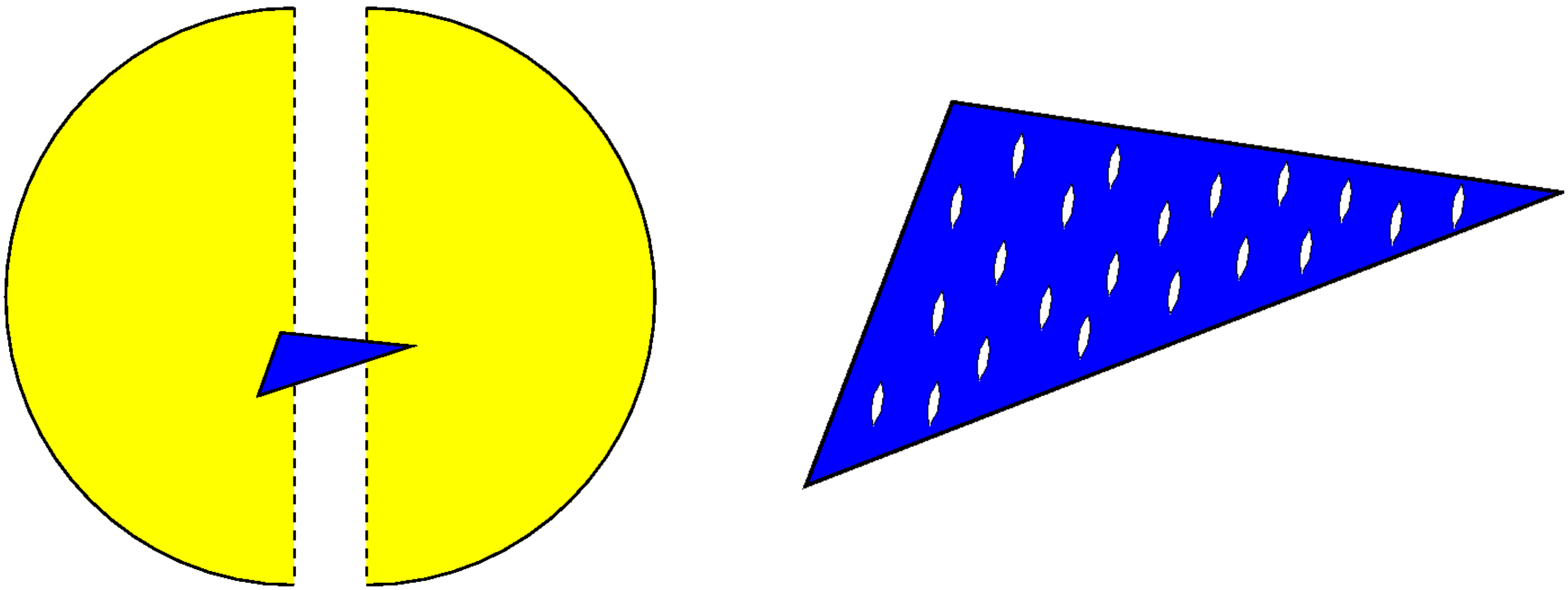
Smeared crack



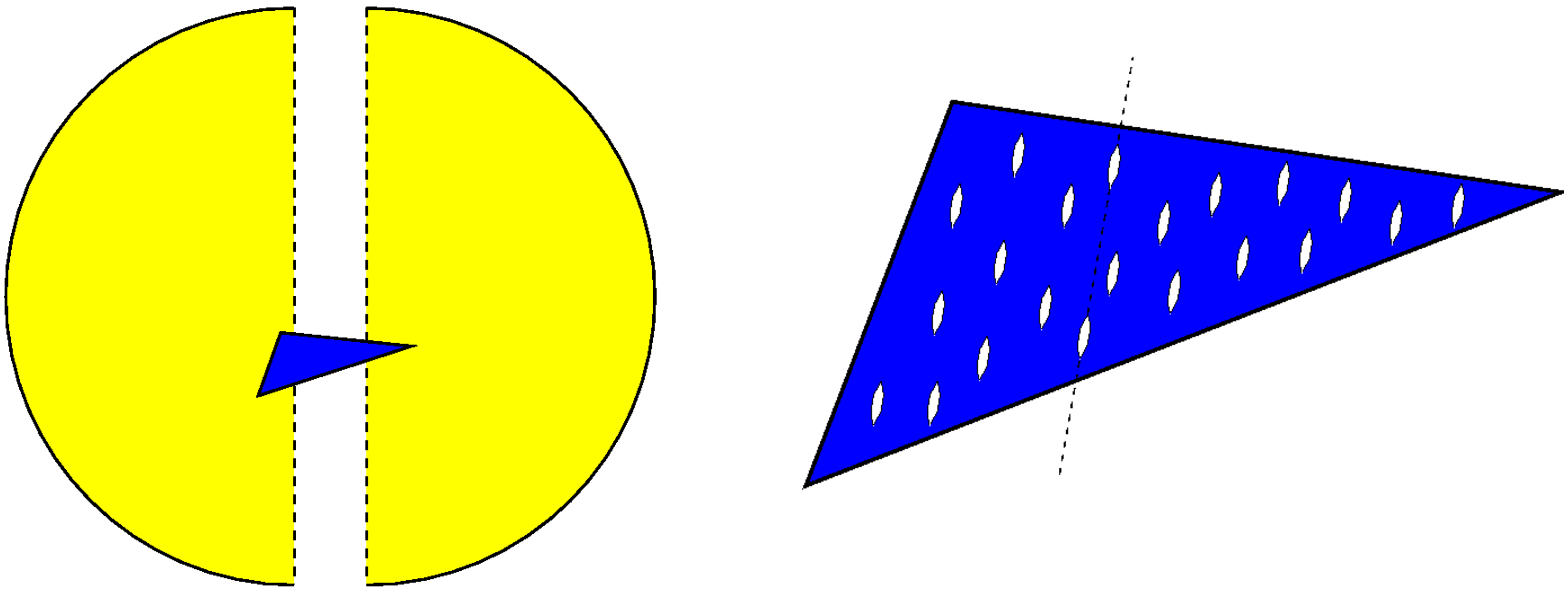
Smeared crack



Smeared crack

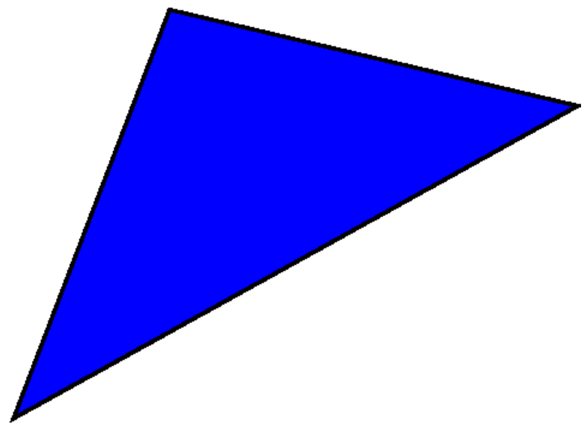


Smeared crack

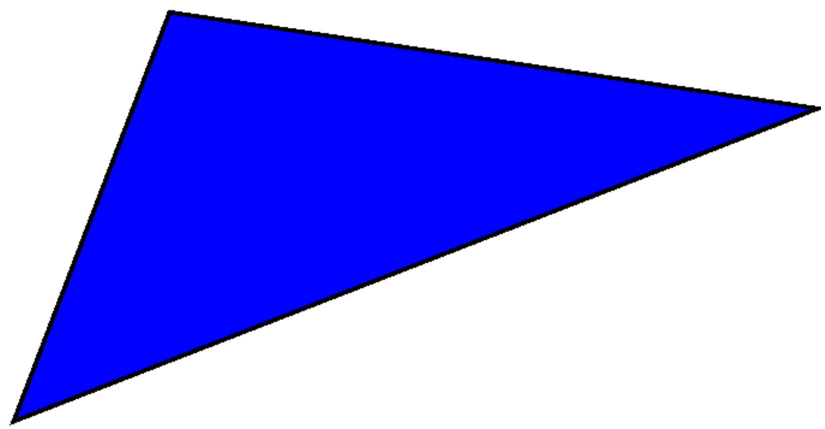


- Misalignment between crack and element
- Distorted principal directions
- Stress locking

Embedded crack (EAS approach)



Embedded crack (EAS approach)



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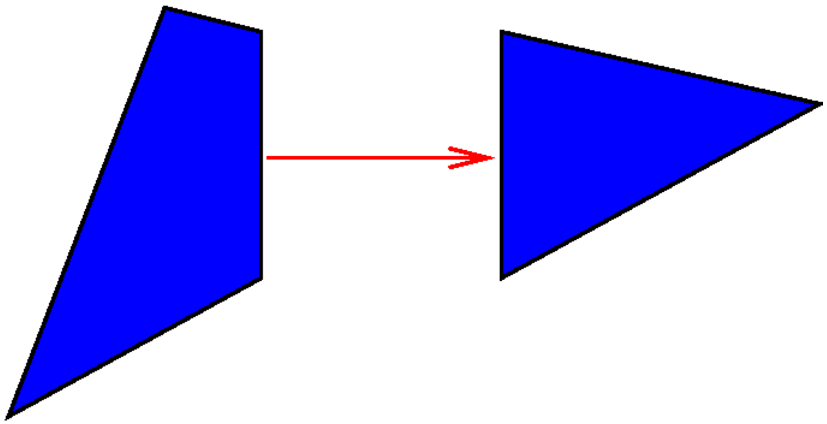
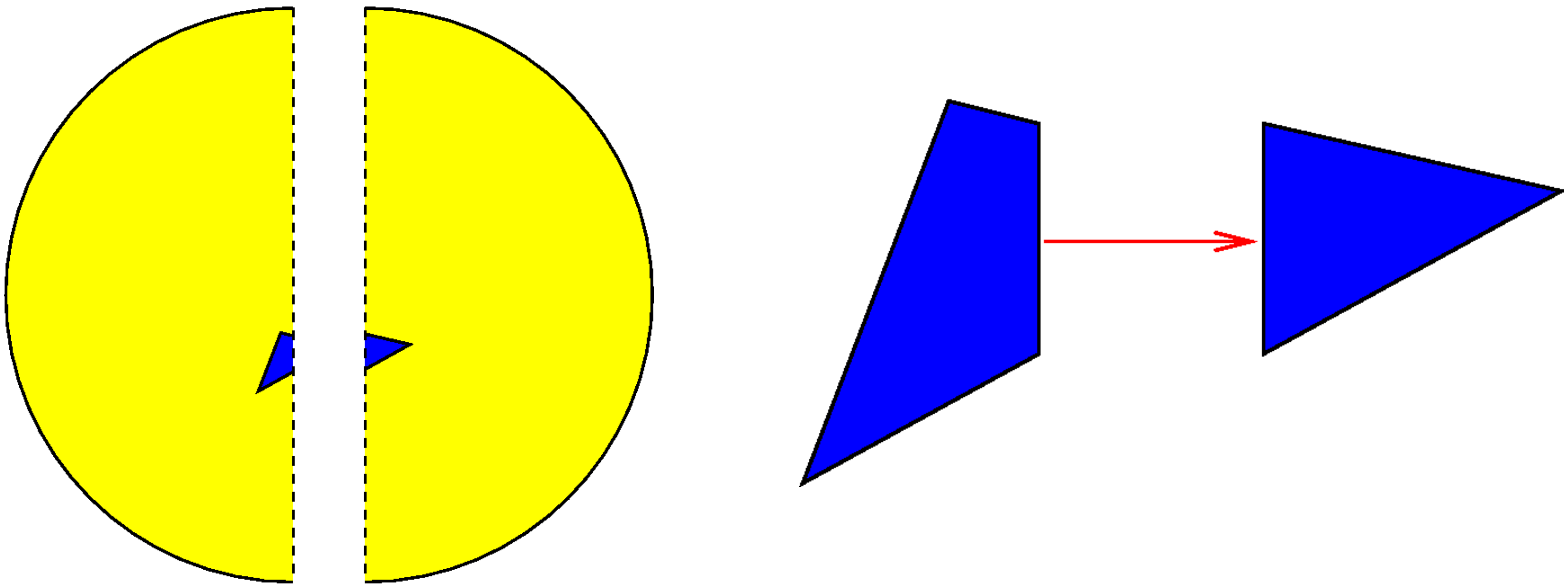
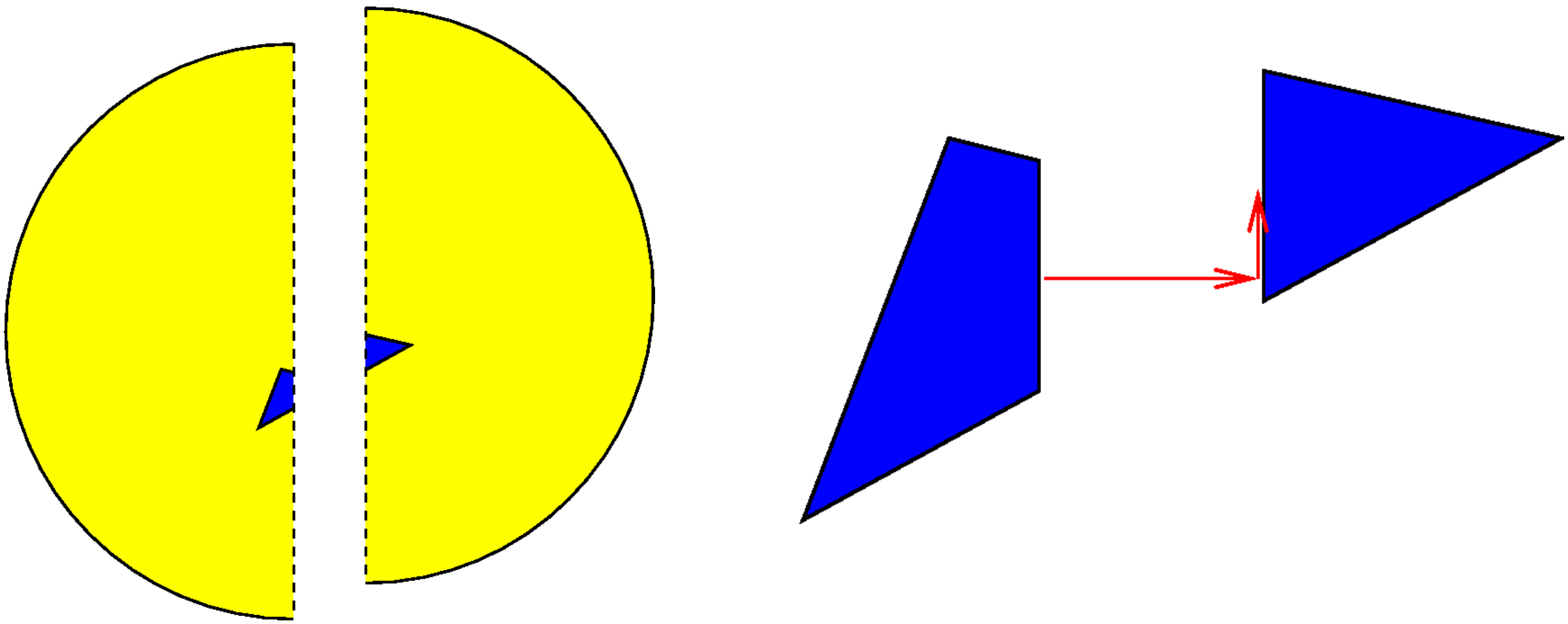


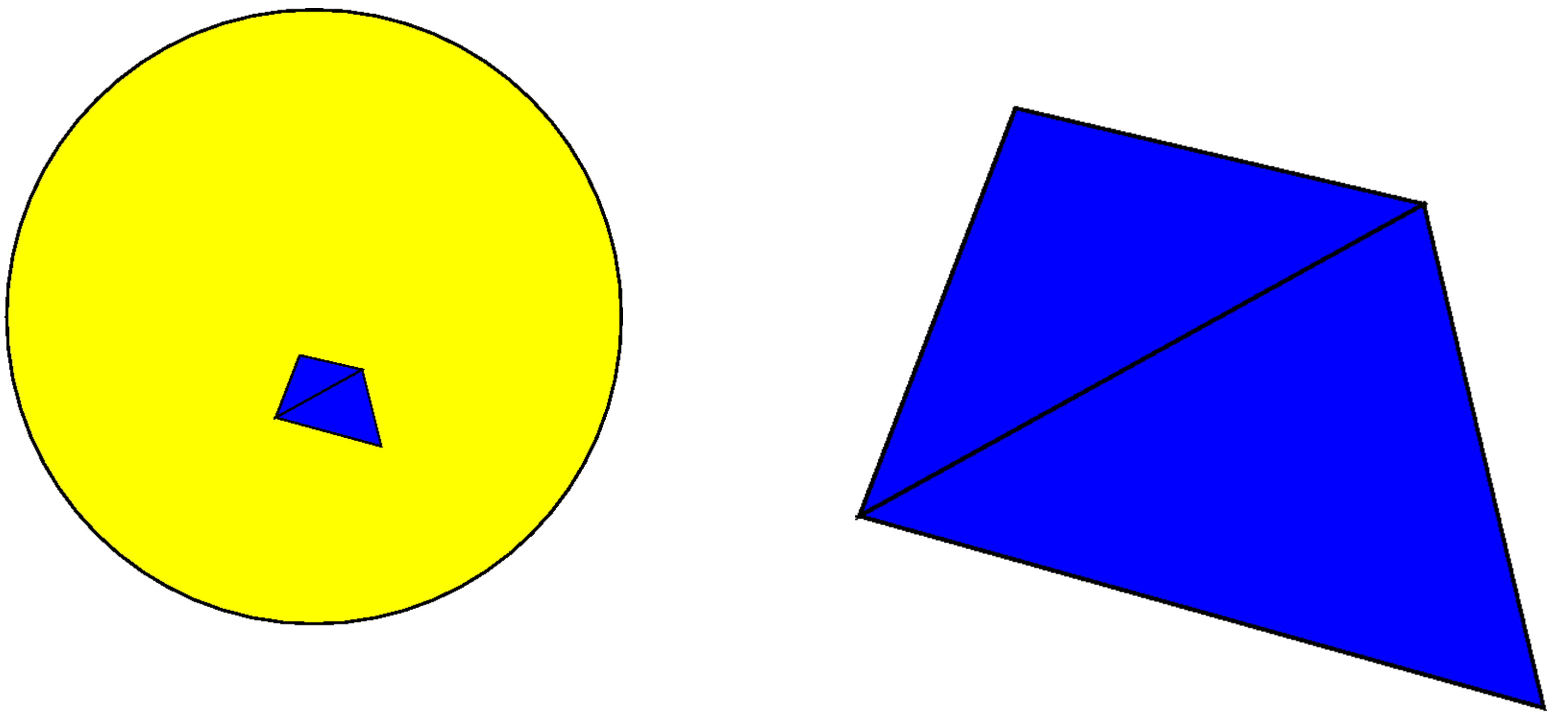
Embedded crack (EAS approach)



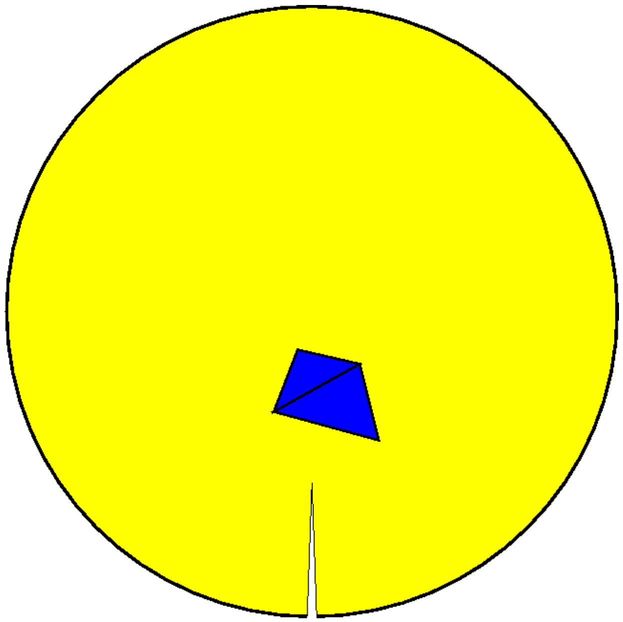
Embedded crack (EAS approach)

