Modeling of Localized Inelastic Deformation

Milan Jirásek

General outline:

A. Introduction
B. Elastoplasticity
C. Damage mechanics
D. Strain localization
E. Regularized continuum models
F. Strong discontinuity models

F. Strong discontinuity models

F.1 Introduction

F.2 Embedded discontinuities (EED-EAS)

F.3 Extended finite elements (XFEM-PUM)

F.4 Comparative evaluation

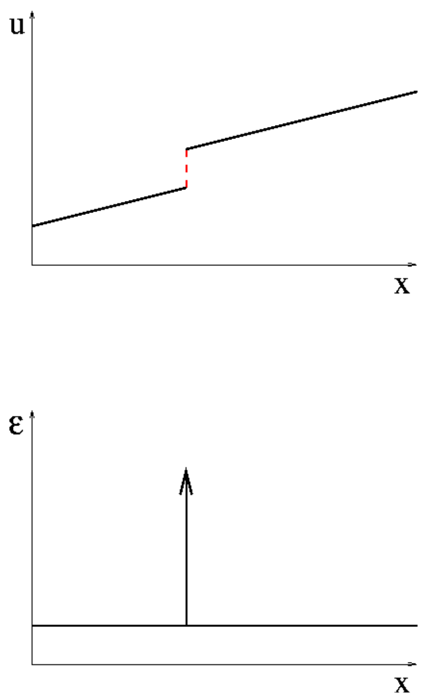
F.5 Regularized continua with strong discont.

F.1

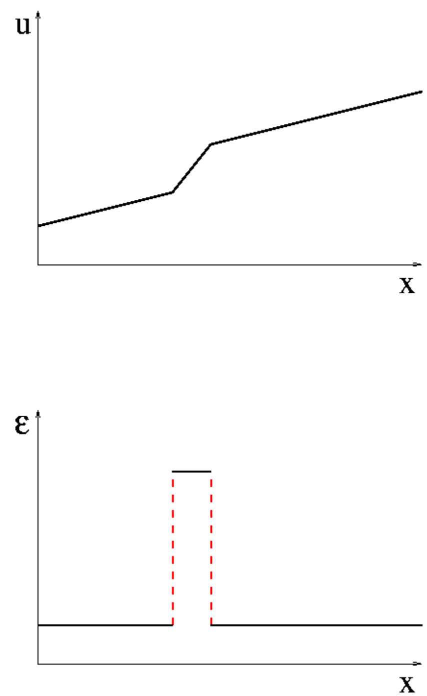
Introduction

Classification of models: kinematic aspects

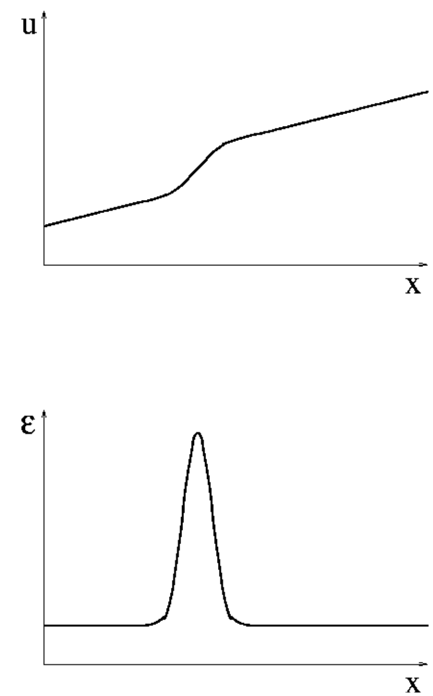
Strong
discontinuity



Weak
discontinuity

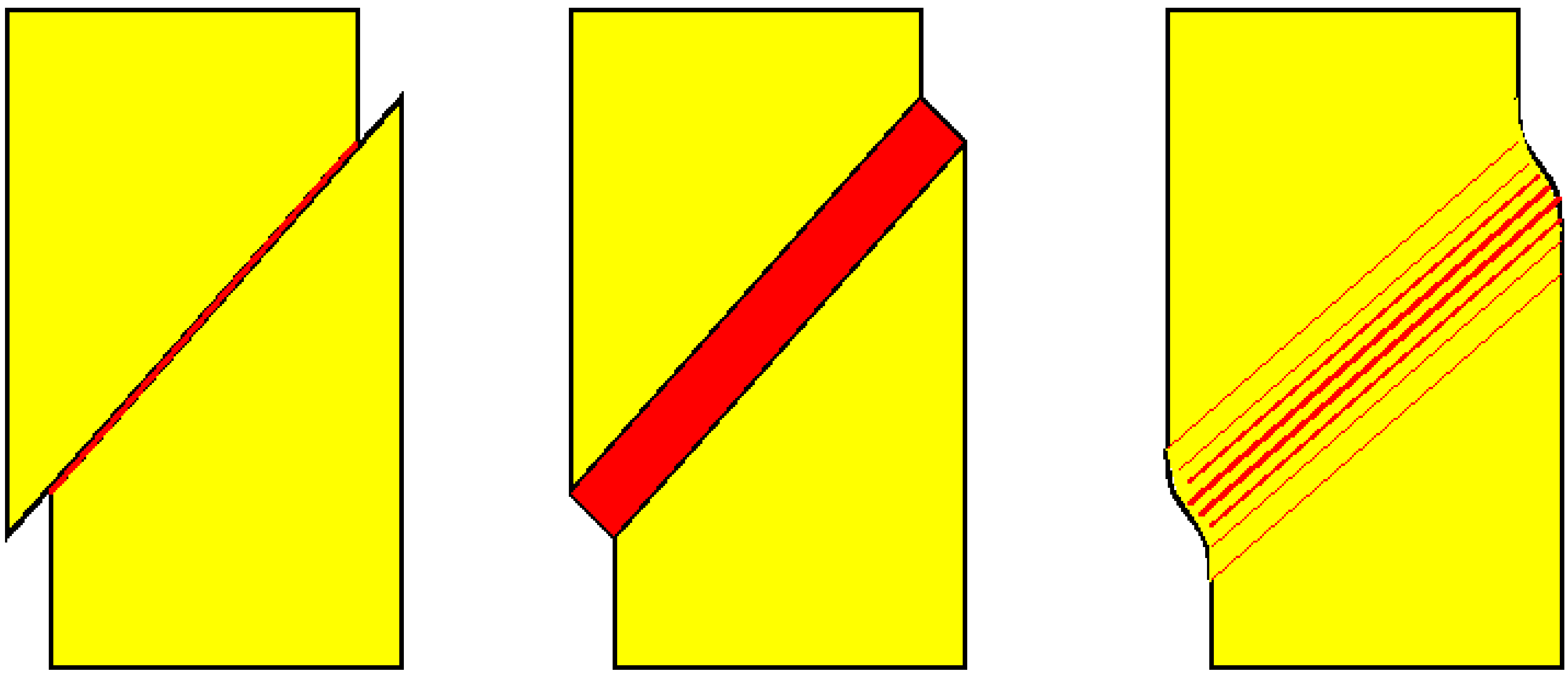


Regularized
localization zone



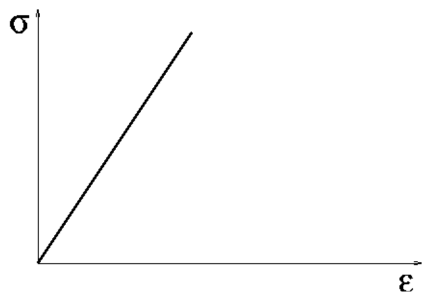
Classification of models: kinematic aspects

Strong discontinuity | Weak discontinuity | Regularized localization zone

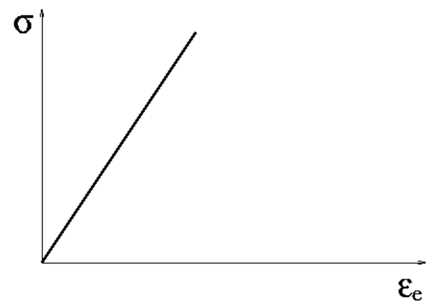


Classification of models: material laws

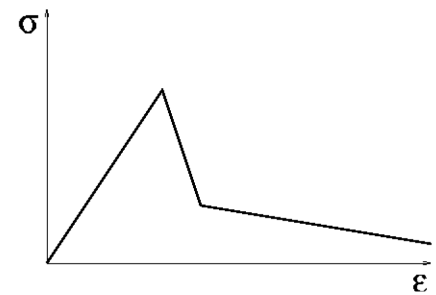
Stress-strain law



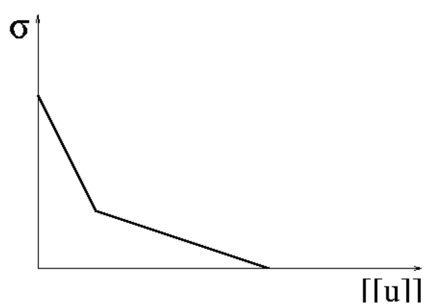
Stress-strain law
(pre-localization part)



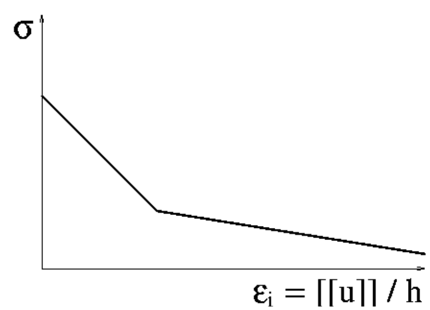
Stress-strain law



Traction-separation law



Stress-strain law
(post-localization part)

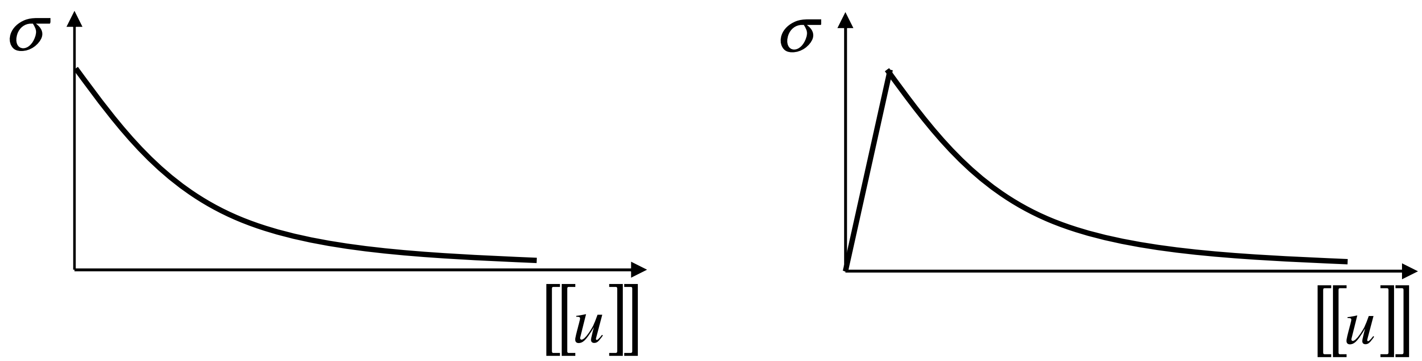


Enrichment acting
as localization limiter:

- nonlocal
- gradient
- Cosserat
- viscosity

Traction-separation laws

1) Formulated directly in the traction-separation space
   a) with nonzero elastic compliance (elasto-plastic, …)
   b) with zero elastic compliance (rigid-plastic, …)



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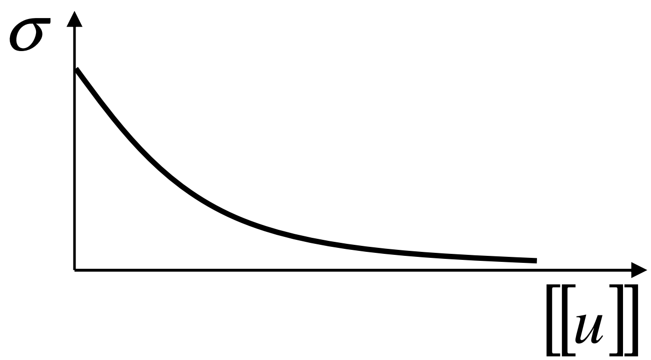
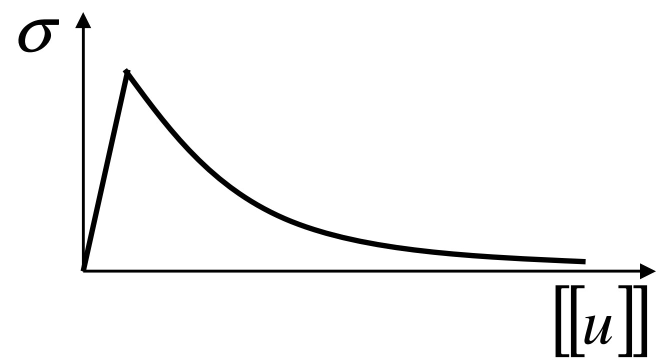
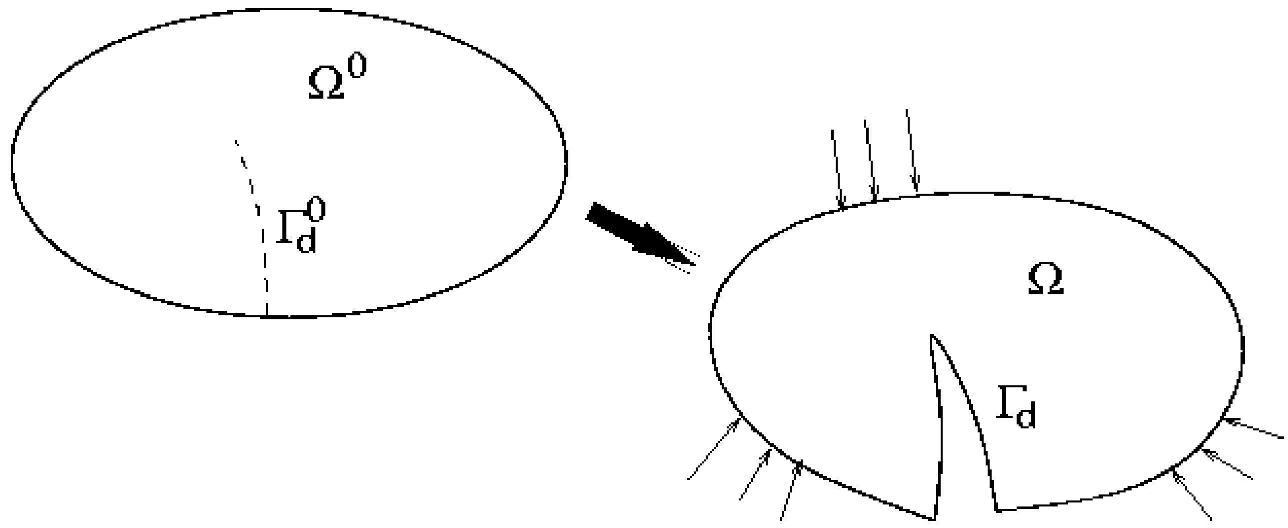




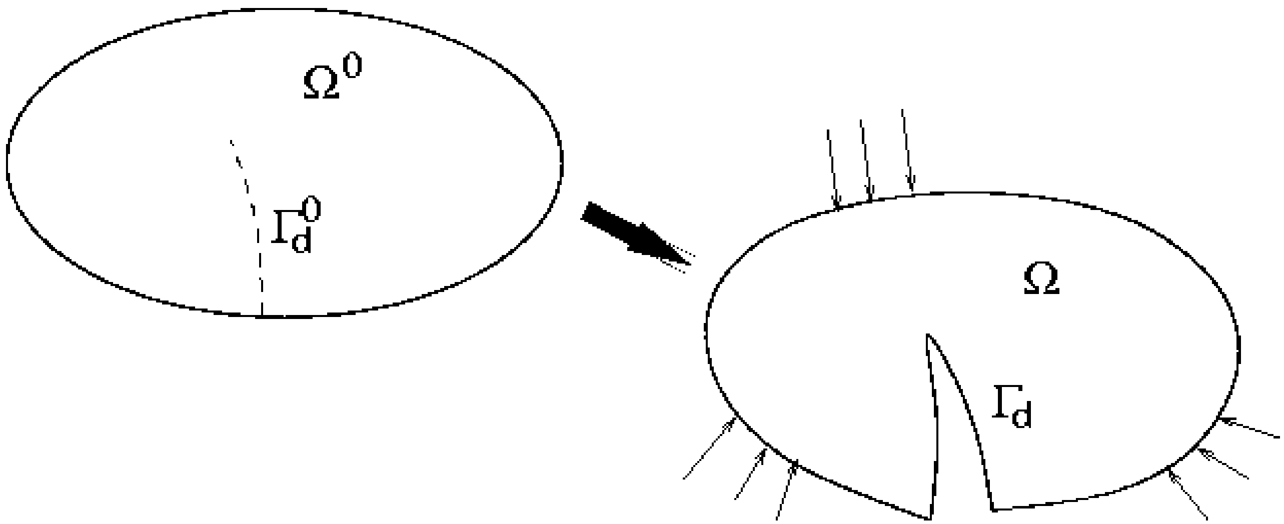
2) "Derived" from a stress-strain law (softening continuum) using the strong discontinuity approach

Finite element representation of strong discontinuities



1) Discontinuities at element interfaces:
   a) Remeshing
   b) Interspersed potential discontinuities

Finite element representation of strong discontinuities

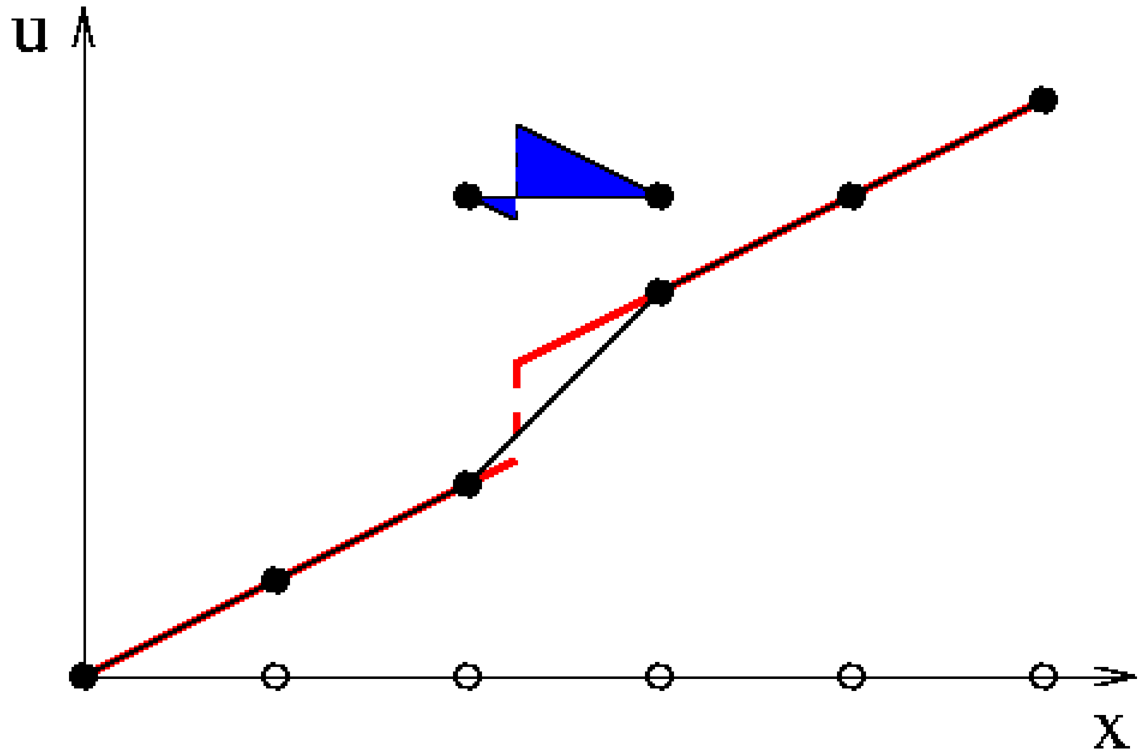


2) Arbitrary discontinuities across elements:

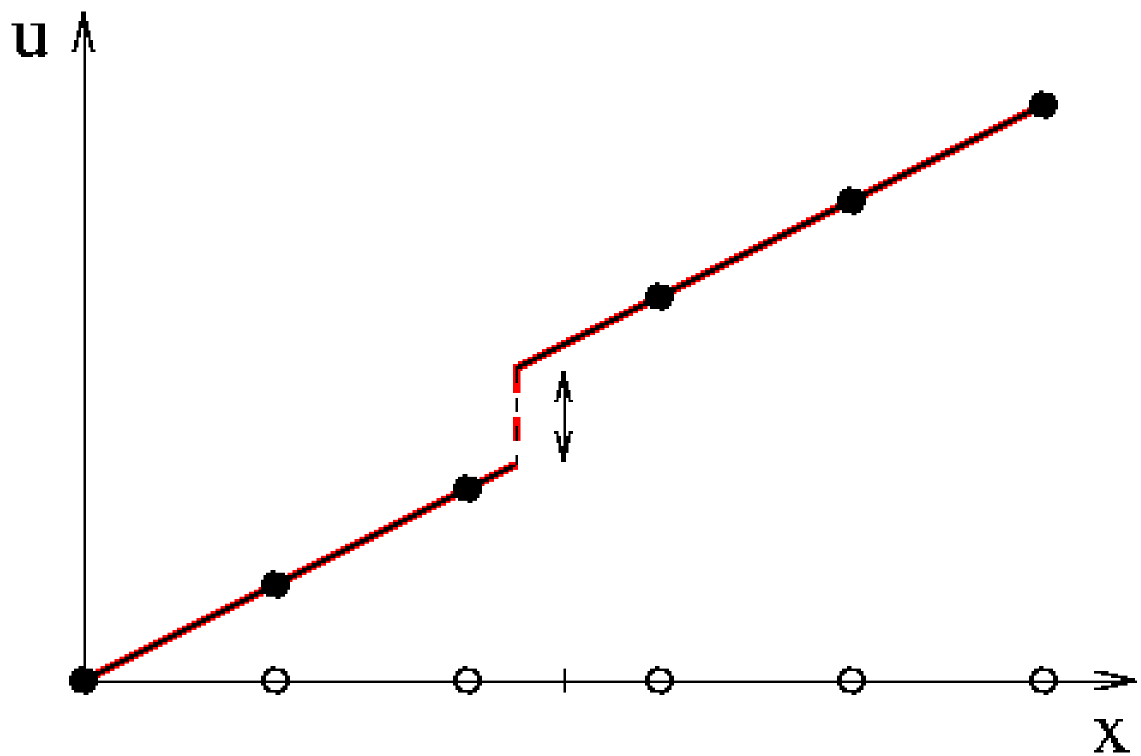
   a) Elements with embedded discontinuities using the enhanced assumed strain formulation (EED-EAS)

   b) Extended finite elements based on the partition-of-unity concept (XFEM-PUM)

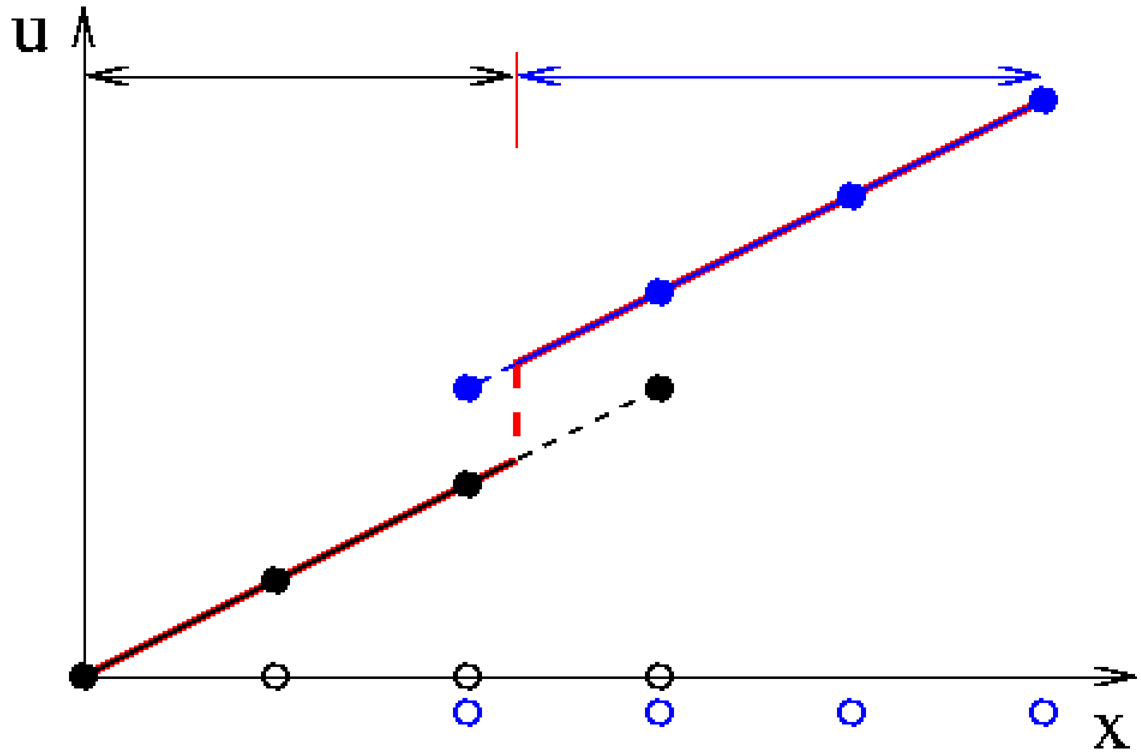
Embedded discontinuity (enhanced assumed strain)



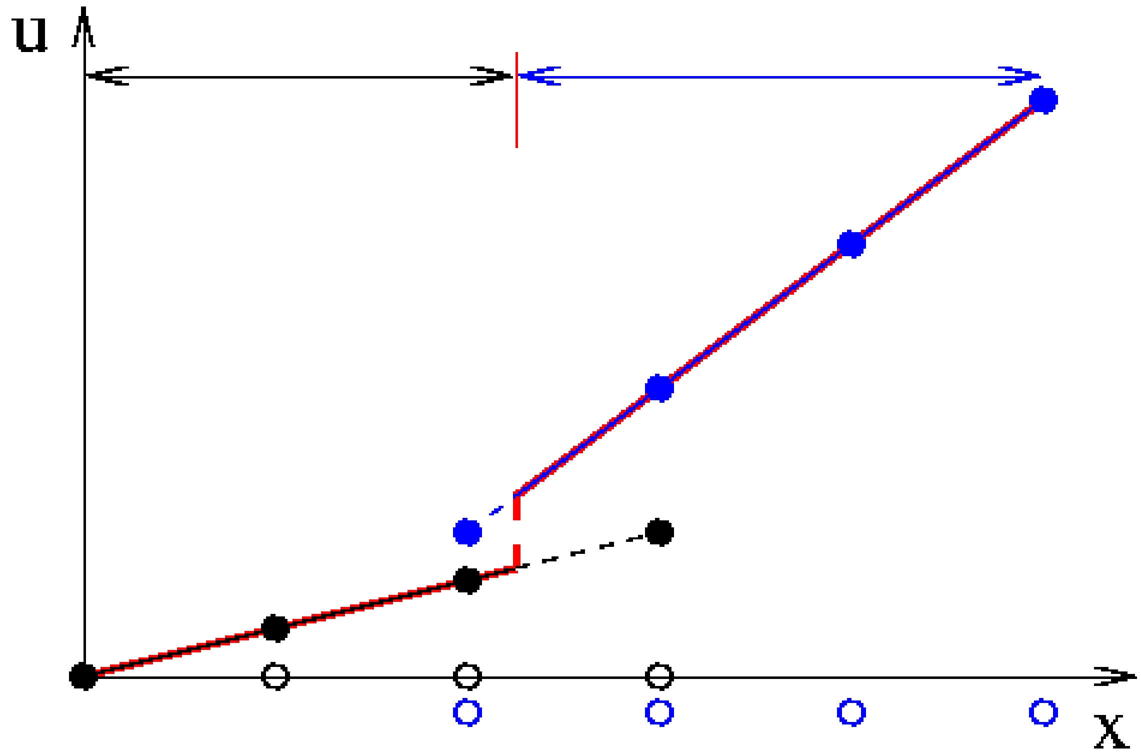
Embedded discontinuity (enhanced assumed strain)



Approximation on two overlapping meshes (XFEM)

