# 1D ELASTICITY

- FEM seeks for the minimum potential energy of the system
- equilibrium found using variational principles

- FEM steps:
  1) subdividing the problem domain (= discretization, e.g. subdivision of truss to members connected at nodes) ⇒ into "finite elements"

  2) element formulation: development of equations for elements (e.g. stiffness of each element – i.e. its response to loading)

  3) assembly: obtaining the equations of the entire system from the equations of individual elements (creating "global matrix")

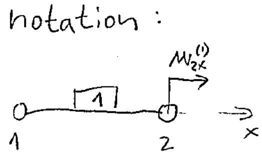
  4) solving the equations

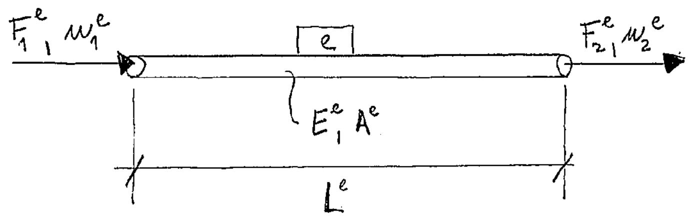
  5) postprocessing: determining quantities of interest (e.g. stresses) + visualization

## BAR ELEMENT (direct stiffness method)

- slender → no resistance to torsion, bending and shear
  ↘ only axial internal forces
- equivalent to spring

- notation: $u_{2x}^{(1)} = u_2^{(1)}$ = x-component of displacement at node 2 of element 1






internal force : $F^e$

stress : $\sigma^e = \frac{F^e}{A^e}$ (tension = positive)

bar behavior : a) equilibrium (sum of nodal forces is zero) :

$$F_1^e + F_2^e = 0$$

b) Hooke's law : $\sigma^e = E^e \varepsilon^e$

c) structure is continuous (no gaps or overlaps) :

$$\varepsilon^e = \frac{u_2^e - u_1^e}{L^e}$$

nodal forces:

$$F_1^e = -F_e \text{ (internal force)} = -A^e \sigma^e = -A^e E^e \varepsilon^e = -\frac{E^e A^e}{L^e}\left(u_2^e - u_1^e\right) = \frac{E^e A^e}{L^e}\left(u_1^e - u_2^e\right)$$

$$F_2^e = \frac{E^e A^e}{L^e}\left(u_2^e - u_1^e\right)$$

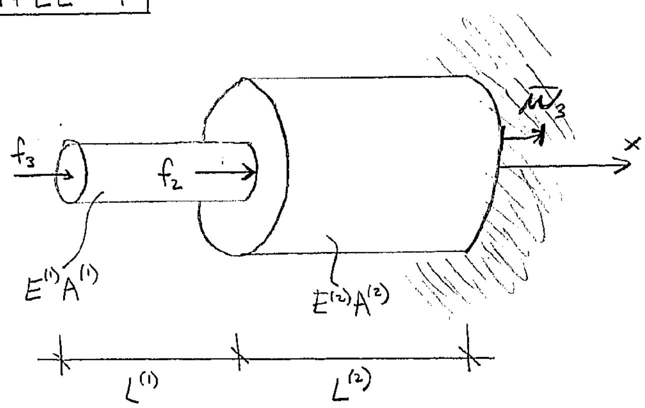
$\Downarrow$

in matrix form : $\underbrace{\begin{bmatrix} F_1^e \\ F_2^e \end{bmatrix}}_{\underline{F}^e} = \underbrace{\frac{E^e A^e}{L^e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}_{\underline{K}^e} \underbrace{\begin{bmatrix} u_1^e \\ u_2^e \end{bmatrix}}_{\underline{u}^e}$

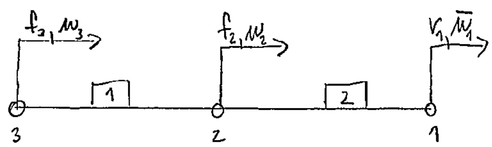
$\underline{K}^e \downarrow$ symmetric ($\underline{K}^e = (\underline{K}^e)^T$)