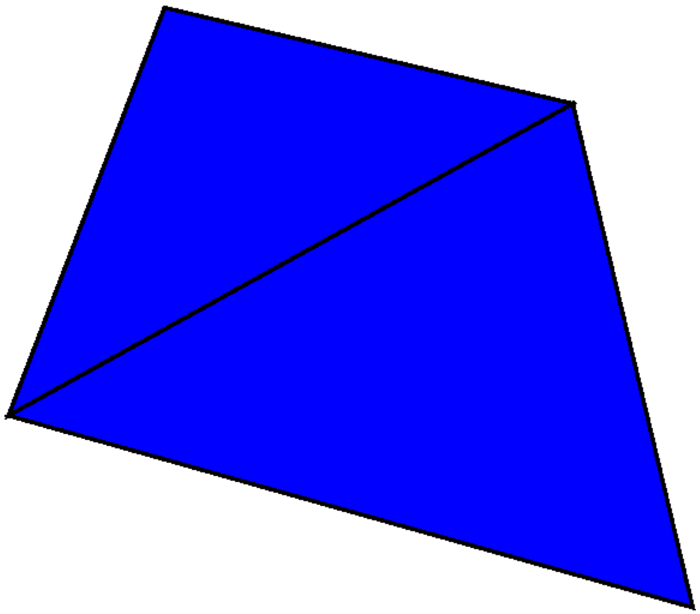


EED-EAS approach: discontinuous interpolation



EED- EAS approach: discontinuous interpolation





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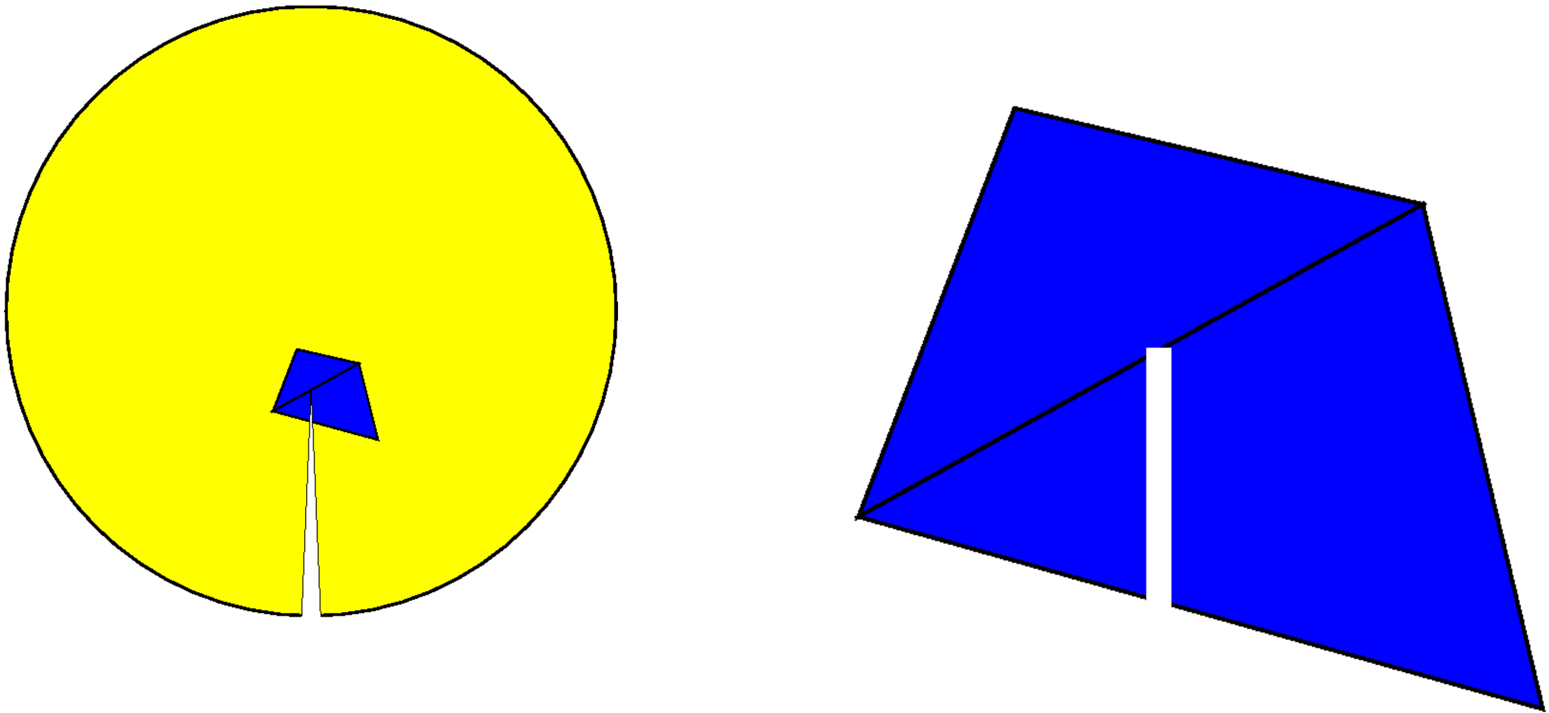
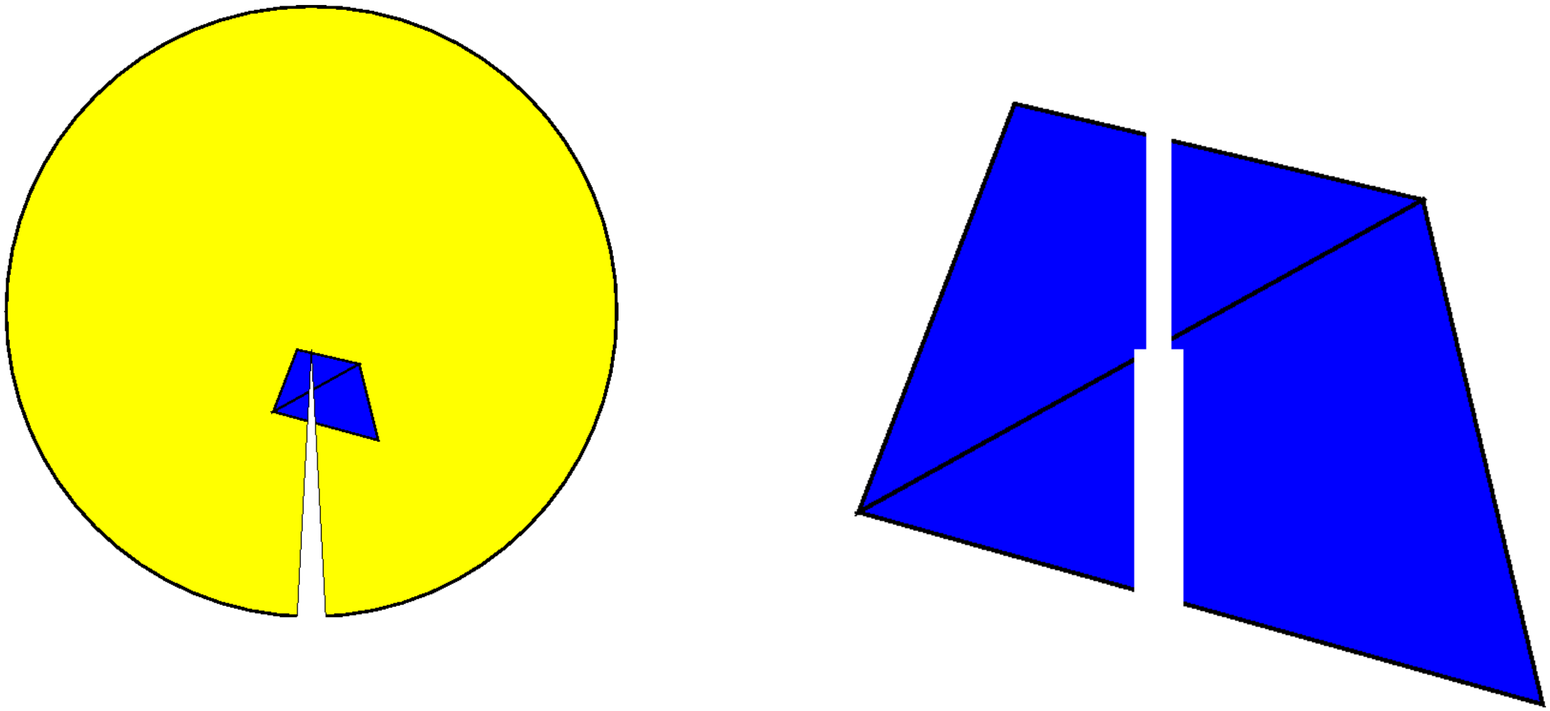


EED- EAS approach: discontinuous interpolation



F.3

Extended Finite Elements (XFEM)

Based on Partition of Unity

Partition of Unity Method

Standard finite element approximation:

$$\mathbf{u}(\mathbf{x}) = \sum_{I=1}^{Nnod} N_I(\mathbf{x})\,\mathbf{d}_I$$

The shape functions are a partition of unity:

$$\sum_{I=1}^{Nnod} N_I(\mathbf{x}) = 1$$

Partition of Unity Method

Standard finite element approximation:

$$\mathbf{u}(\mathbf{x}) = \sum_{I=1}^{Nnod} N_I(\mathbf{x})\,\mathbf{d}_I$$

The shape functions are a partition of unity:

$$\sum_{I=1}^{Nnod} N_I(\mathbf{x}) = 1$$

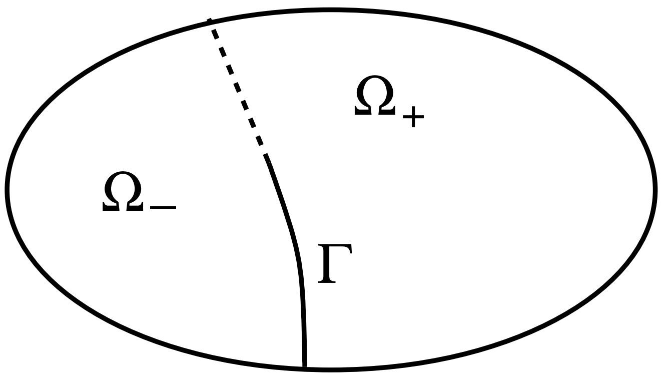
Enriched approximation:

$$\mathbf{u}(\mathbf{x}) = \sum_{I=1}^{Nnod} N_I(\mathbf{x})\left[\mathbf{d}_I + \sum_{i \in L_I} G_i(\mathbf{x})\mathbf{e}_{iI}\right]$$

selected enrichment functions (G_i)

Partition of Unity Method – eXtended Finite Elements

Enrichment by Heaviside function:



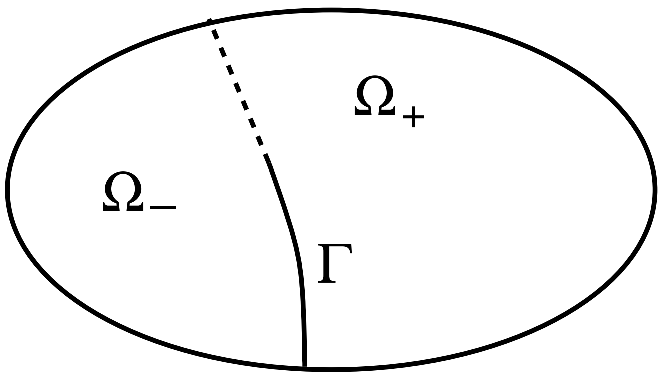
$$H_\Gamma(\mathbf{x}) = \begin{cases} 1 & \text{for } x \in \Omega^+ \\ 0 & \text{for } x \in \Omega^- \end{cases}$$

$$\mathbf{u}(\mathbf{x}) = \sum_{I=1}^{Nnod} N_I(\mathbf{x})\left[\mathbf{d}_I + H_\Gamma(\mathbf{x})\mathbf{e}_I\right] =$$

$$= \sum_{I=1}^{Nnod} N_I(\mathbf{x})\mathbf{d}_I + \sum_{I=1}^{Nnod} N_I(\mathbf{x}) H_\Gamma(\mathbf{x})\mathbf{e}_I$$

Partition of Unity Method – eXtended Finite Elements

If the support of N_I is contained in Ω^+, then $N_I H_\Gamma = N_I$



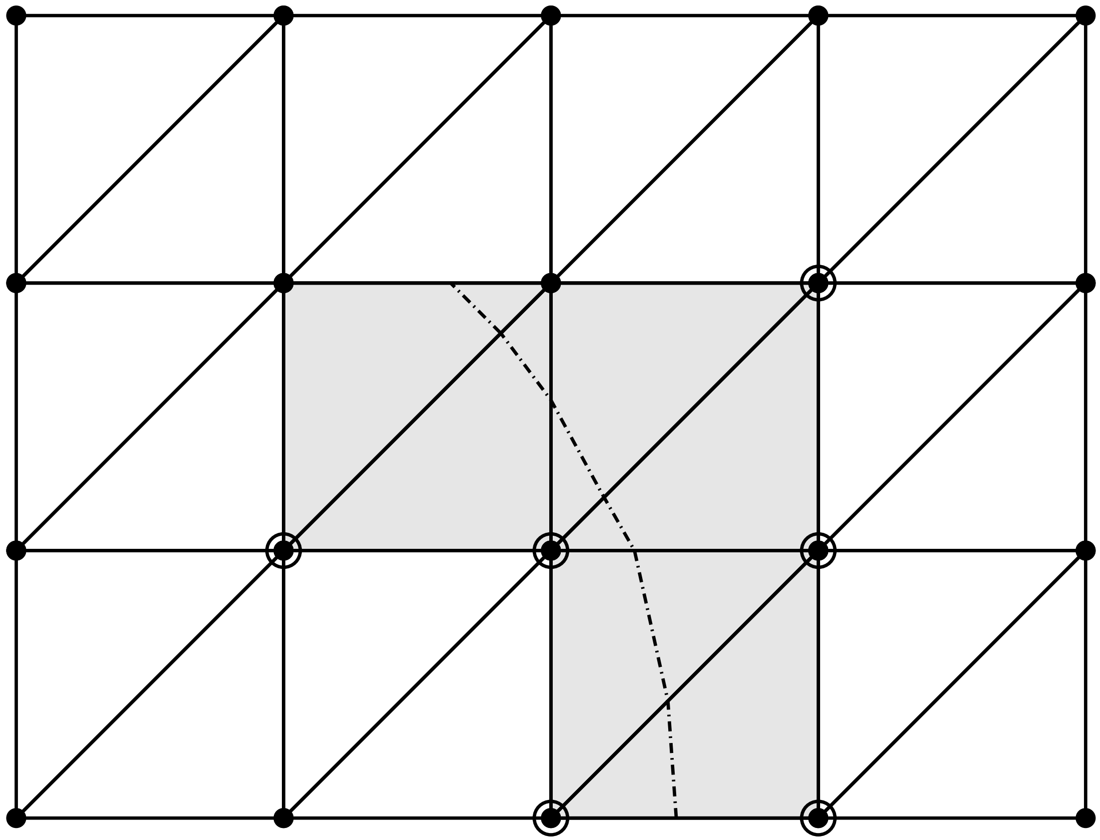
If the support of N_I is contained in Ω^-, then $N_I H_\Gamma = 0$

Only if the support of N_I is cut by Γ ,
then the function $N_I H_\Gamma$ really enriches the basis.

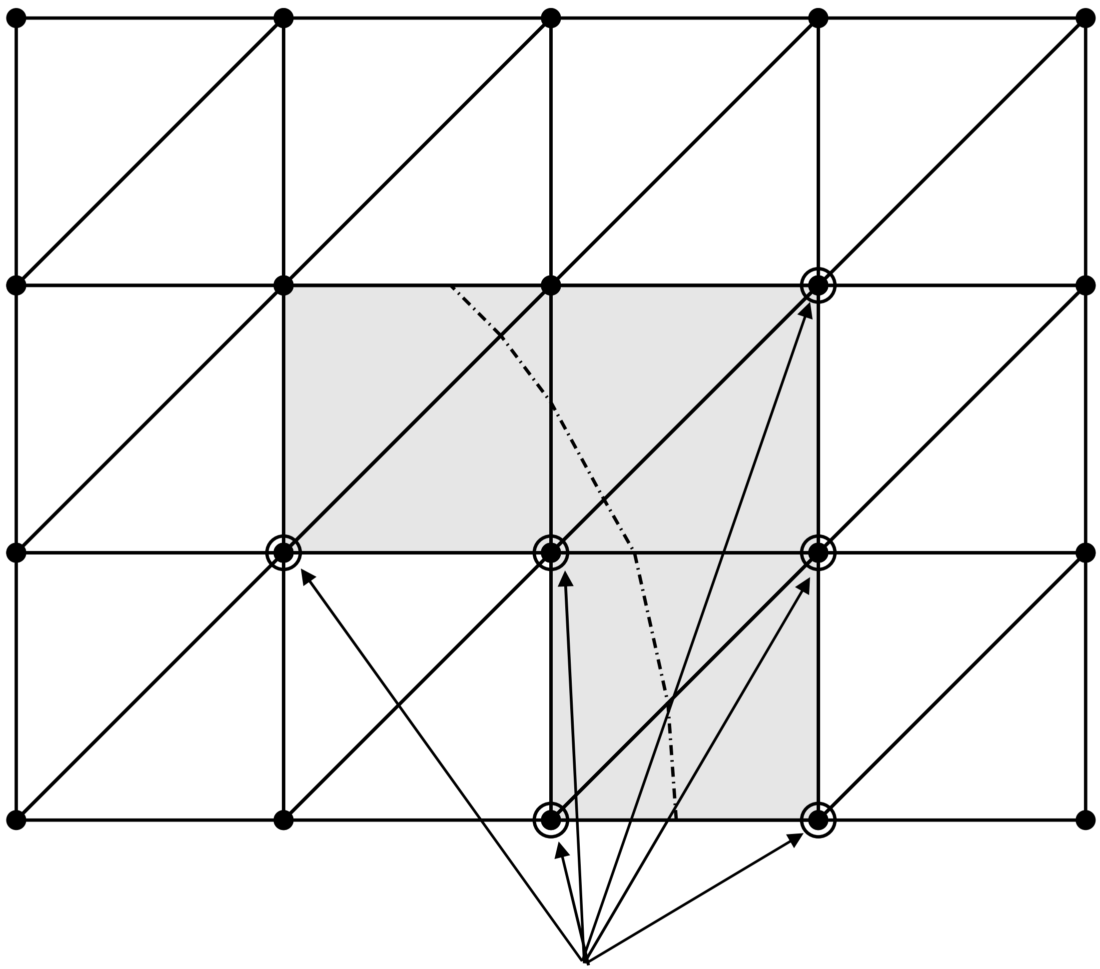
$$\mathbf{u}(\mathbf{x}) = \sum_{I=1}^{Nnod} N_I(\mathbf{x})\mathbf{d}_I + \sum_{I \in S_H} N_I(\mathbf{x}) H_\Gamma(\mathbf{x})\mathbf{e}_I$$

set of nodes with Heaviside enrichment

Partition of Unity Method – eXtended Finite Elements



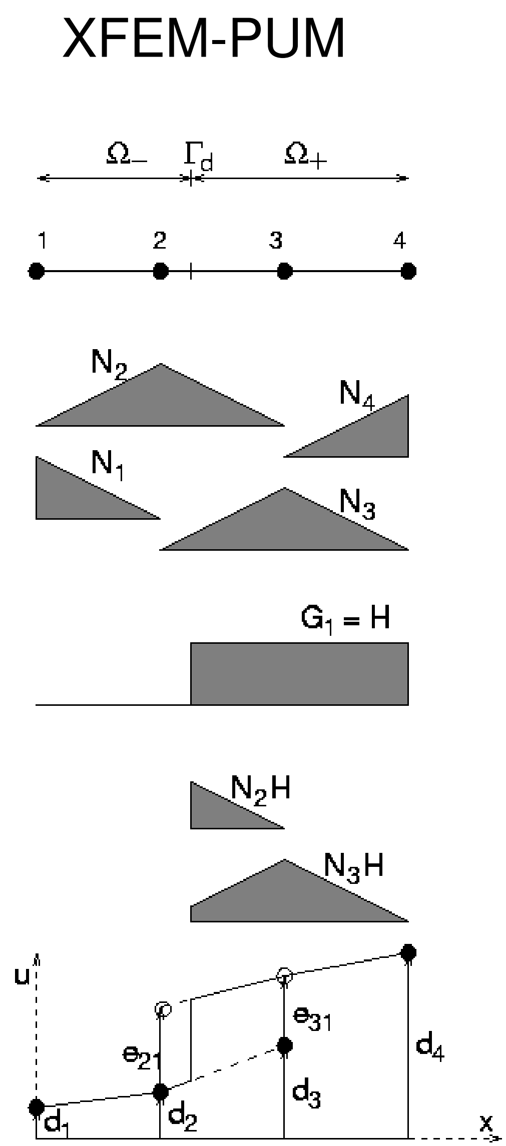
Partition of Unity Method – eXtended Finite Elements

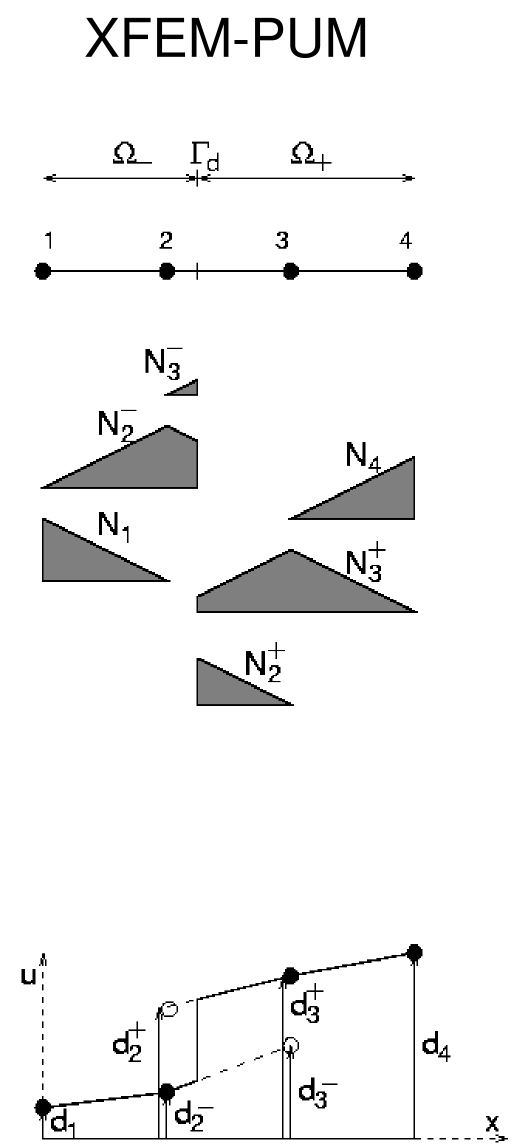


nodes with Heaviside enrichment

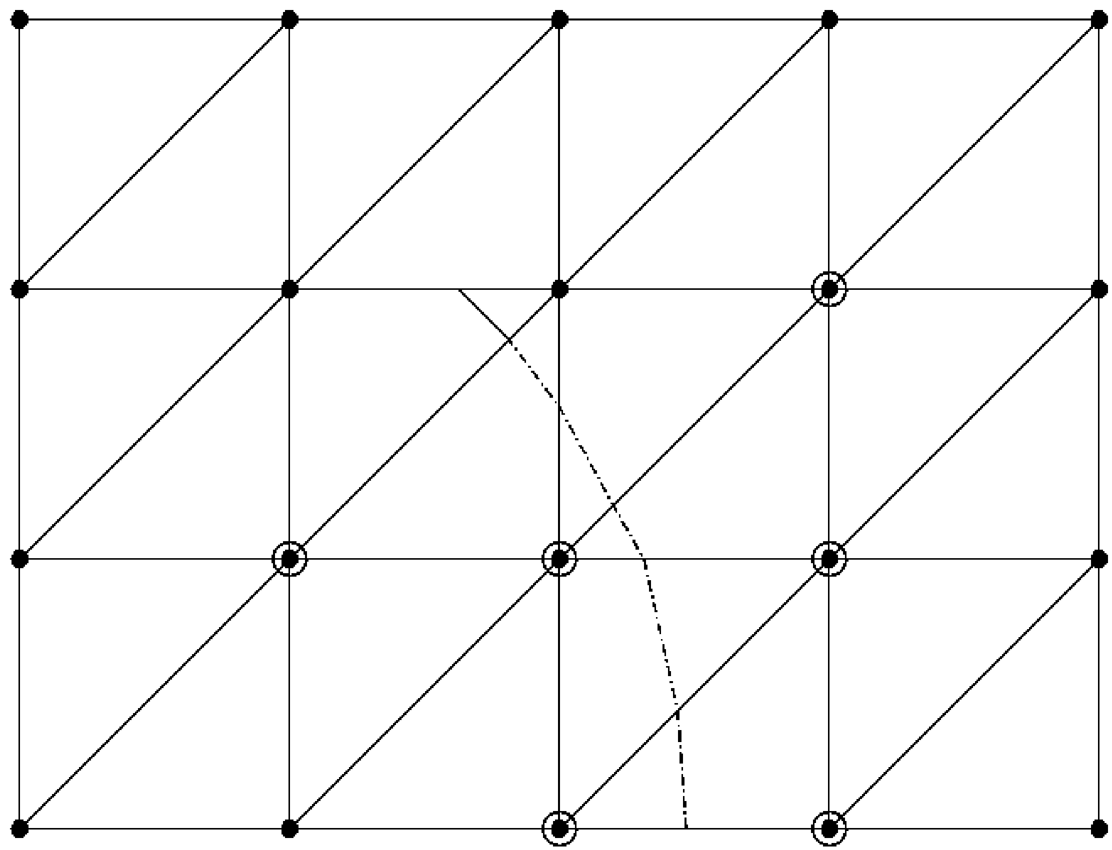
Partition of Unity Method – eXtended Finite Elements