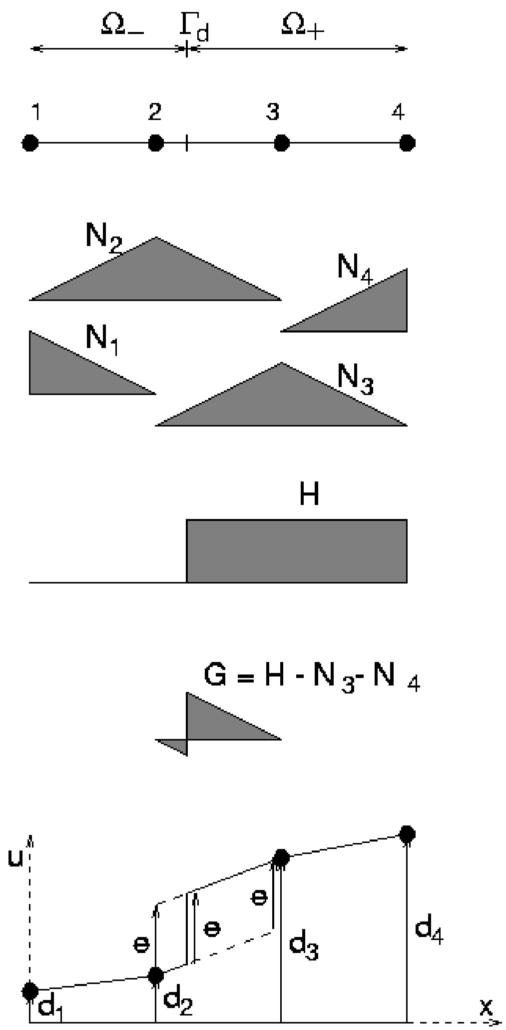


Approximation on two overlapping meshes (XFEM)



Enrichment of interpolation functions in one dimension

EED-EAS



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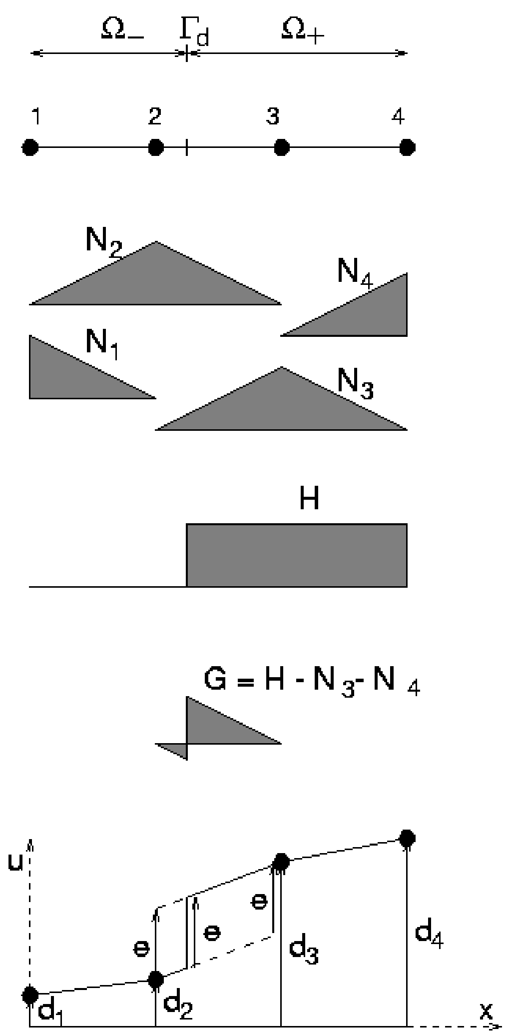
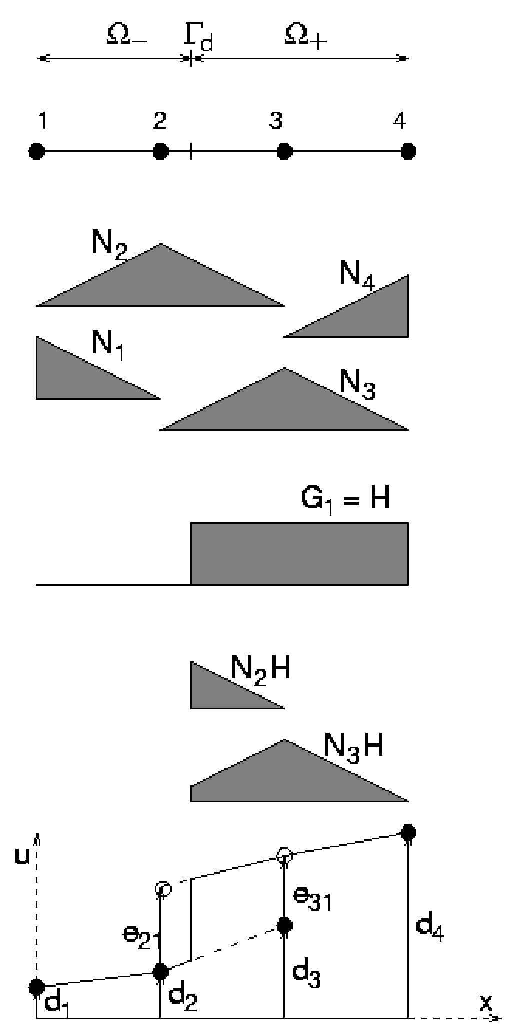


XFEM-PUM



Enrichment of interpolation functions in one dimension

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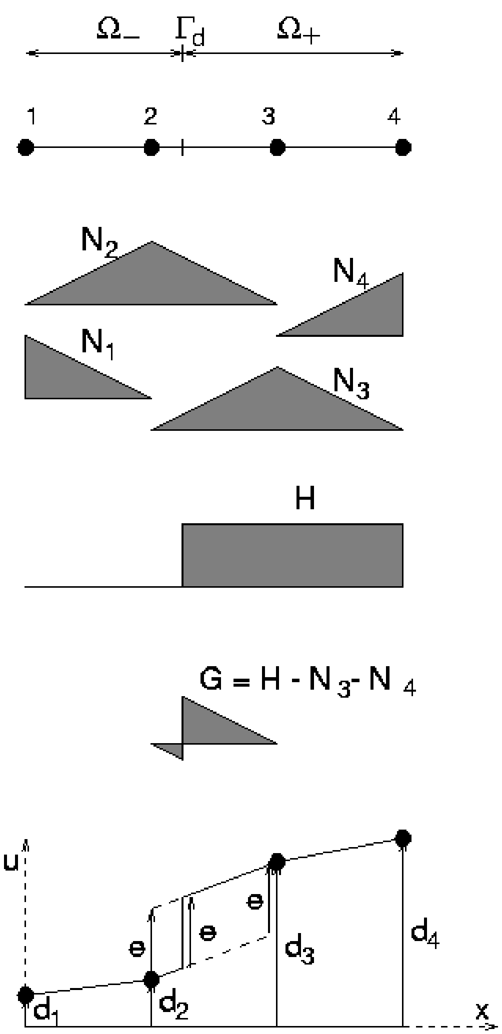
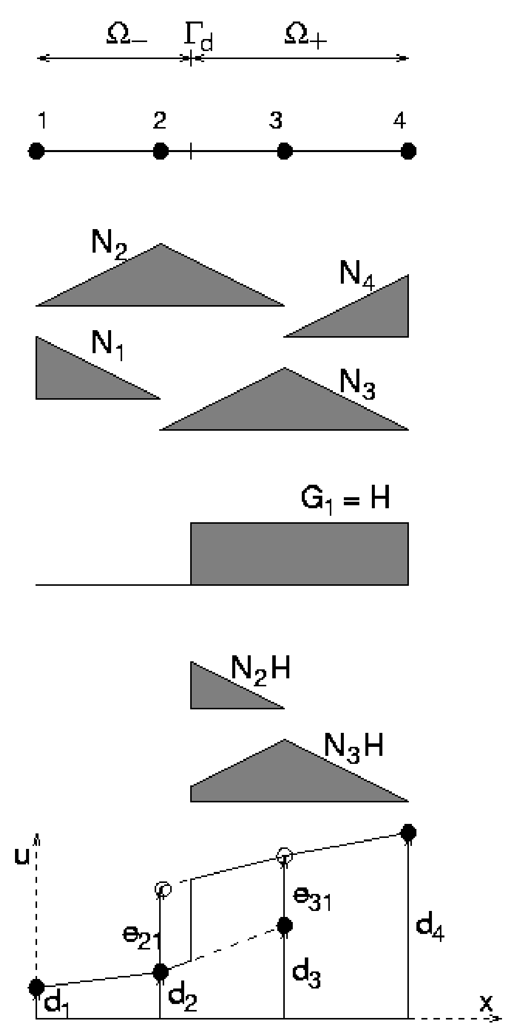


XFEM-PUM



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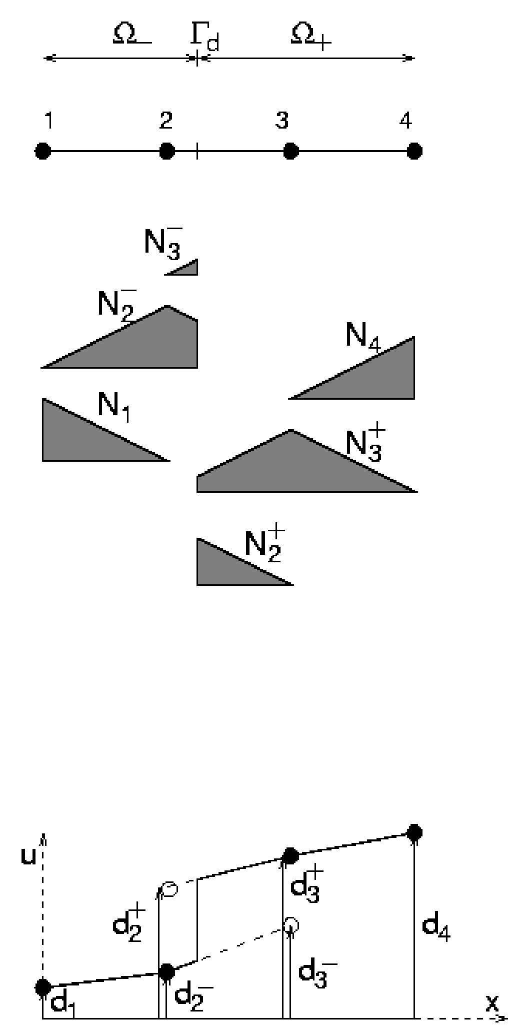


F.2

Elements with Embedded Discontinuities (EAS)

Elements with embedded discontinuities

$$\mathbf{d} \rightarrow \boldsymbol{\varepsilon} \rightarrow \boldsymbol{\sigma} \rightarrow \mathbf{f}$$

Elements with embedded discontinuities

$\mathbf{d}$

$\boldsymbol{\varepsilon}$ $\mathbf{e}$ … separation (displacement jump)

$\boldsymbol{\sigma}$ $\mathbf{t}$ … traction

$\mathbf{f}$

Elements with embedded discontinuities

$\mathbf{d}$

? kinematics ?

$\boldsymbol{\varepsilon}$ $\mathbf{e}$

↓ material ↓

$\boldsymbol{\sigma}$ $\mathbf{t}$

? equilibrium ?

$\mathbf{f}$

Elements with embedded discontinuities

$\mathbf{d}$

kinematics

$\boldsymbol{\varepsilon}$ $\quad$ $\mathbf{e}$

$\downarrow$ material $\downarrow$

$\boldsymbol{\sigma}$ $\quad$ $\mathbf{t}$

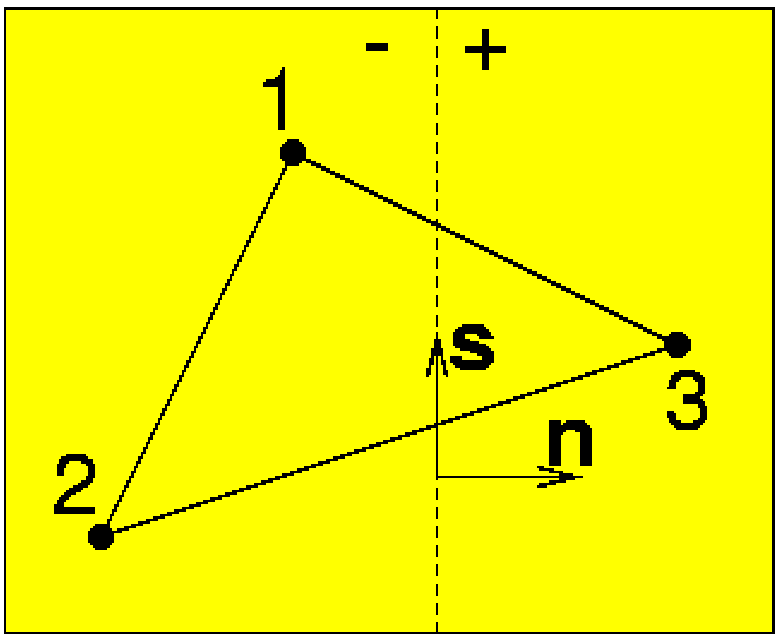
equilibrium

$\mathbf{f}$

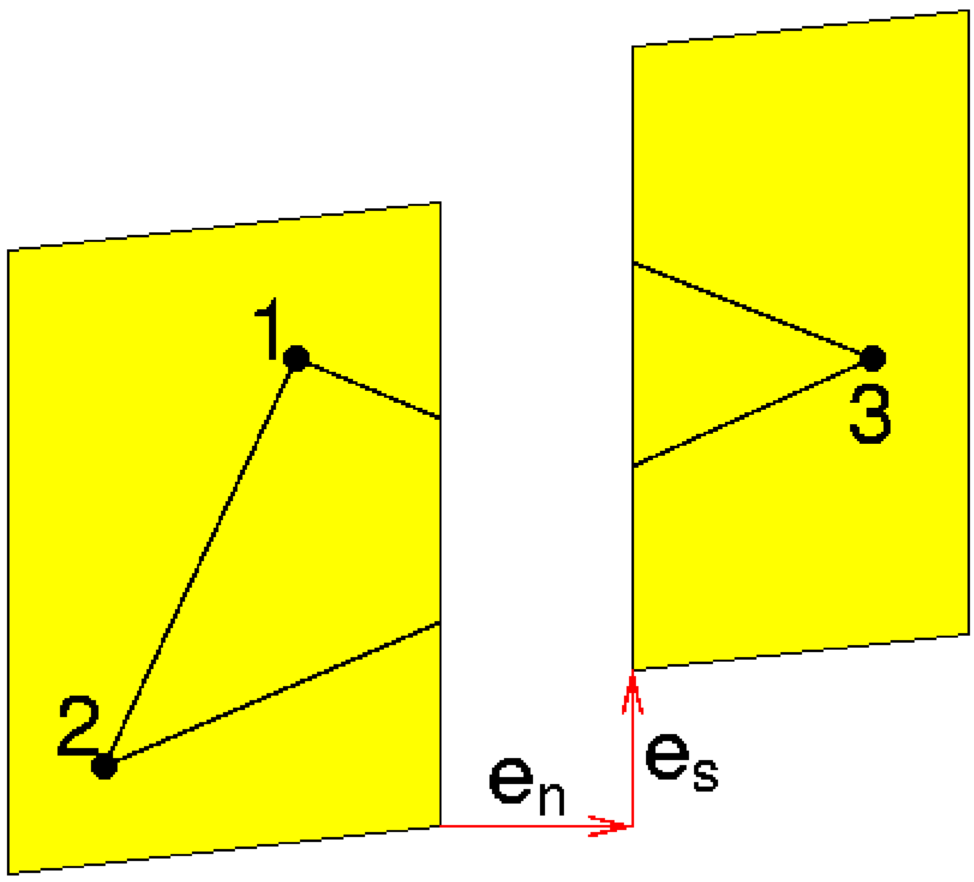
Three types of formulations:

- KOS … kinematically optimal symmetric
- SOS … statically optimal symmetric
- **SKON … kinematically and statically optimal nonsymmetric**

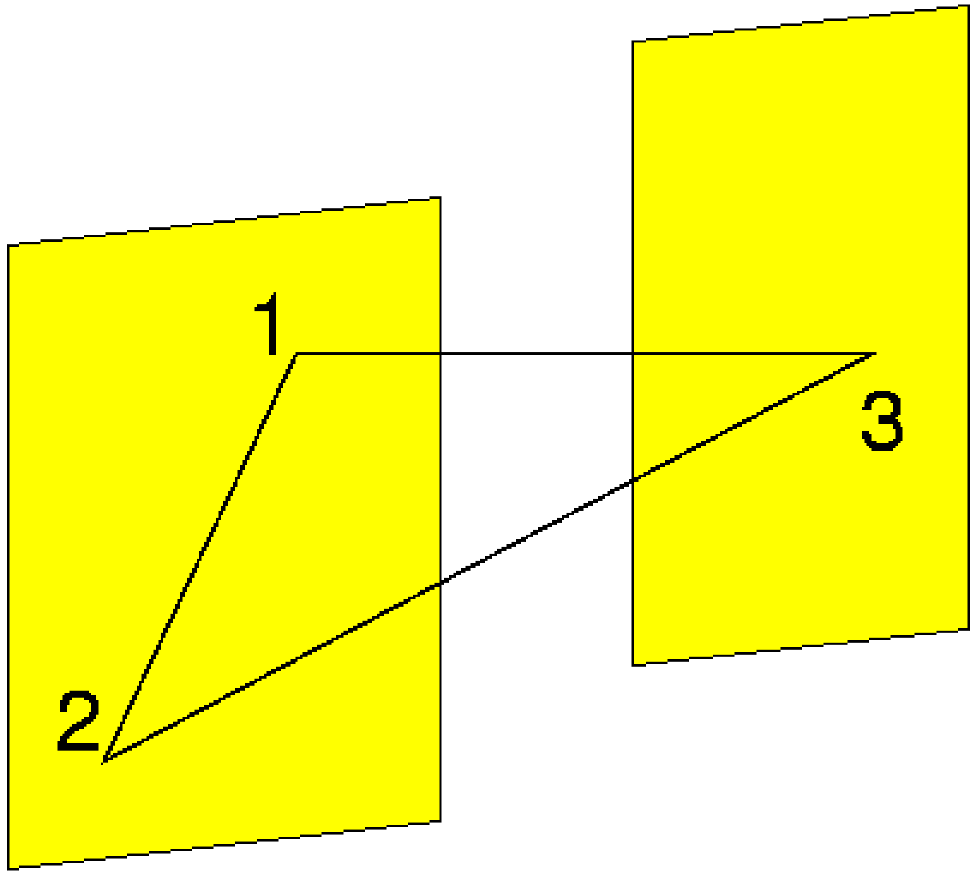
Elements with embedded discontinuities



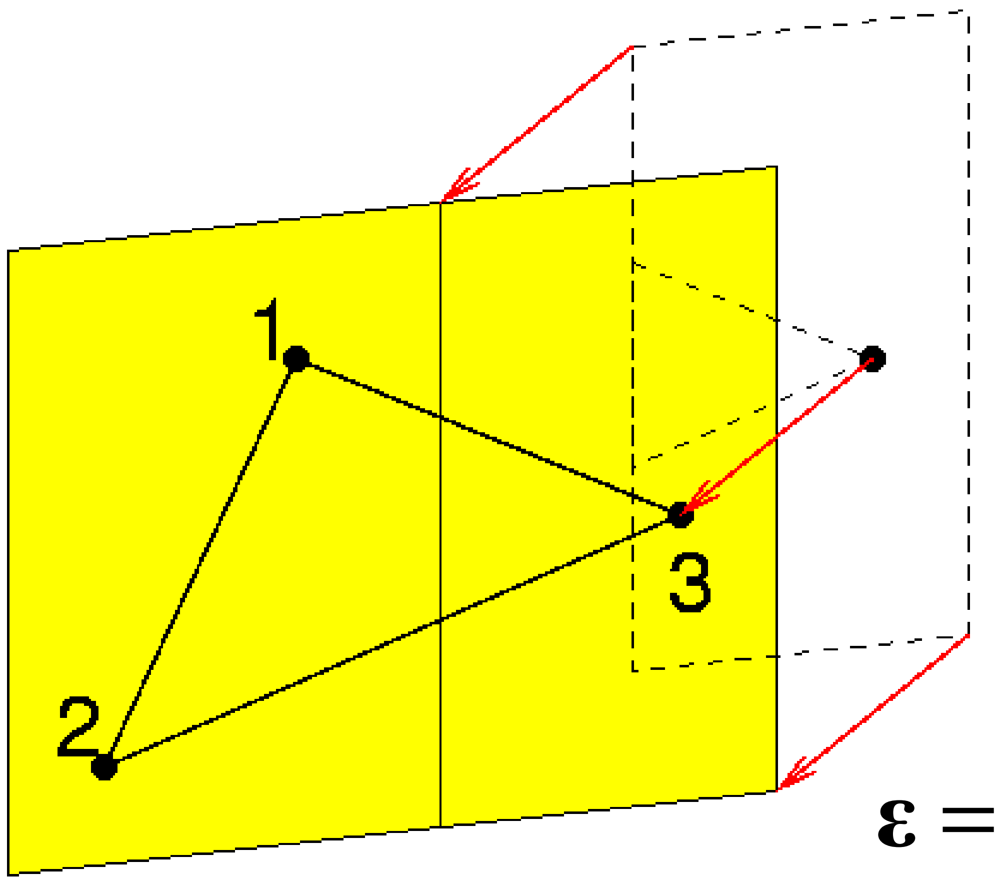
Elements with embedded discontinuities



Elements with embedded discontinuities



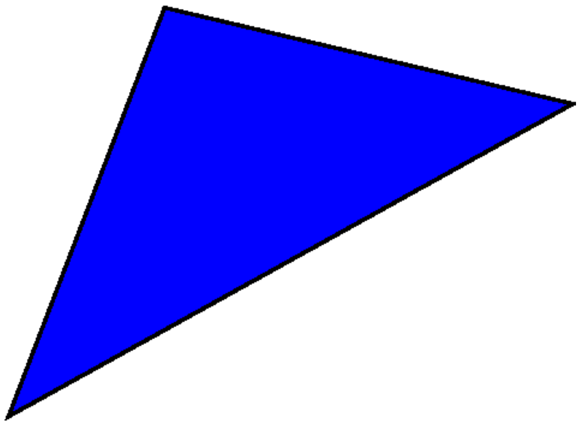
Elements with embedded discontinuities



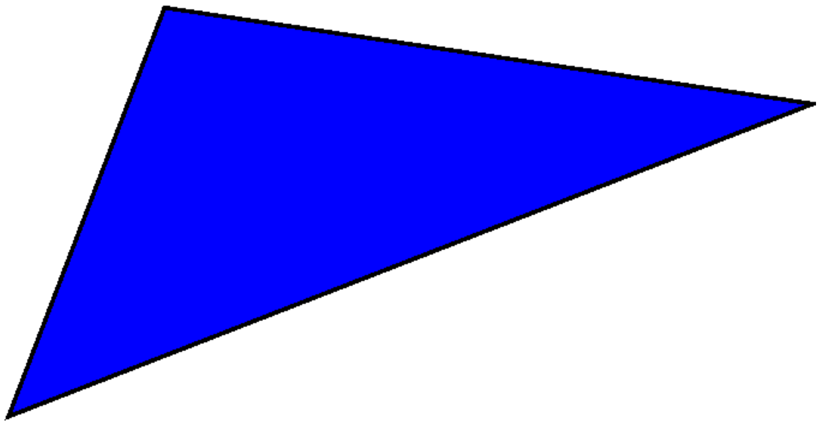
$$\boldsymbol{\varepsilon} = \mathbf{B}\left(\mathbf{d} - \mathbf{He}\right)$$