- the equation $\underline{F}^e = \underline{K}^e \underline{u}^e$ describes the behavior of an element
- because of linearity in Hooke's law and strain-displacement relationship the system is linear

EXAMPLE 1




scheme :



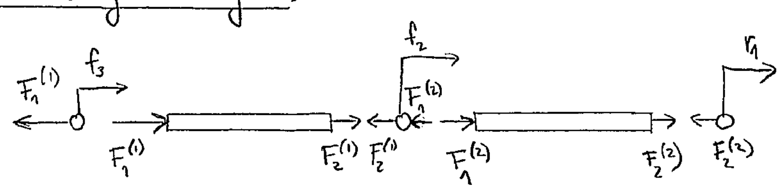

1) discretization : nodes where load is applied and properties of the structure change

2) at nodes with prescribed displacement $\bar{w}$ the force is referred to as a reaction (it is unknown)

3) at nodes with known force the displacement must be unknown

4) for each bar element the nodal forces are related to nodal displacements via stiffness matrix $\underline{F}^e = \underline{k}^e \underline{d}^e$

free body diagram :




- global system of equations from compatibility between elements and nodal equilibrium
- the forces exerted by the elements on the nodes are equal and opposite

contributions of

$$
\begin{array}{l}
\\ \text{node 1} \\ \text{node 2} \\ \text{node 3}
\end{array}
\overset{\text{element 1:}}{\begin{bmatrix} 0 \\ F_2^{(1)} \\ F_1^{(1)} \end{bmatrix}} + \overset{\text{element 2:}}{\begin{bmatrix} F_2^{(2)} \\ F_1^{(2)} \\ 0 \end{bmatrix}} = \underbrace{\begin{bmatrix} 0 \\ f_2 \\ f_3 \end{bmatrix}}_{\underline{f}\text{ - prescribed at nodes}} + \overbrace{\begin{bmatrix} r_1 \\ 0 \\ 0 \end{bmatrix}}^{\text{unknown forces (=reactions)}}
$$

- element stiffness equation for element 1: $\begin{bmatrix} F_1^{(1)} \\ F_2^{(1)} \end{bmatrix} = \begin{bmatrix} k^{(1)} & -k^{(1)} \\ -k^{(1)} & k^{(1)} \end{bmatrix} \begin{bmatrix} u_3 \\ u_2 \end{bmatrix}$

$$
\underbrace{\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & k^{(1)} & -k^{(1)} \\ 0 & -k^{(1)} & k^{(1)} \end{bmatrix} \end{array}}_{\underline{k}^{(1)}} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}}_{\underline{d}} + \underbrace{\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} k^{(2)} & -k^{(2)} & 0 \\ -k^{(2)} & k^{(2)} & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}}_{\underline{k}^{(2)}} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}}_{\underline{d}} = \underbrace{\begin{bmatrix} 0 \\ f_2 \\ f_3 \end{bmatrix}}_{\underline{f}} + \underbrace{\begin{bmatrix} r_1 \\ 0 \\ 0 \end{bmatrix}}_{\underline{r}}
$$

$$
\underbrace{\left(\underline{k}^{(1)} + \underline{k}^{(2)}\right)}_{\text{global stiffness matrix}} \underline{d} = \underline{f} + \underline{r}
$$

$$
\underline{K} = \sum_{e=1}^{n_{el}} \underline{k}^e = \underbrace{\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} k^{(2)} & -k^{(2)} & 0 \\ -k^{(2)} & k^{(1)}+k^{(2)} & -k^{(1)} \\ 0 & -k^{(1)} & k^{(1)} \end{bmatrix} \end{array}\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}}_{\text{singular matrix} \rightarrow \text{ we need boundary conditions}}
$$