The enriched approximation can be rearranged
to give better physical meaning to the degrees of freedom:

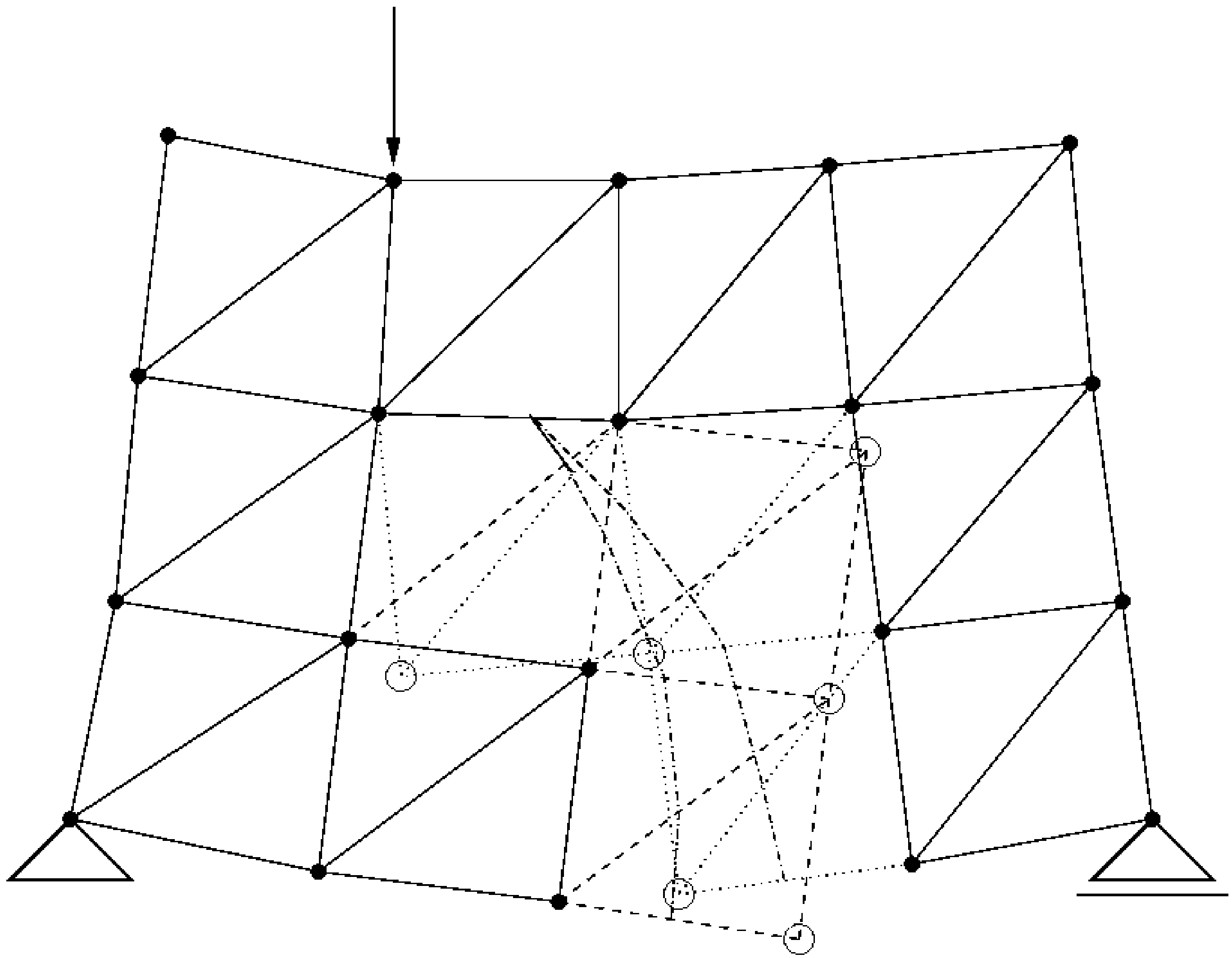




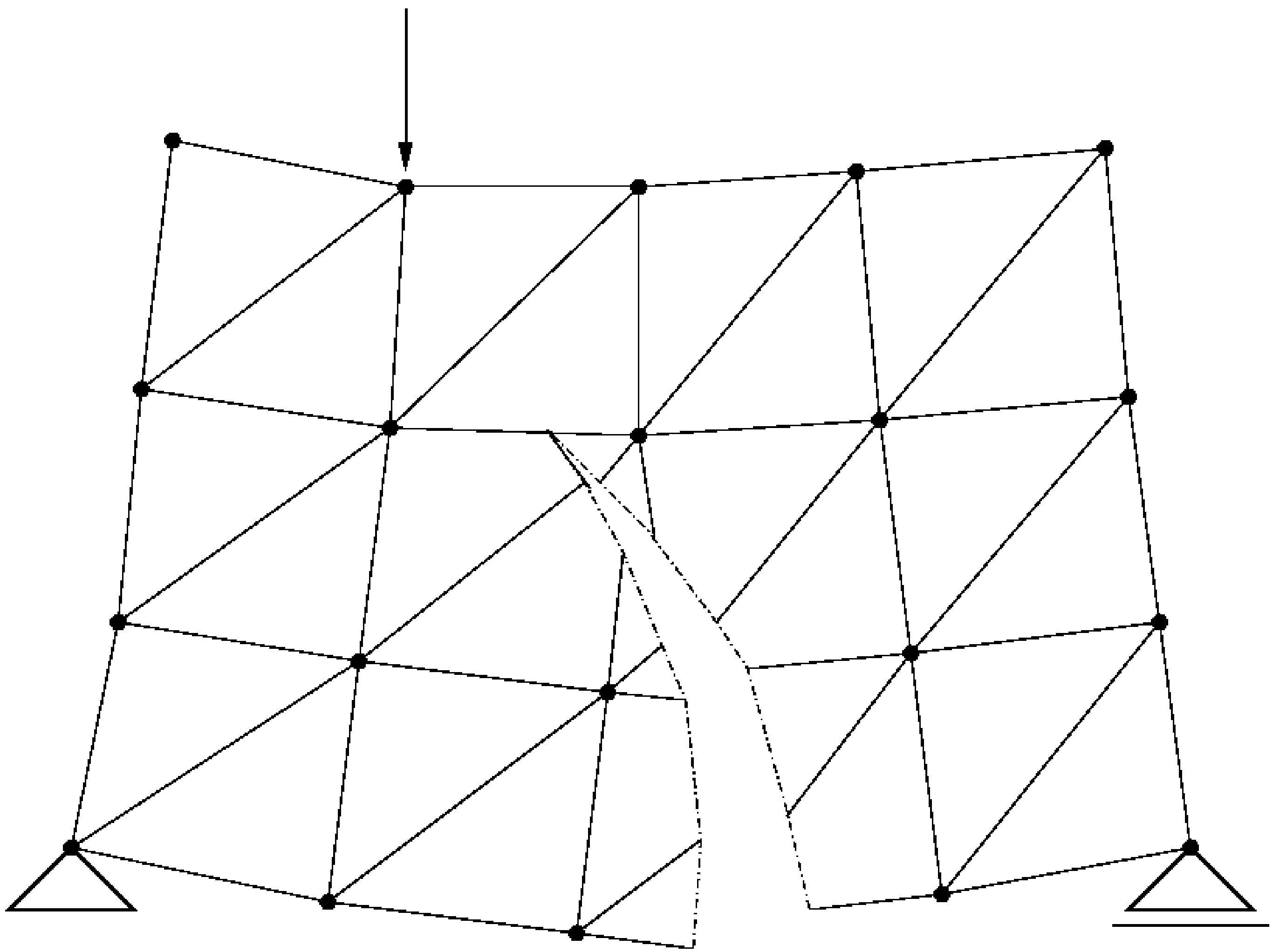
XFEM – enrichment by step function



XFEM – enrichment by step function

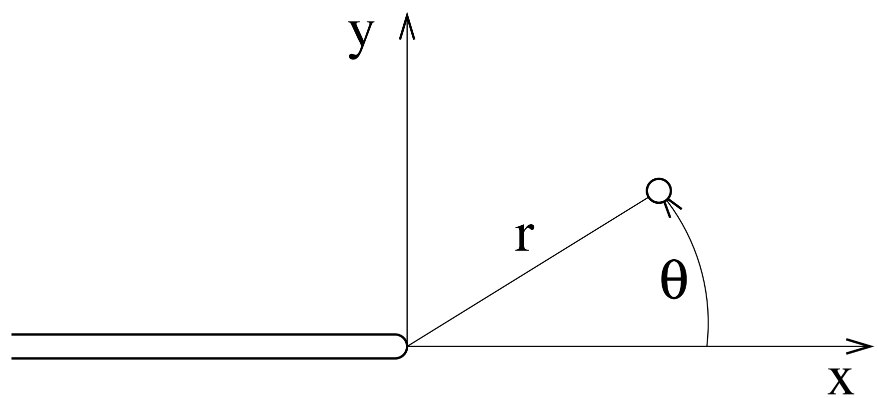


XFEM – enrichment by step function



XFEM – tip enrichment

Additional enrichment improving the approximation around the crack tip:



Functions that appear in the analytical near-tip solution:

$$B_1(r,\theta) = \sqrt{r}\sin\frac{\theta}{2} \qquad B_3(r,\theta) = \sqrt{r}\sin\frac{\theta}{2}\sin\theta$$

$$B_2(r,\theta) = \sqrt{r}\cos\frac{\theta}{2} \qquad B_4(r,\theta) = \sqrt{r}\cos\frac{\theta}{2}\sin\theta$$

XFEM – tip enrichment

Additional enrichment improving the approximation around the crack tip:

$$\mathbf{u}(\mathbf{x}) = \sum_{I=1}^{Nnod} N_I(\mathbf{x})\mathbf{d}_I + \sum_{I\in S_H} N_I(\mathbf{x}) H_\Gamma(\mathbf{x})\mathbf{e}_{0I} +$$

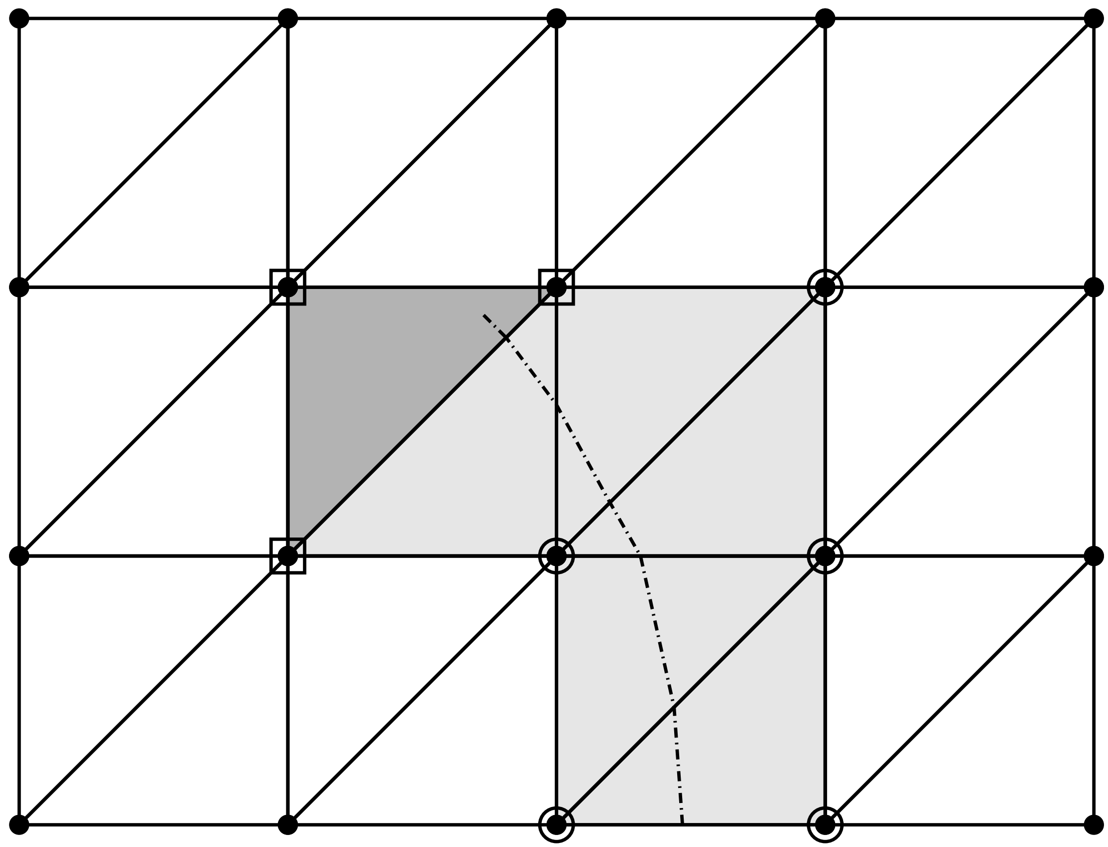
$$+ \sum_{I\in S_B} \sum_{i=1}^{4} N_I(\mathbf{x}) B_i(r(\mathbf{x}), \theta(\mathbf{x}))\mathbf{e}_{iI}$$

Functions that appear in the analytical near-tip solution:

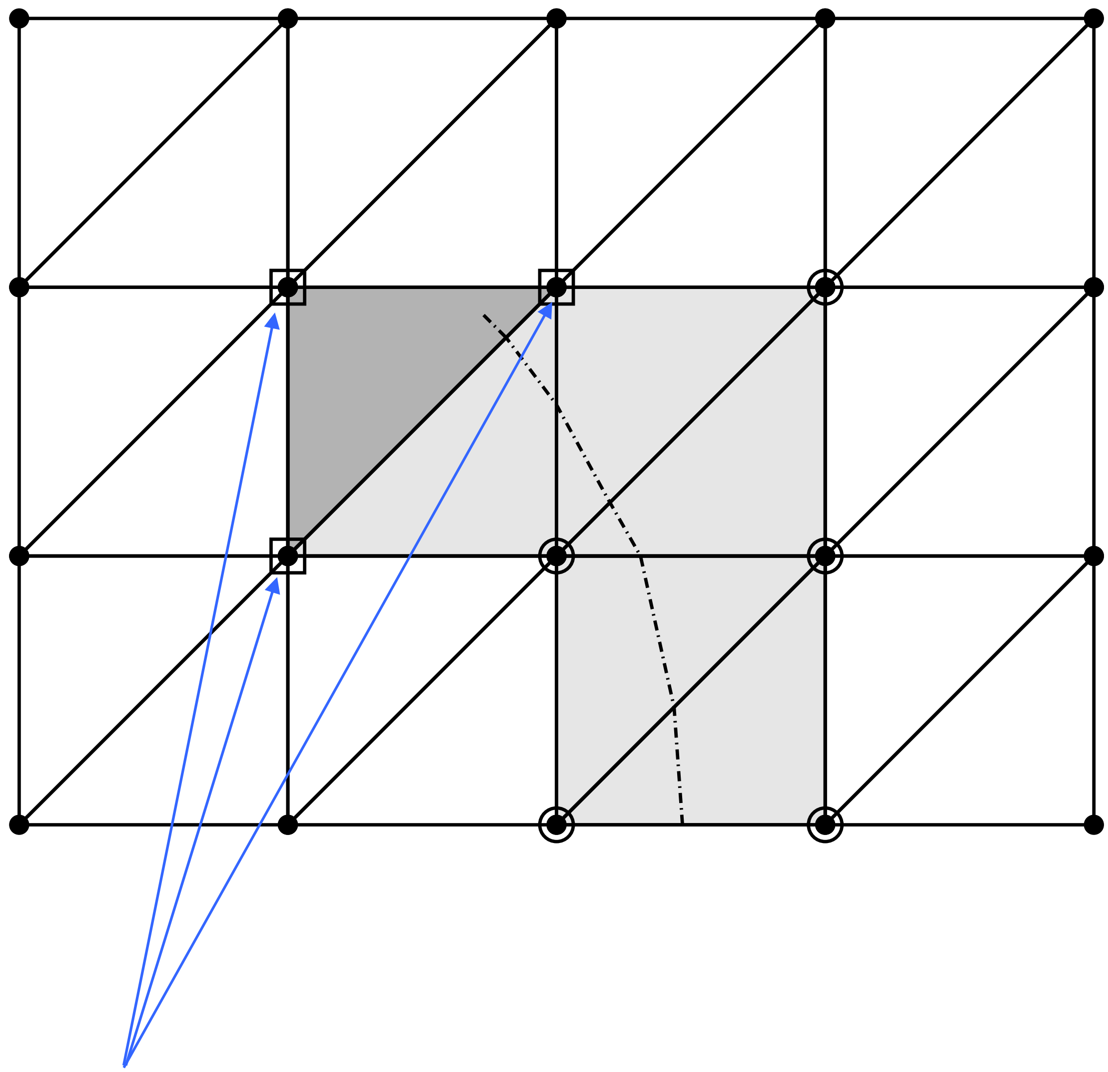
$$B_1(r,\theta) = \sqrt{r}\sin\frac{\theta}{2} \qquad B_3(r,\theta) = \sqrt{r}\sin\frac{\theta}{2}\sin\theta$$

$$B_2(r,\theta) = \sqrt{r}\cos\frac{\theta}{2} \qquad B_4(r,\theta) = \sqrt{r}\cos\frac{\theta}{2}\sin\theta$$