$$\mathbf{t} = \mathbf{P}^T \boldsymbol{\sigma}$$

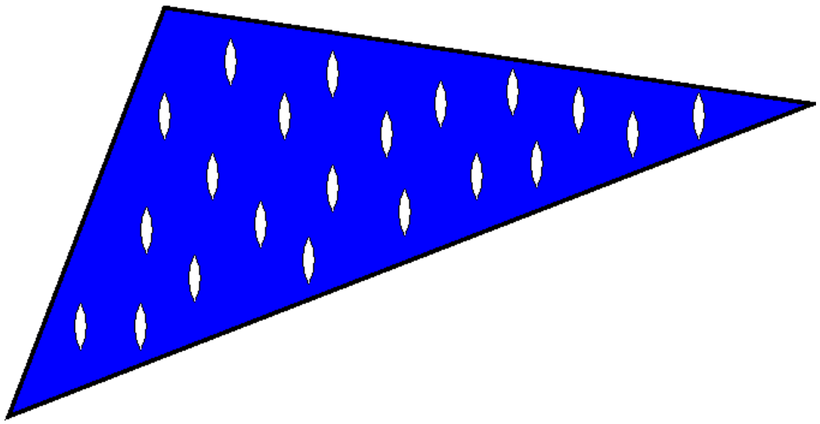
Smeared crack



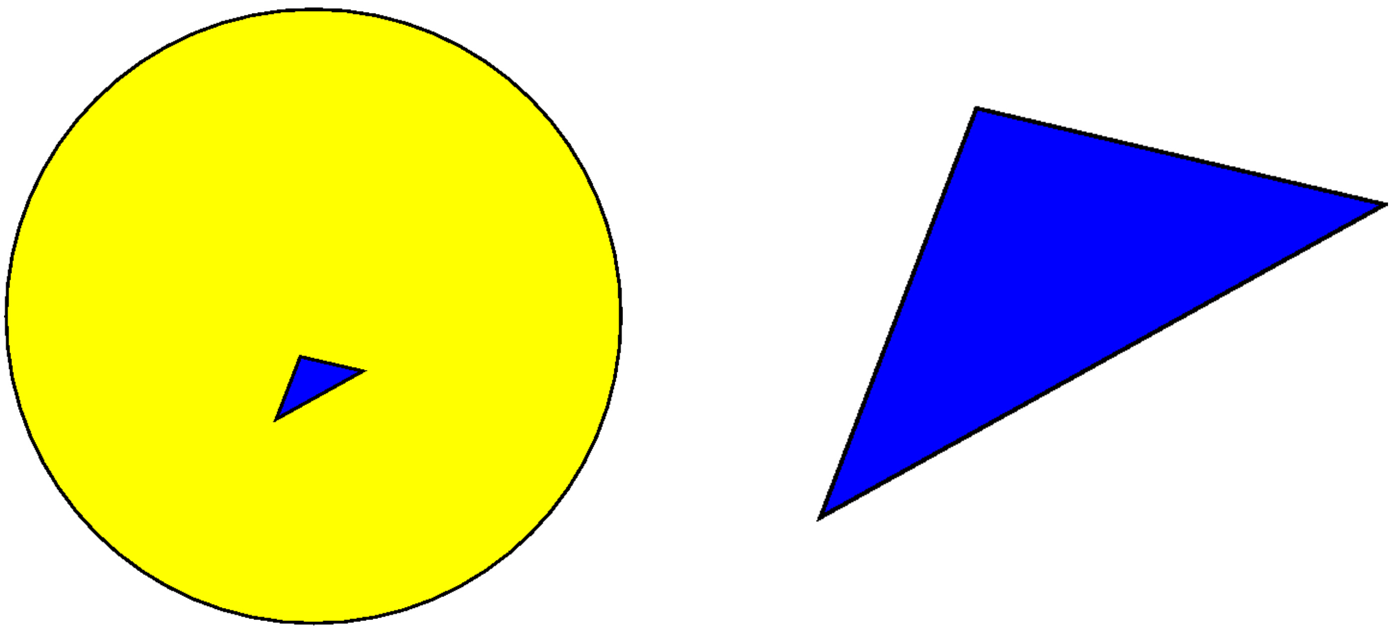
Smeared crack



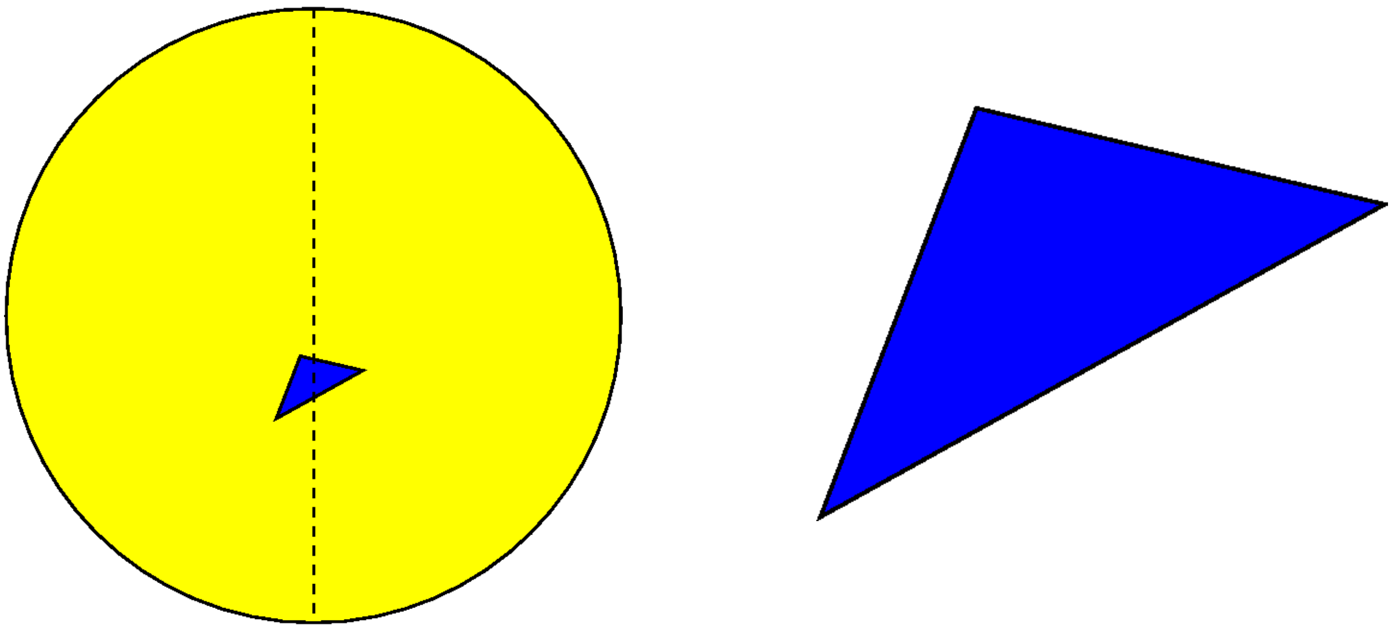
Smeared crack



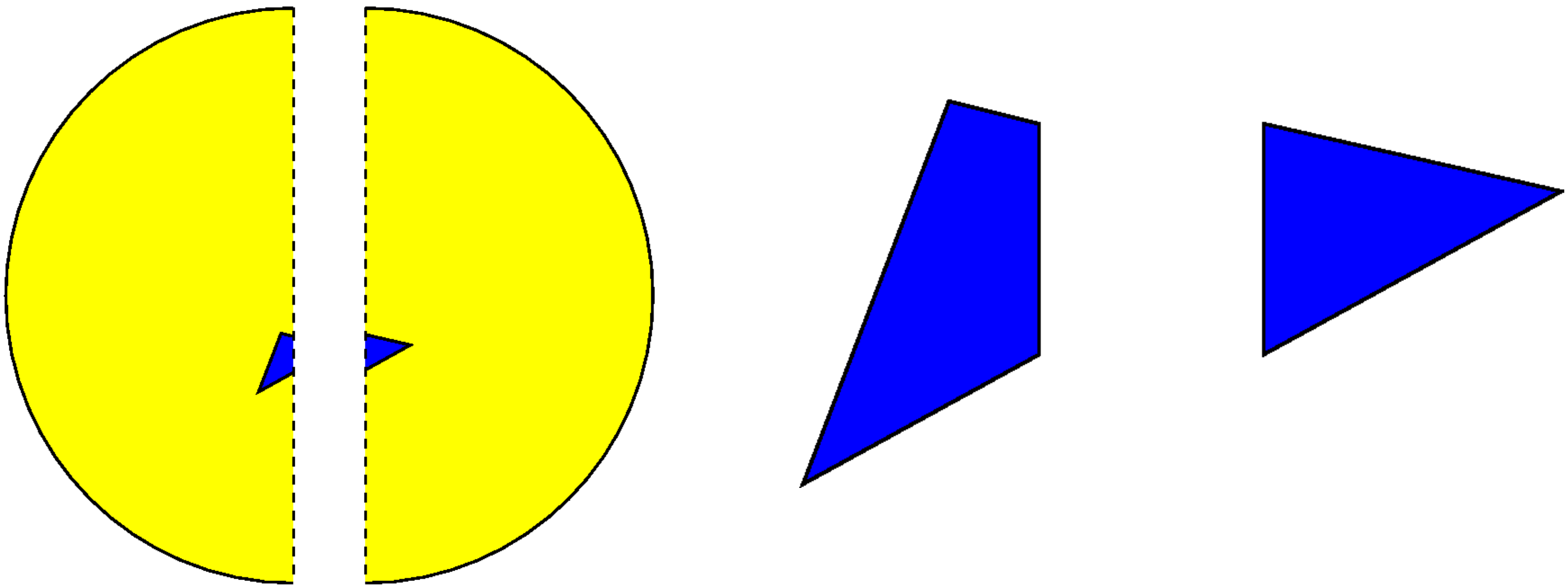
Smeared crack



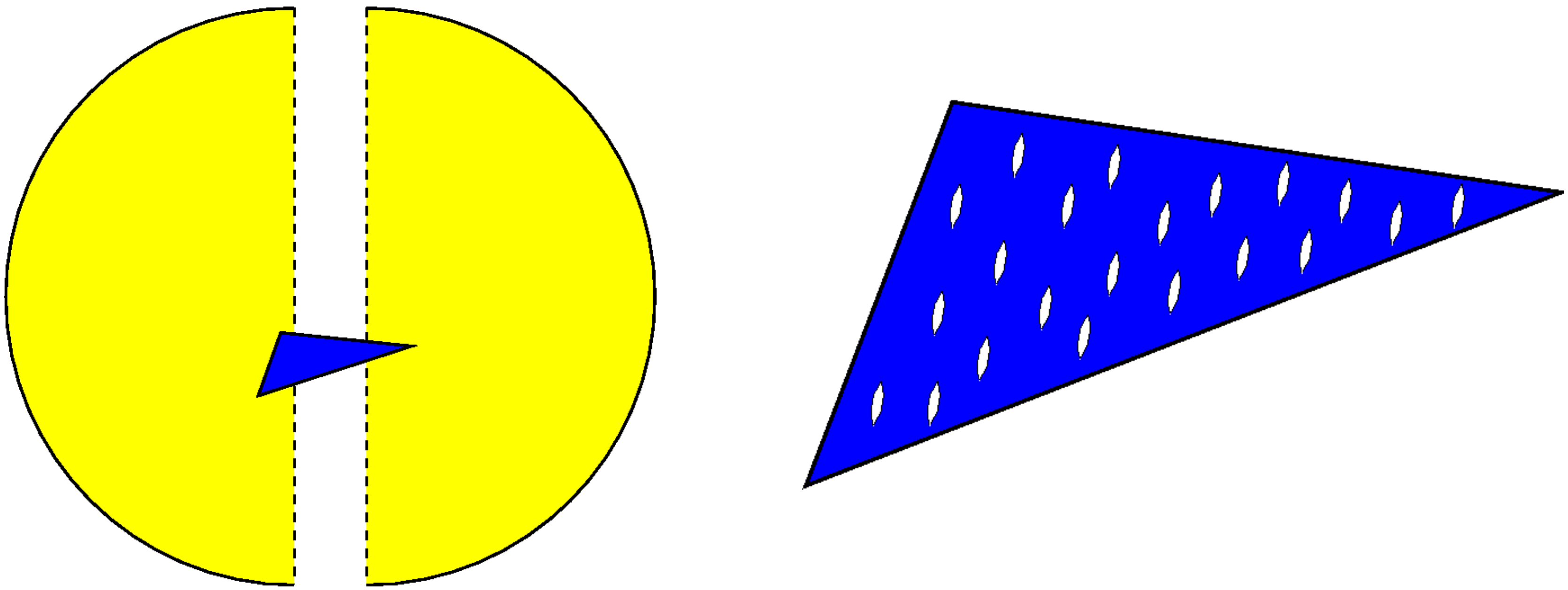
Smeared crack



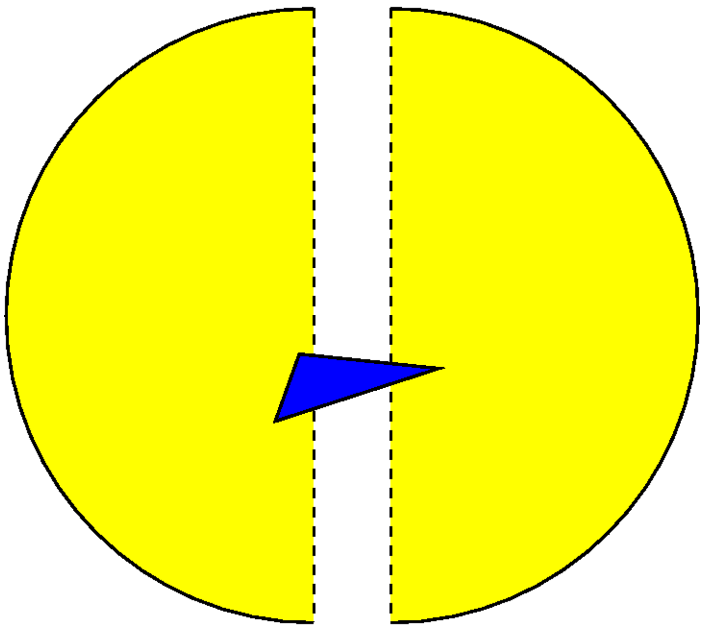
Smeared crack

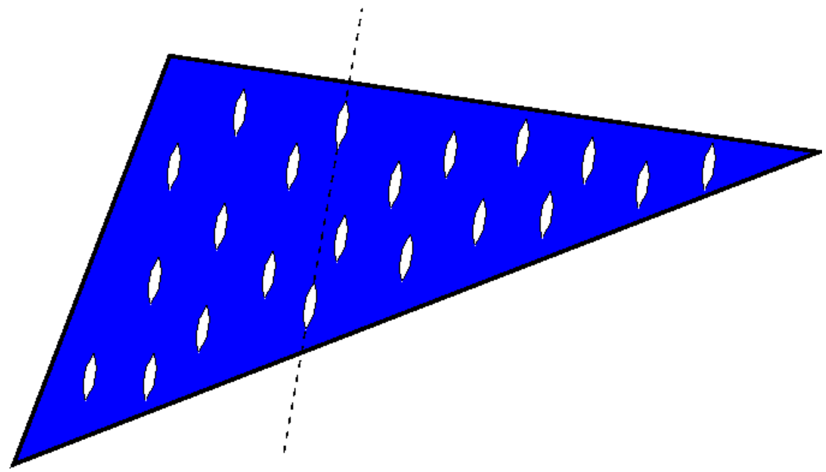


Smeared crack



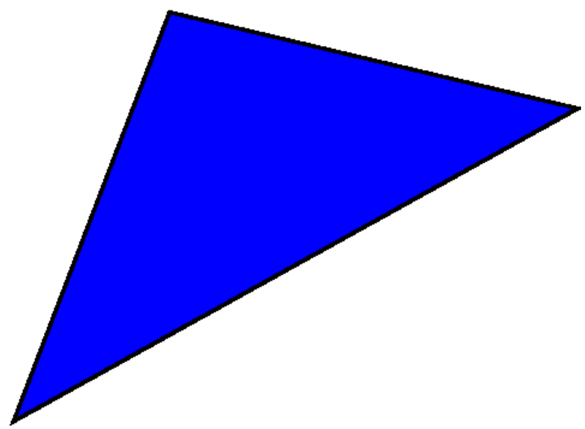
Smeared crack



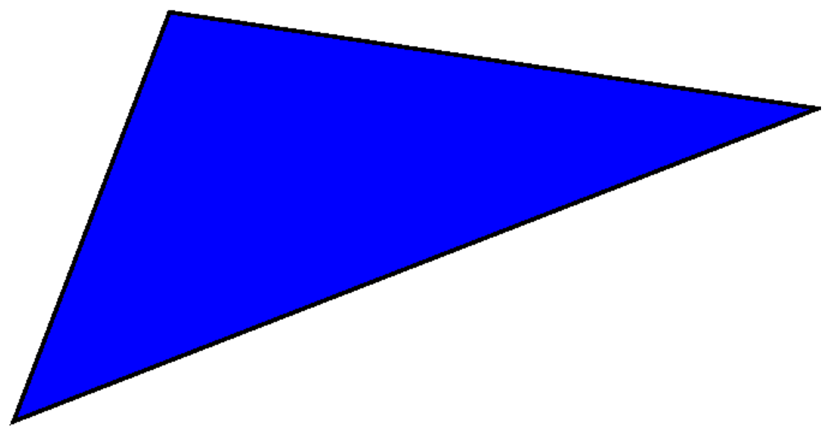


- Misalignment between crack and element
- Distorted principal directions
- Stress locking

Embedded crack (EAS approach)



Embedded crack (EAS approach)



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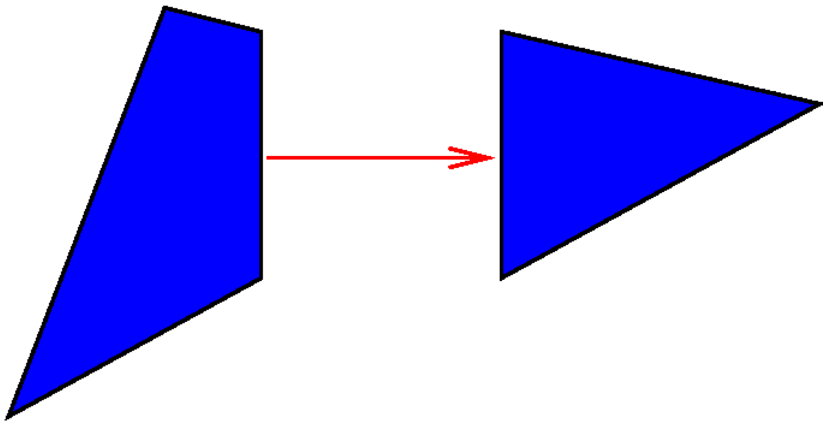

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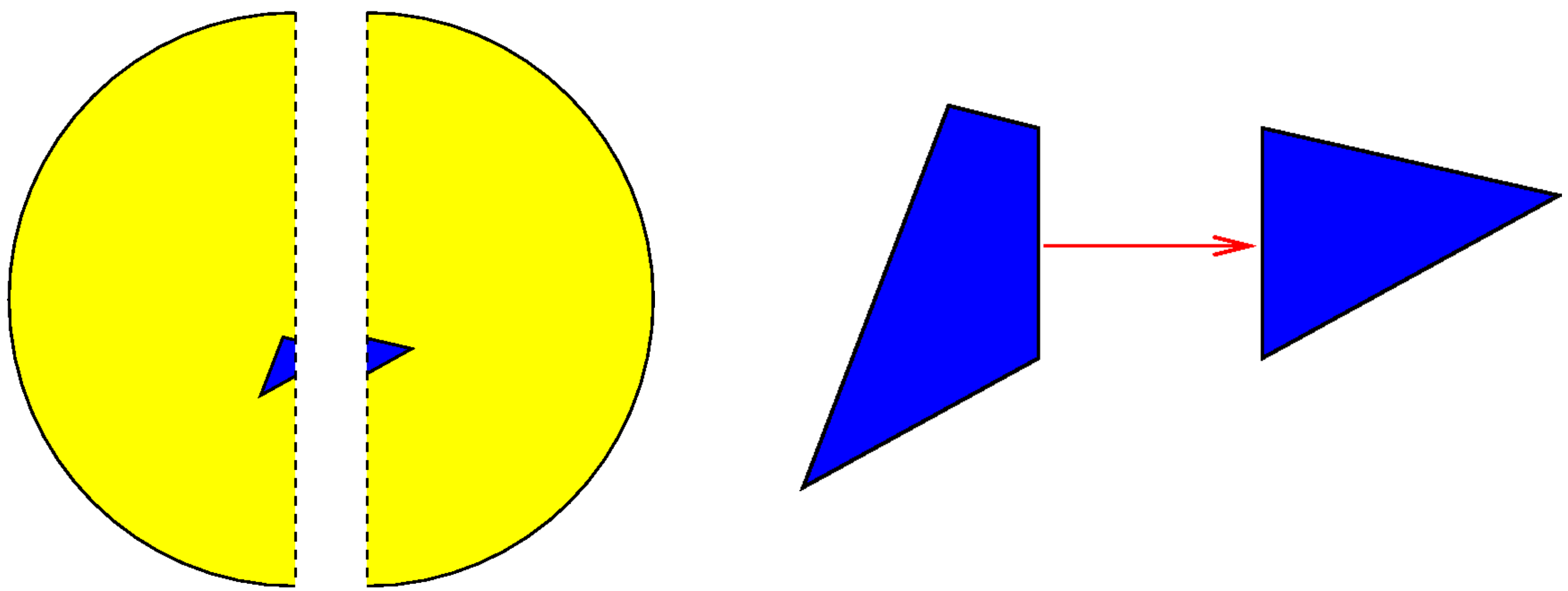
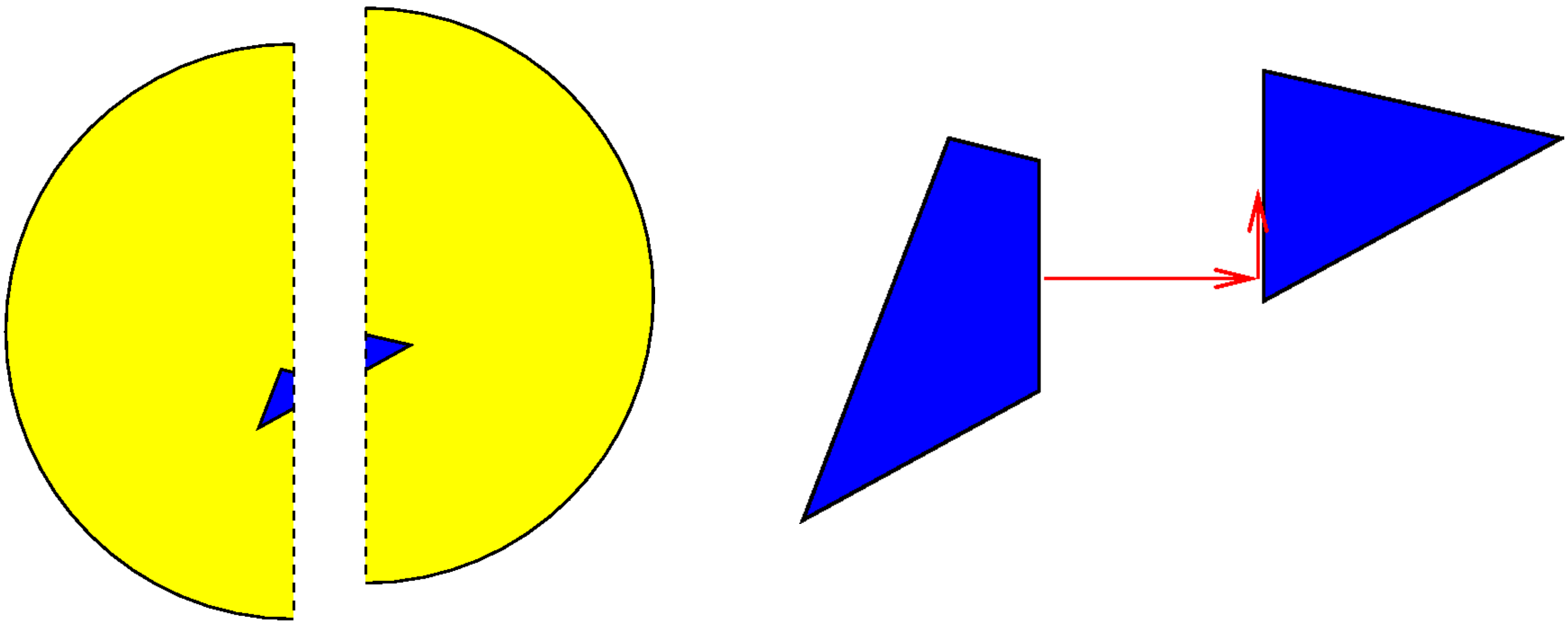
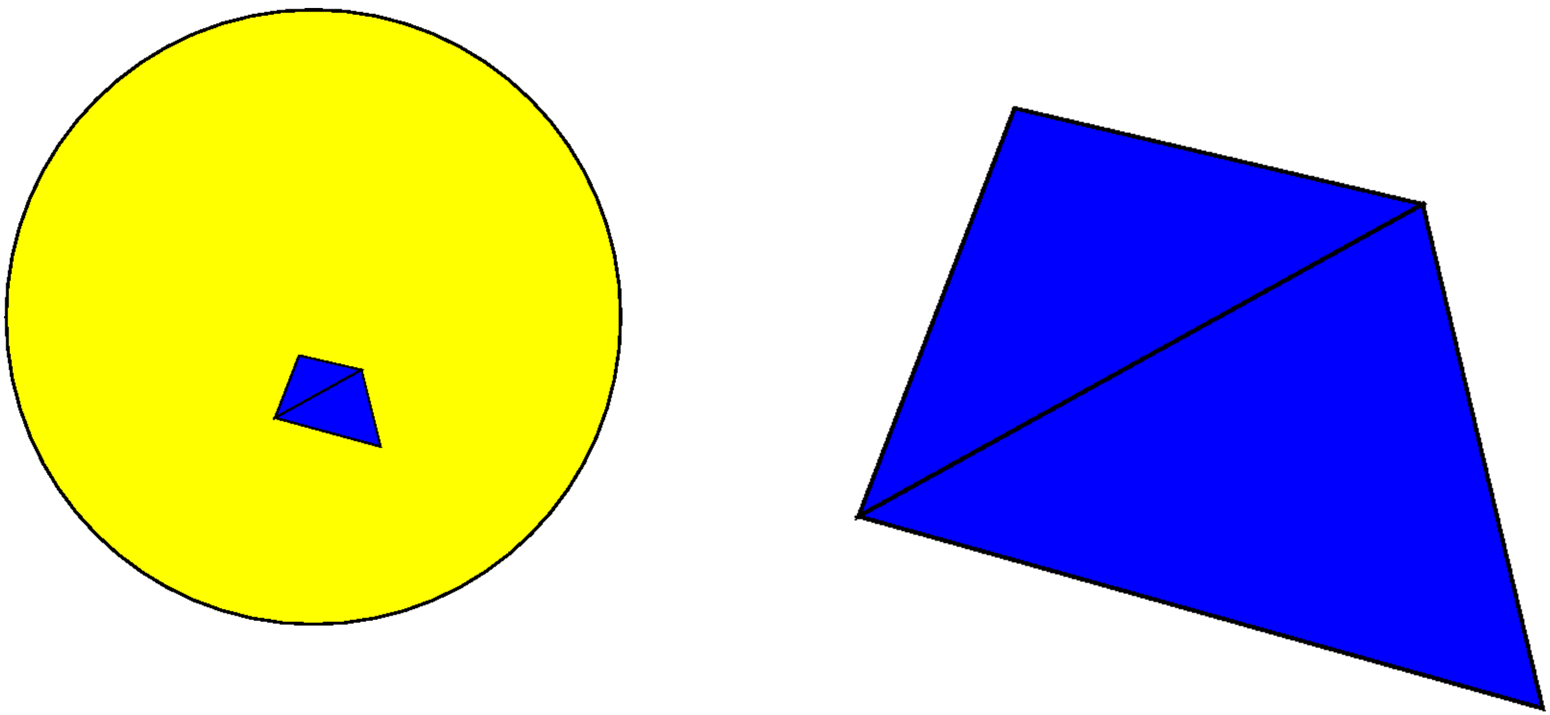


Embedded crack (EAS approach)



EED-EAS approach: discontinuous interpolation



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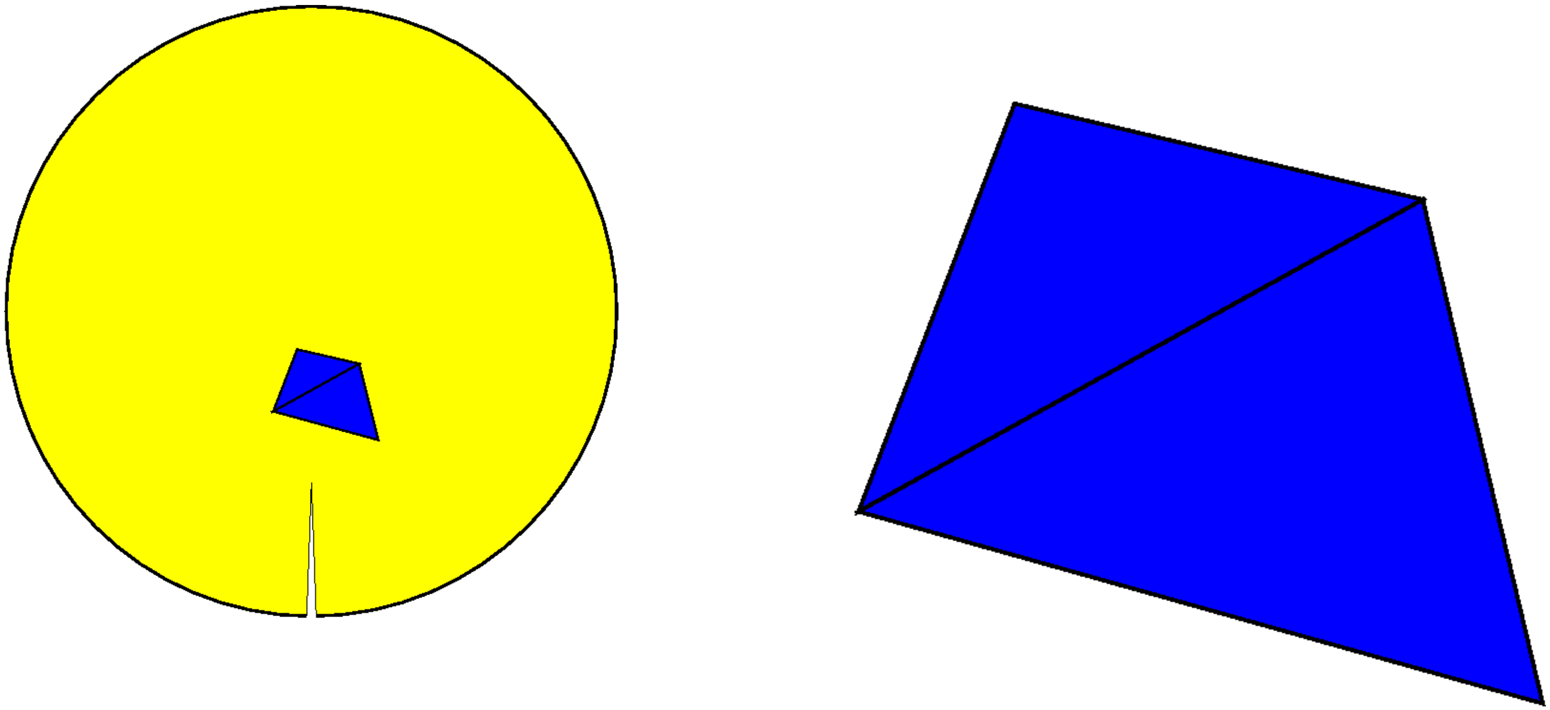