- essential boundary conditions on "E-nodes" → prescribed displ.
- forces prescribed at free "F-nodes"

$$
\underline{d} = \begin{bmatrix} \underline{d}_E \\ \underline{d}_F \end{bmatrix} \qquad \underline{f} = \begin{bmatrix} \cancel{\underline{f}_E} \rightarrow \emptyset \\ \underline{f}_F \end{bmatrix} \qquad \underline{r} = \begin{bmatrix} \underline{r}_E \\ \cancel{\underline{r}_F} \end{bmatrix}
$$

∅ no reactions at free nodes

$$
\begin{bmatrix} k^{(2)} & -k^{(2)} & 0 \\ -k^{(2)} & k^{(1)}+k^{(2)} & -k^{(1)} \\ 0 & -k^{(1)} & k^{(1)} \end{bmatrix} \begin{bmatrix} \bar{u}_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} r_1 \\ -4 \\ 10 \end{bmatrix} \longrightarrow \begin{bmatrix} \underline{K}_E & \underline{K}_{EF} \\ \underline{K}_{EF}^T & \underline{K}_F \end{bmatrix} \begin{bmatrix} \underline{d}_E \\ \underline{d}_F \end{bmatrix} = \begin{bmatrix} \underline{r}_E \\ \underline{f}_F \end{bmatrix}
$$

$$\Rightarrow \quad \underline{k}_{EF}^{T}\,\bar{\underline{d}}_E + \underline{k}_F\,\underline{d}_F = \underline{f}_F \quad \Rightarrow \quad \boxed{\begin{array}{l} \text{displacements: } \underline{d}_F = \underline{k}_F^{-1}\left(\underline{f}_F - \underline{k}_{EF}^{T}\,\bar{\underline{d}}_E\right) \\ \text{reactions: } \underline{r}_E = \underline{k}_E\,\bar{\underline{d}}_E + \underline{k}_{EF}\,\underline{d}_F \end{array}}$$

## Penalty method

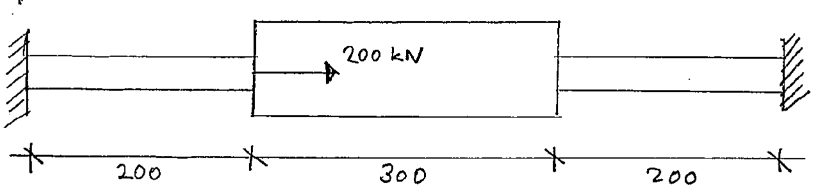
- for matrices up to 1000 unknowns it can be faster than partitioning
- prescribed displacements substituted by very large numbers:

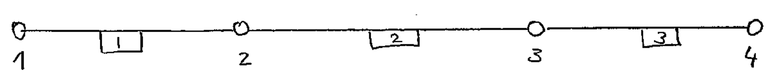
like very stiff spring between node 1 and support

$$\begin{bmatrix} \beta & -k^{(2)} & 0 \\ -k^{(2)} & k^{(1)}+k^{(2)} & -k^{(1)} \\ 0 & -k^{(1)} & k^{(1)} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \beta\,\bar{u}_1 \\ -4 \\ 10 \end{bmatrix}$$

$\beta \sim$ average diagonal value in $\underline{k}$ multiplied by $10^7 \Rightarrow$ the other terms in $\underline{k}_{EF}$ become negligible and irrelevant

## Example 2




$A^{(1)} = A^{(3)} = 2400\ mm^2$
$E^{(1)} = E^{(3)} = 70\ GPa$

$$k^{(1)} = \frac{E^{(1)}A^{(1)}}{L^{(1)}} = \frac{70\cdot10^{9}\cdot 2400\cdot10^{-6}}{200} = 84\cdot10^{4}\ Nm^{-1}$$