XFEM – tip enrichment

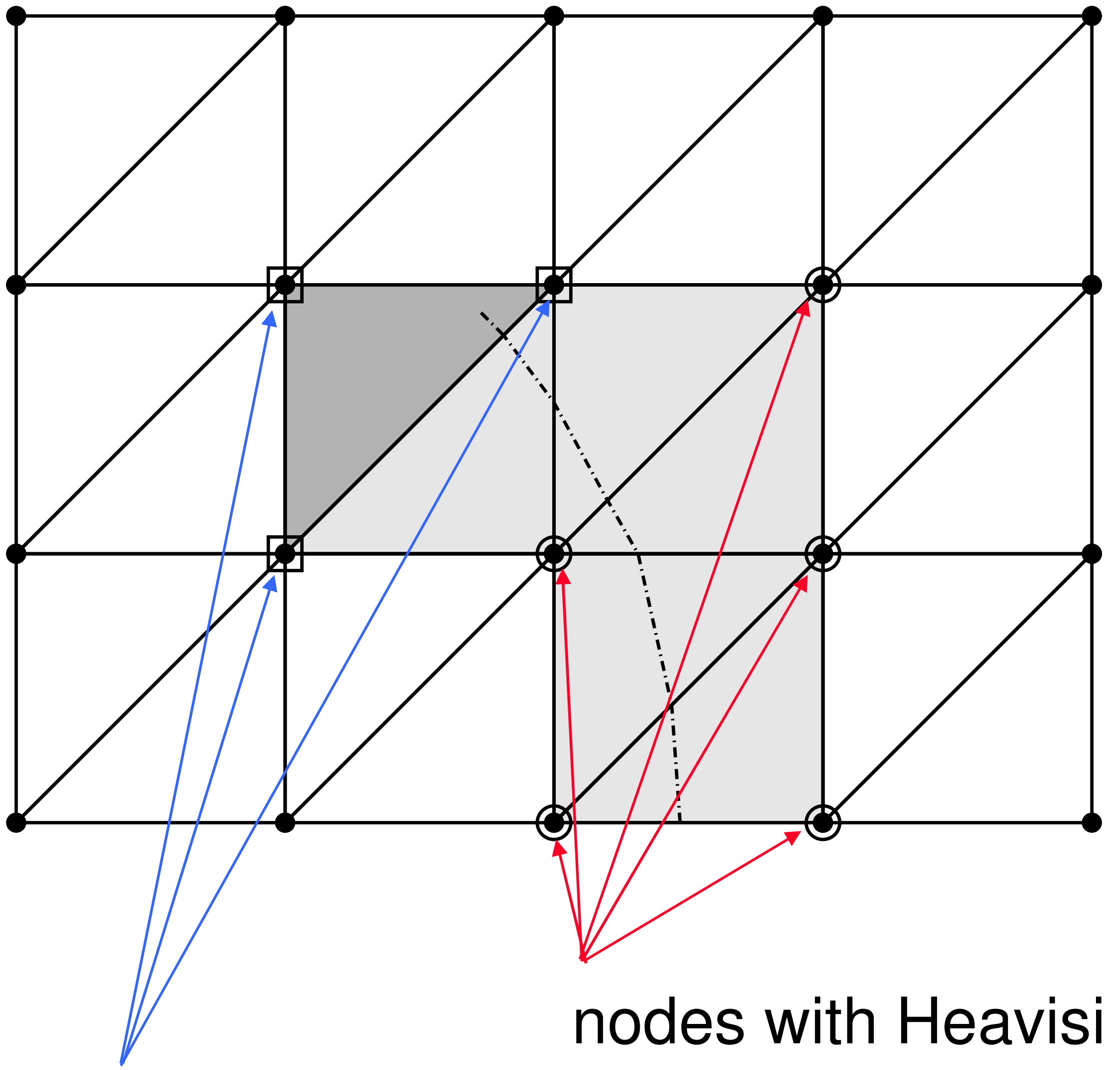


XFEM – tip enrichment

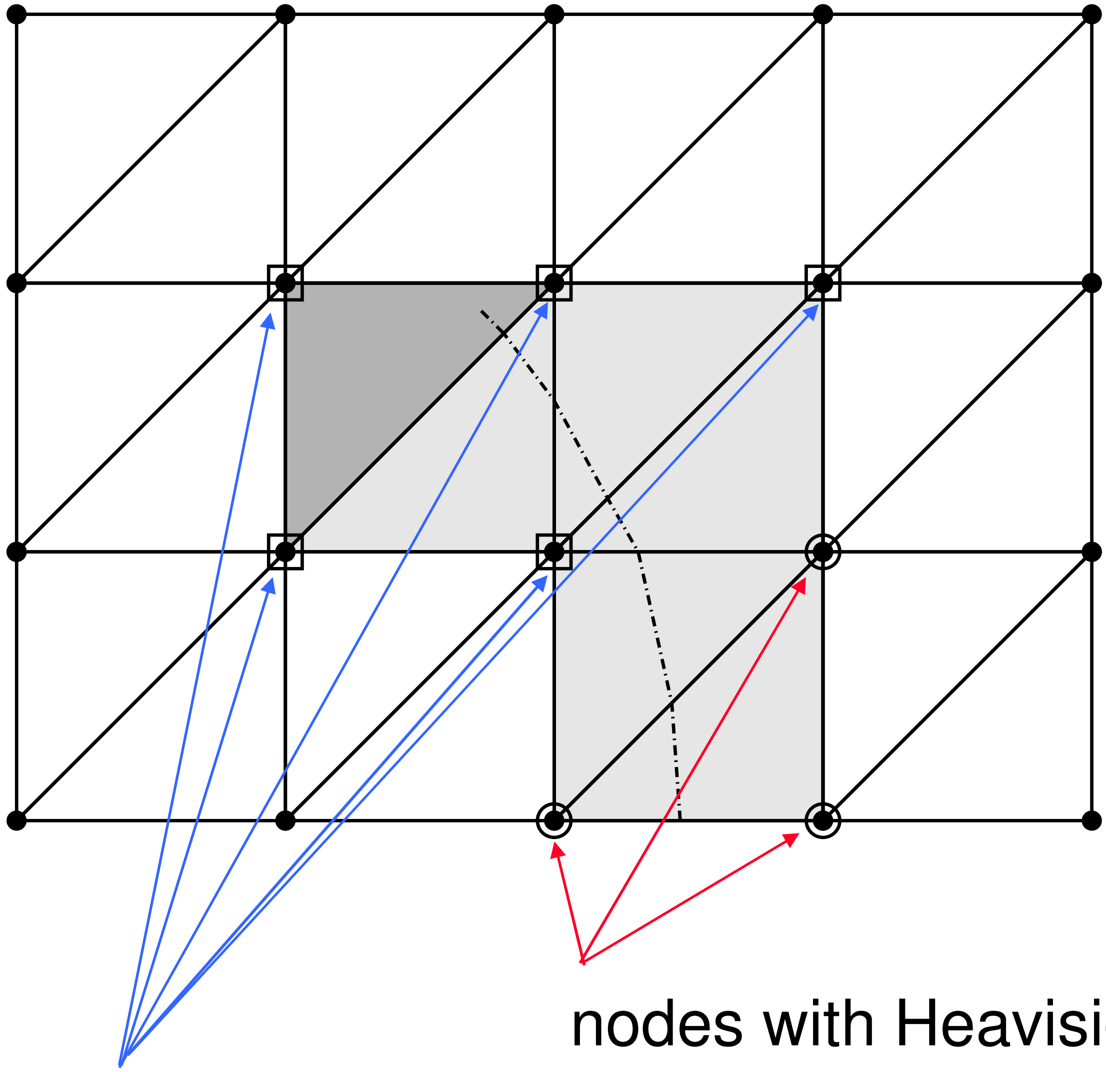


nodes with enrichment by near-tip functions

XFEM – tip enrichment



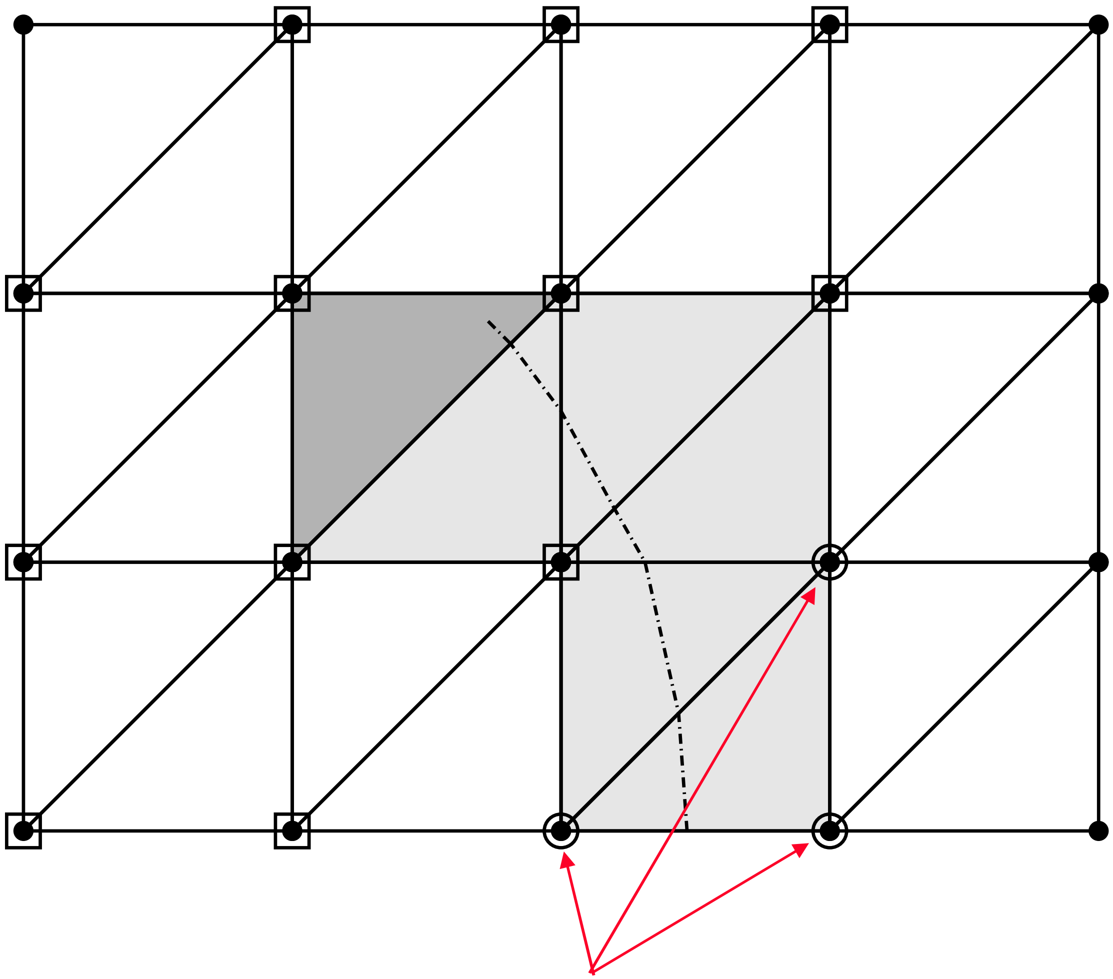
XFEM – tip enrichment



nodes with Heaviside enrichment

nodes with enrichment by near-tip functions

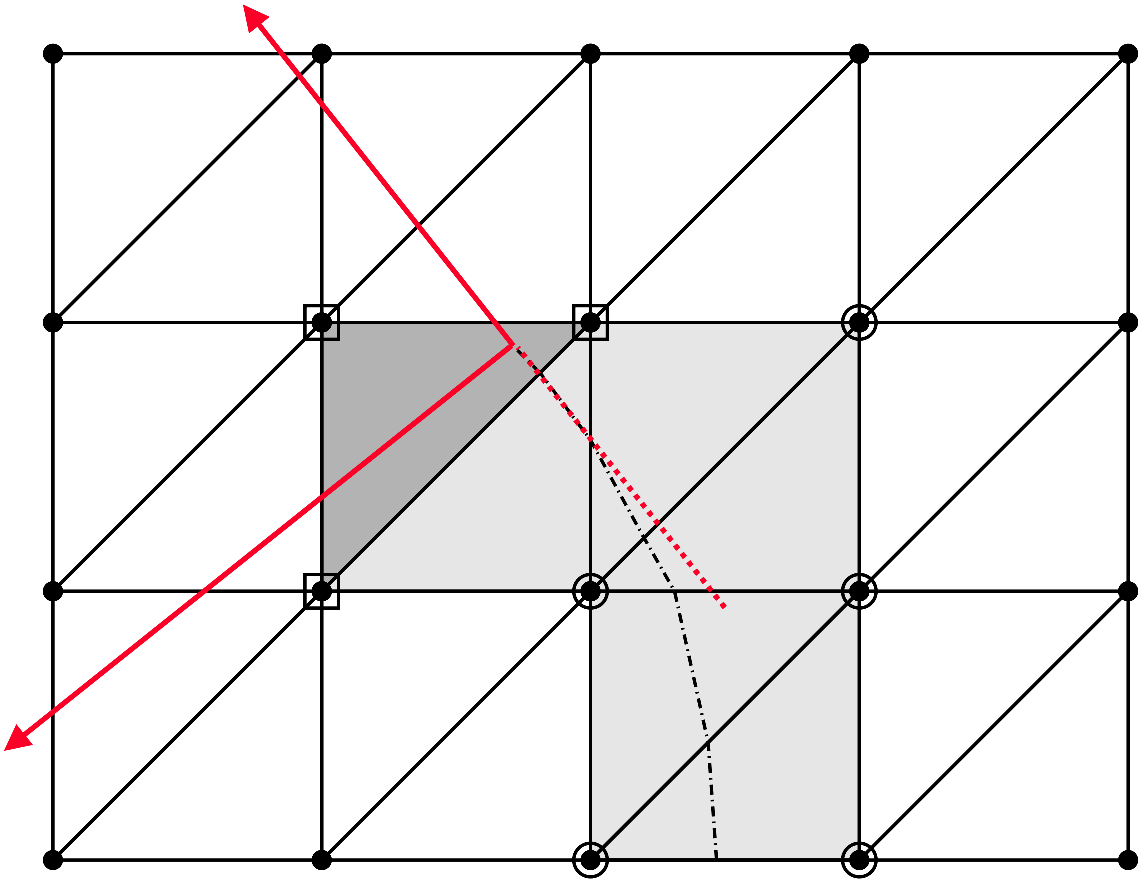
XFEM – tip enrichment



nodes with Heaviside enrichment

nodes with enrichment by near-tip functions

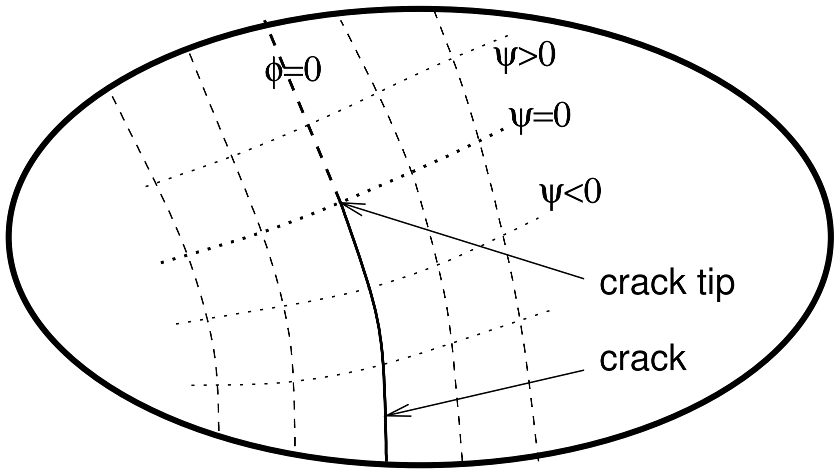
XFEM – tip enrichment



But if the crack is curved, we cannot define functions B_i in terms of the standard polar coordinates because B_1 would not be discontinuous across the crack but across the dotted line.

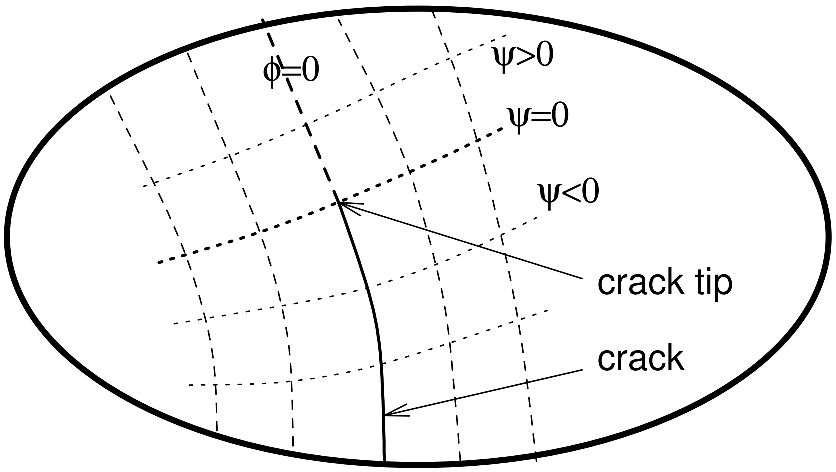
XFEM – level set functions

Remedy:
Construct curvilinear coordinates φ and ψ such that the crack is characterized by $\varphi = 0$ and $\psi \leq 0$



XFEM – level set functions

Remedy:
Construct curvilinear coordinates φ and ψ such that the crack is characterized by $\varphi = 0$ and $\psi \leq 0$



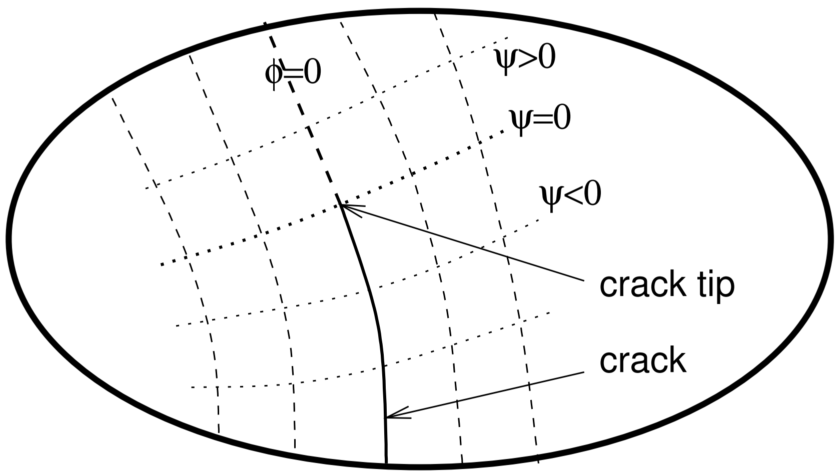
and define B_i in terms of the pseudo-polar coordinates

$$r(\psi,\varphi) = \sqrt{\psi^2 + \varphi^2}$$

$$\theta(\psi,\varphi) = \operatorname{sgn}(\varphi)\arccos\frac{\psi}{\sqrt{\psi^2 + \varphi^2}}$$

XFEM – level set functions

Functions φ and ψ are the so-called **level set functions.**

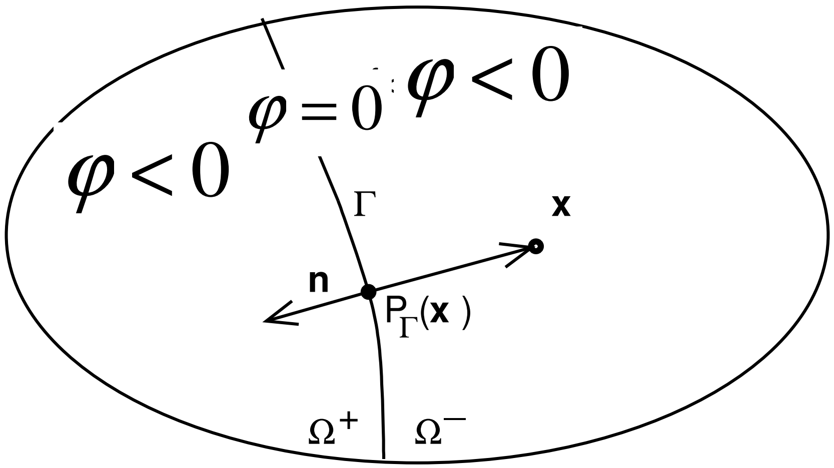


They are defined by their values at nodes around the crack and interpolated using the standard shape functions:

$$\varphi(\mathbf{x}) = \sum_I N_I(\mathbf{x})\,\varphi_I, \quad \psi(\mathbf{x}) = \sum_I N_I(\mathbf{x})\,\psi_I$$

XFEM – level set functions

For an existing crack, function φ can be constructed as the signed distance function:



$$\varphi(\mathbf{x}) = \|\mathbf{x} - P_\Gamma(\mathbf{x})\| \operatorname{sgn}\left[(\mathbf{x} - P_\Gamma(\mathbf{x})) \cdot \mathbf{n}(P_\Gamma(\mathbf{x}))\right]$$

F.4

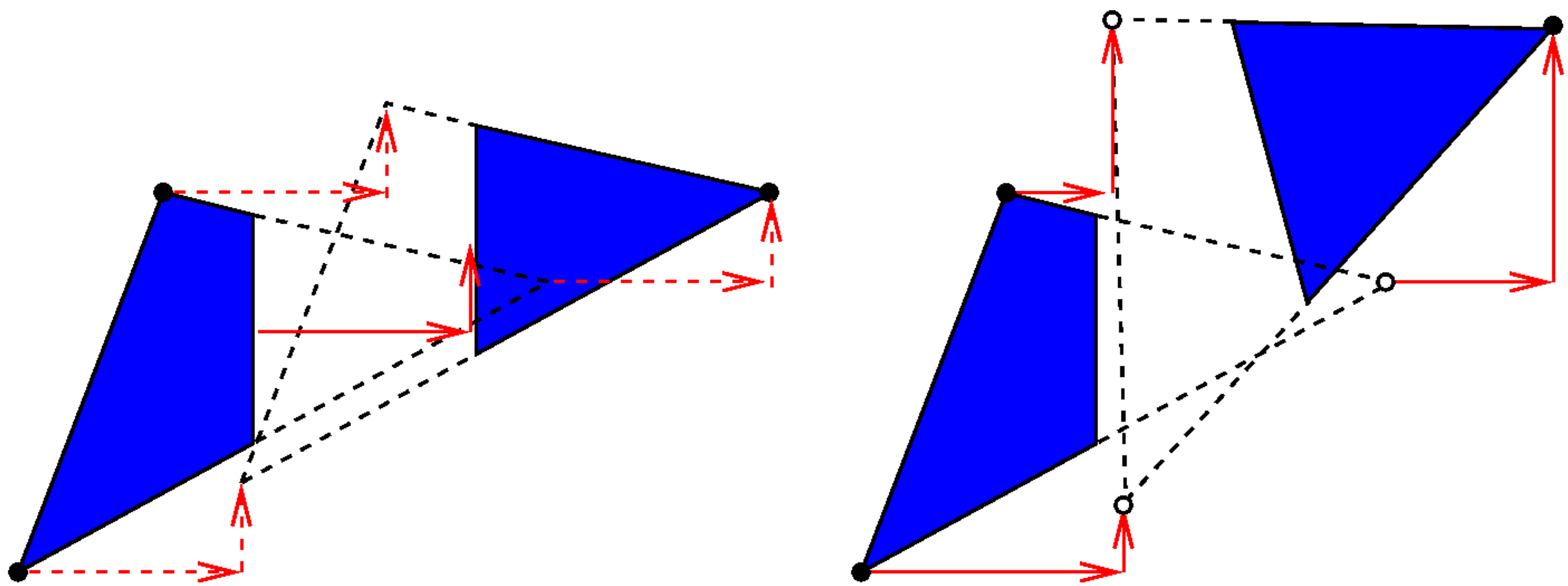
Comparison:

EED-EAS versus XFEM-PUM

Comparison of EED-EAS and XFEM-PUM

Embedded discontinuity

Extended finite elements



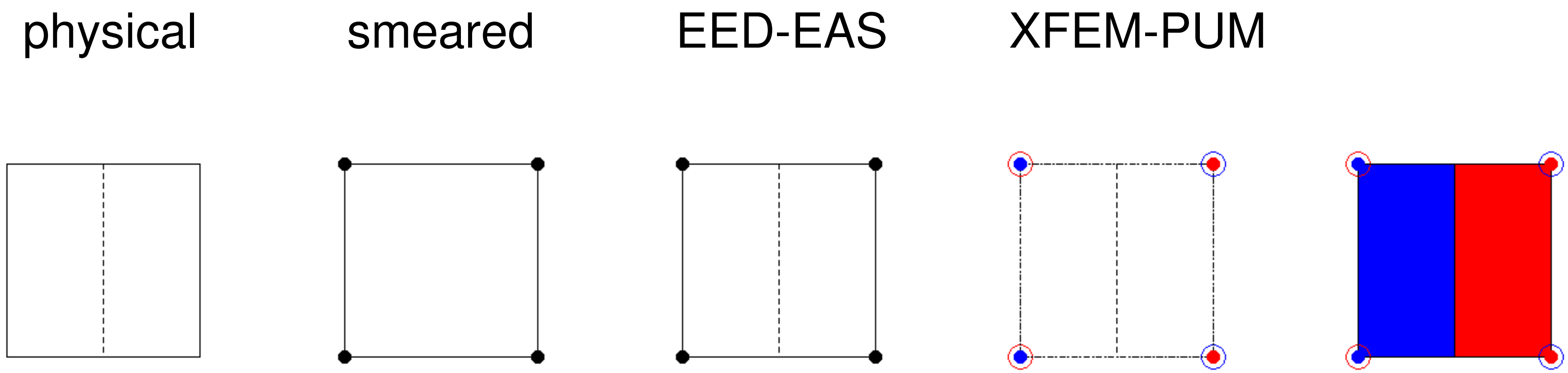
Comparison of EED-EAS and XFEM-PUM

	Embedded discontinuity	Extended finite elements
DOF‘s added	locally	globally
and related to	elements	nodes