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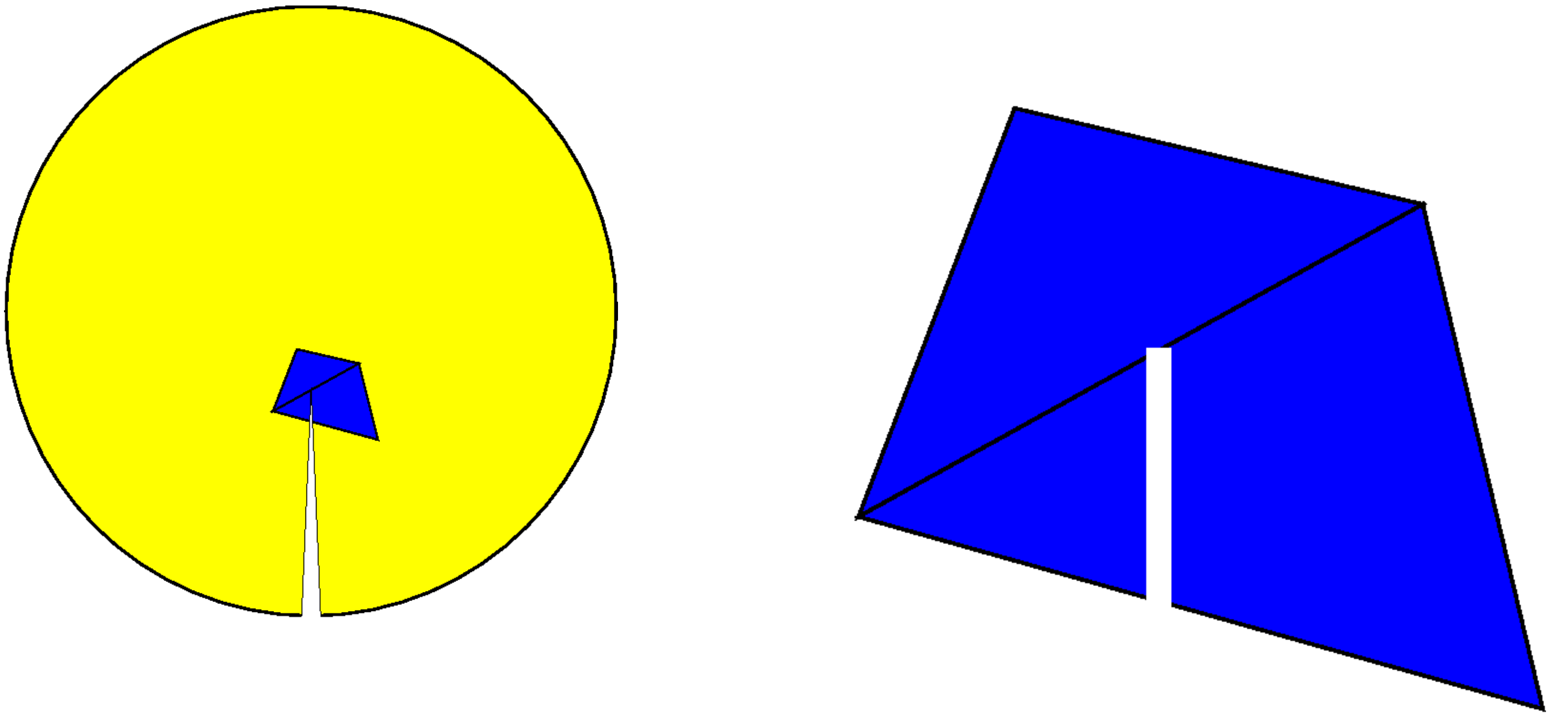

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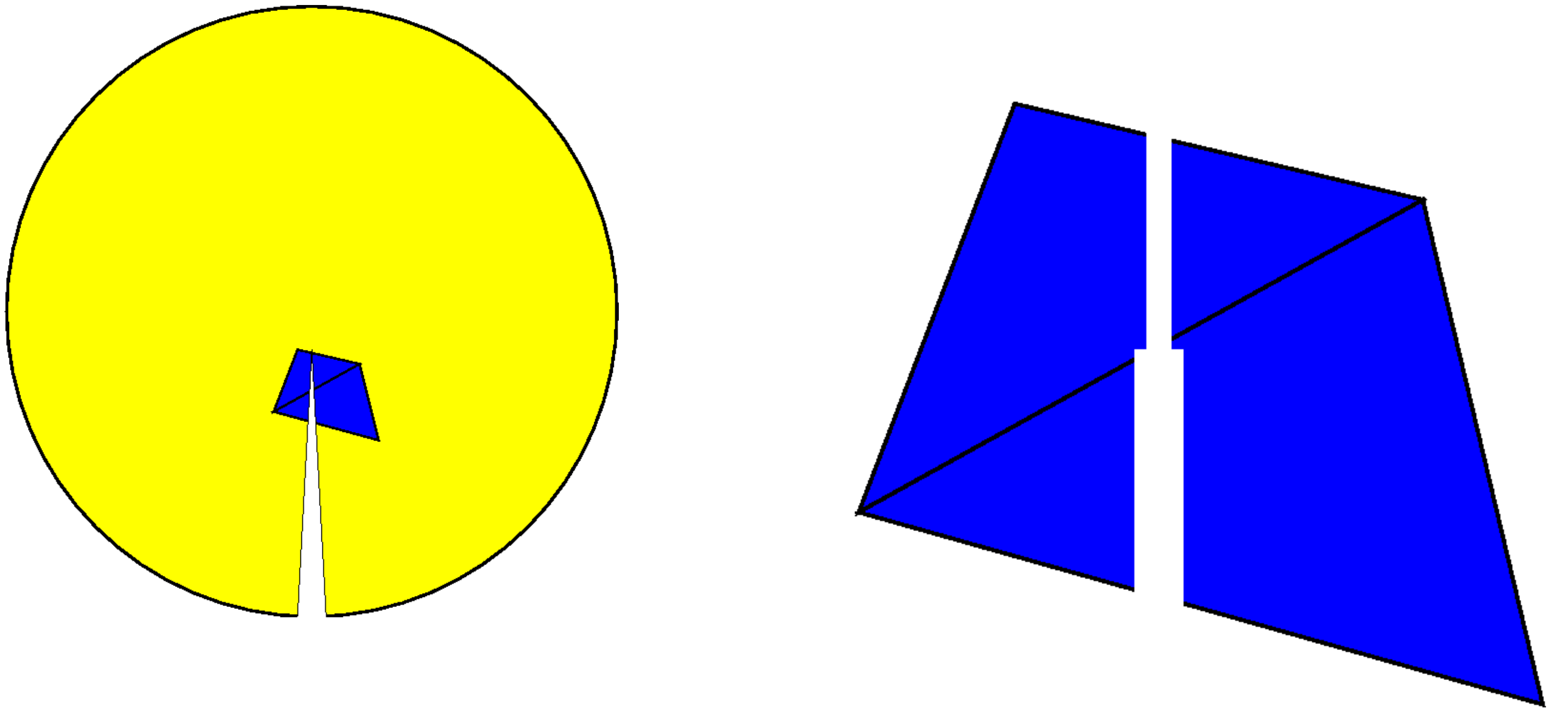
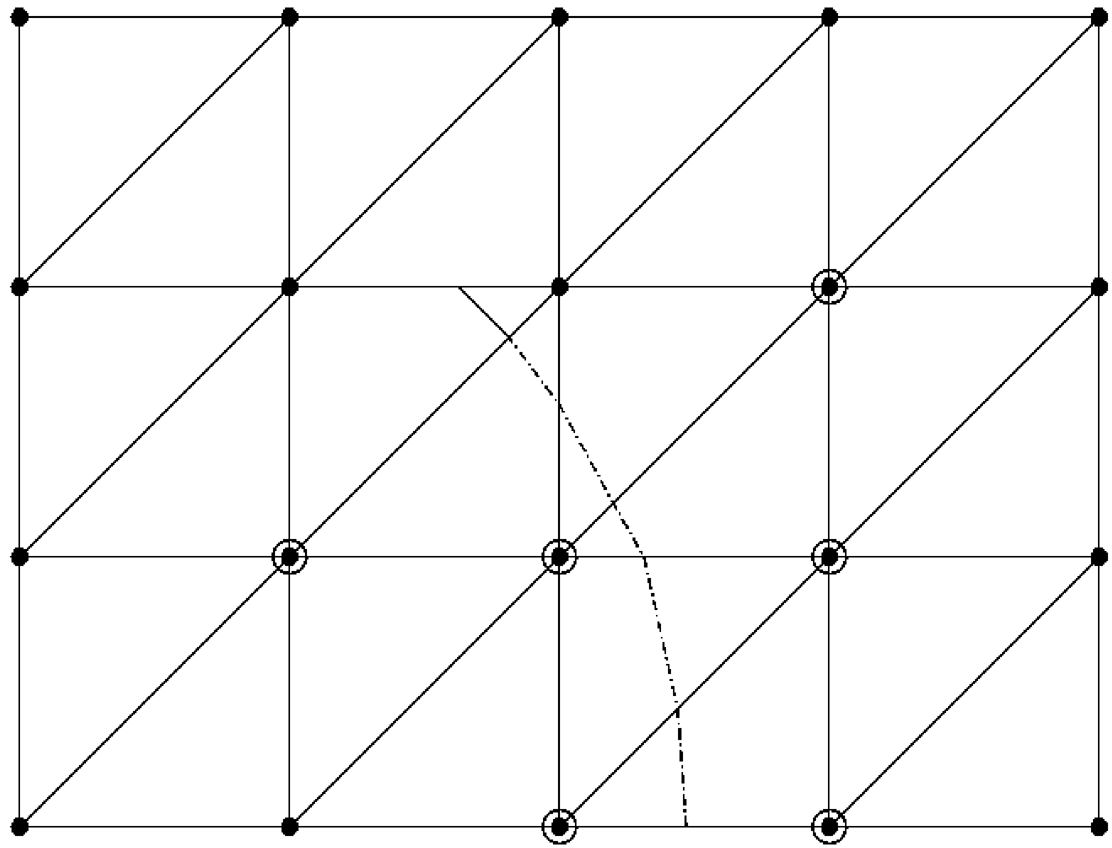


F.3

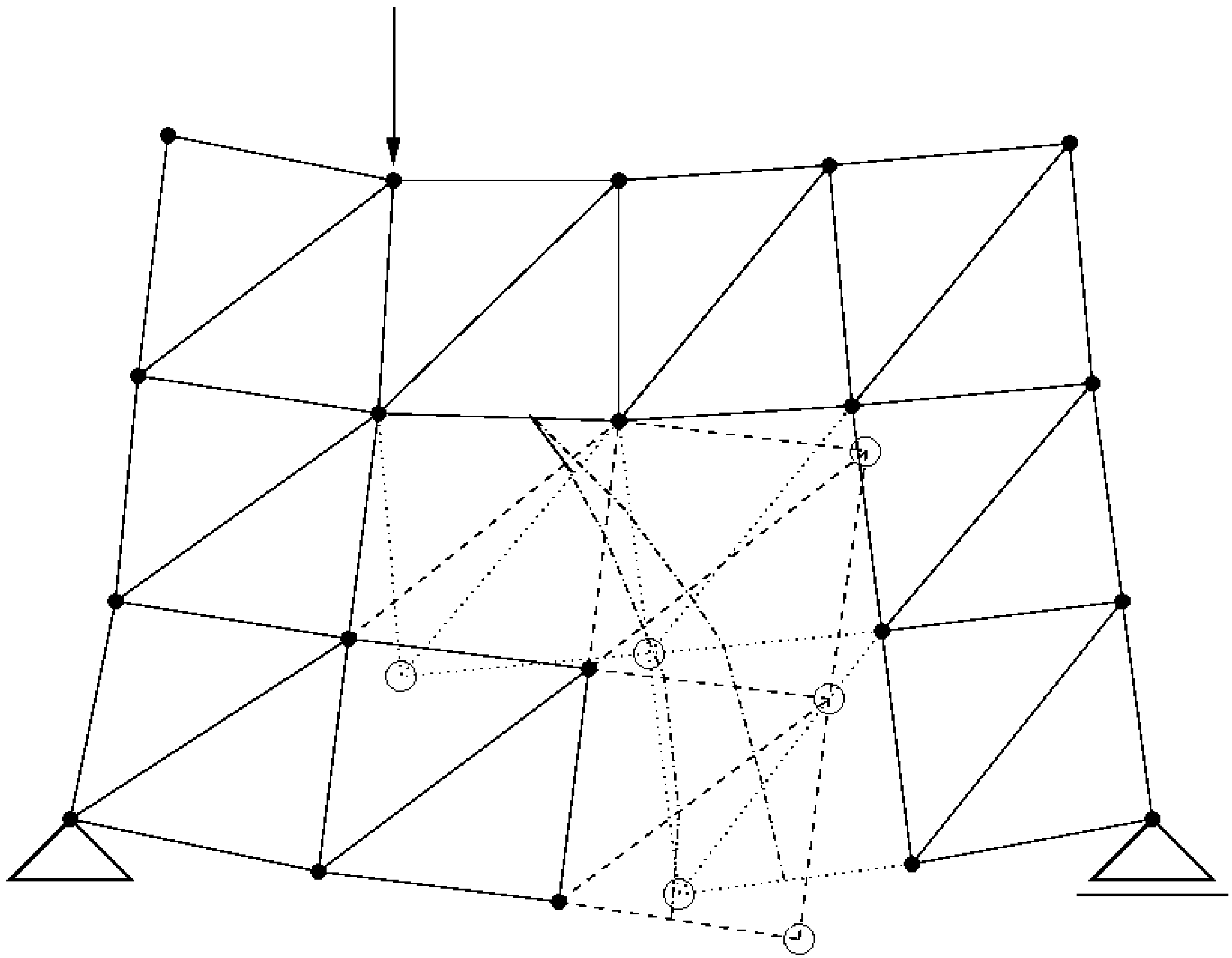
Extended Finite Elements (XFEM)

Based on Partition of Unity

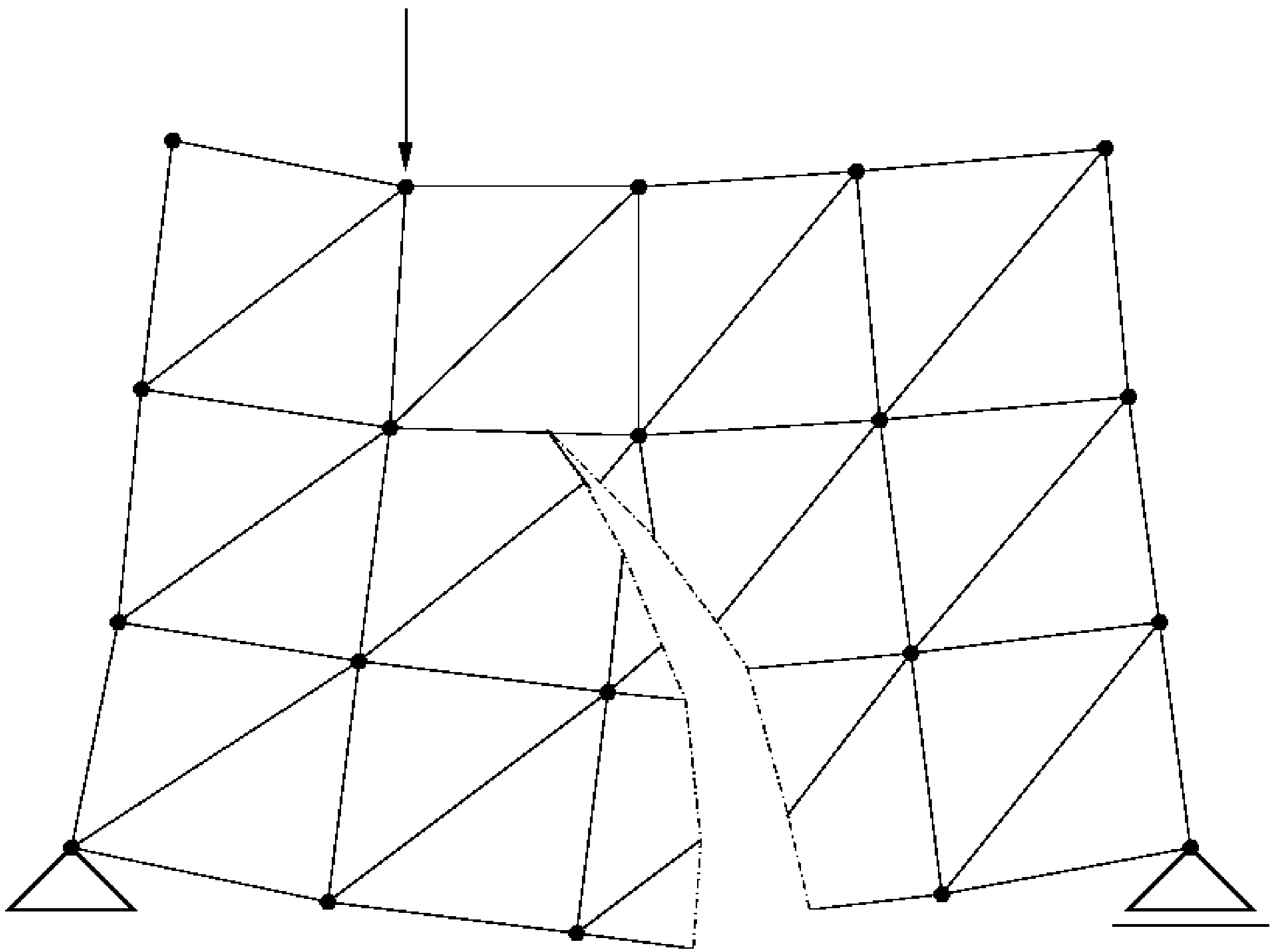
XFEM-PUM



XFEM-PUM



XFEM-PUM



Partition of Unity Method – eXtended Finite Elements

Enrichment by Heaviside function:

$$u(x) = \sum_{I=1}^{N} N_I(x) d_I + \sum_{I \in \mathcal{E}} H(x) N_I(x) e_I$$

Enrichment by arbitrary functions:

$$u(x) = \sum_{I=1}^{N} N_I(x) \left[d_I + \sum_{J=1}^{M} e_{IJ} H_J(x) \right]$$

All enrichment functions can be exactly reproduced, since for $d_I = 0$ and $e_{IJ} = \delta_{JK}$ we have

$$u(x) = \sum_{I=1}^{N} N_I(x) H_K(x) = H_K(x)$$

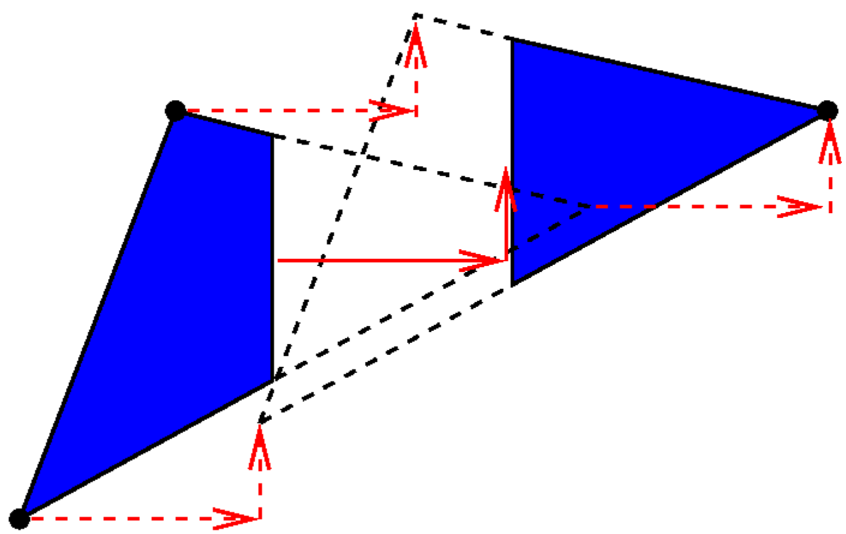
F.4

Comparison:

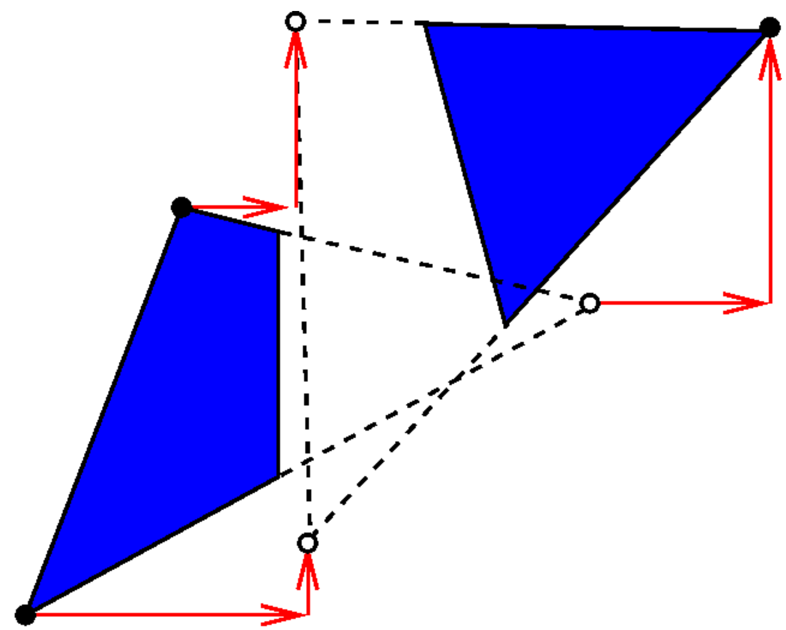
EED-EAS versus XFEM-PUM

Comparison of EED-EAS and XFEM-PUM

Embedded discontinuity



Extended finite elements



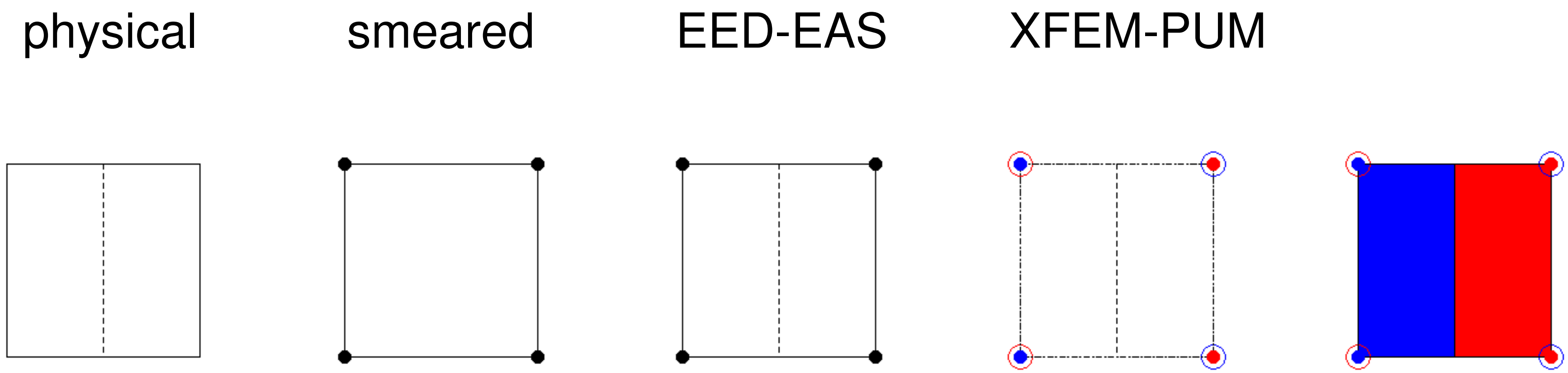
Comparison of EED-EAS and XFEM-PUM

	Embedded discontinuity	Extended finite elements
DOF‘s added	locally	globally
and related to	elements	nodes

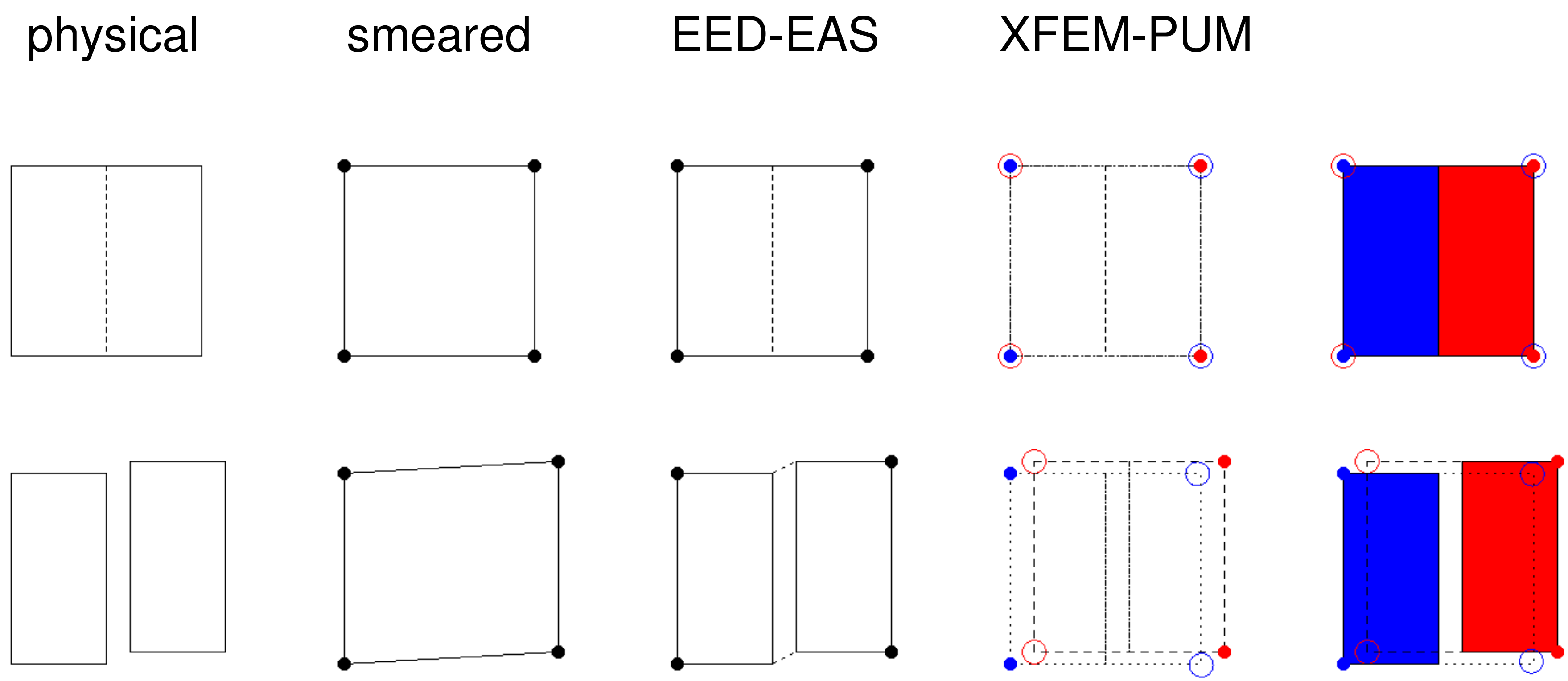
Comparison of EED-EAS and XFEM-PUM

	Embedded discontinuity	Extended finite elements
DOF's added	locally	globally
and related to	elements	nodes
Approximation of crack opening	discontinuous	continuous
Enrichment	incompatible	compatible