$A^{(2)} = 3000\ mm^2$
$E^{(2)} = 40\ GPa$

$$k^{(3)} = k^{(1)}, \quad k^{(2)} = \frac{E^{(2)}A^{(2)}}{L^{(2)}} = 40\cdot10^{4}\ Nm^{-1}$$

$$\underline{k}^{(1)} = \underline{k}^{(3)} = 84\cdot10^{4}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \underline{k}^{(2)} = 40\cdot10^{4}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

assembly of global matrices:

$$
\begin{array}{cc}
 & \begin{array}{cccc} ① & ② & ③ & ④ \end{array} \\
\underline{K} = & \left[\begin{array}{c|cc|c} 84 & -84 & 0 & 0 \\ \hline -84 & 84+40 & -40 & 0 \\ 0 & -40 & 40+84 & -84 \\ \hline 0 & 0 & -84 & 84 \end{array}\right] \cdot 10^4
\end{array}
$$

$$\underline{K}_E = \begin{bmatrix} 84 & 0 \\ 0 & 84 \end{bmatrix} \cdot 10^4 \qquad \underline{K}_{EF}^T = \begin{bmatrix} -84 & 0 \\ 0 & -84 \end{bmatrix} \cdot 10^4 \qquad \underline{K}_F = \begin{bmatrix} 124 & -40 \\ -40 & 124 \end{bmatrix} \cdot 10^4 \qquad [N m^{-1}]$$

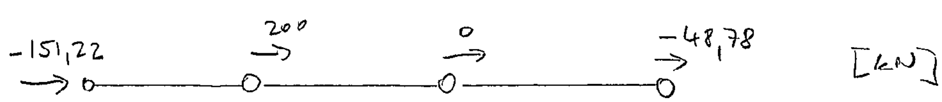
$$\underline{f} = \begin{bmatrix} 0 \\ \hline 200 \\ 0 \\ \hline 0 \end{bmatrix} \cdot 10^3 \; \Big\} f_F \quad [N] \qquad \underline{d} = \begin{bmatrix} 0 \\ \hline w_2 \\ w_3 \\ \hline 0 \end{bmatrix} \Big\} d_F \quad [m] \qquad \underline{r} = \begin{bmatrix} r_1 \\ \hline 0 \\ 0 \\ \hline r_4 \end{bmatrix} \quad [N]$$

$$\underline{d}_F = \underline{K}_F^{-1}\left(\underline{f}_F - \underline{K}_{EF}^T \underline{d}_E\right) = \begin{bmatrix} 124 & -40 \\ -40 & 124 \end{bmatrix}^{-1} \cdot 10^{-4} \cdot \left( \begin{bmatrix} 200 \\ 0 \end{bmatrix} \cdot 10^3 - \cancel{\begin{bmatrix} -84 & 0 \\ 0 & -84 \end{bmatrix} 10^4 \cdot \begin{bmatrix} 0 \\ 0 \end{bmatrix}} \right) =$$

$$= \begin{bmatrix} 0.180023 \\ 0.058072 \end{bmatrix} m$$

$$\underline{r}_E = \underline{K}_E \underline{d}_E + \underline{K}_{EF} \underline{d}_F = \begin{bmatrix} 84 & 0 \\ 0 & 84 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -84 & 0 \\ 0 & -84 \end{bmatrix} \cdot 10^4 \begin{bmatrix} 0.180023 \\ 0.058072 \end{bmatrix} =$$

$$= \begin{bmatrix} -151{,}22 \\ -48{,}78 \end{bmatrix} kN$$




$$\overrightarrow{X}: \quad -151{,}22 + 200 - 48{,}78 = 0 \quad ✓$$

strains: $\varepsilon^{(i)} = \dfrac{w_2^{(i)} - w_1^{(i)}}{L^{(i)}}$ , stresses: $\sigma^{(i)} = E^{(i)} \varepsilon^{(i)}$

$$\sigma^{(1)} = \frac{w_2^{(1)} - w_1^{(1)}}{L^{(1)}} E^{(1)} = \frac{0.180023}{200} \cdot 70 \cdot 10^9 = 63 \text{ MPa} \quad (\text{tension})$$

$$\sigma^{(2)} = \frac{w_2^{(2)} - w_1^{(2)}}{L^{(2)}} E^{(2)} = \frac{0.058072 - 0.180023}{300} \cdot 40 \cdot 10^9 = -16{,}3 \text{ MPa} \quad (\text{compression})$$

$\sigma^{(3)}$ ... accordingly