Comparison of EED-EAS and XFEM-PUM

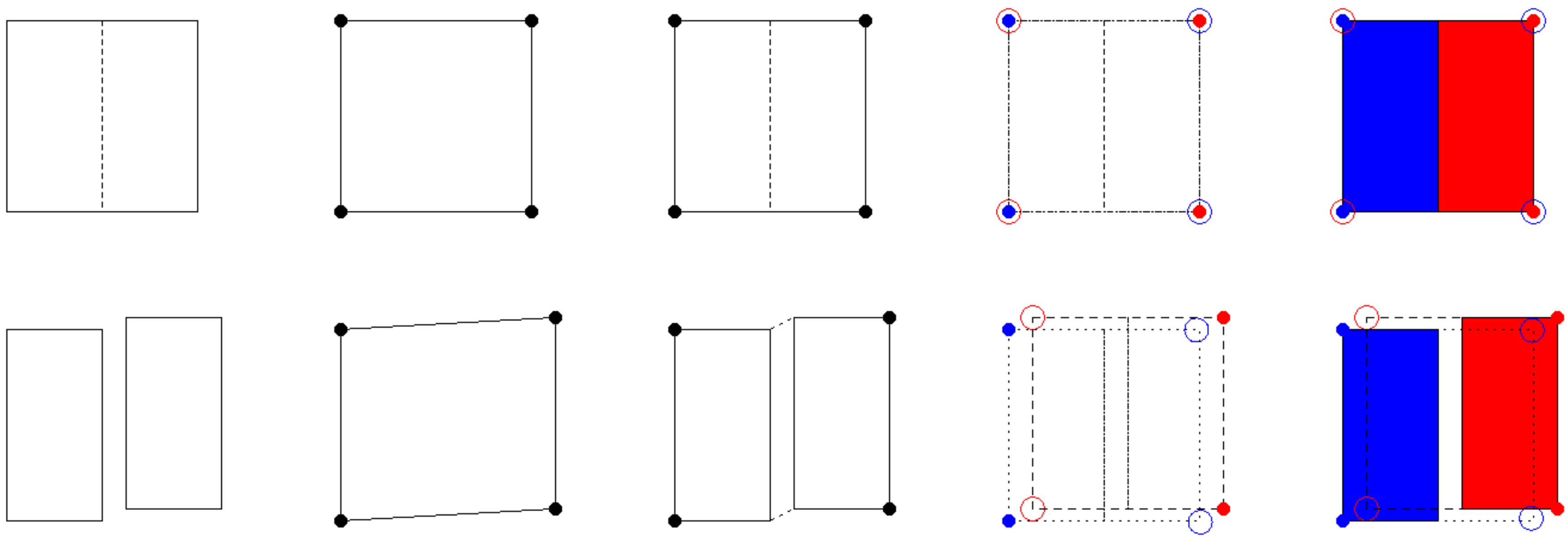
	Embedded discontinuity	Extended finite elements
DOF's added	locally	globally
and related to	elements	nodes
Approximation of crack opening	discontinuous	continuous
Enrichment	incompatible	compatible