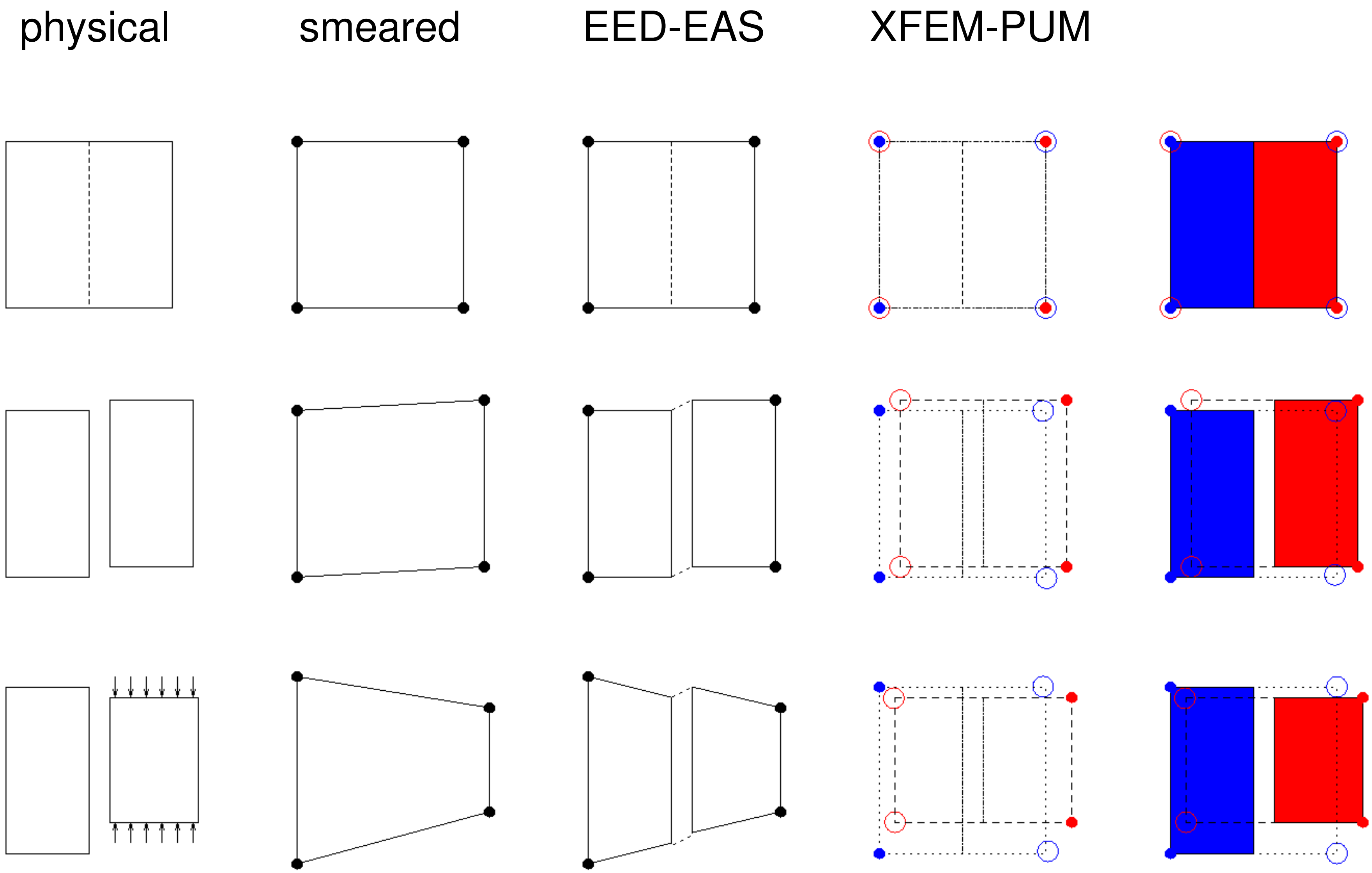
Separation test



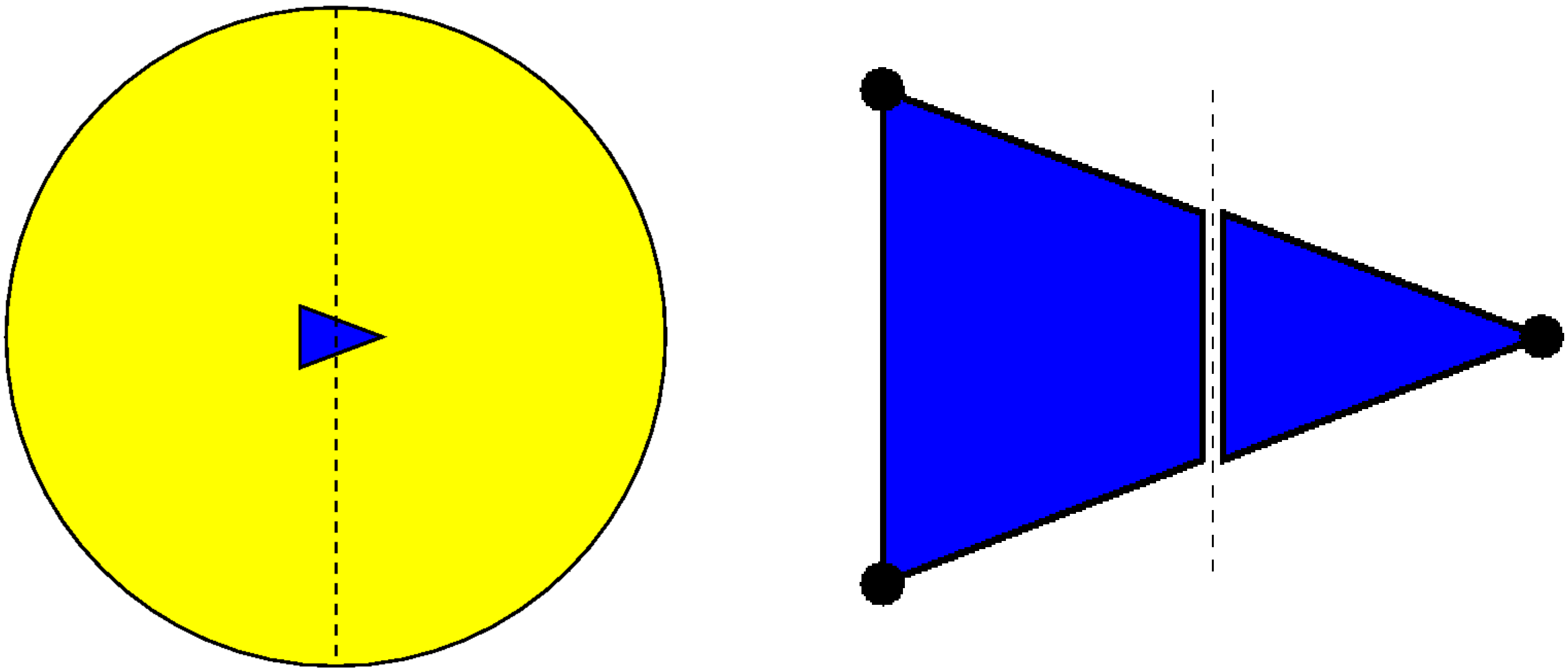
Separation test



Separation test



EED-EAS approach: partial coupling



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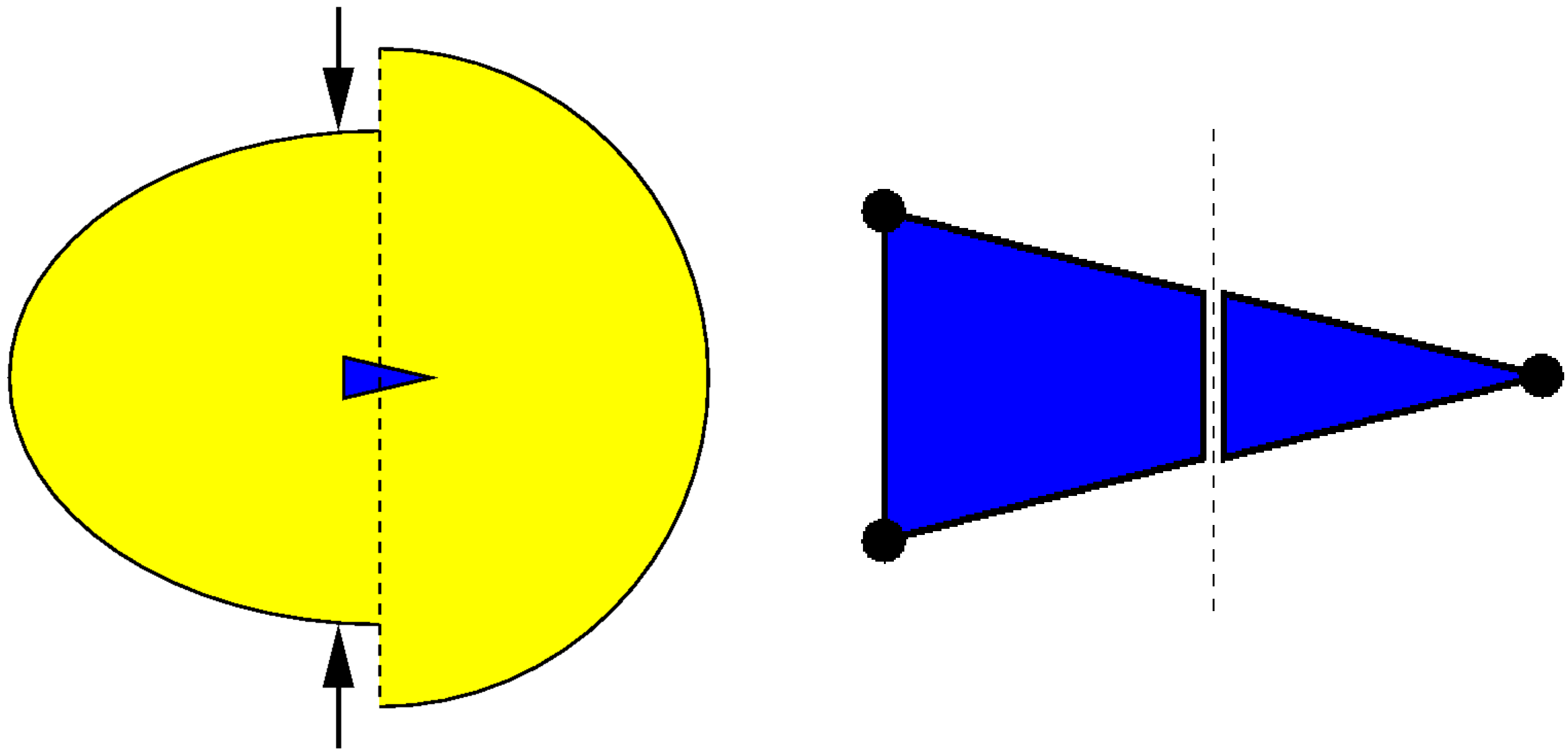

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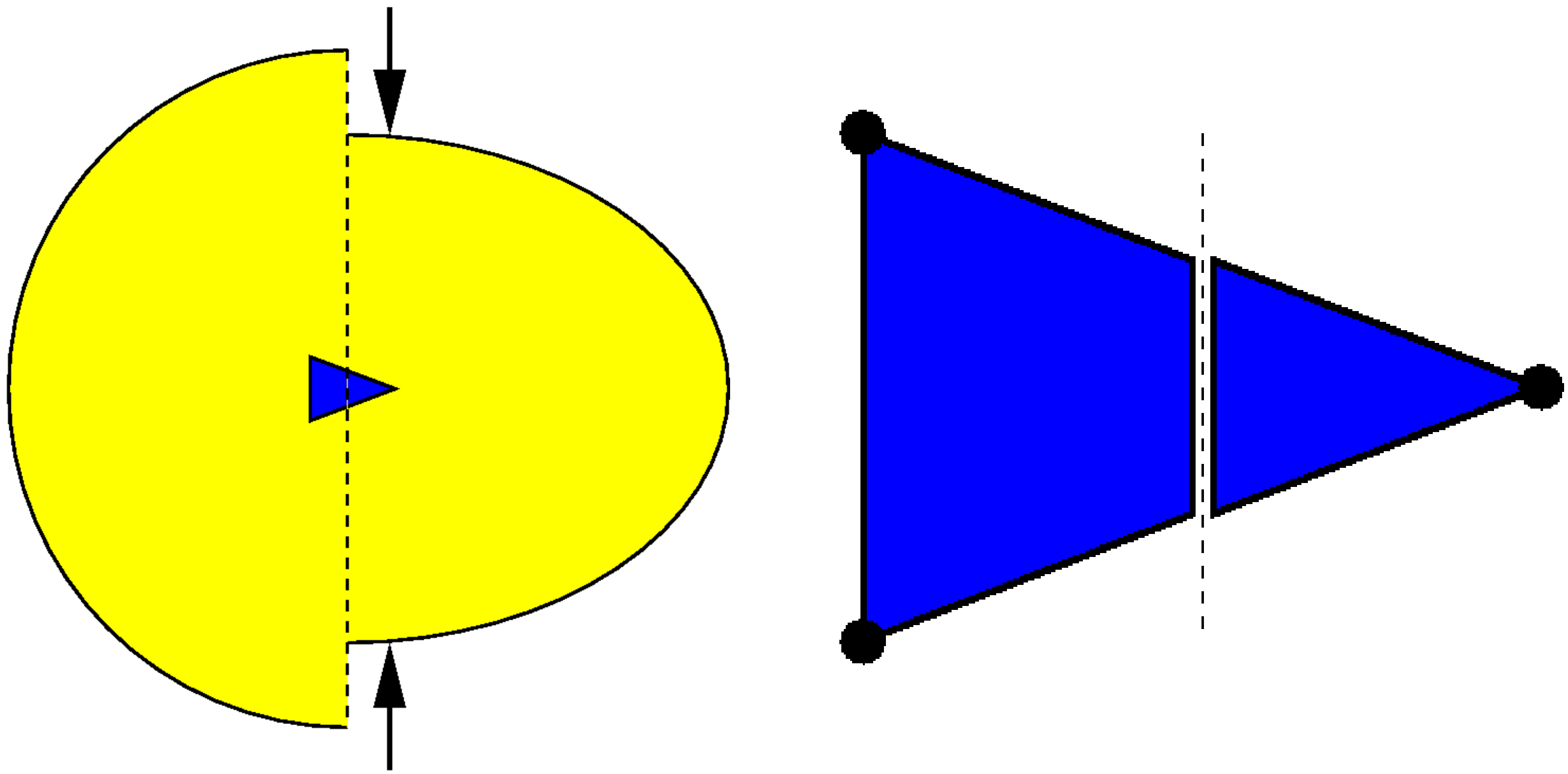
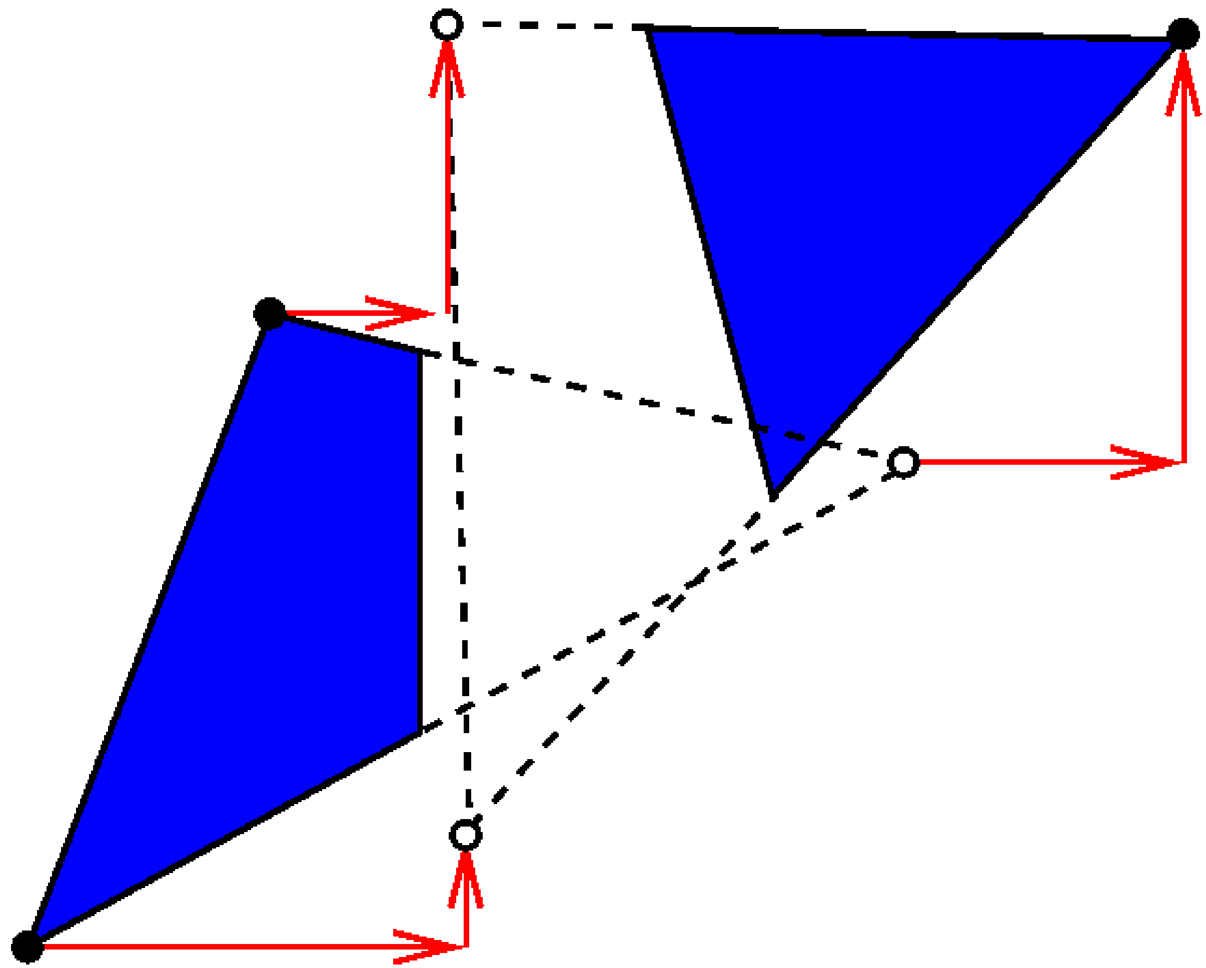


XFEM-PUM approach: complete decoupling



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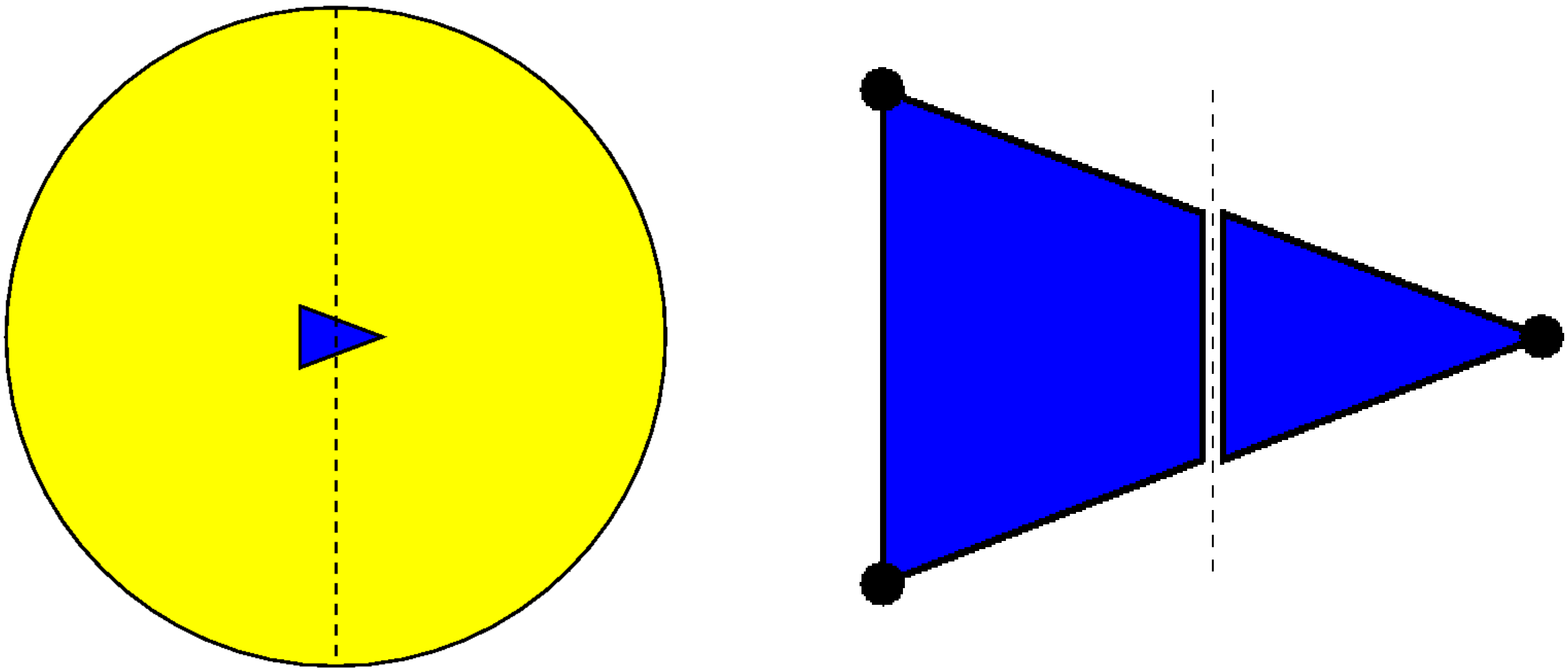

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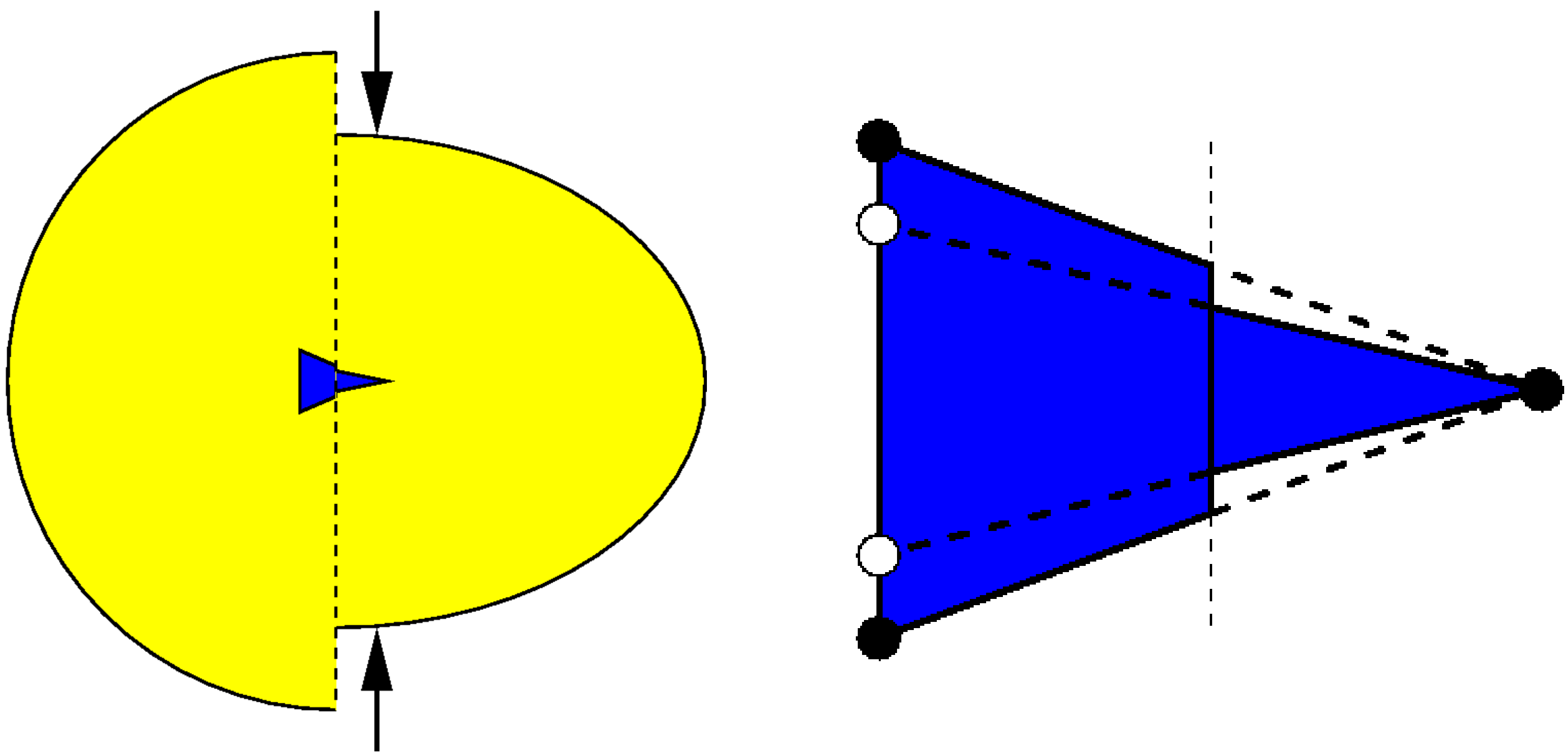

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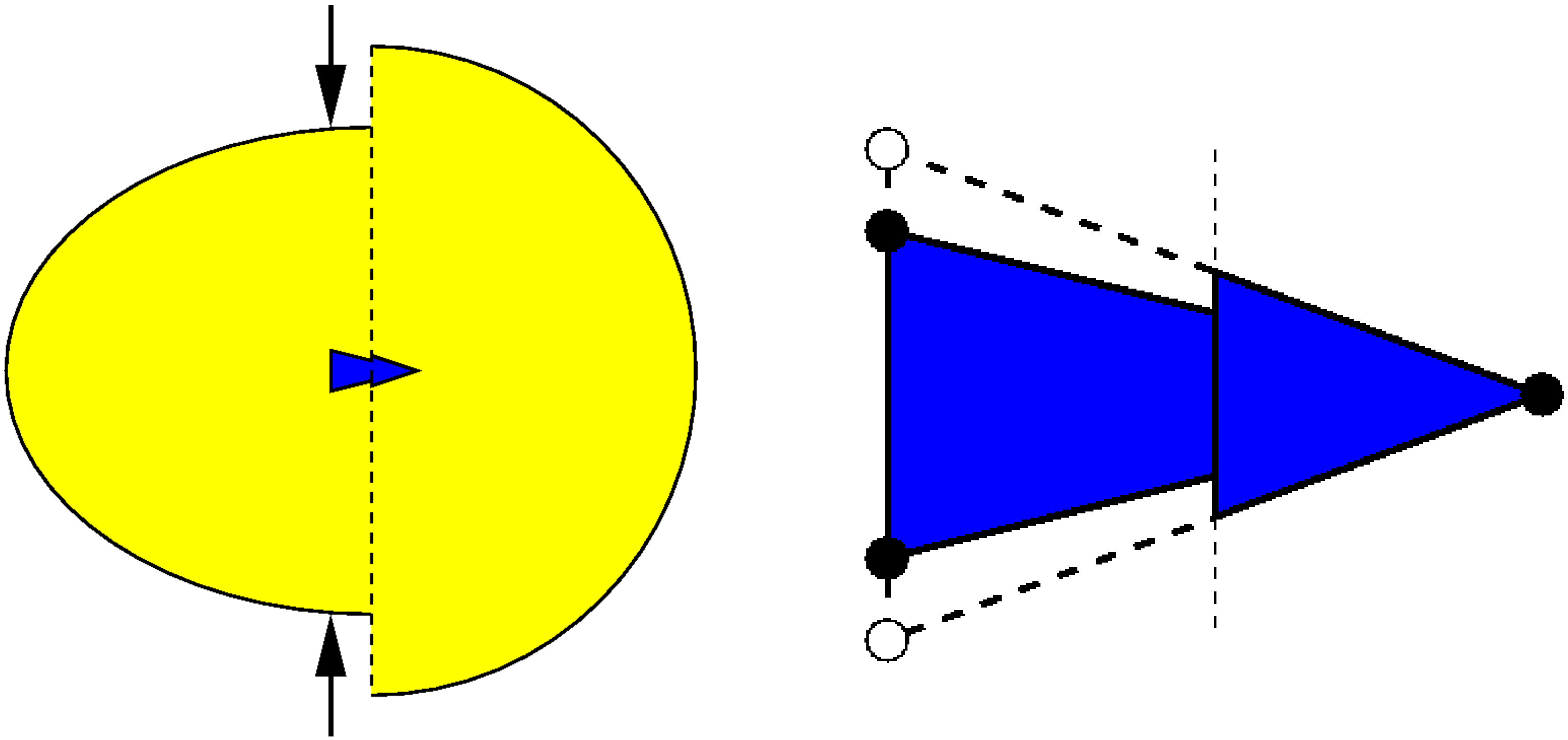
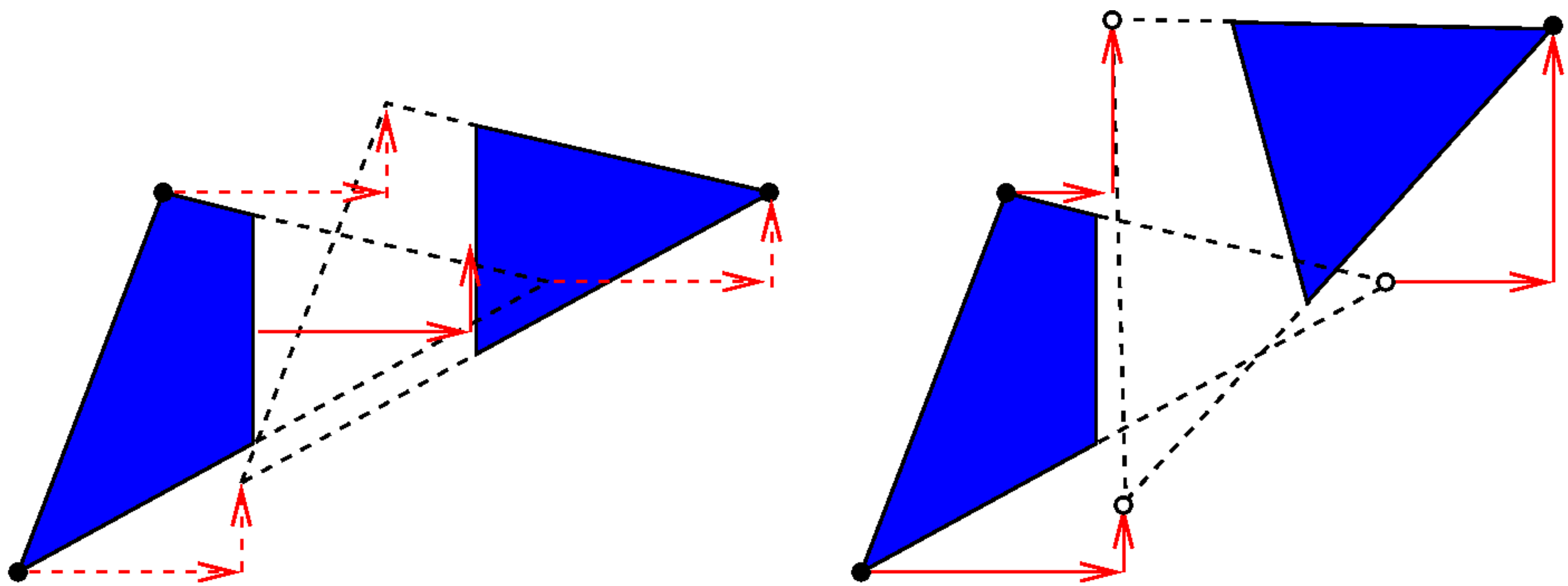


Comparison of EED-EAS and XFEM-PUM

Embedded discontinuity

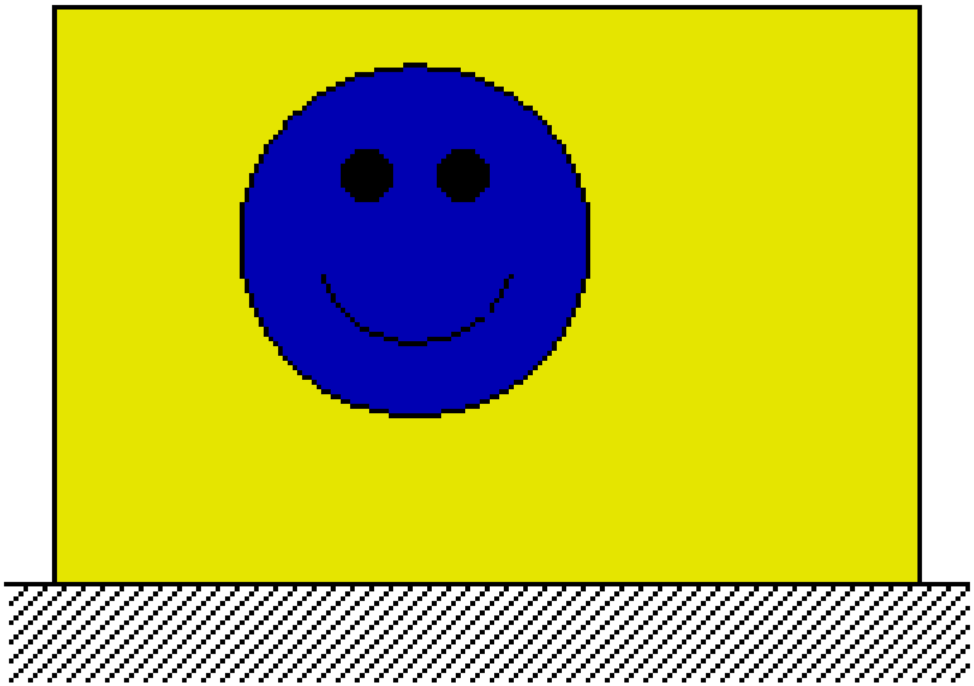
Extended finite elements



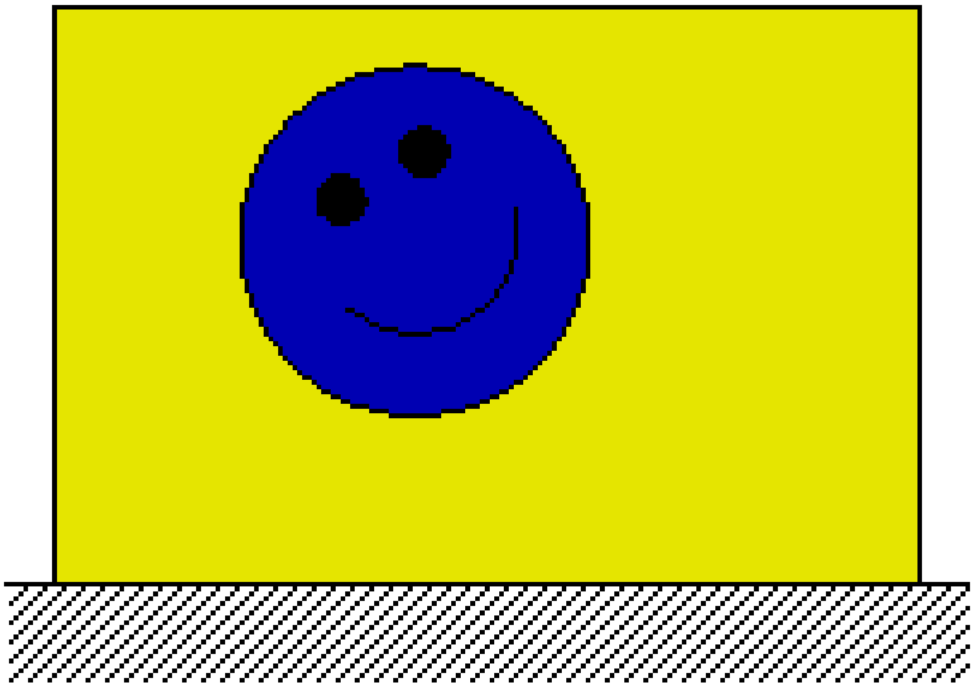
Comparison of EED-EAS and XFEM-PUM

	Embedded discontinuity	Extended finite elements
DOF‘s added	locally	globally
and related to	elements	nodes
Approximation of crack opening	discontinuous	continuous
Enrichment	incompatible	compatible
Separated parts	**partially coupled**	**fully decoupled**

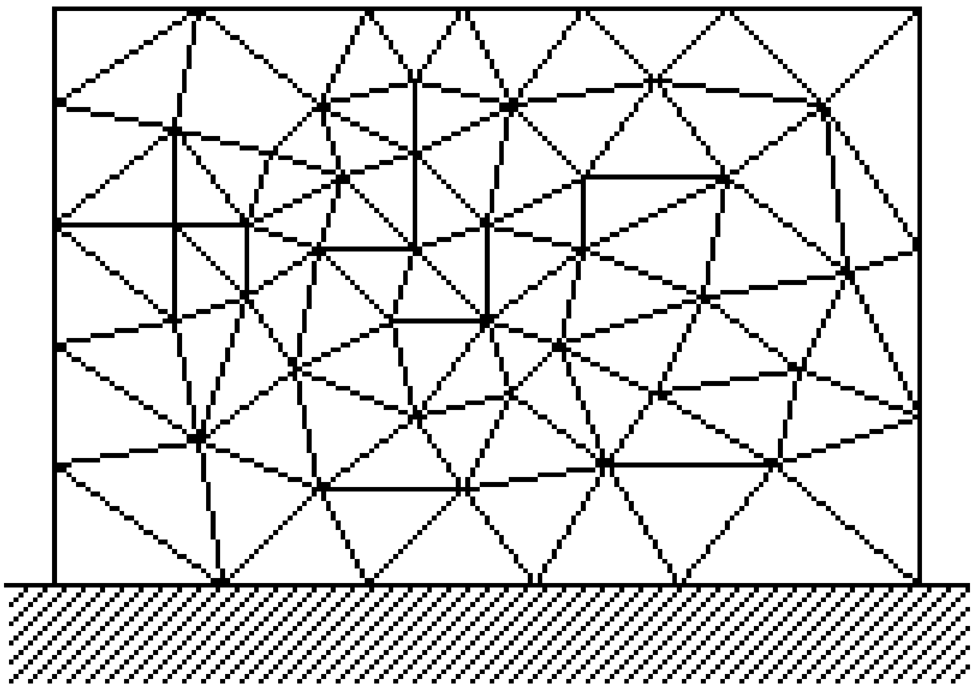
Journal bearing: Physical process



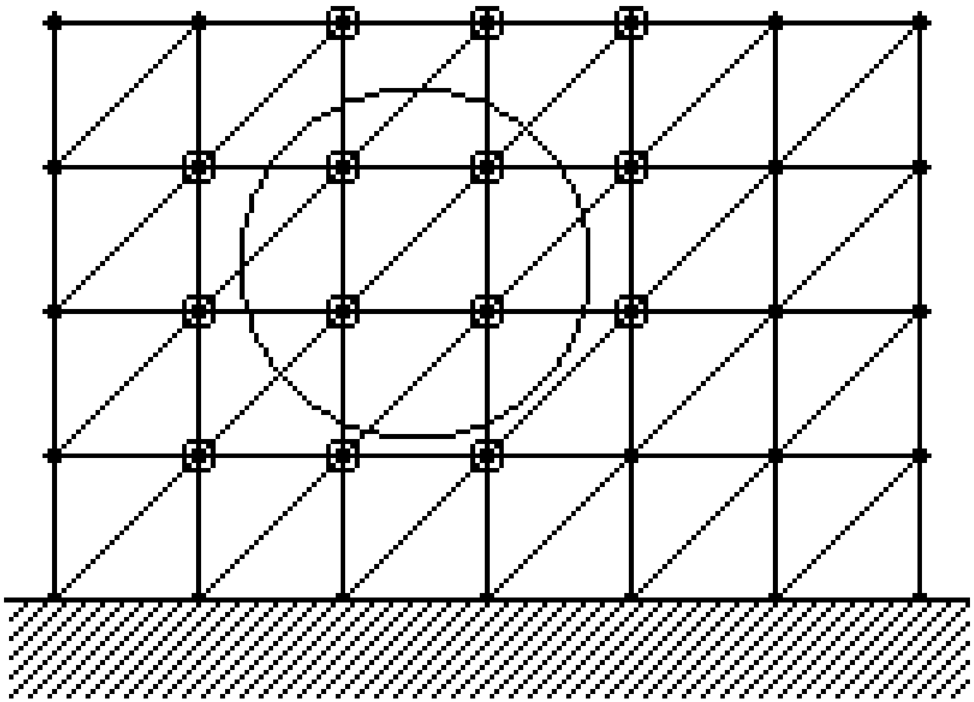
Journal bearing: Physical process



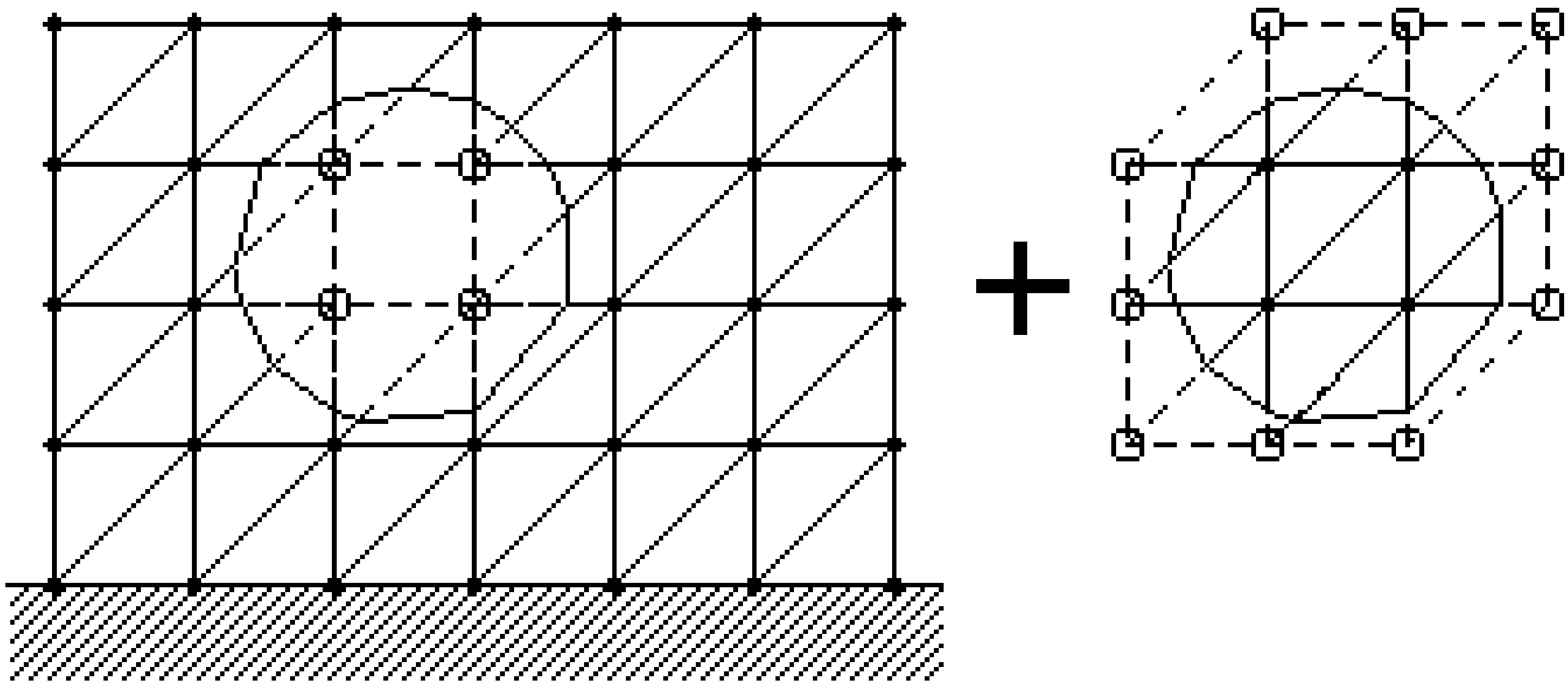
Journal bearing: Mesh respecting material boundaries



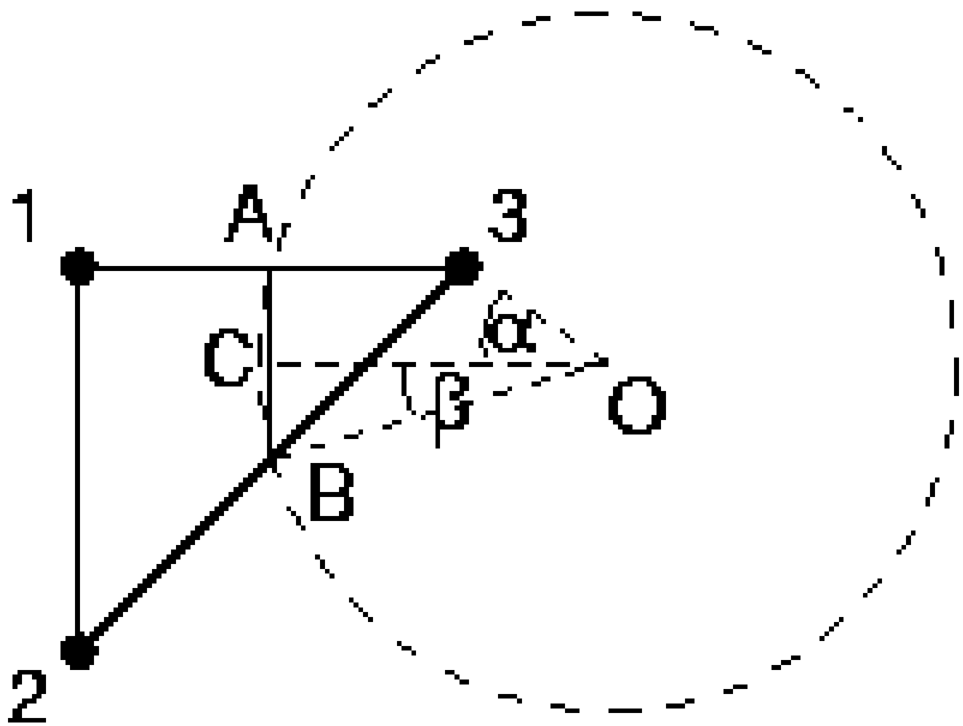
Journal bearing: Structured mesh with enrichment



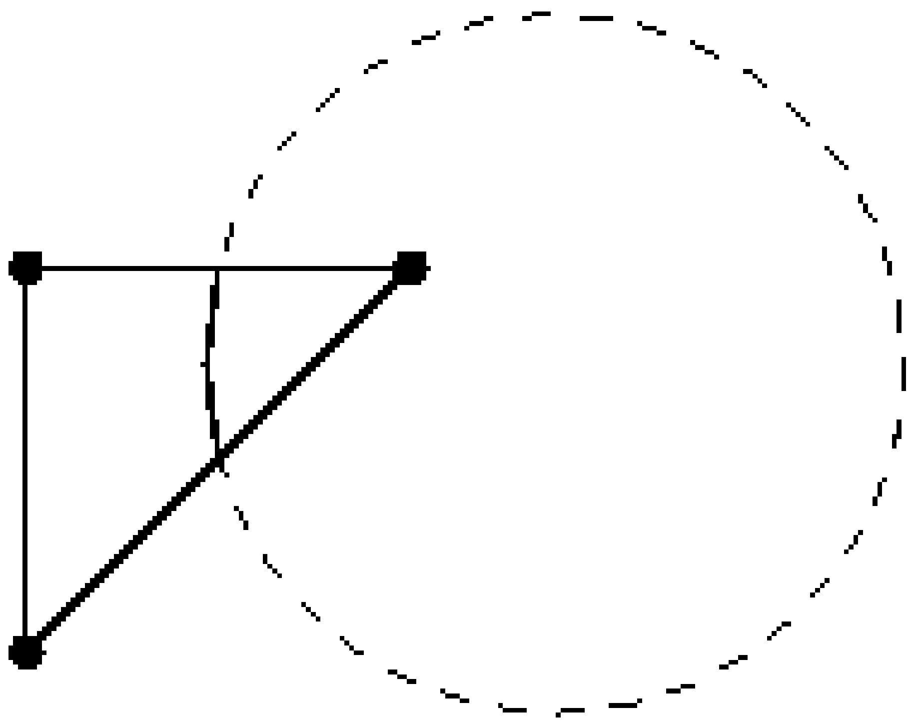
Journal bearing: Structured mesh with enrichment



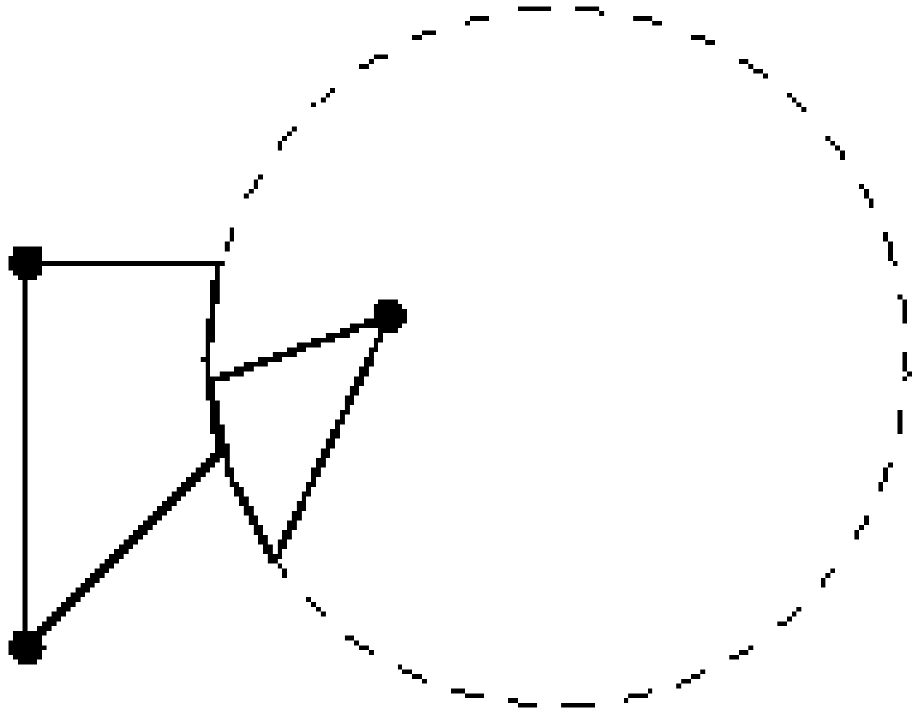
One element crossed by pre-existing discontinuity