Separation test



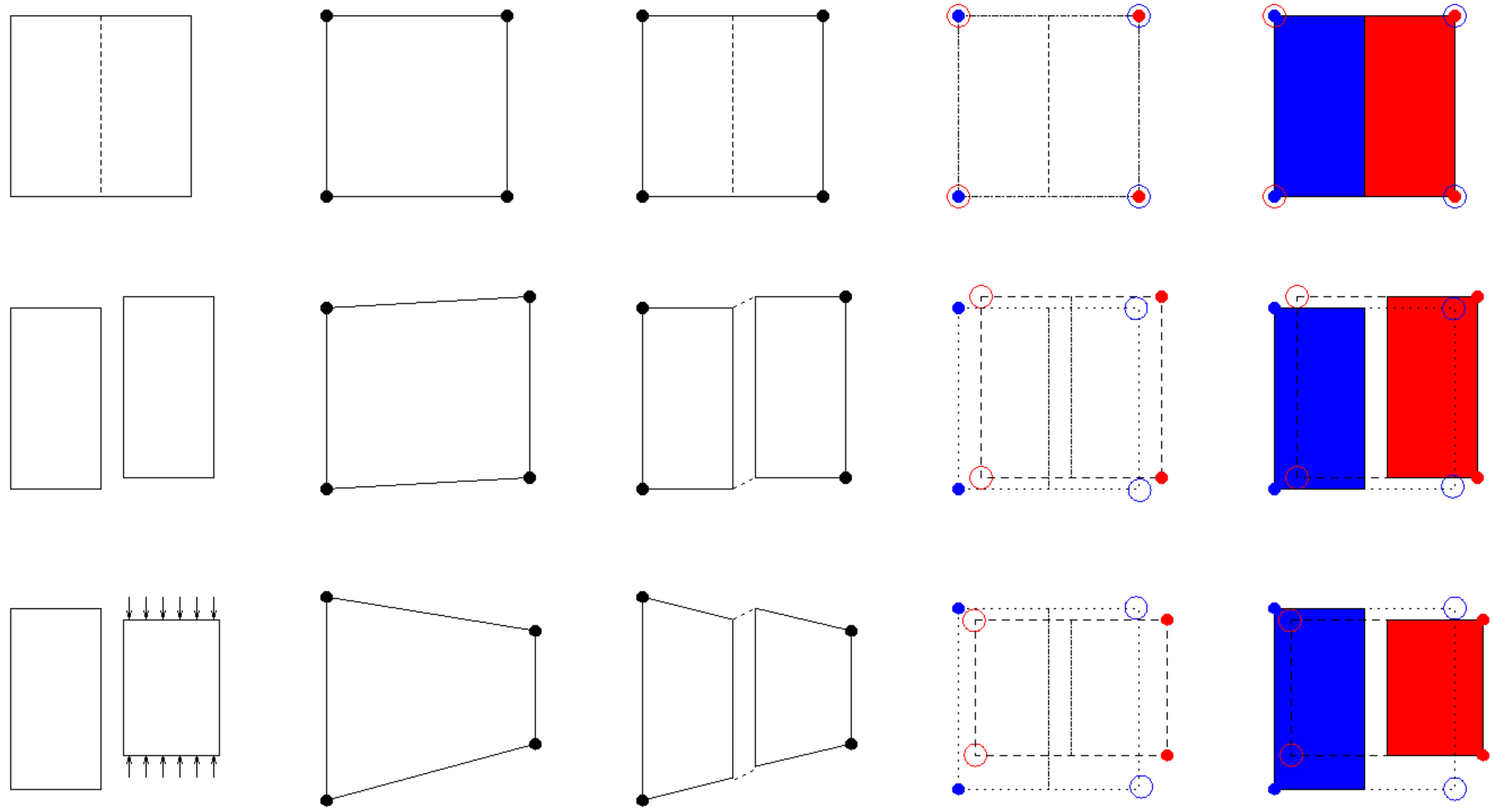
Separation test

physical smeared EED-EAS XFEM-PUM

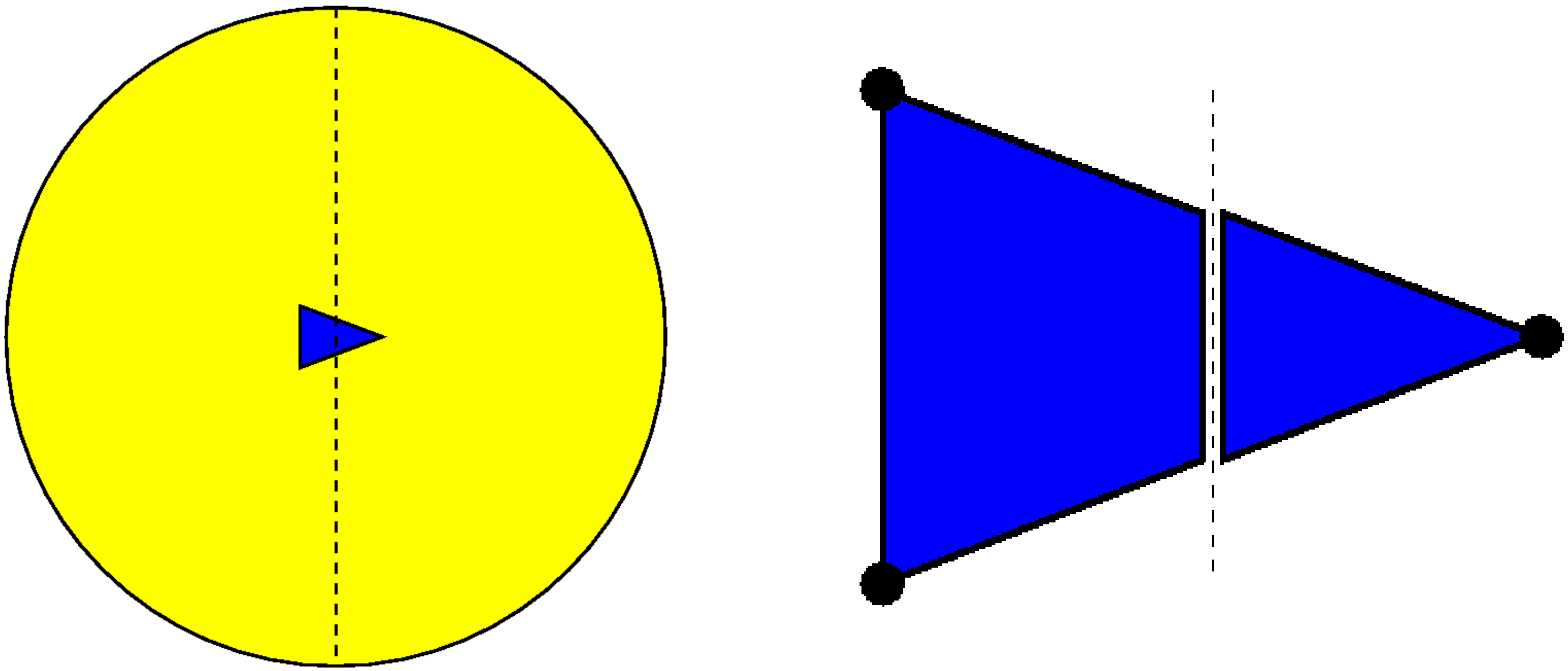


Separation test

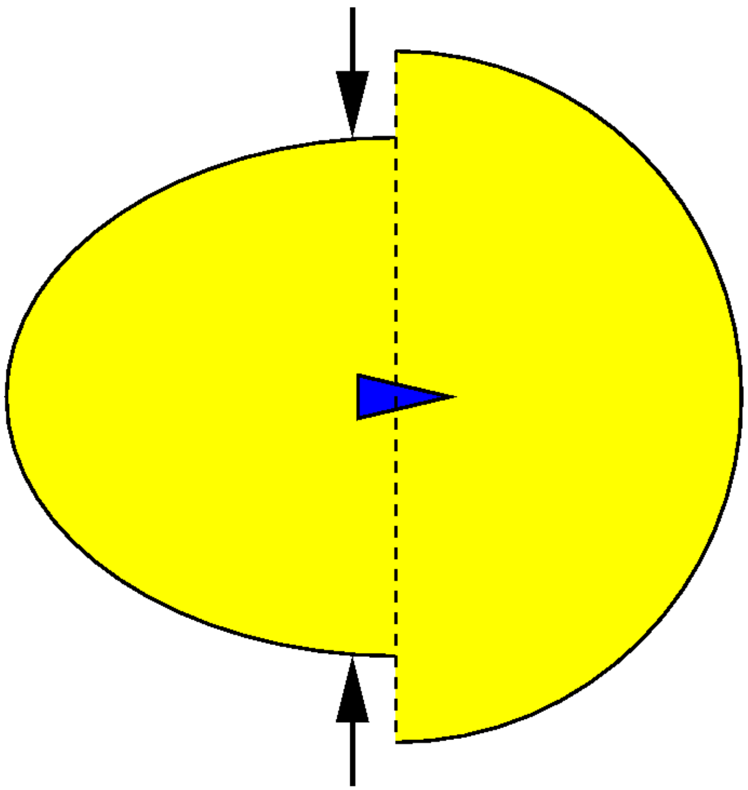
physical smeared EED-EAS XFEM-PUM

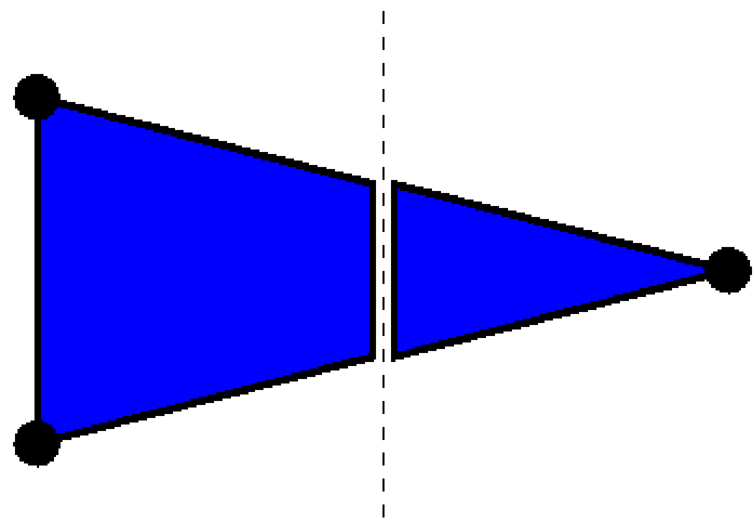


EED-EAS approach: partial coupling

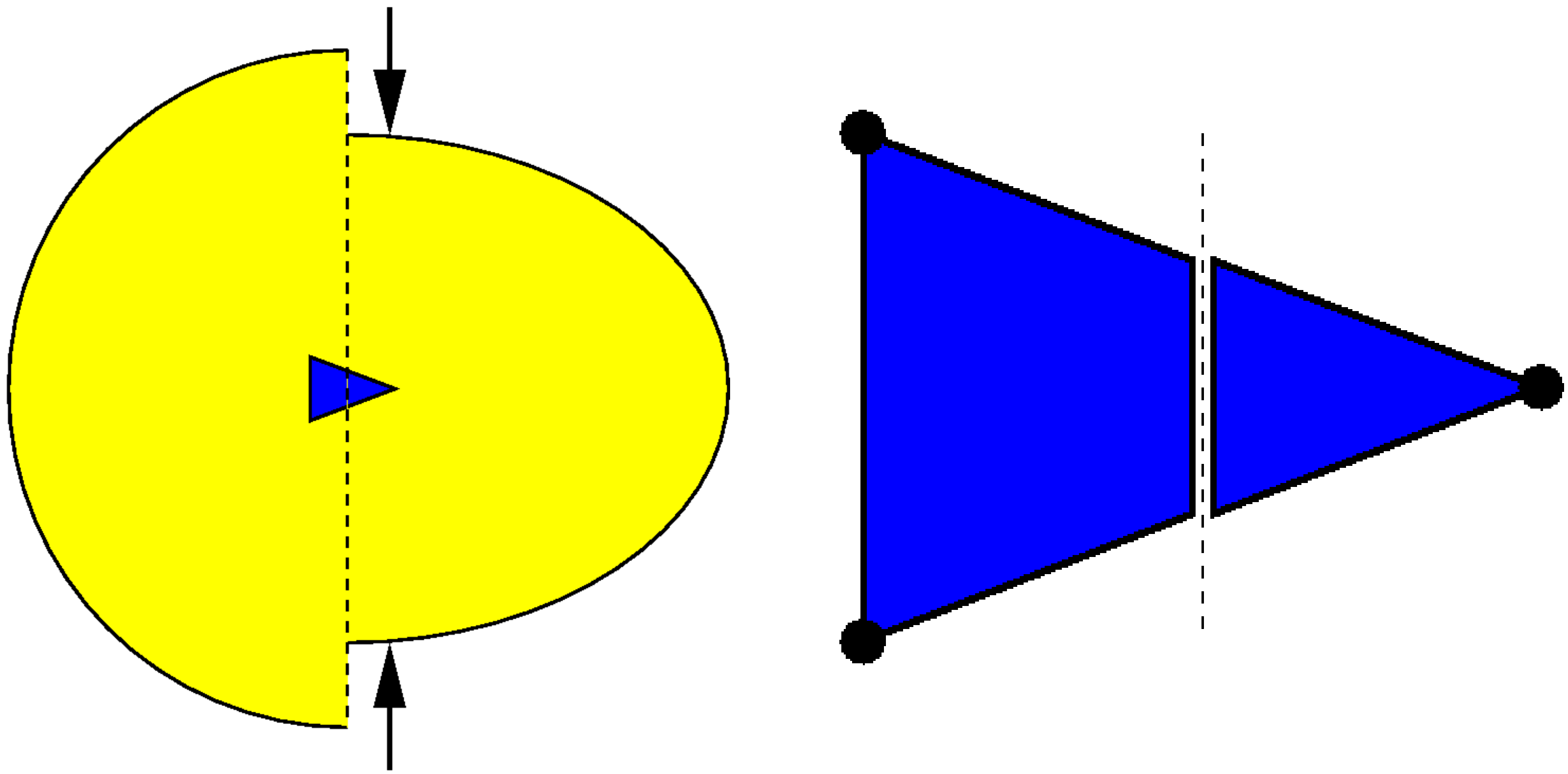


EED- EAS approach: partial coupling

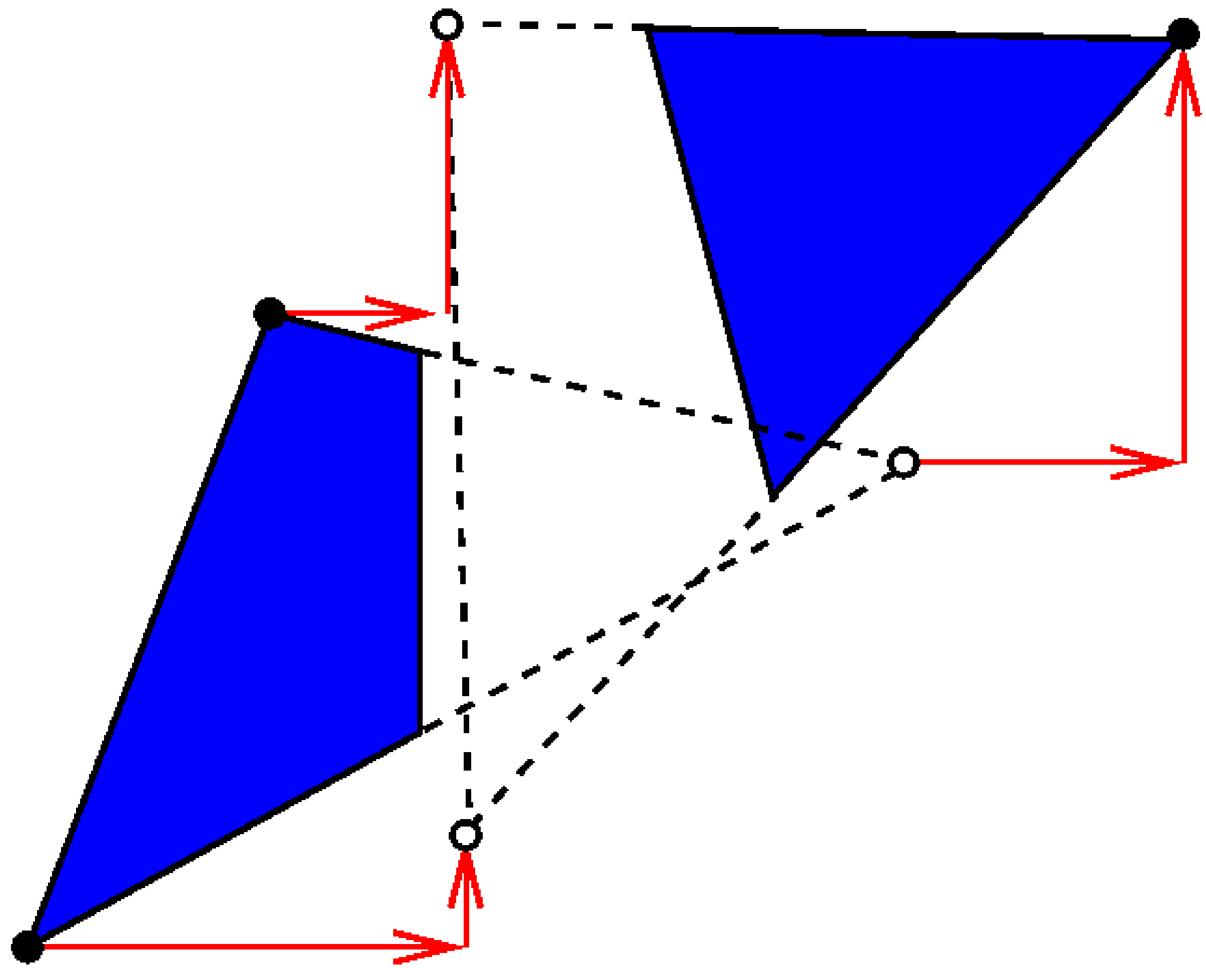




EED- EAS approach: partial coupling



XFEM-PUM approach: complete decoupling



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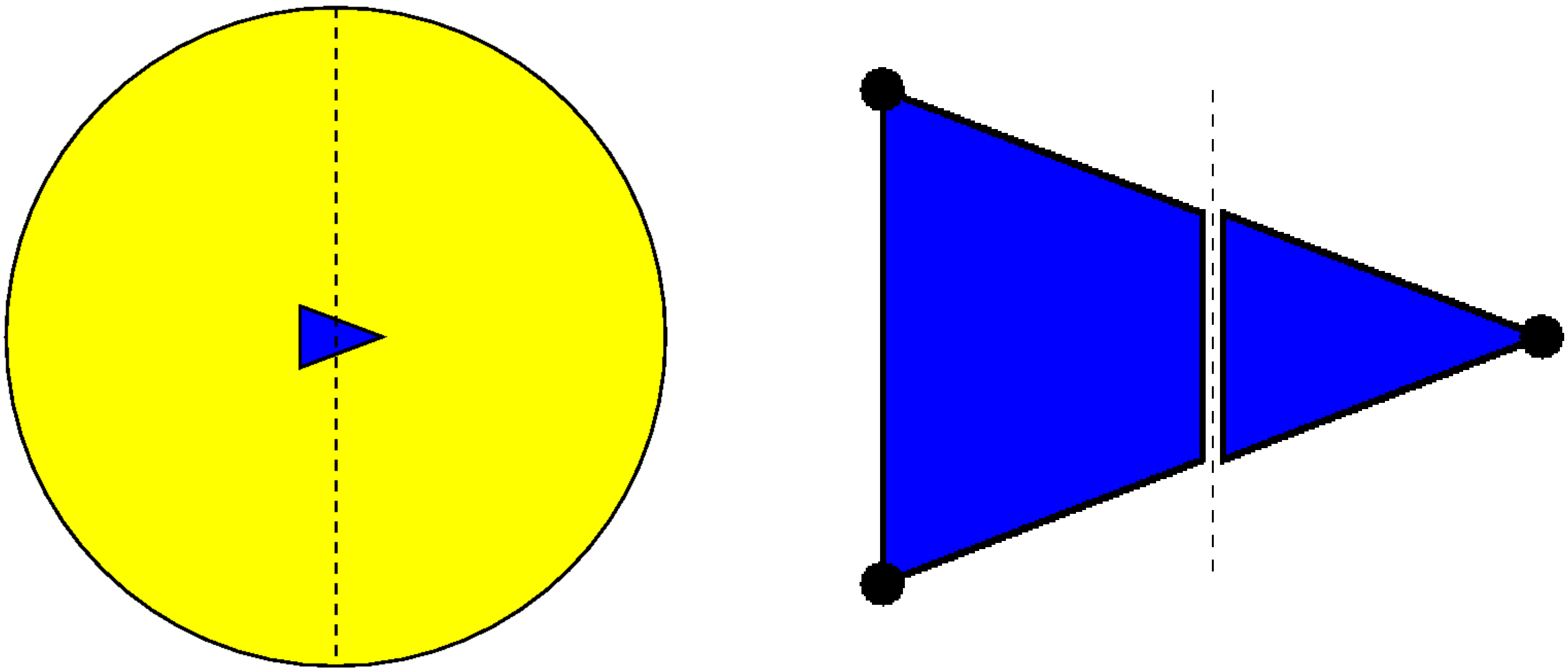
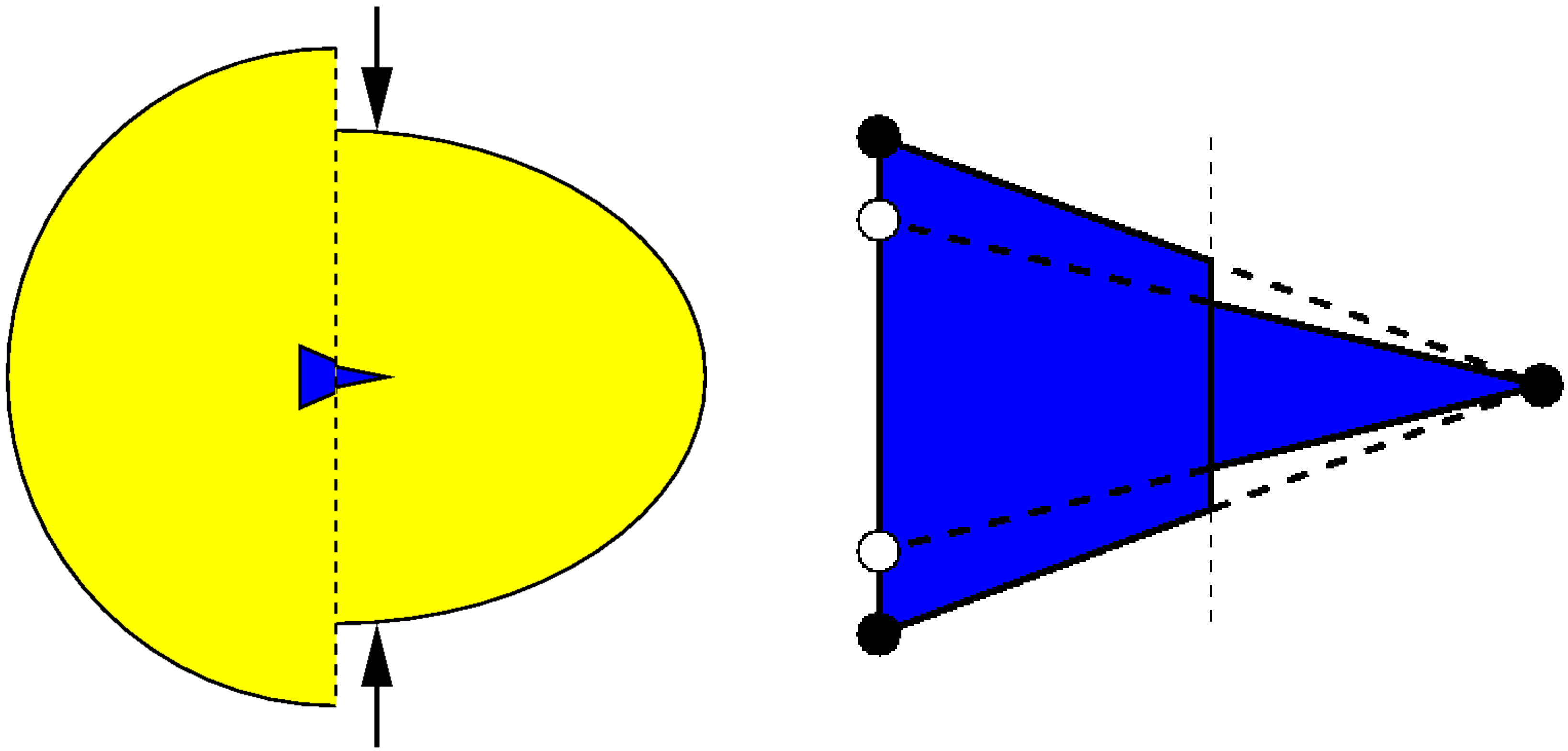
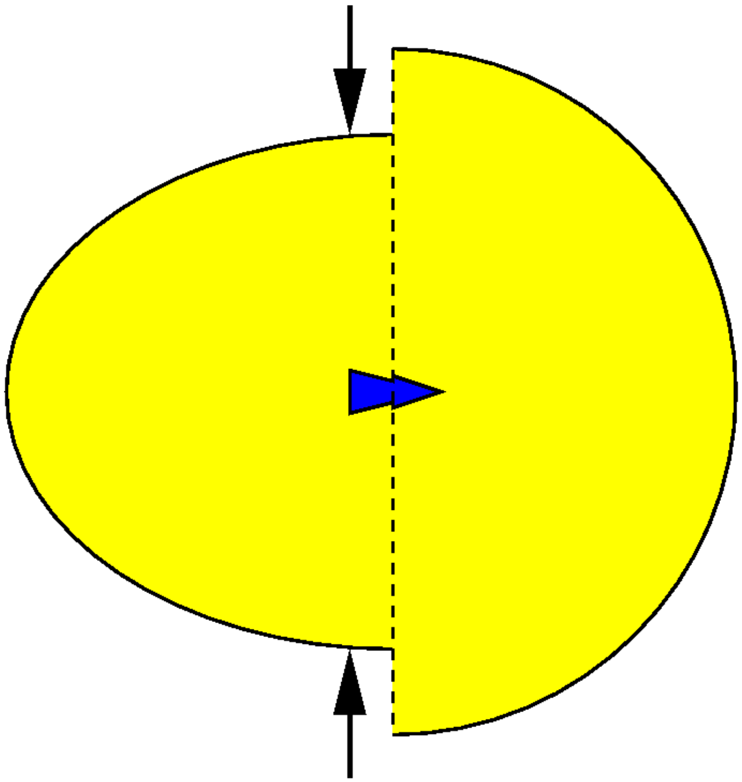
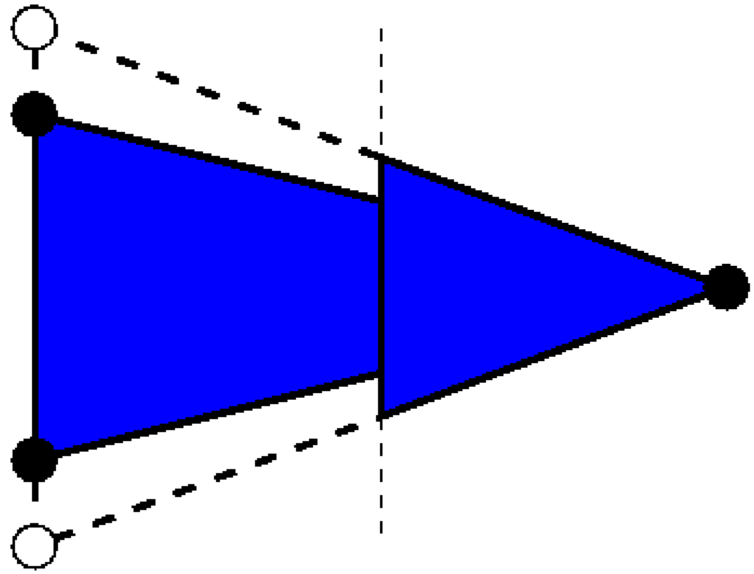


XFEM-PUM approach: complete decoupling



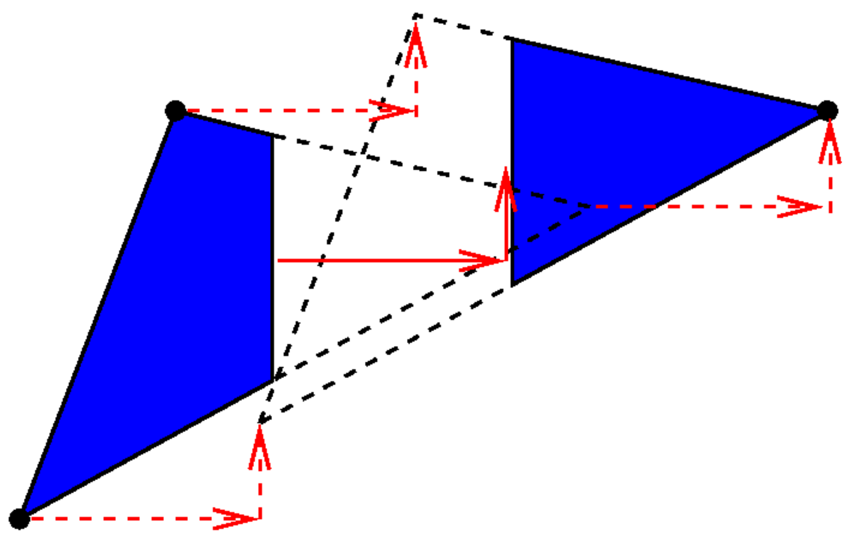
XFEM-PUM approach: complete decoupling

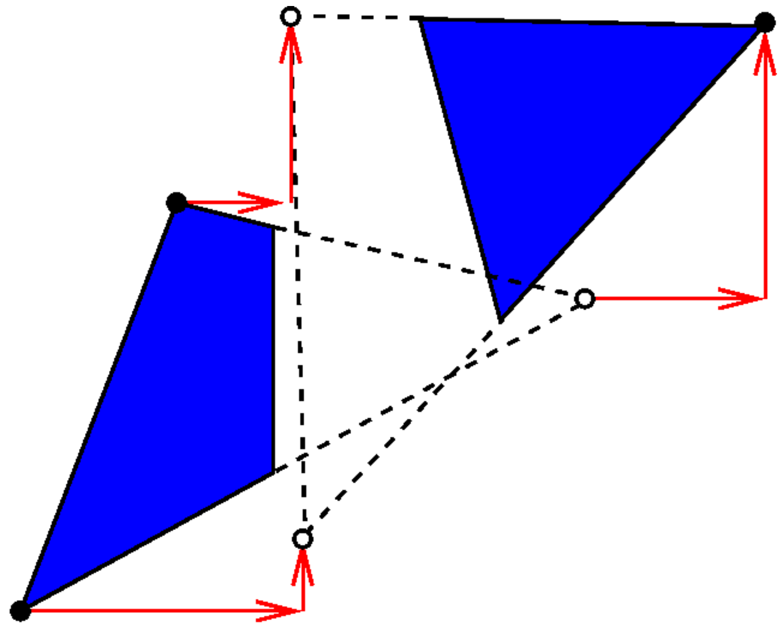




Comparison of EED-EAS and XFEM-PUM

Embedded discontinuity | Extended finite elements

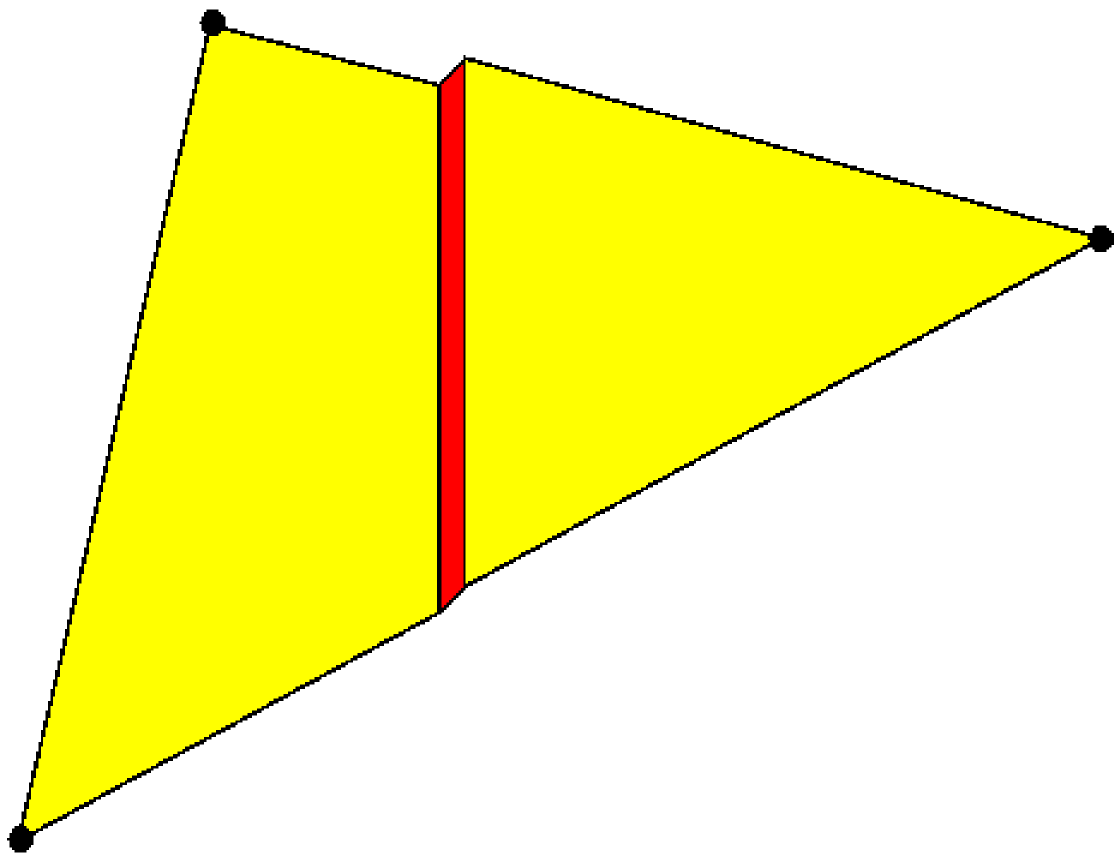




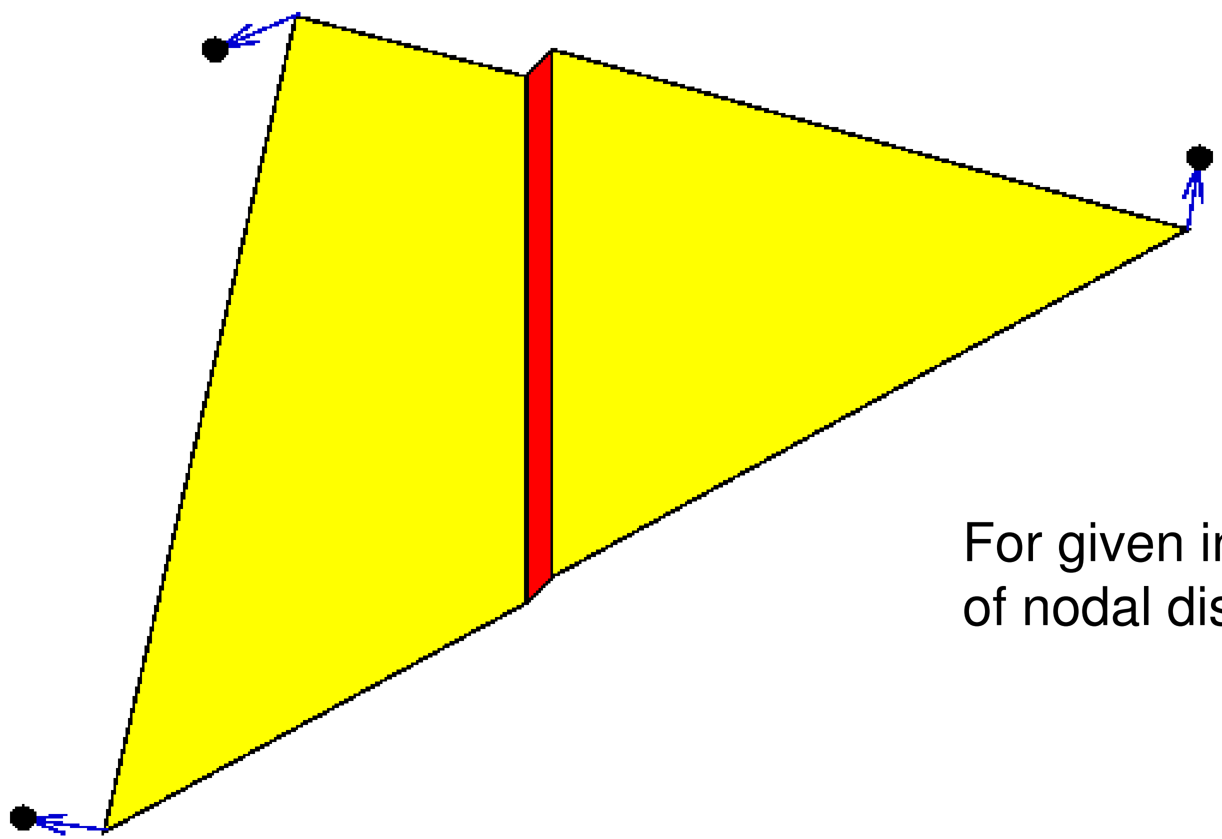
Comparison of EED-EAS and XFEM-PUM

	Embedded discontinuity	Extended finite elements
DOF's added	locally	globally
and related to	elements	nodes
Approximation of crack opening	discontinuous	continuous
Enrichment	incompatible	compatible
Separated parts	**partially coupled**	**fully decoupled**

Uniqueness of the element response (EED-EAS)

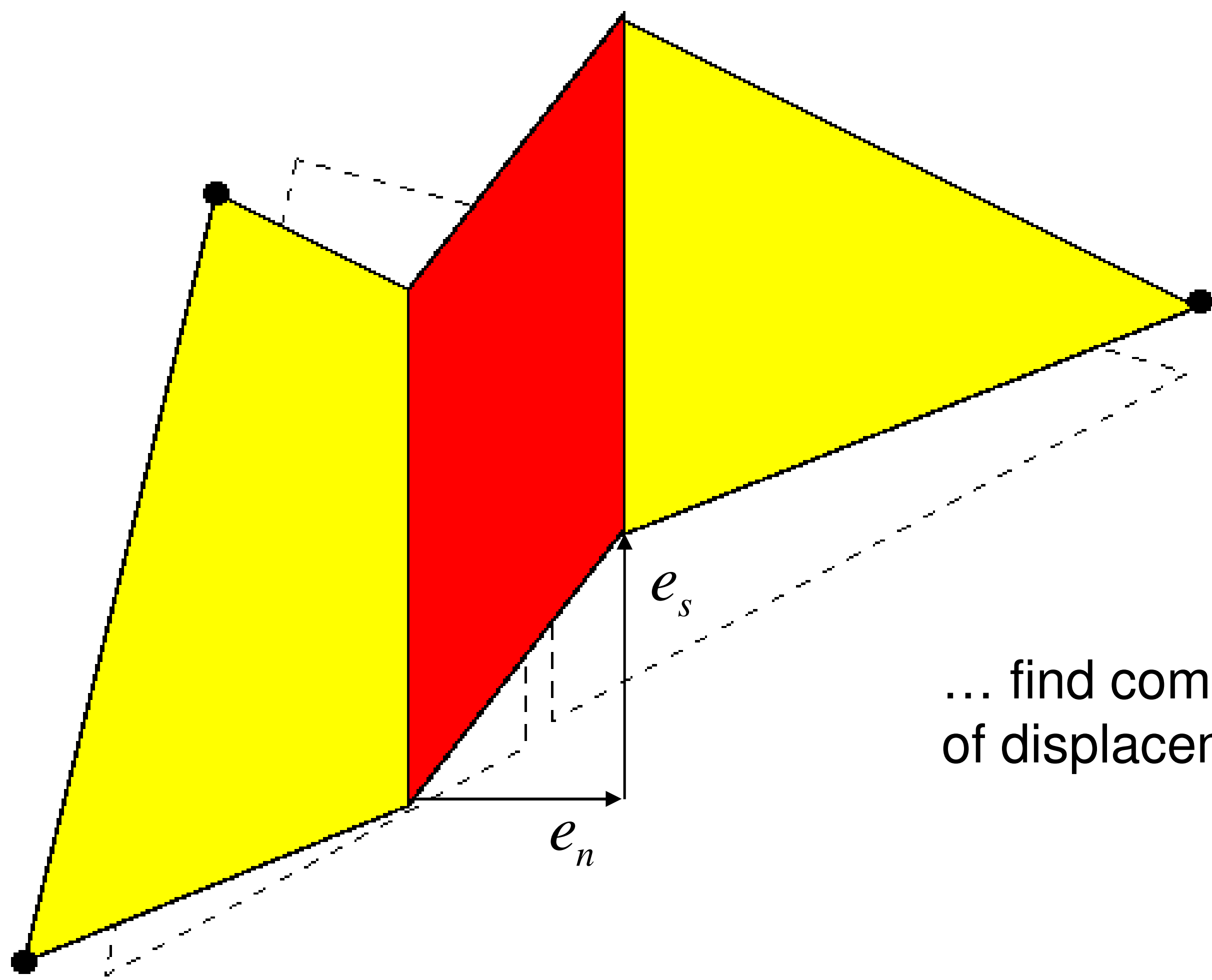


Uniqueness of the element response



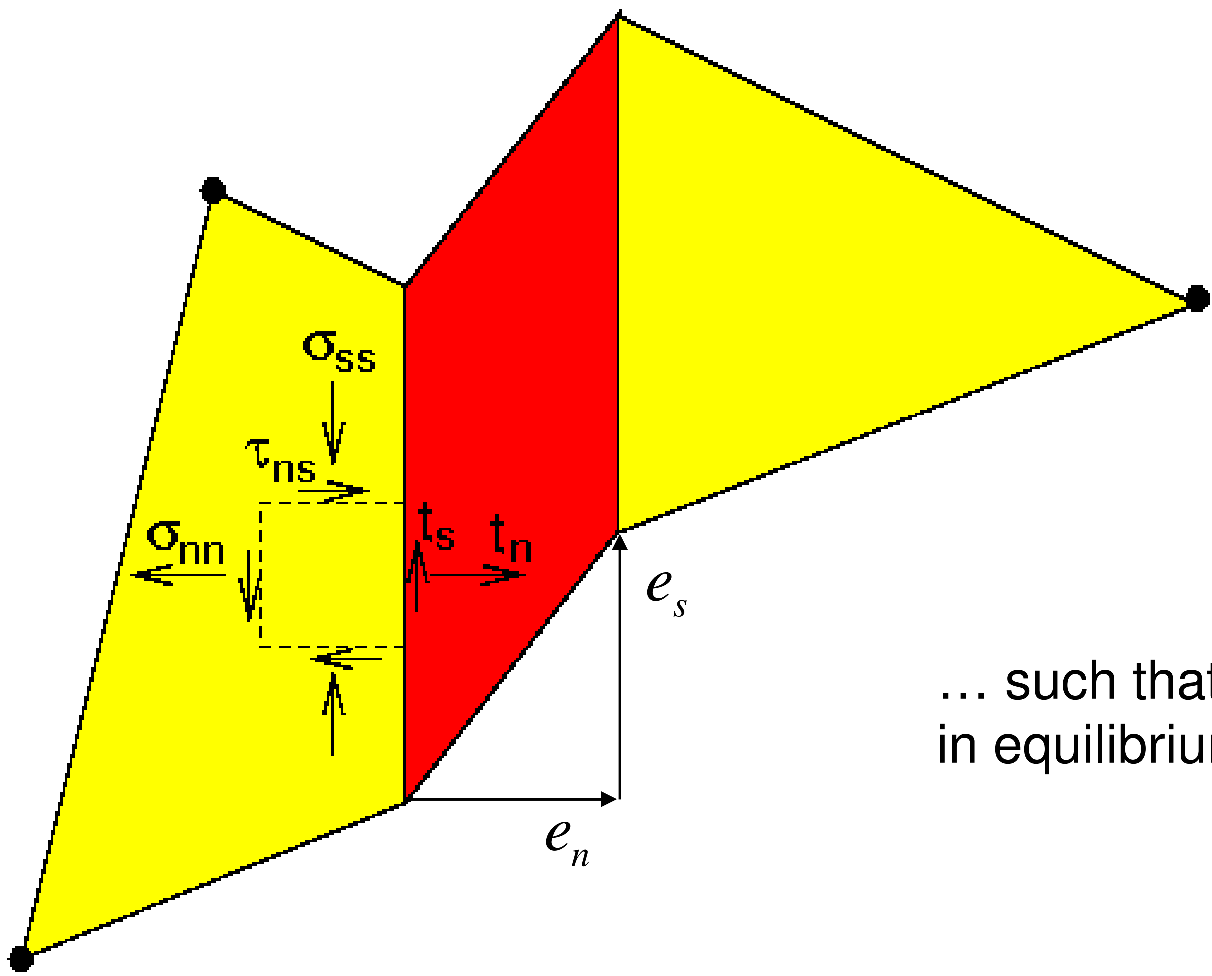
For given increments
of nodal displacements …

Uniqueness of the element response



… find components
of displacement jump …

Uniqueness of the element response



… such that tractions are
in equilibrium with stresses.