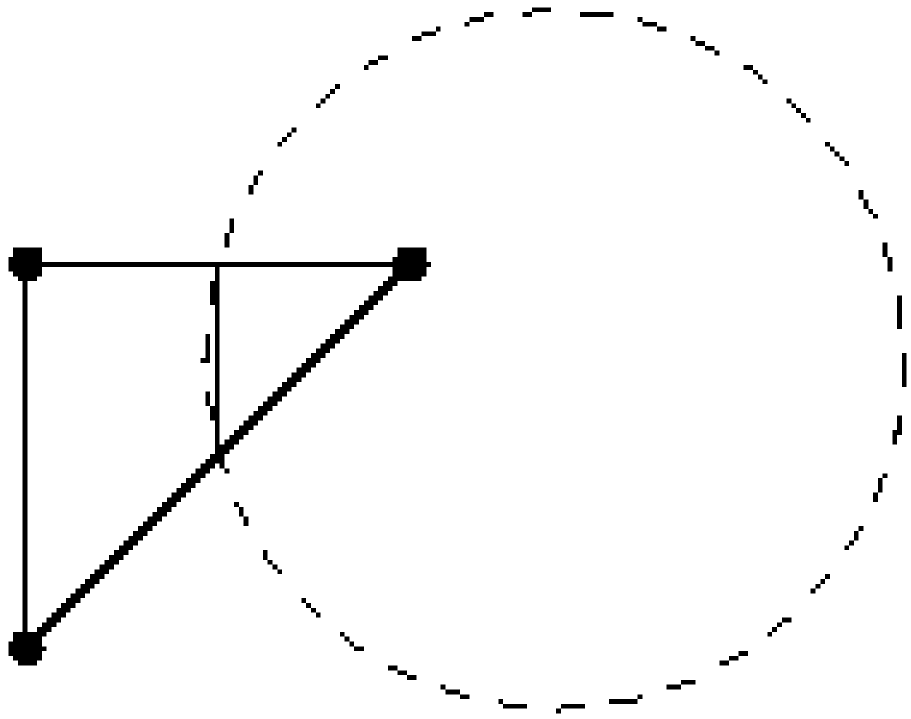
One element: Physical process



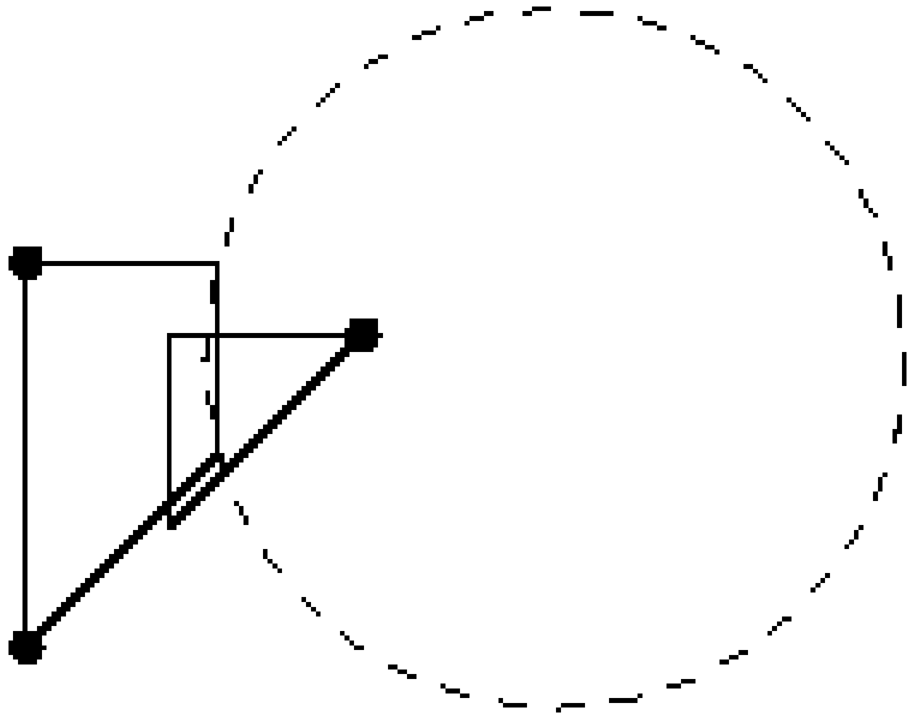
One element: Physical process



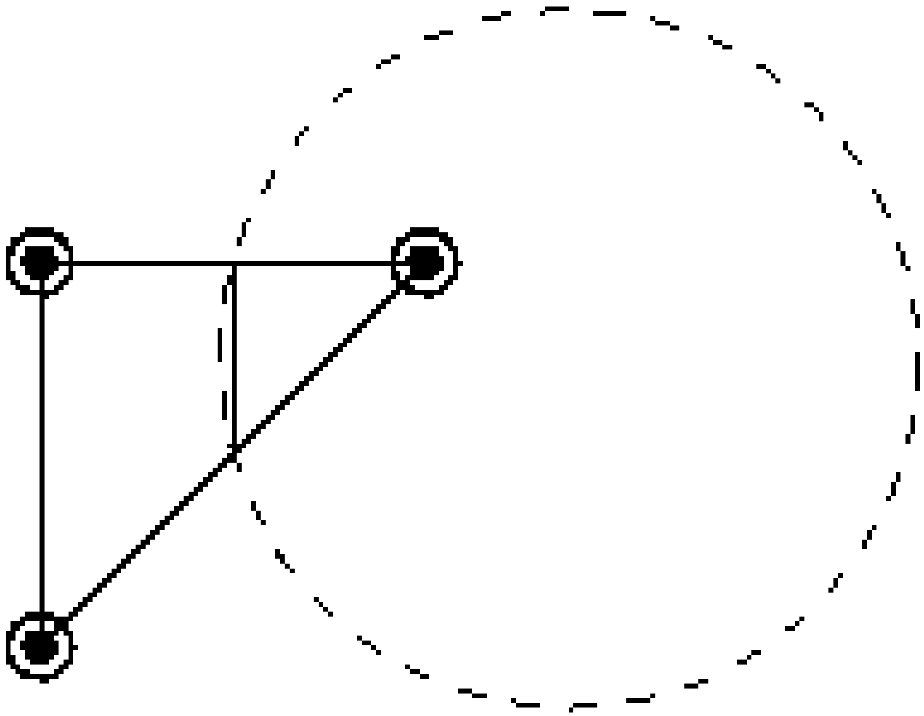
One element: EED-EAS



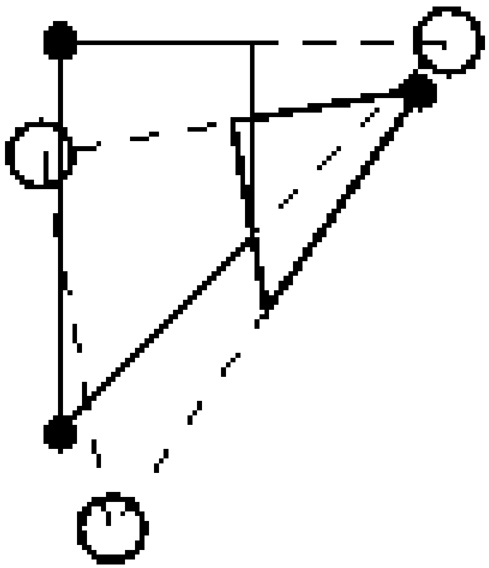
One element: EED-EAS



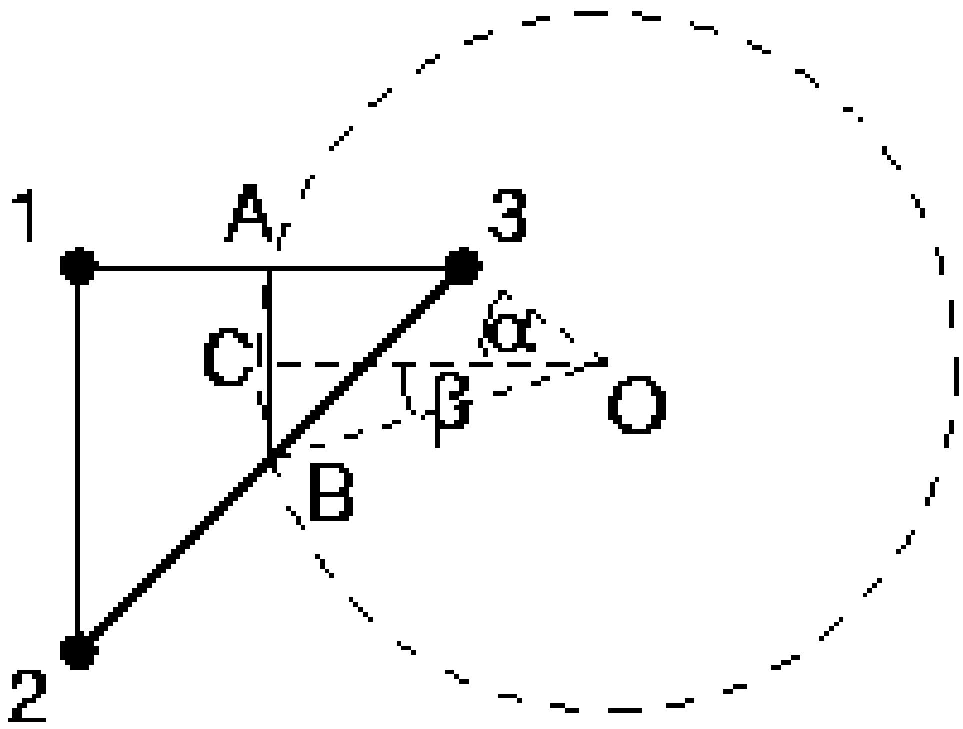
One element: XFEM-PUM



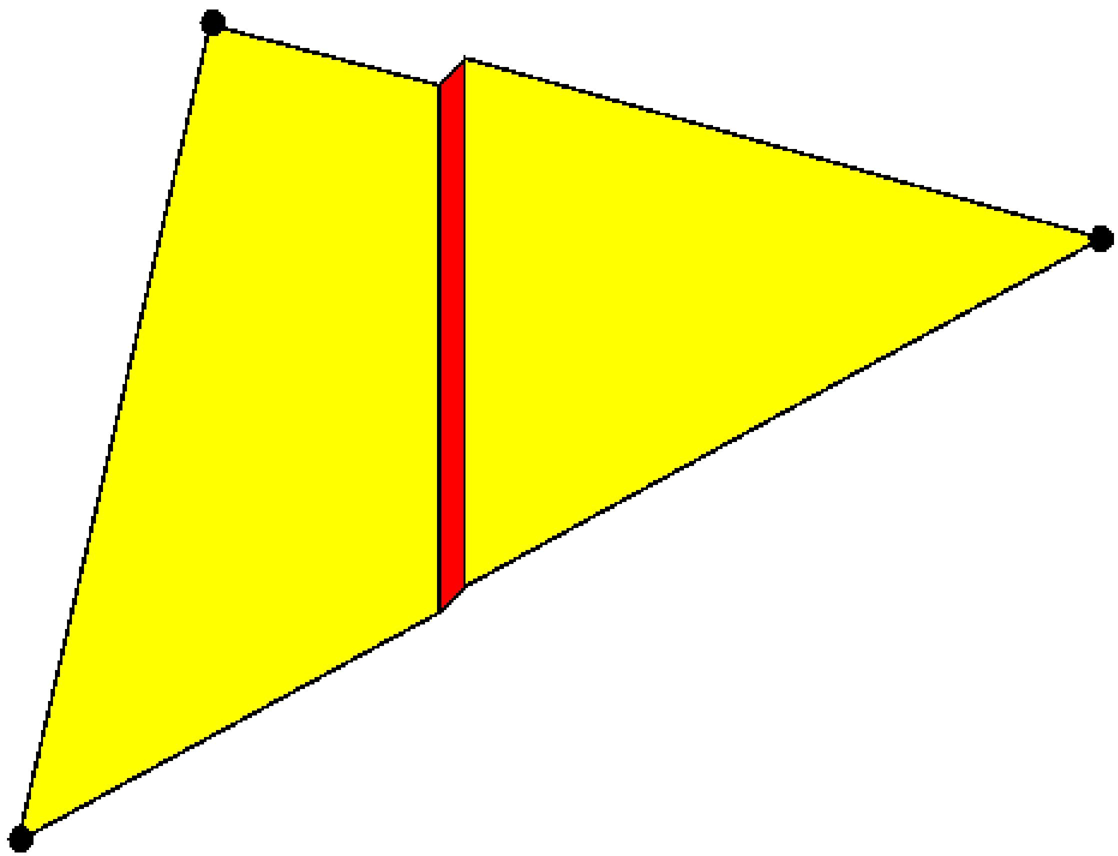
One element: XFEM-PUM



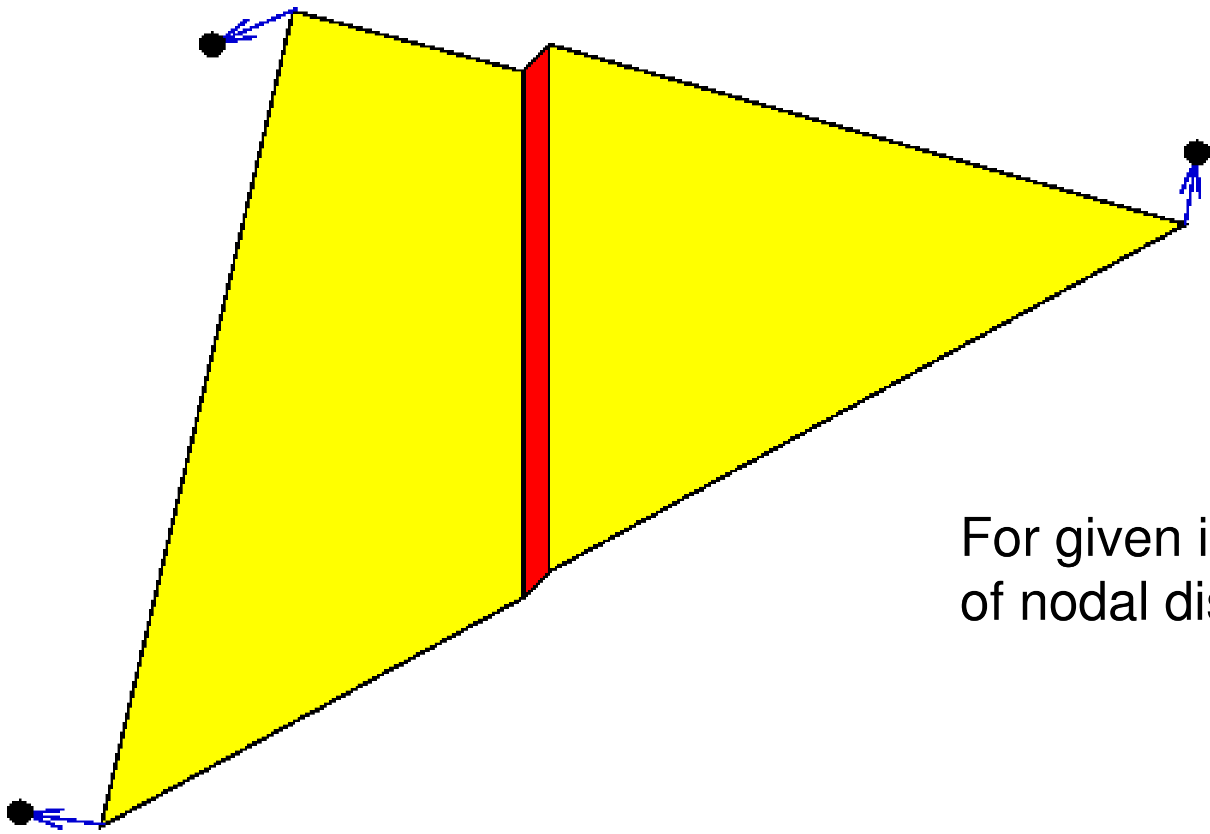
One element crossed by pre-existing discontinuity



Uniqueness of the element response (EED-EAS)

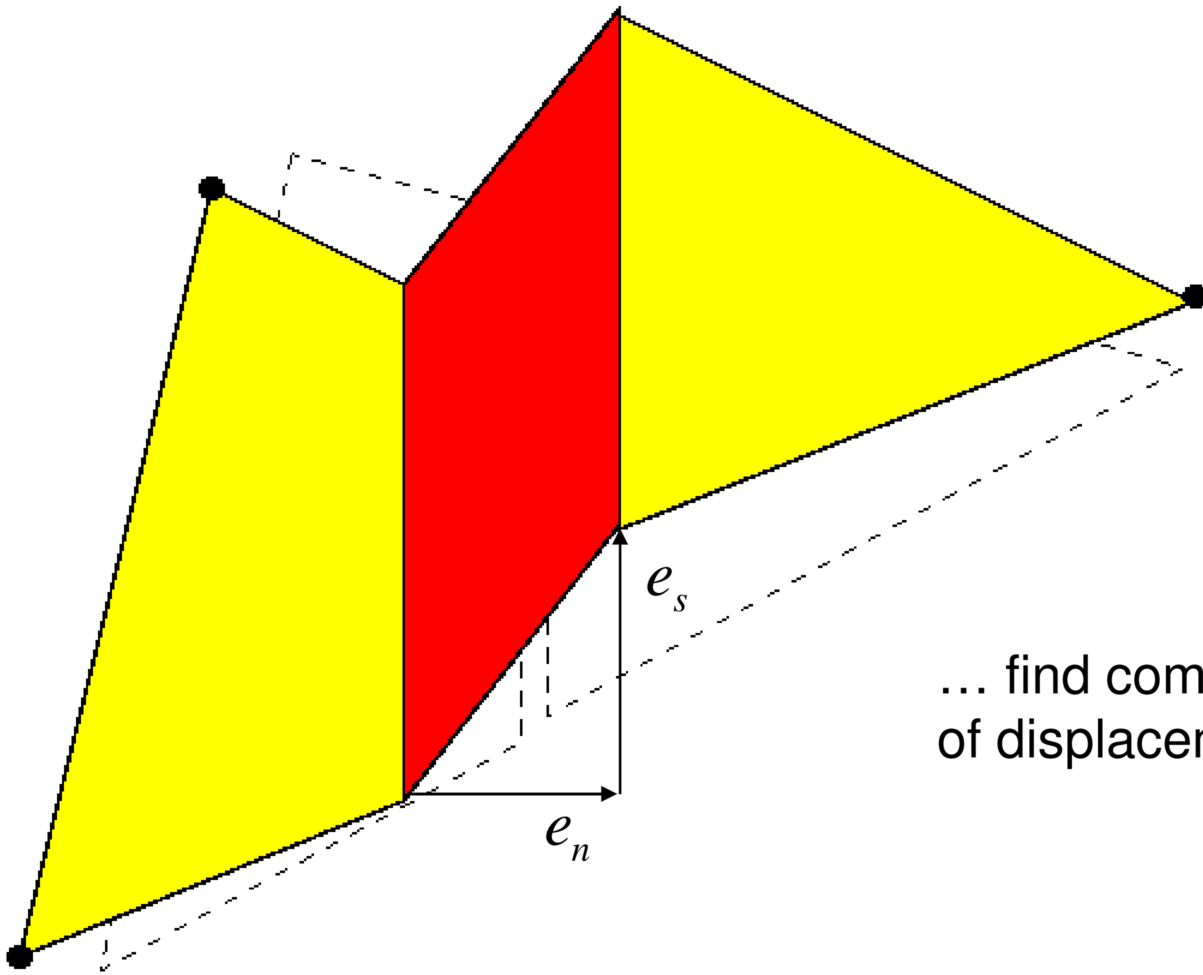


Uniqueness of the element response



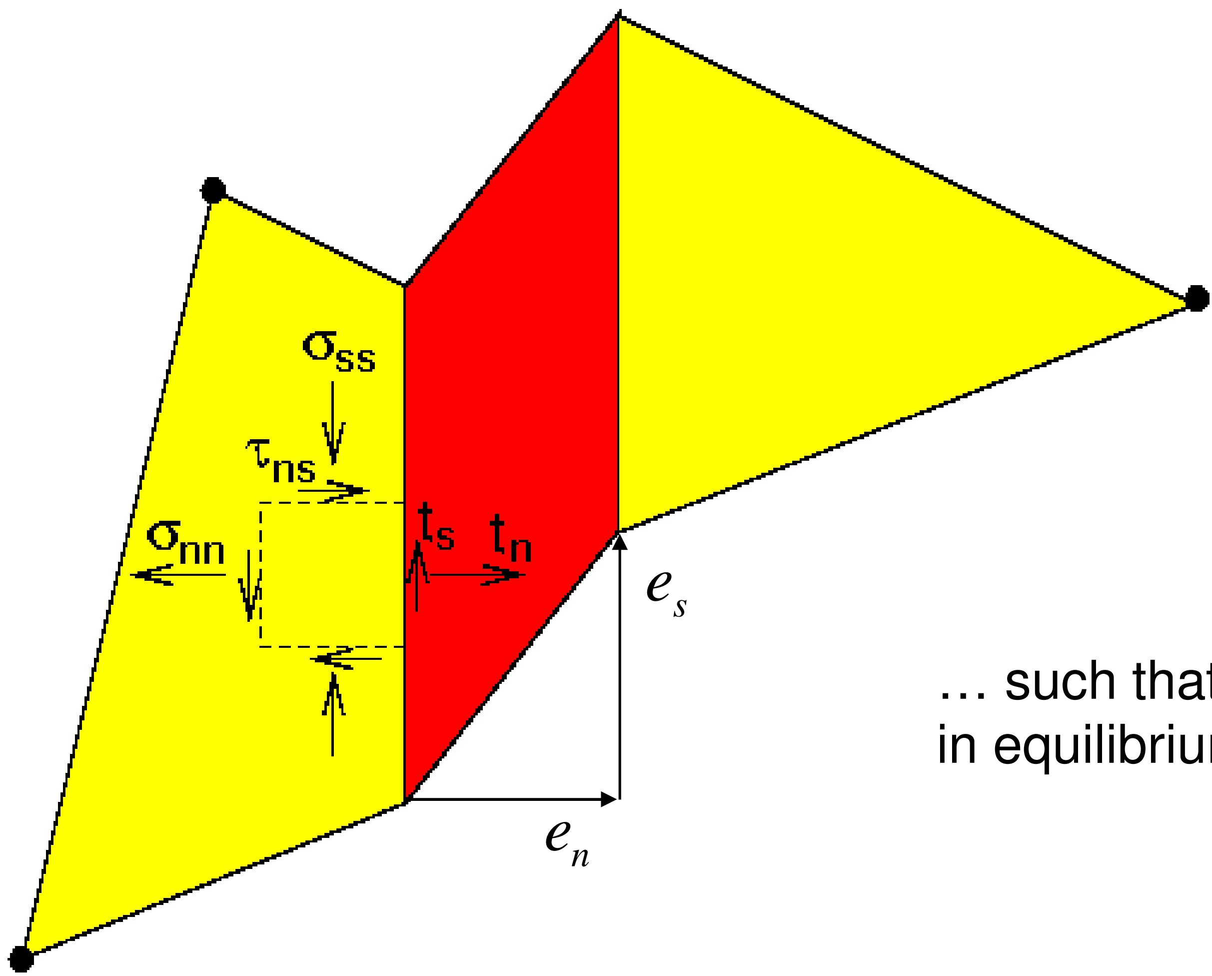
For given increments
of nodal displacements …

Uniqueness of the element response



… find components
of displacement jump …

Uniqueness of the element response



… such that tractions are in equilibrium with stresses.

Uniqueness of the element response

The solution is unique for infinitesimal displacement increments of an arbitrary direction if

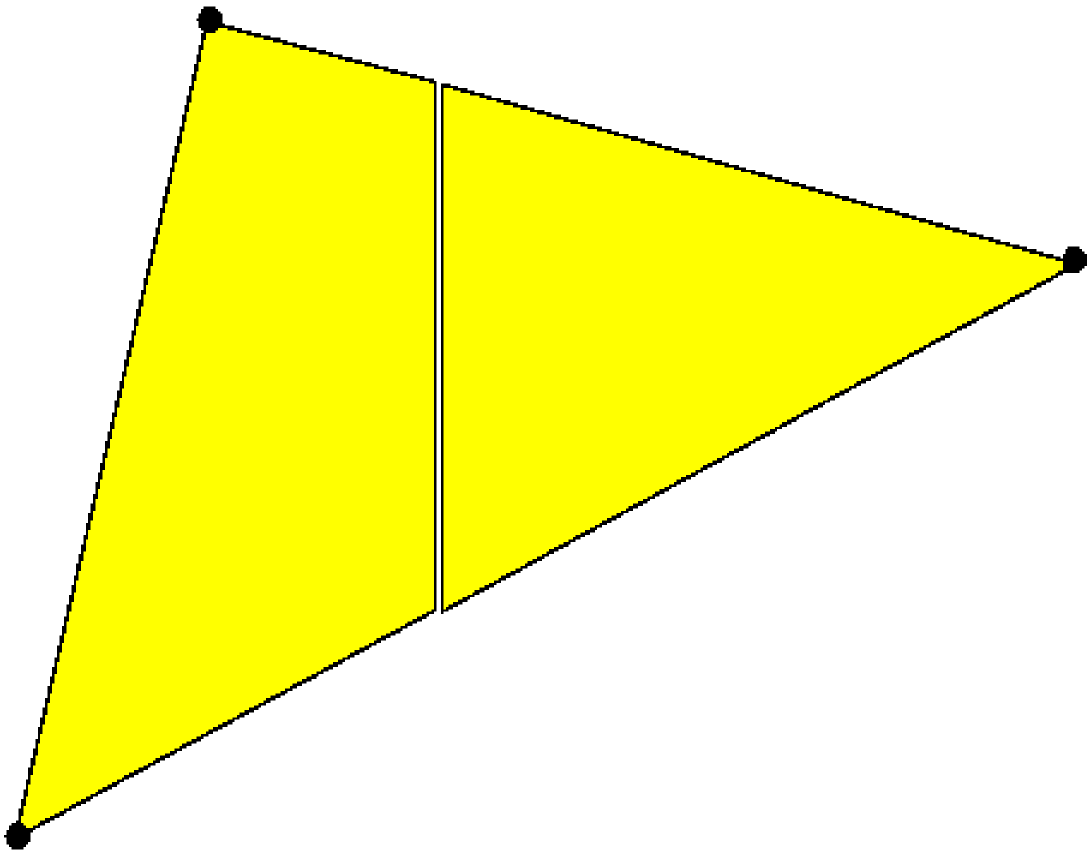
$$\lambda_{\min}\left(\mathbf{Q}_{sym}\right) > -H_{\min}$$

where $\mathbf{Q}_{sym}$ is the symmetric part of $\mathbf{Q} = \mathbf{P}^T \mathbf{D}_e \mathbf{B} \mathbf{H}$

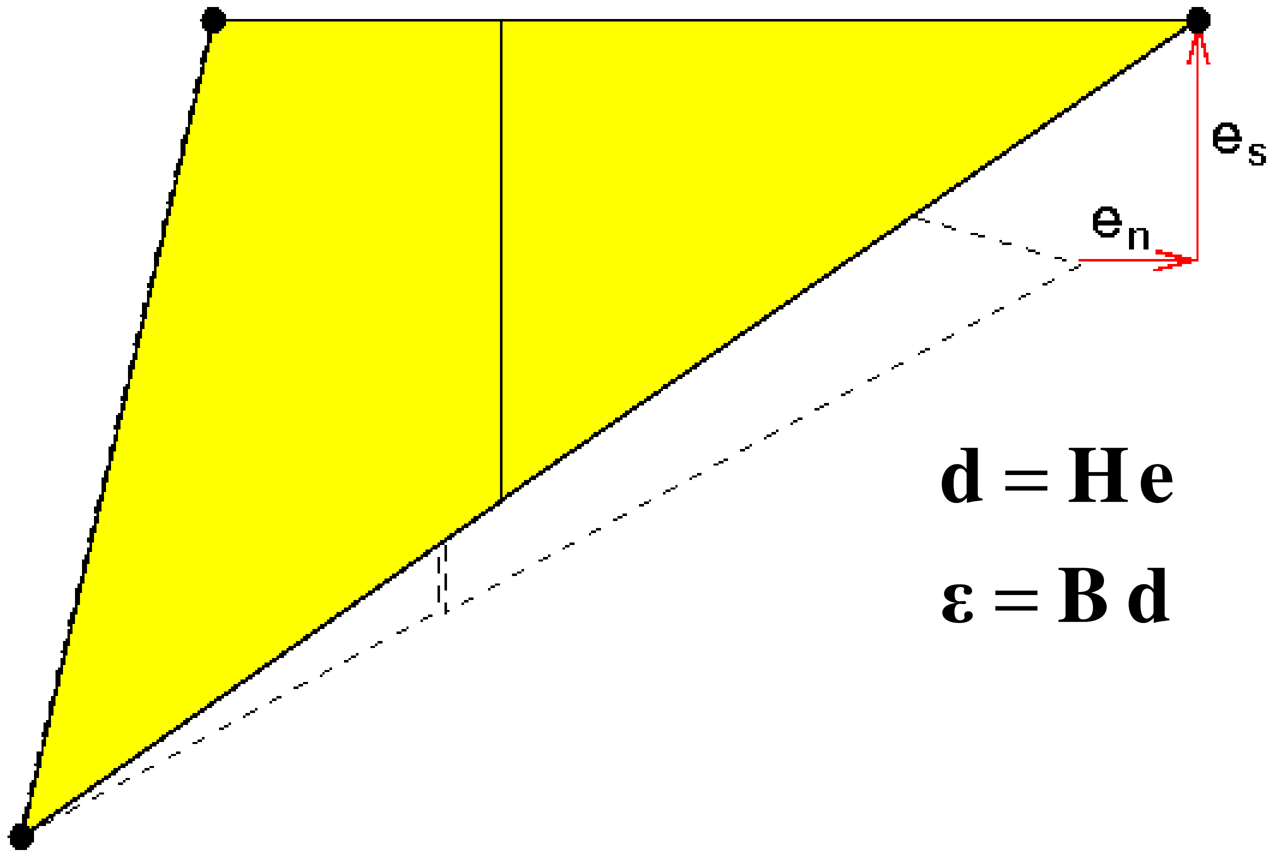
and $H_{\min} < 0$ is the minimum value of discrete softening modulus.

Physical meaning of $\mathbf{Q}$ …

Uniqueness of the element response



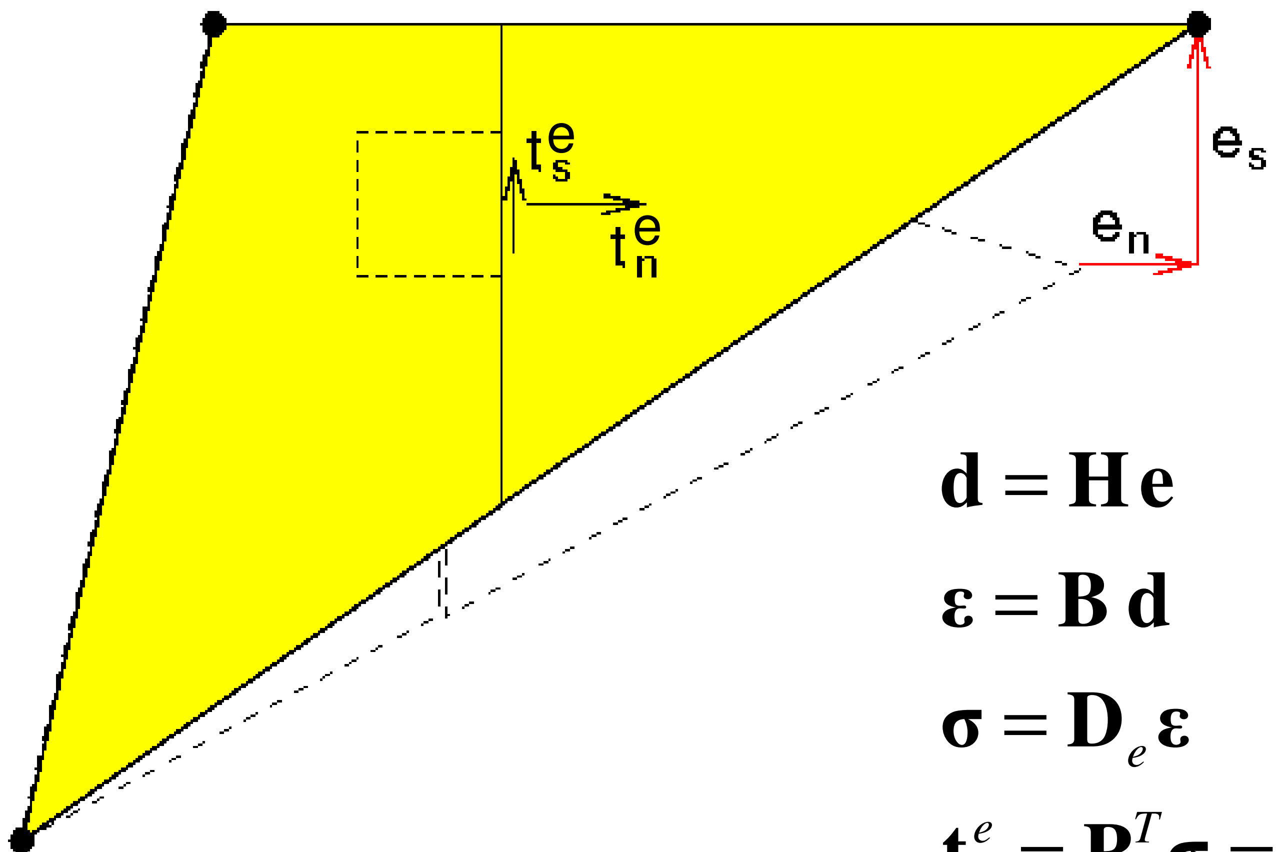
Uniqueness of the element response



$$\mathbf{d} = \mathbf{H}\,\mathbf{e}$$

$$\boldsymbol{\varepsilon} = \mathbf{B}\,\mathbf{d}$$

Uniqueness of the element response



$$\mathbf{d} = \mathbf{H}\,\mathbf{e}$$

$$\boldsymbol{\varepsilon} = \mathbf{B}\,\mathbf{d}$$

$$\boldsymbol{\sigma} = \mathbf{D}_e\,\boldsymbol{\varepsilon}$$

$$\mathbf{t}^e = \mathbf{P}^T\boldsymbol{\sigma} = \mathbf{P}^T\mathbf{D}_e\,\mathbf{B}\,\mathbf{H}\,\mathbf{e}$$

Uniqueness of the element response

$$\lambda_{\min}\left(\mathbf{Q}_{sym}\right) > -H_{\min}$$

$$\mathbf{Q} = \mathbf{P}^T \mathbf{D}_e \mathbf{B} \mathbf{H}$$

is proportional to the elastic modulus
and inversely proportional to the element size

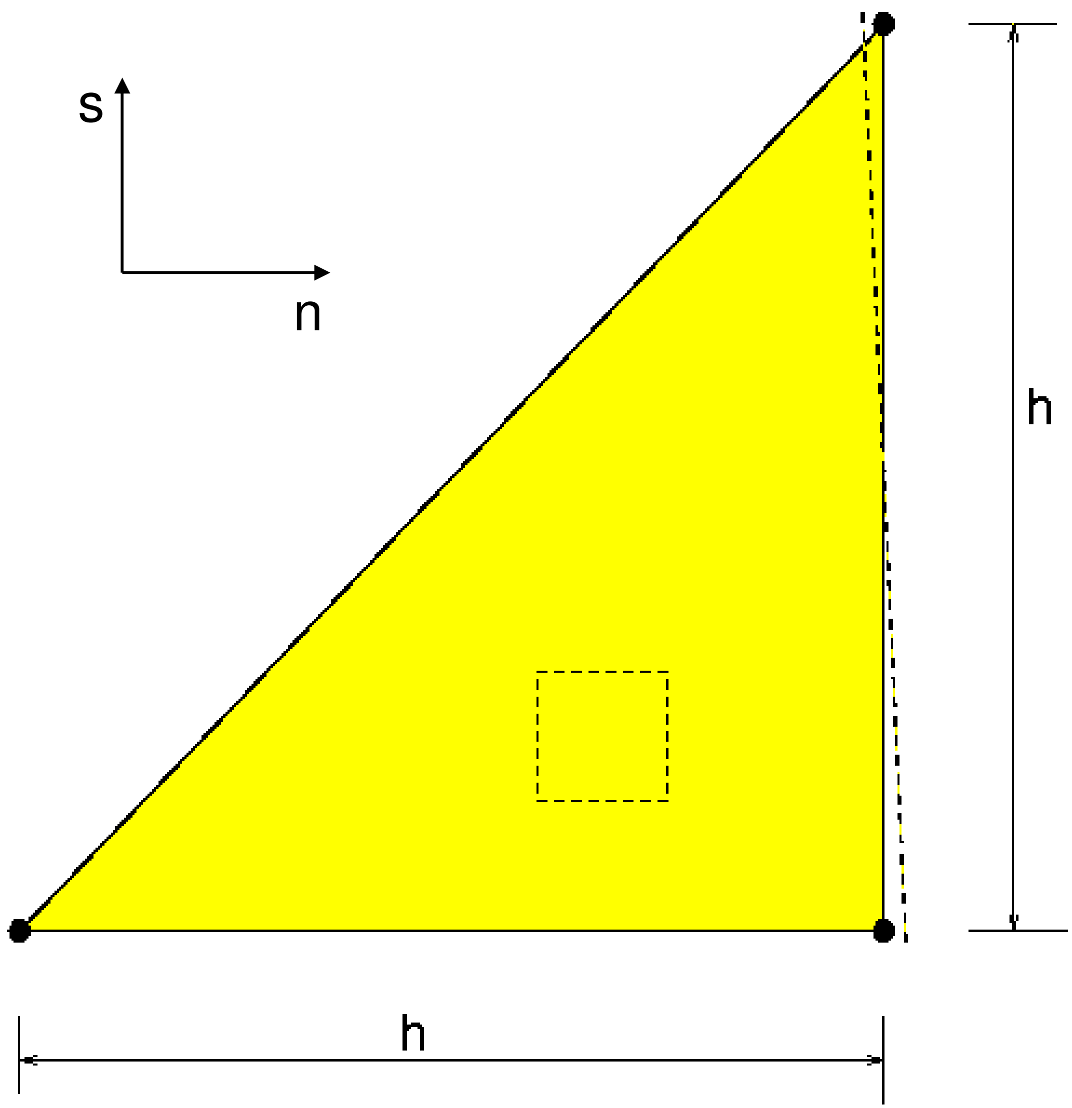
Uniqueness of the element response

$$\lambda_{\min}\left(\mathbf{Q}_{sym}\right) > -H_{\min}$$

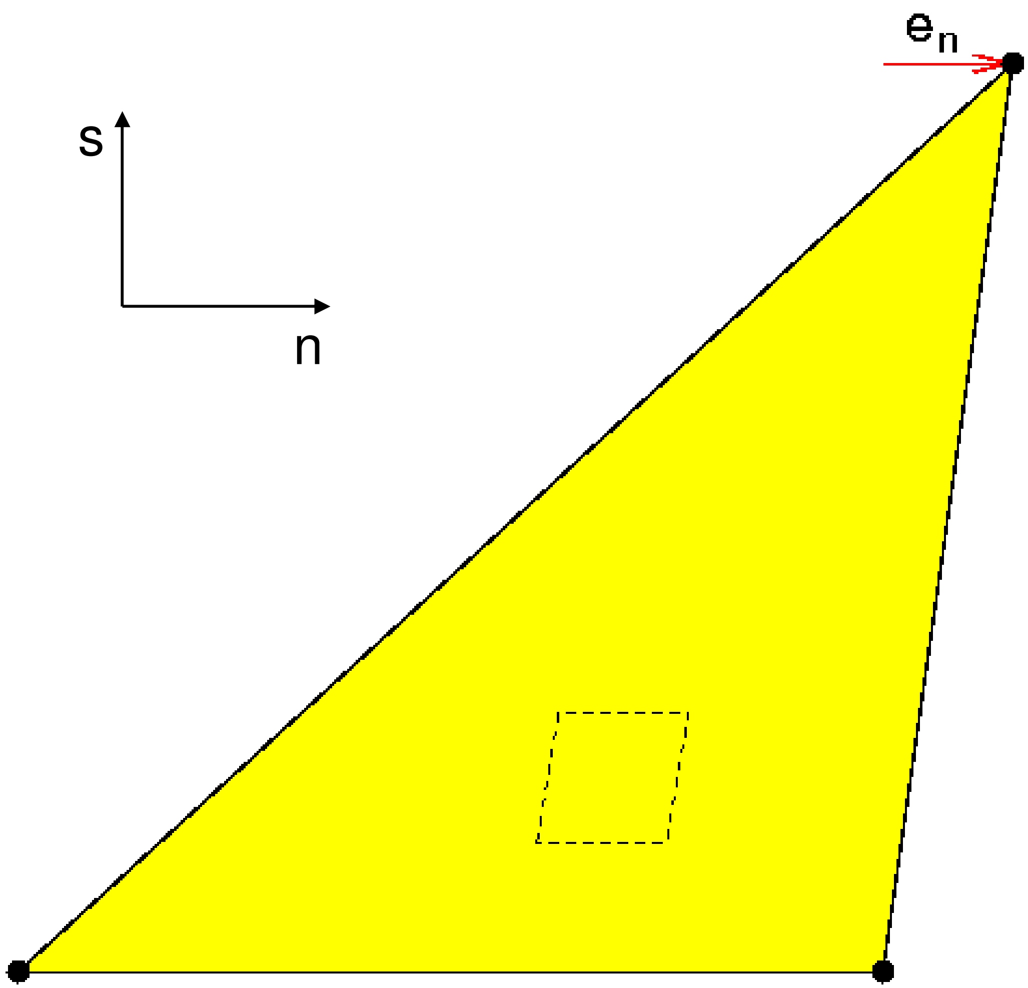
$$\mathbf{Q} = \mathbf{P}^T \mathbf{D}_e \mathbf{B} \mathbf{H}$$ is proportional to the elastic modulus and inversely proportional to the element size

$$\mathbf{e}^T \mathbf{Q}_{sym} \mathbf{e} = \mathbf{e}^T \mathbf{Q} \mathbf{e} = \mathbf{e}^T \mathbf{t}^e < 0$$ can happen

Uniqueness of the element response



Uniqueness of the element response



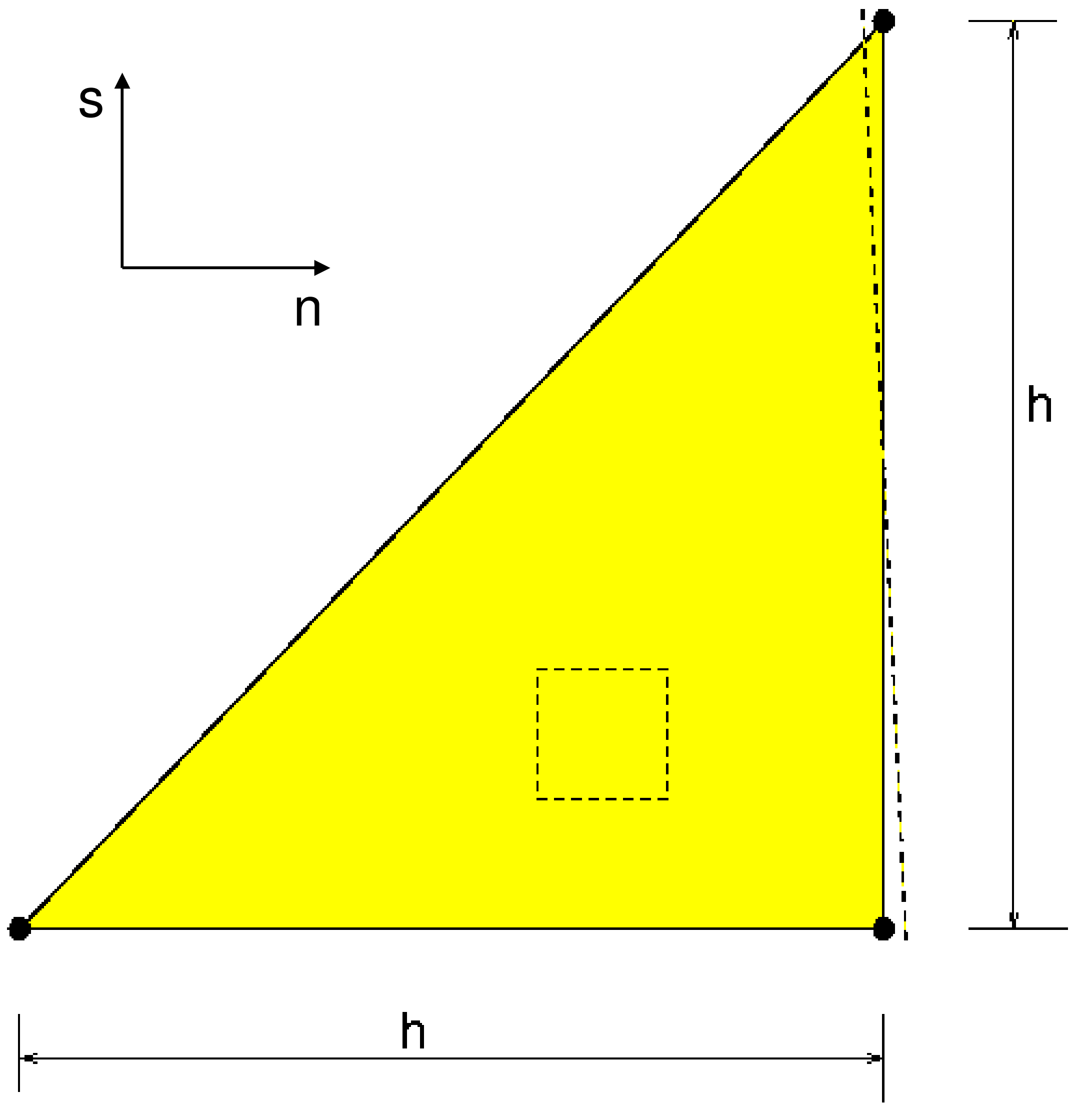
$$\gamma = e_n / h$$

$$\tau = G\gamma$$

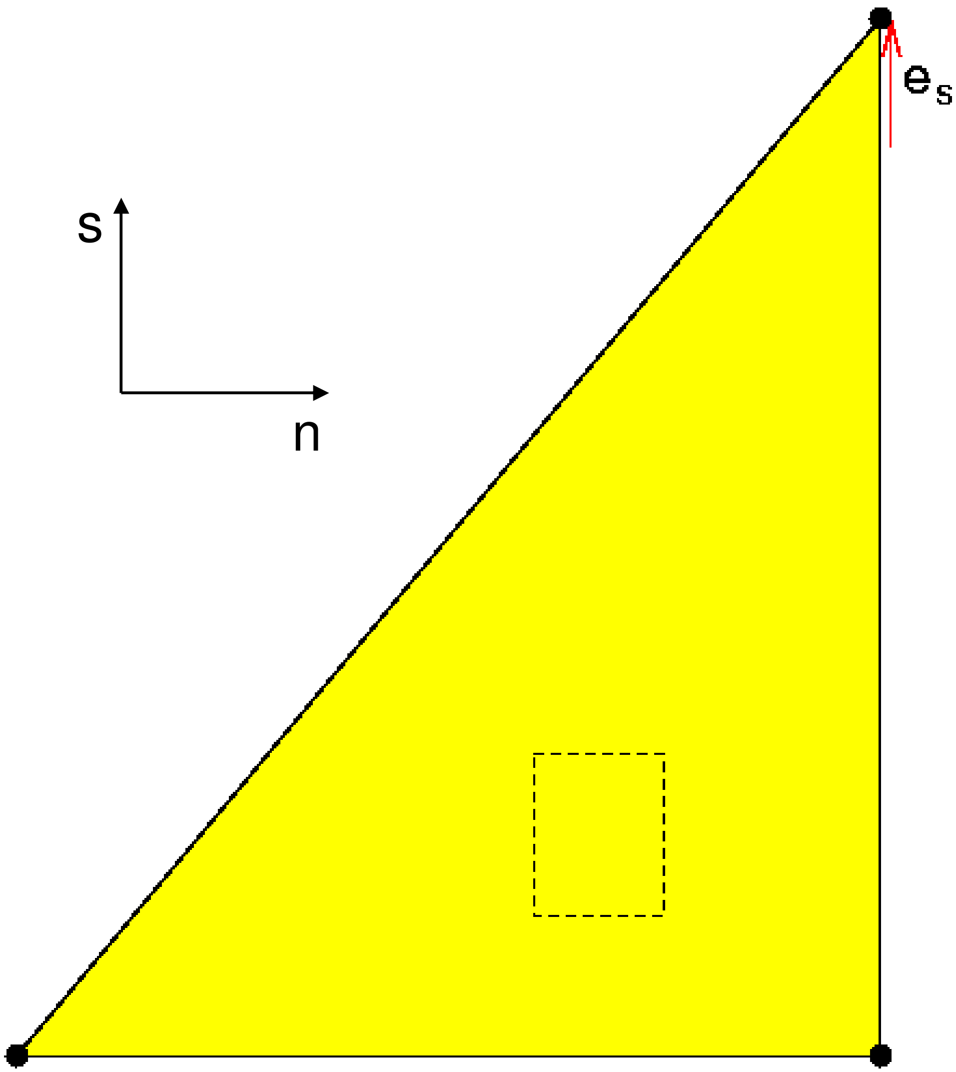
$$t_n = 0$$

$$t_s = \tau = Ge_n / h$$

Uniqueness of the element response



Uniqueness of the element response



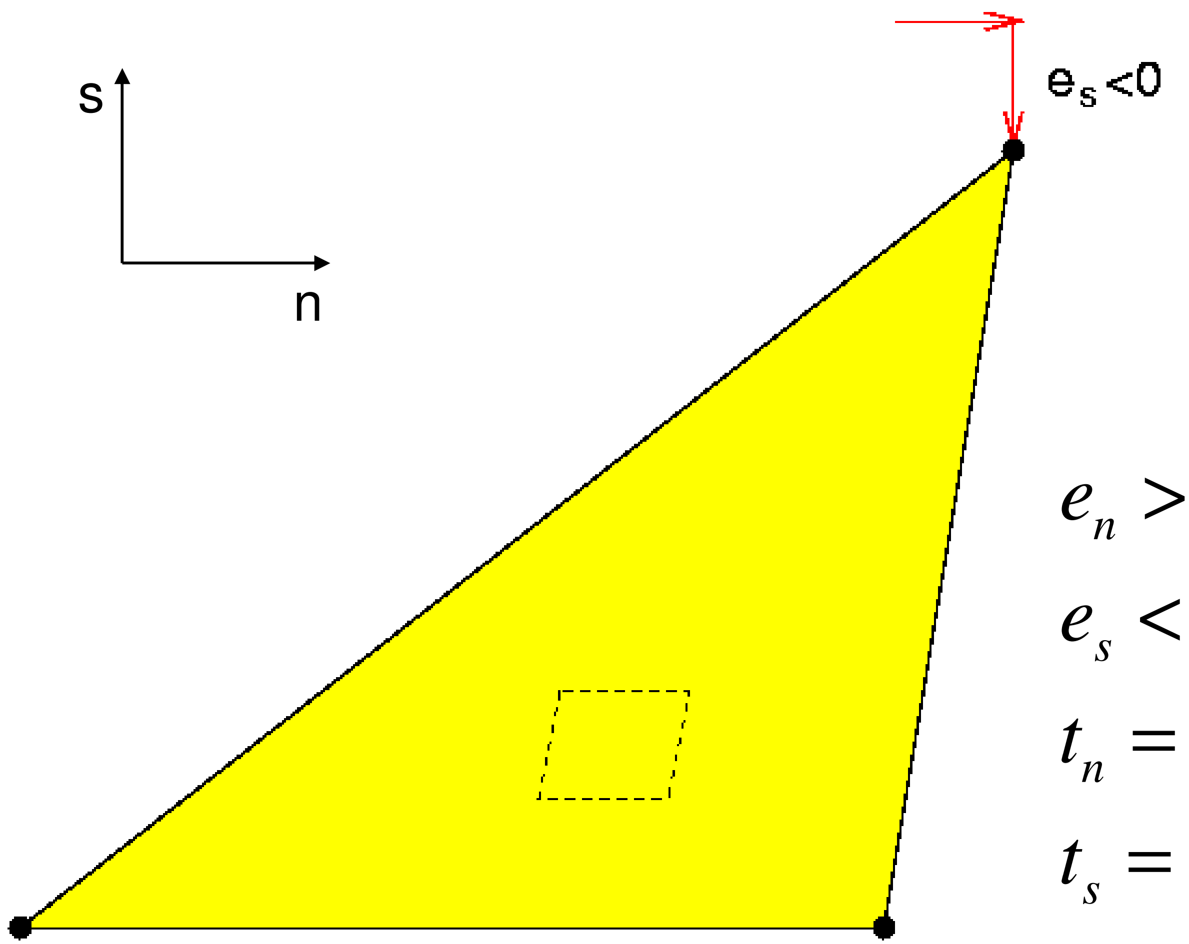
$$\varepsilon_{ss} = e_s / h$$

$$\sigma_{nn} = D_{12}\varepsilon_{ss}$$

$$t_n = \sigma_{nn} = D_{12} e_s / h$$

$$t_s = 0$$

Uniqueness of the element response



$$e_n > 0$$

$$e_s < 0$$

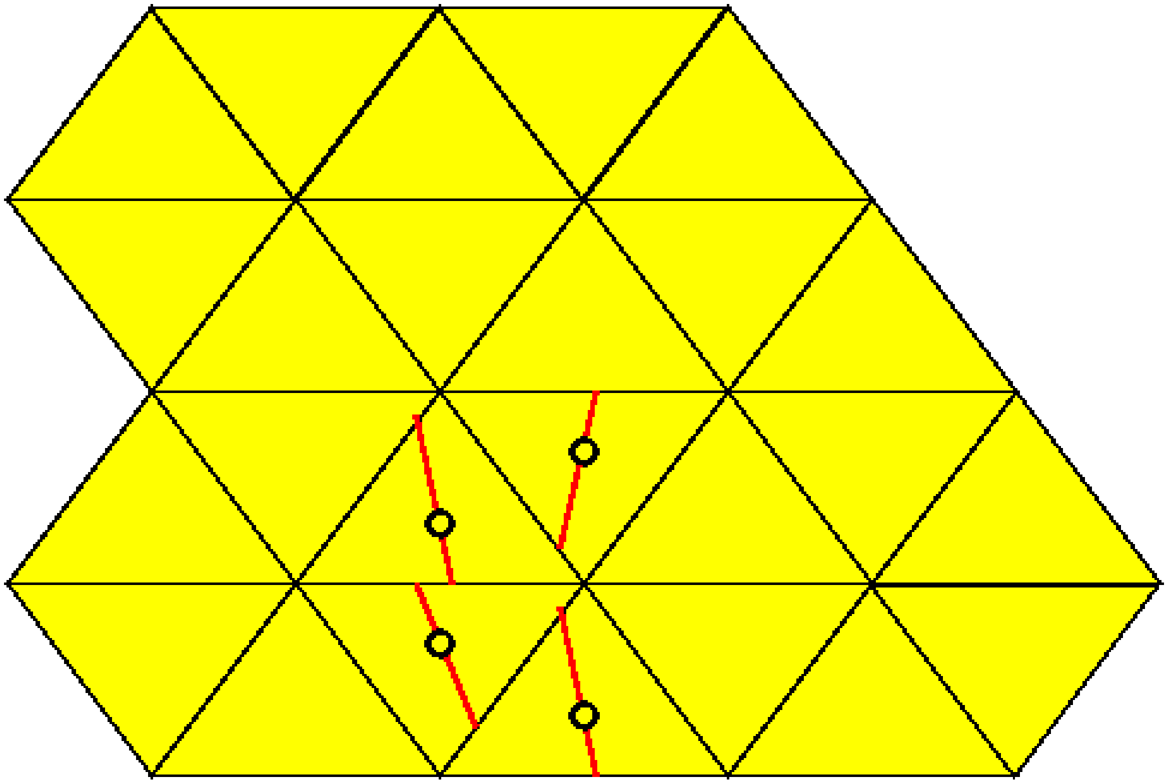
$$t_n = D_{12} e_s / h < 0$$

$$t_s = G e_n / h > 0$$

$$\mathbf{e}^T \mathbf{t}^e < 0$$

Uniqueness of the element response

discontinuity segments placed at element centers



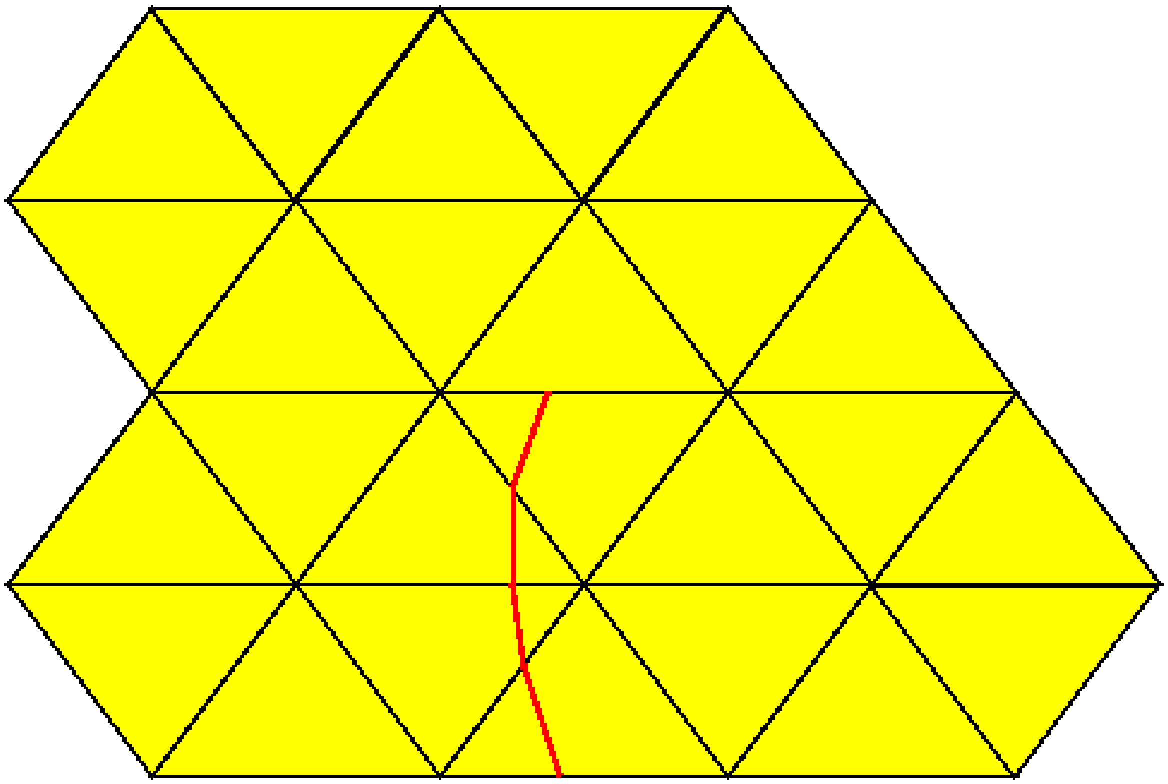
Uniqueness of the element response

discontinuity segments placed at element centers

maximum deviation α between element side and discontinuity is limited (e.g., 30 degrees for an equilateral triangle)

Uniqueness of the element response

discontinuity segments form a continuous path



Uniqueness of the element response

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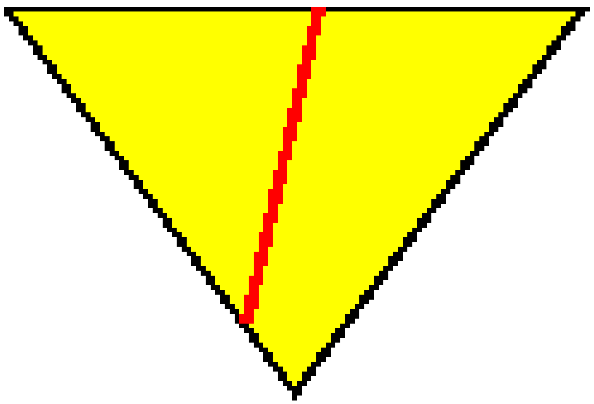

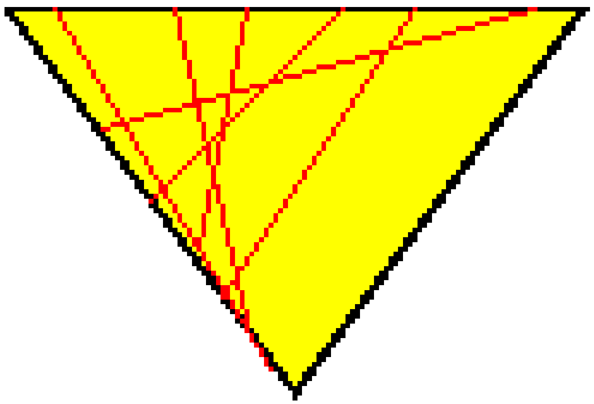

maximum deviation α between element side and discontinuity
is given by the largest angle of the triangle
(e.g., 60 degrees for an equilateral triangle)

Uniqueness of the element response

Condition under which uniqueness can be guaranteed if the element is sufficiently small:

plane stress … $\cos\alpha > \dfrac{1+\nu}{3-\nu}$

true only if $\nu < 1/3$ and the element is close to equilateral

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plane strain … $\cos\alpha > \dfrac{1}{3-4\nu}$

true only if $\nu < 1/4$ and the element is close to equilateral

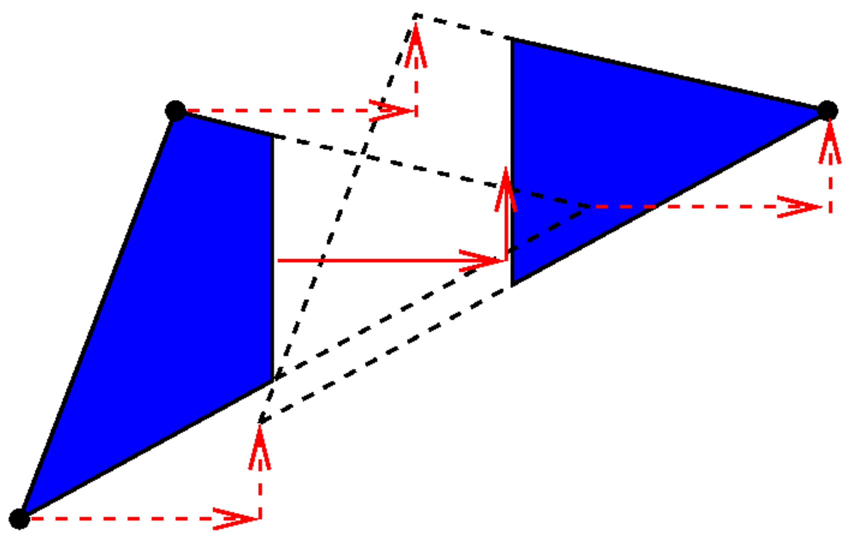
three dimensions … $\cos\alpha > \dfrac{1}{3-4\nu}$

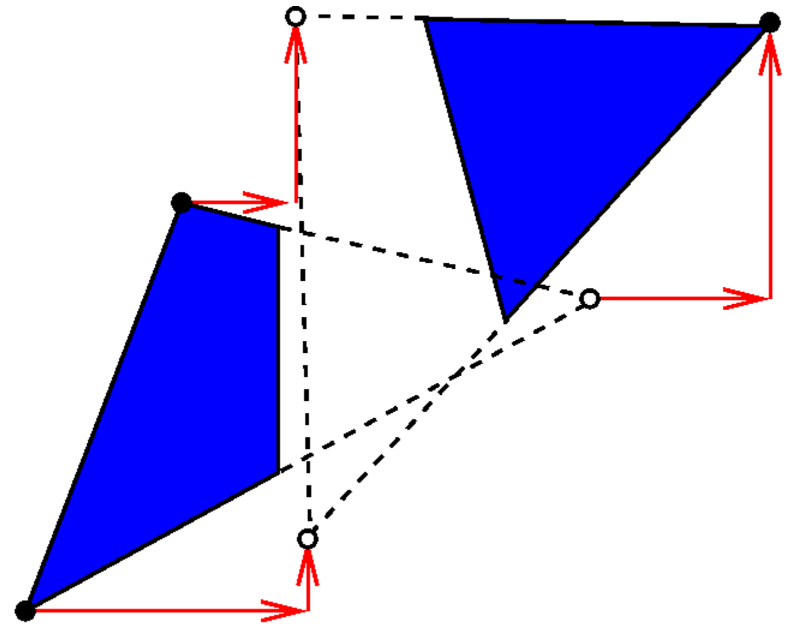
impossible even if the tetrahedral element is regular

Comparison of EED-EAS and XFEM-PUM

Embedded discontinuity

Extended finite elements





Comparison of EED-EAS and XFEM-PUM

	Embedded discontinuity	Extended finite elements
DOF‘s added	locally	globally
and related to	elements	nodes
Approximation of crack opening	discontinuous	continuous
Enrichment	incompatible	compatible
Separated parts	partially interacting	independent
Numerical behavior	**rather fragile**	**more robust**

Comparison of EED-EAS and XFEM-PUM

	Embedded discontinuity	Extended finite elements
Stiffness matrix	always nonsymmetric	can be symmetric
Integration scheme for continuous part	remains standard	must be modified
Global degrees of freedom	do not change	added during simulation
Implementation effort	smaller	larger

Comparison of EED-EAS and XFEM-PUM

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Stiffness matrix	always nonsymmetric	can be symmetric
Integration scheme for continuous part	remains standard	must be modified
Global degrees of freedom	do not change	added during simulation
Implementation effort	smaller	larger
		... but it pays off

The End