Uniqueness of the element response

The solution is unique for infinitesimal displacement increments of an arbitrary direction if

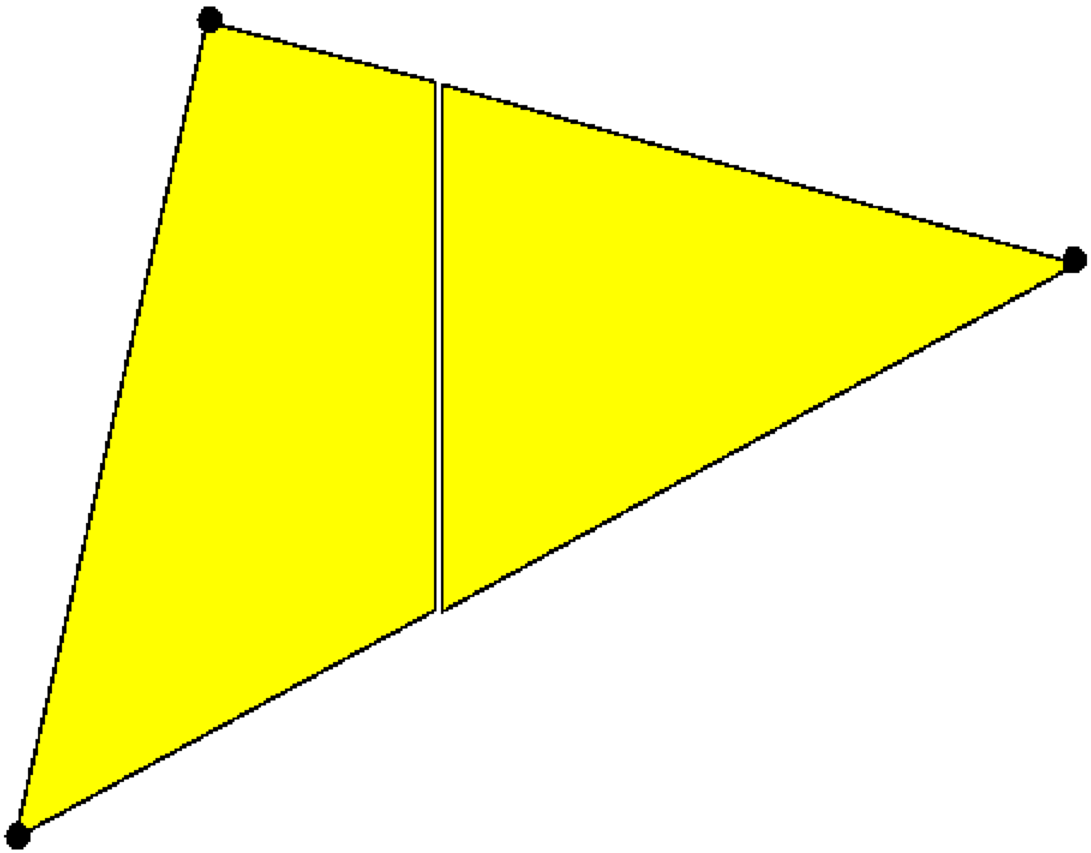
$$\lambda_{\min}\left(\mathbf{Q}_{sym}\right) + H > 0$$

where $\mathbf{Q}_{sym}$ is the symmetric part of $\mathbf{Q} = \mathbf{P}^T \mathbf{D}_e \mathbf{B} \mathbf{H}$

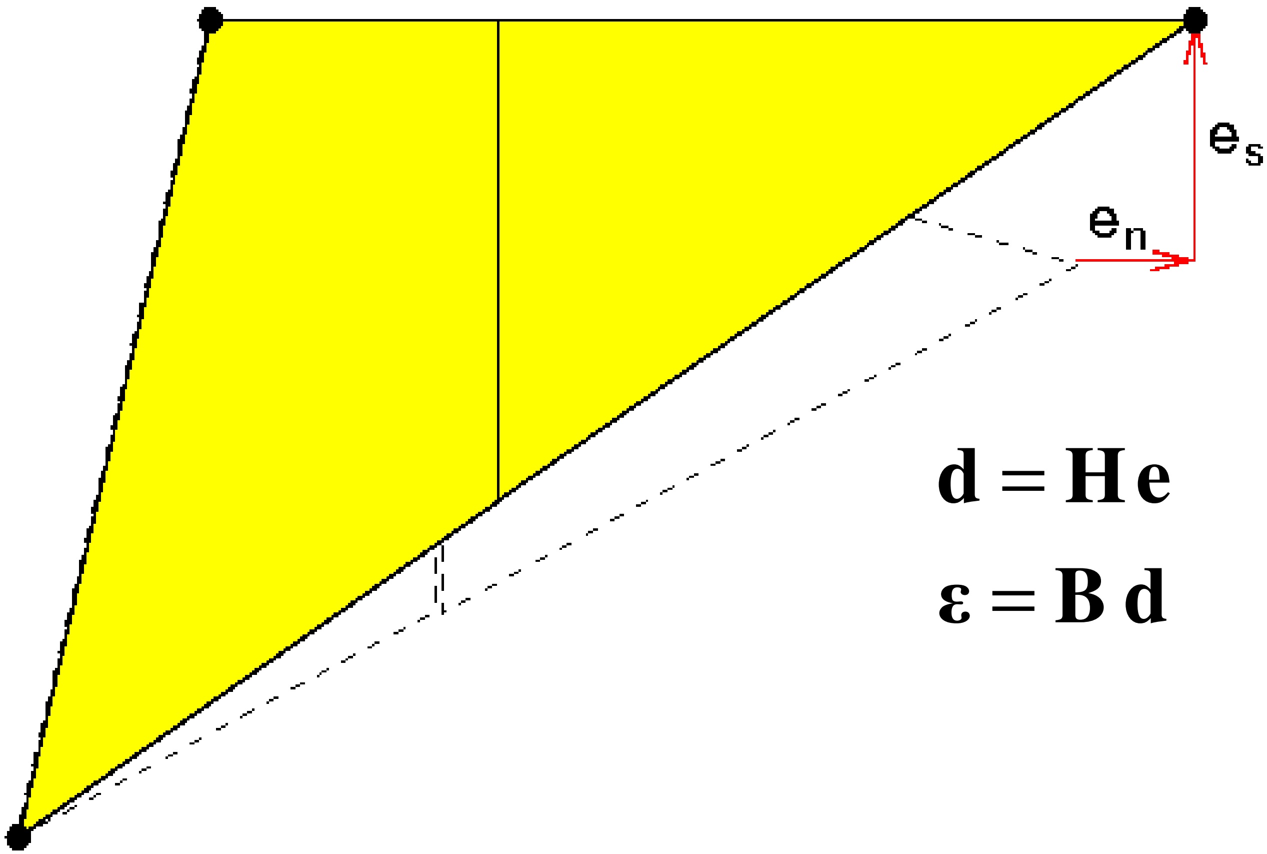
and $H < 0$ is the discrete softening modulus.

Physical meaning of $\mathbf{Q}$ …

Uniqueness of the element response



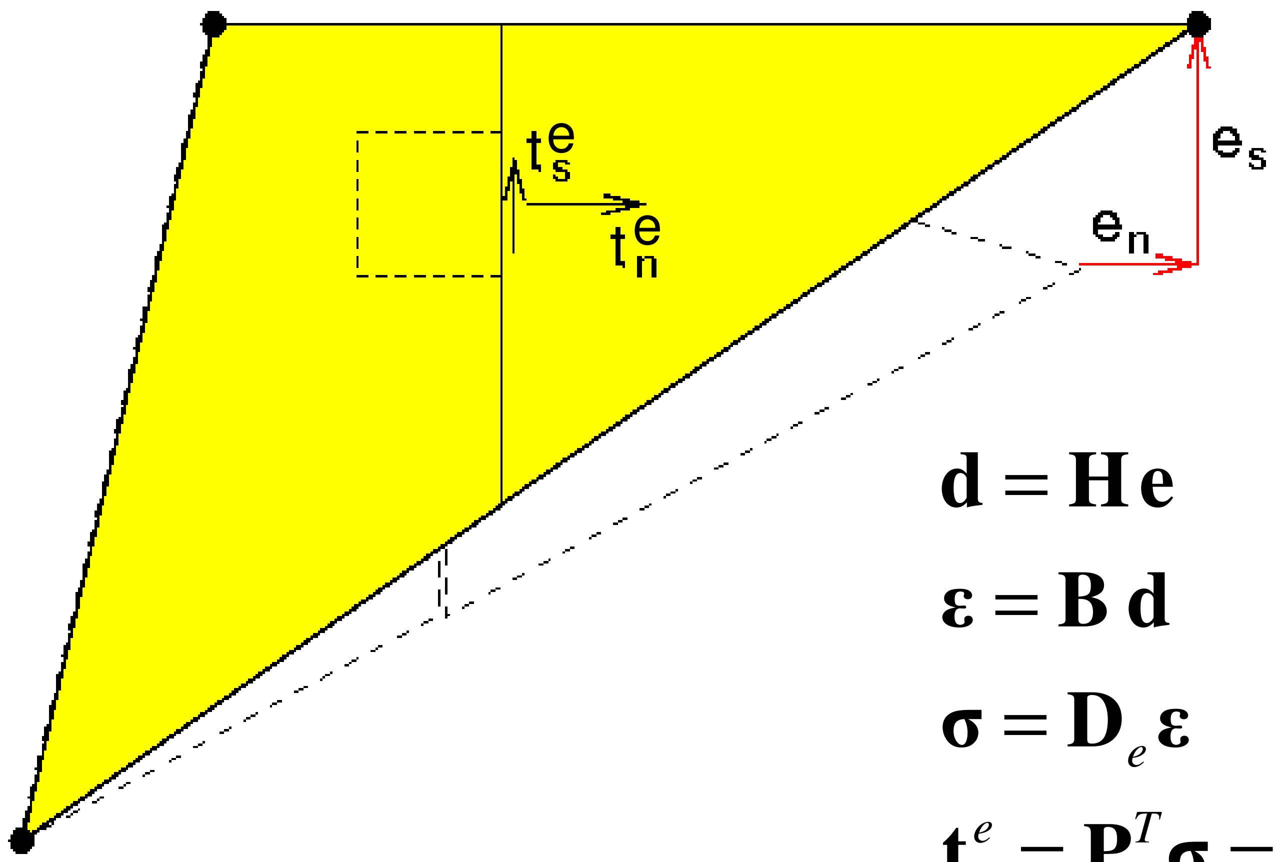
Uniqueness of the element response



$$\mathbf{d} = \mathbf{H}\,\mathbf{e}$$

$$\boldsymbol{\varepsilon} = \mathbf{B}\,\mathbf{d}$$

Uniqueness of the element response



$$\mathbf{d} = \mathbf{H}\,\mathbf{e}$$

$$\boldsymbol{\varepsilon} = \mathbf{B}\,\mathbf{d}$$

$$\boldsymbol{\sigma} = \mathbf{D}_e\,\boldsymbol{\varepsilon}$$

$$\mathbf{t}^e = \mathbf{P}^T\boldsymbol{\sigma} = \mathbf{P}^T\mathbf{D}_e\,\mathbf{B}\,\mathbf{H}\,\mathbf{e}$$

Uniqueness of the element response

$$\lambda_{\min}\left(\mathbf{Q}_{sym}\right) > -H_{\min}$$

$\mathbf{Q} = \mathbf{P}^T \mathbf{D}_e \mathbf{B} \mathbf{H}$ is proportional to the elastic modulus and inversely proportional to the element size

Uniqueness of the element response

$$\lambda_{\min}\left(\mathbf{Q}_{sym}\right) > -H_{\min}$$

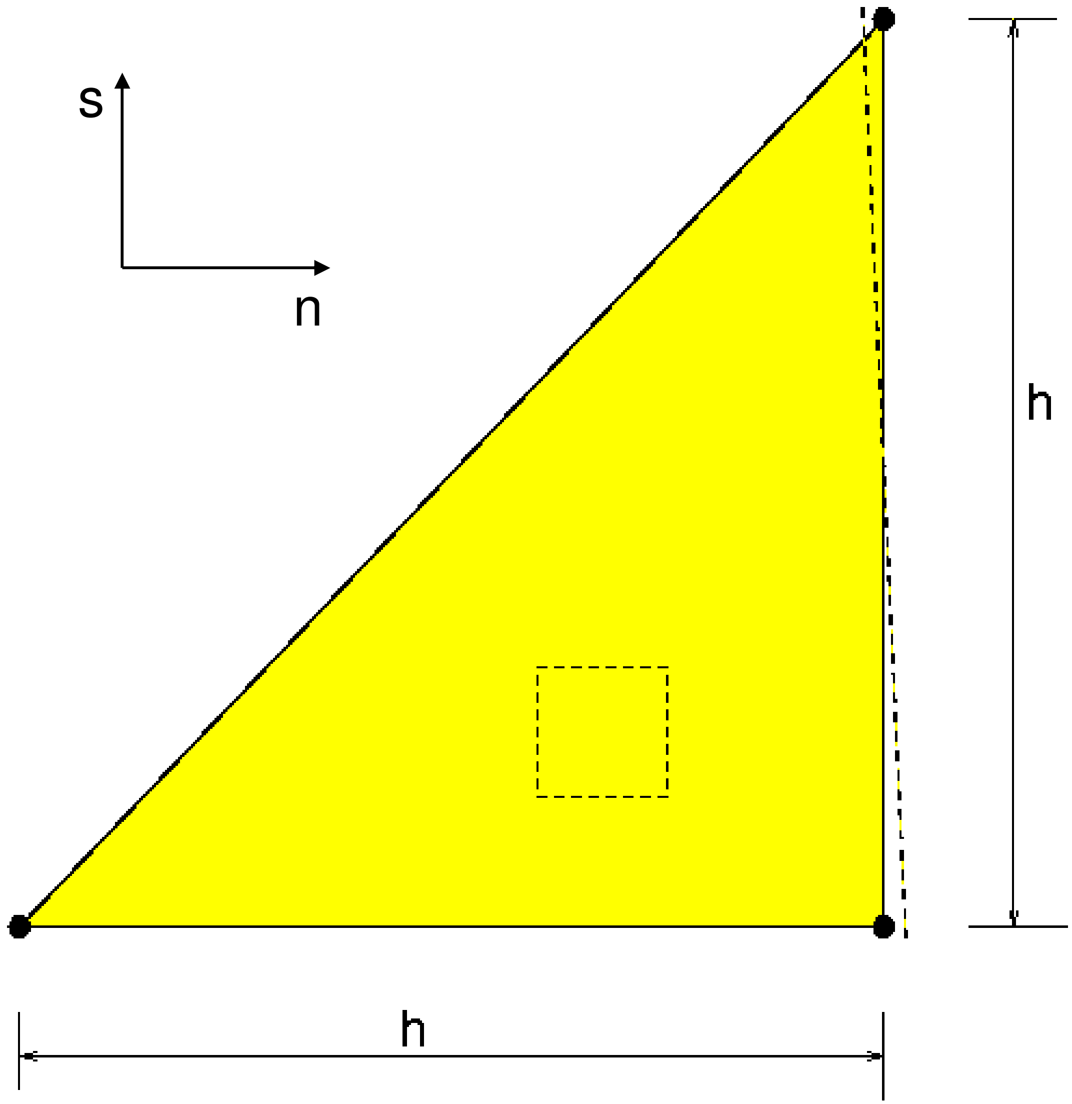
$$\mathbf{Q} = \mathbf{P}^T \mathbf{D}_e \mathbf{B} \mathbf{H}$$

is proportional to the elastic modulus
and inversely proportional to the element size

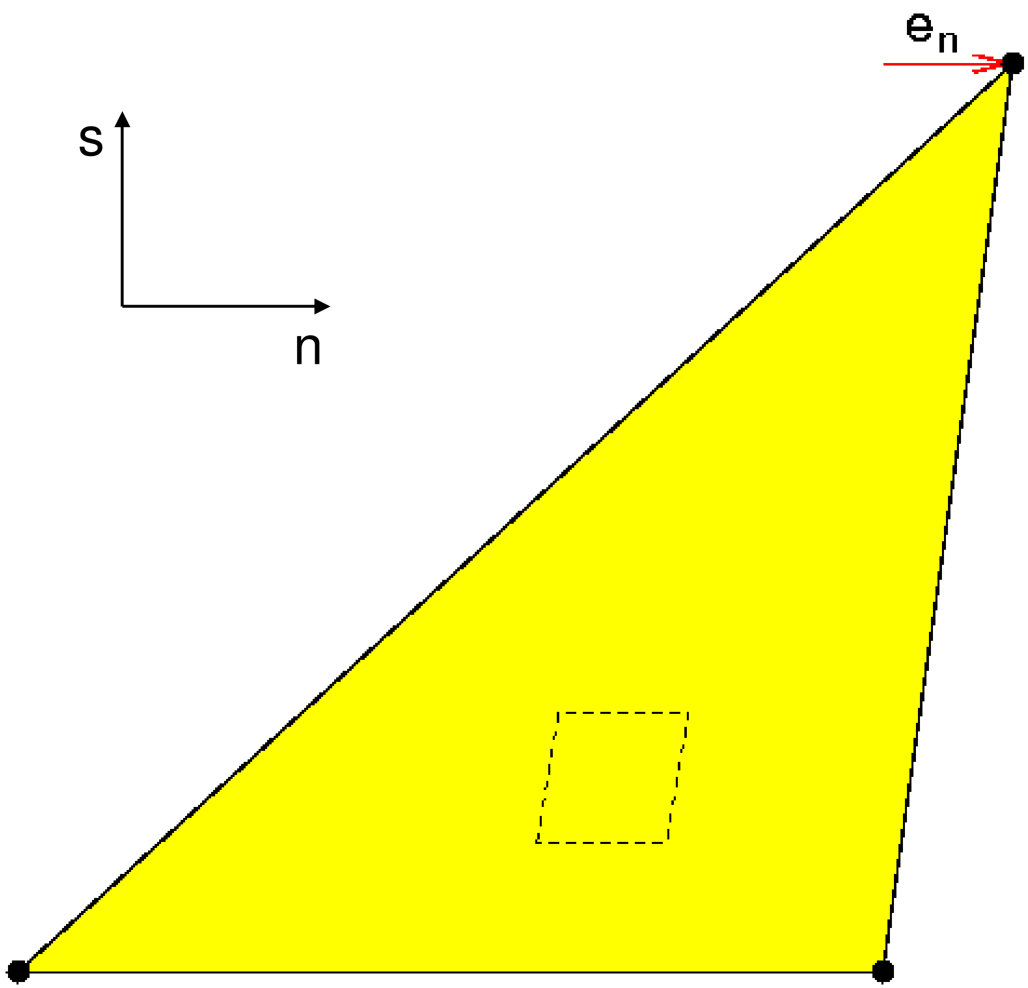
$$\mathbf{e}^T \mathbf{Q}_{sym} \mathbf{e} = \mathbf{e}^T \mathbf{Q} \mathbf{e} = \mathbf{e}^T \mathbf{t}^e < 0$$

can happen

Uniqueness of the element response



Uniqueness of the element response



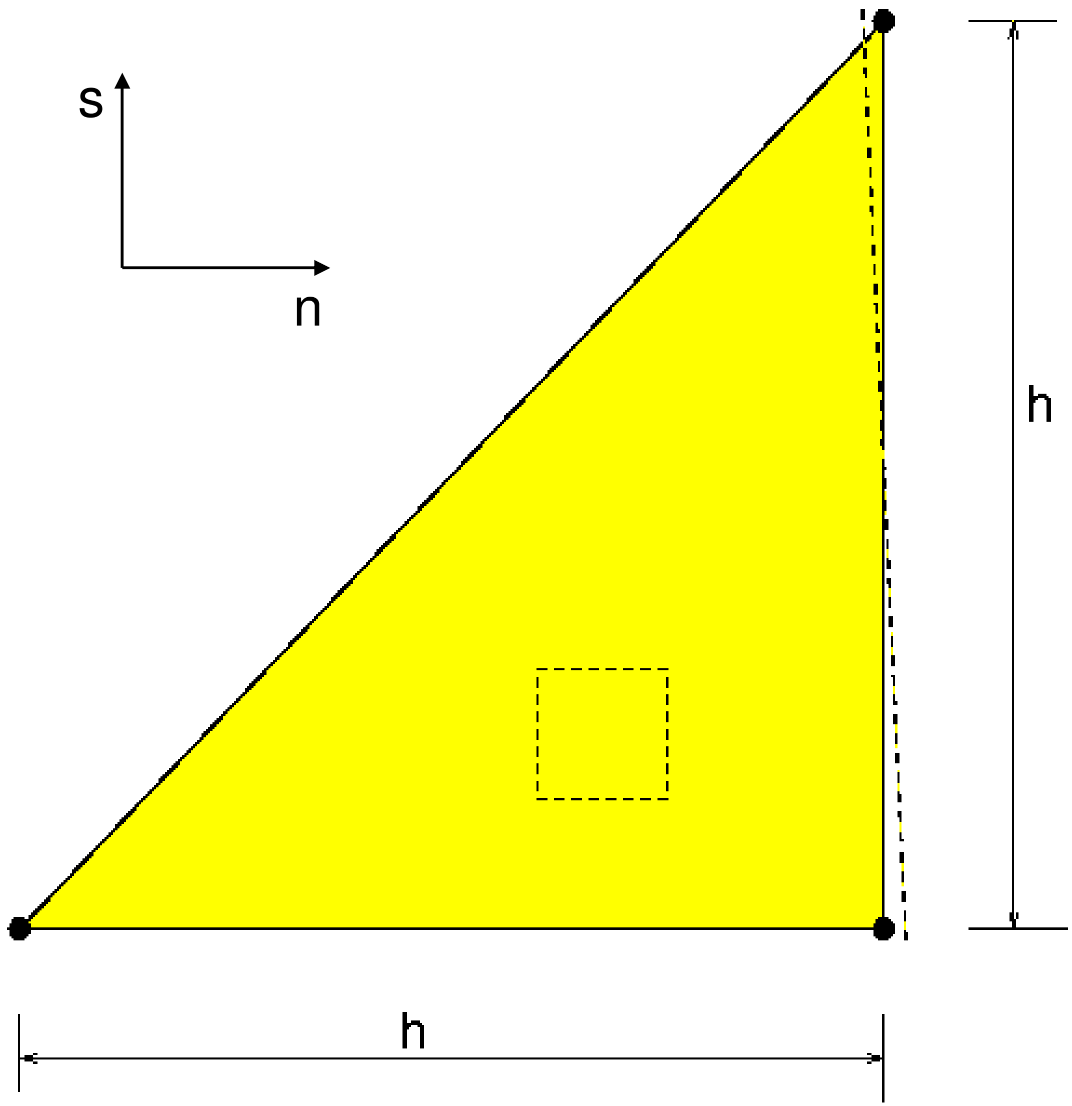
$$\gamma = e_n / h$$

$$\tau = G\gamma$$

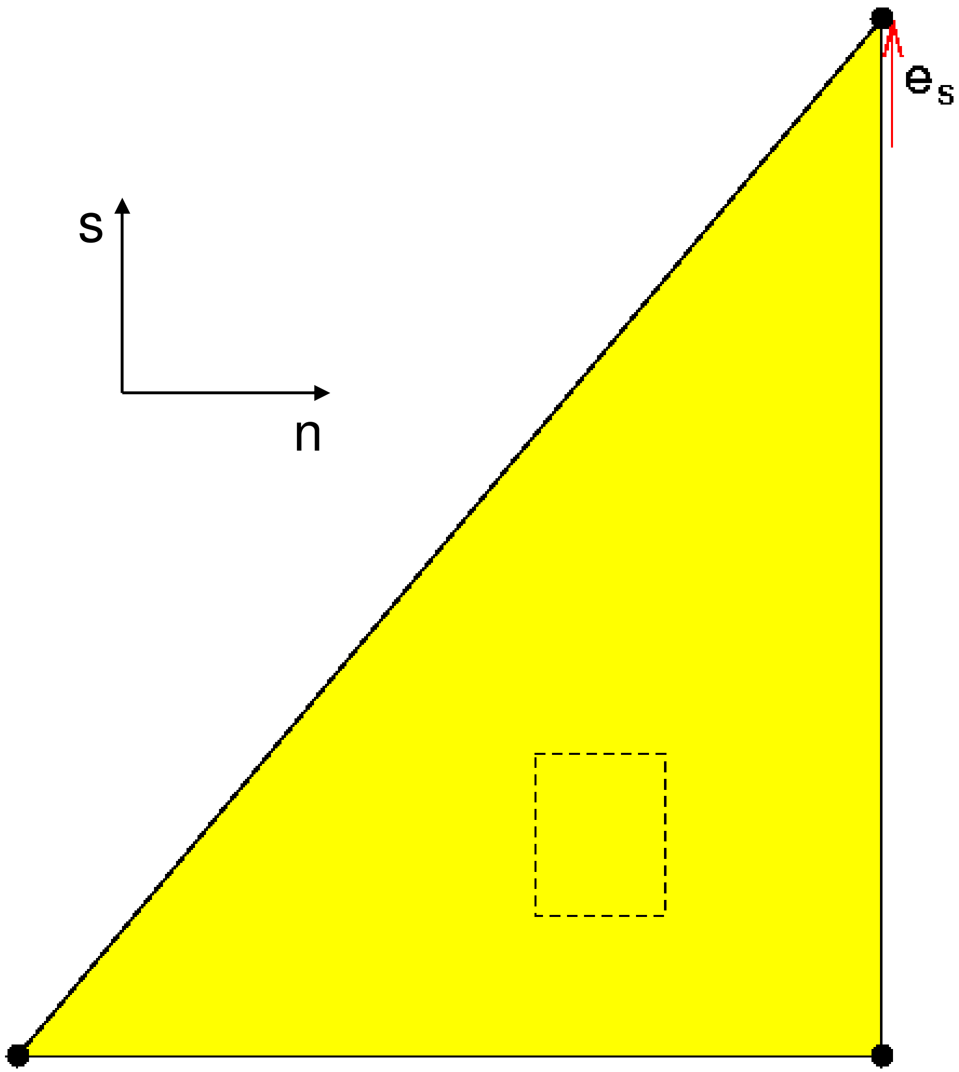
$$t_n = 0$$

$$t_s = \tau = Ge_n / h$$

Uniqueness of the element response



Uniqueness of the element response



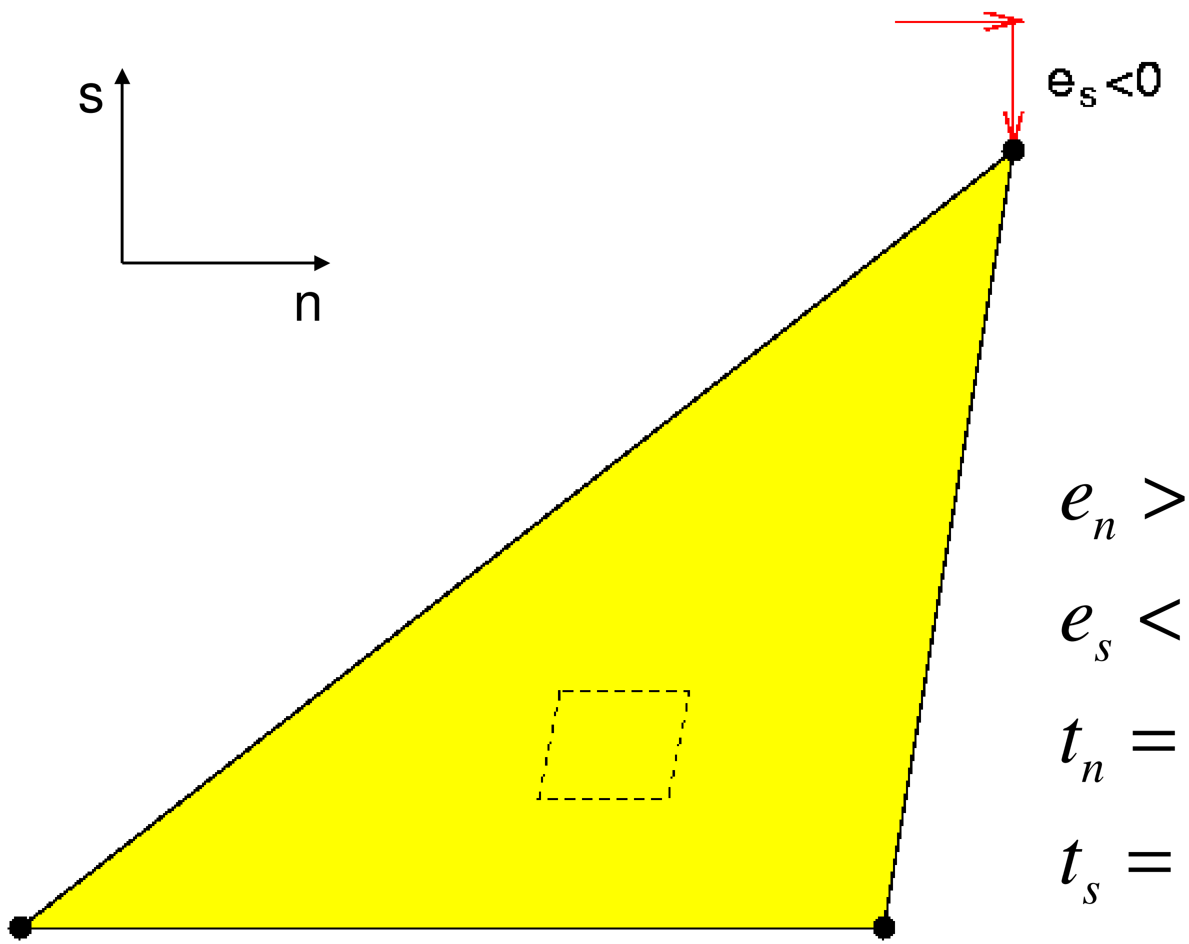
$$\varepsilon_{ss} = e_s / h$$

$$\sigma_{nn} = D_{12}\varepsilon_{ss}$$

$$t_n = \sigma_{nn} = D_{12} e_s / h$$

$$t_s = 0$$

Uniqueness of the element response



$$e_n > 0$$

$$e_s < 0$$

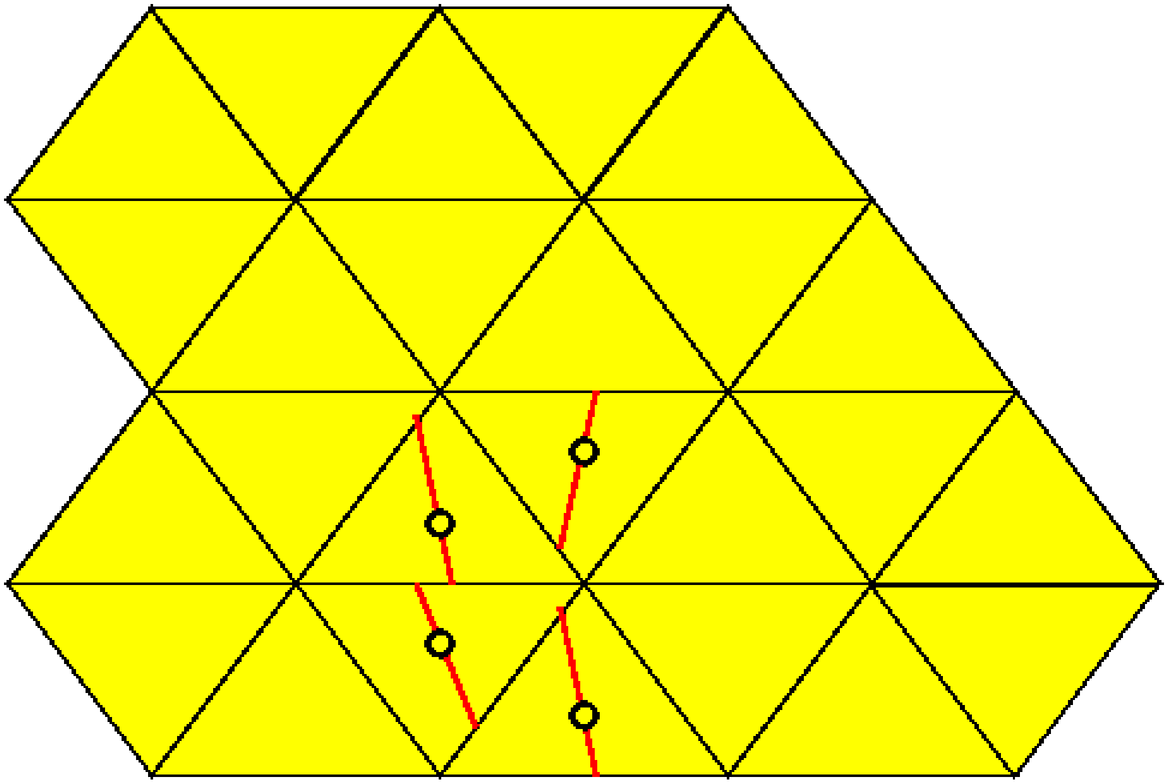
$$t_n = D_{12} e_s / h < 0$$

$$t_s = G e_n / h > 0$$

$$\mathbf{e}^T \mathbf{t}^e < 0$$

Uniqueness of the element response

discontinuity segments placed at element centers



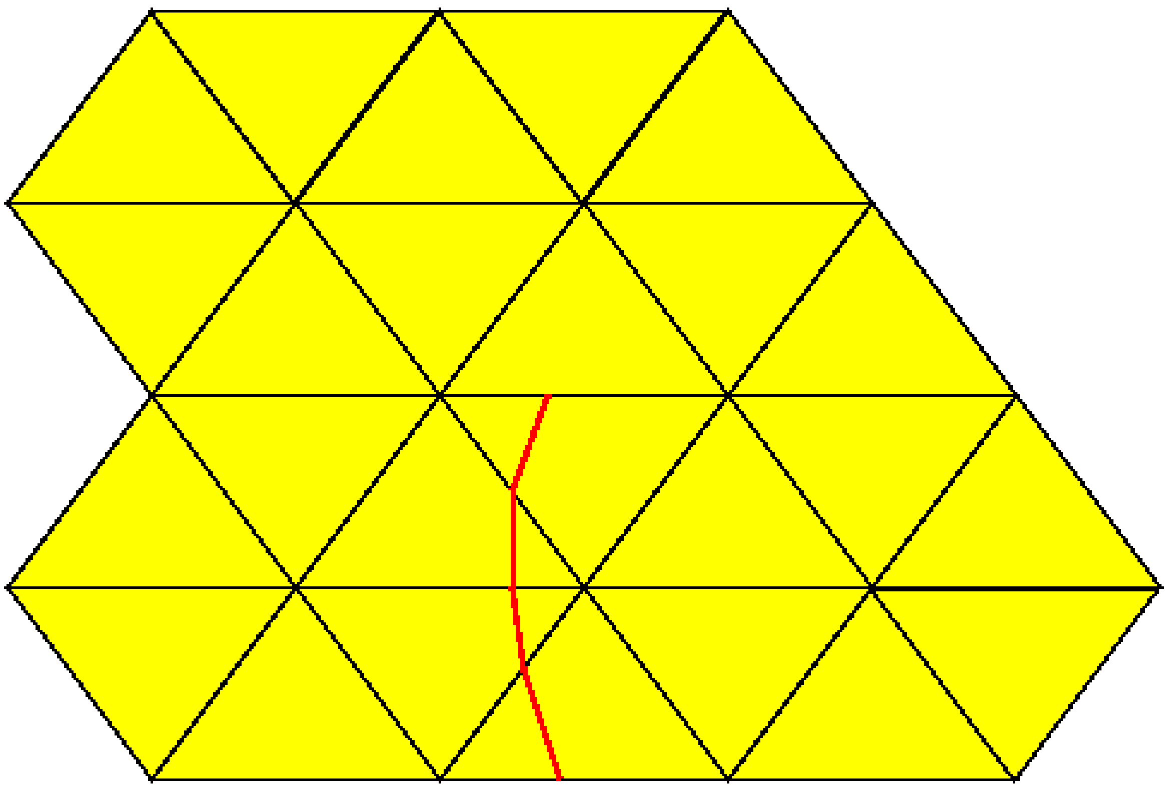
Uniqueness of the element response

discontinuity segments placed at element centers

maximum deviation α between element side and discontinuity is limited (e.g., 30 degrees for an equilateral triangle)

Uniqueness of the element response

discontinuity segments form a continuous path



Uniqueness of the element response

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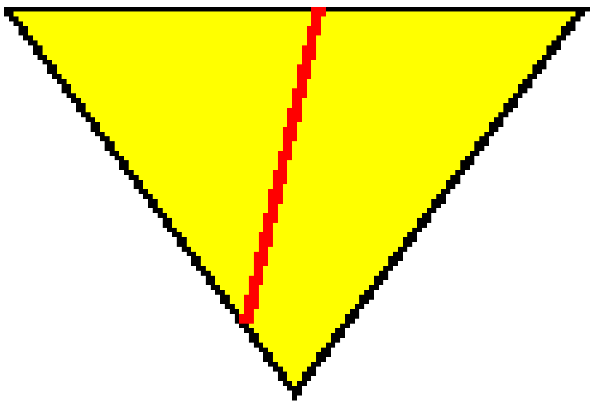

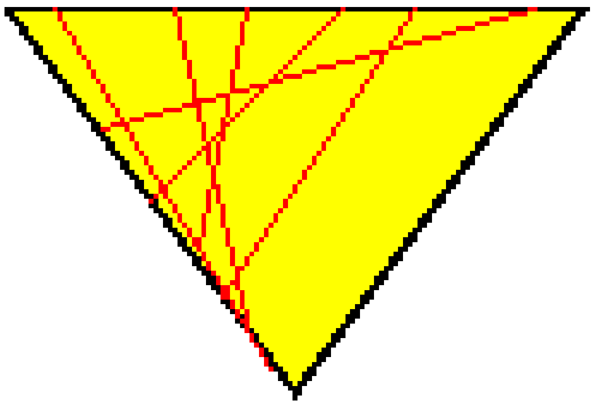

maximum deviation α between element side and discontinuity
is given by the largest angle of the triangle
(e.g., 60 degrees for an equilateral triangle)

Uniqueness of the element response

Condition under which uniqueness can be guaranteed if the element is sufficiently small:

plane stress … $\cos\alpha > \dfrac{1+\nu}{3-\nu}$

true only if $\nu < 1/3$ and the element is close to equilateral

Uniqueness of the element response

Condition under which uniqueness can be guaranteed if the element is sufficiently small:

plane stress …
$$\cos\alpha > \frac{1+\nu}{3-\nu}$$

true only if $\nu < 1/3$ and the element is close to equilateral

plane strain …
$$\cos\alpha > \frac{1}{3-4\nu}$$

true only if $\nu < 1/4$ and the element is close to equilateral

Uniqueness of the element response

Condition under which uniqueness can be guaranteed if the element is sufficiently small:

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true only if $\nu < 1/3$ and the element is close to equilateral

plane strain … $\cos\alpha > \dfrac{1}{3-4\nu}$

true only if $\nu < 1/4$ and the element is close to equilateral

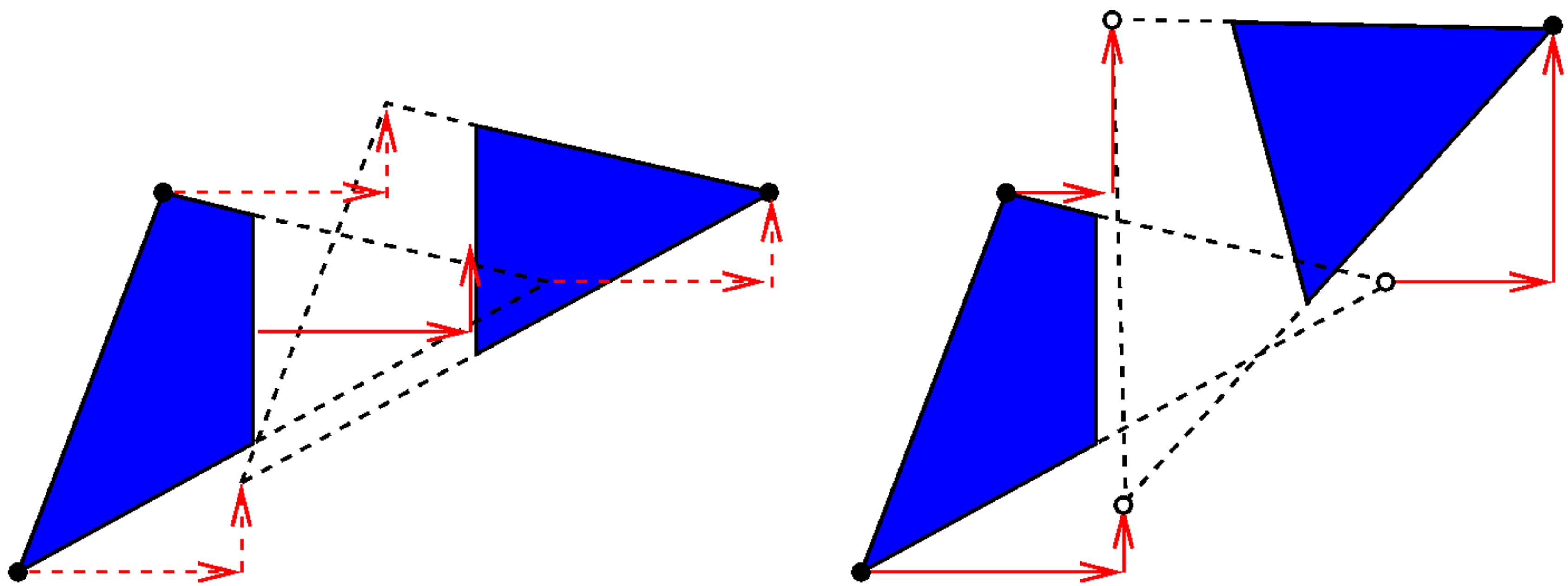
three dimensions … $\cos\alpha > \dfrac{1}{3-4\nu}$

violated even if the tetrahedral element is regular

Comparison of EED-EAS and XFEM-PUM

Embedded discontinuity

Extended finite elements



Comparison of EED-EAS and XFEM-PUM

	Embedded discontinuity	Extended finite elements
DOF‘s added	locally	globally
and related to	elements	nodes
Approximation of crack opening	discontinuous	continuous
Enrichment	incompatible	compatible
Separated parts	partially interacting	independent
Numerical behavior	**rather fragile**	**more robust**

Comparison of EED-EAS and XFEM-PUM

	Embedded discontinuity	Extended finite elements
Stiffness matrix	always nonsymmetric	can be symmetric
Integration scheme for continuous part	remains standard	must be modified
Global degrees of freedom	do not change	added during simulation
Implementation effort	smaller	larger