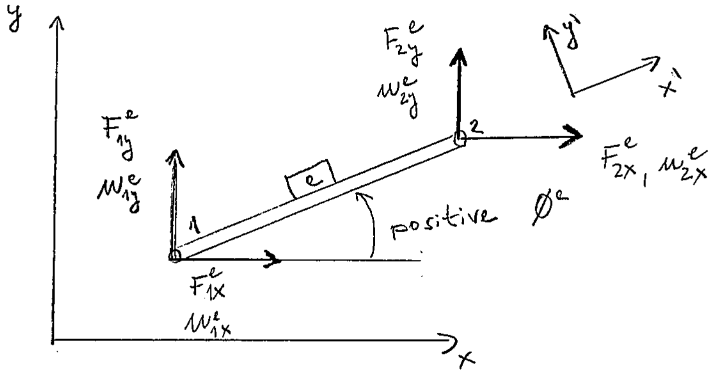
2D TRUSSES

- composed of 1D members
- needed transformation to global coordinate system



$$\underline{F}^e = \begin{bmatrix} F^e_{1x} \\ F^e_{1y} \\ F^e_{2x} \\ F^e_{2y} \end{bmatrix} \qquad \underline{d}^e = \begin{bmatrix} u^e_{1x} \\ u^e_{1y} \\ u^e_{2x} \\ u^e_{2y} \end{bmatrix}$$

- vector transformations: $u'^e_{1x} = u^e_{1x} \cos\phi^e + u^e_{1y} \sin\phi^e$

$$u'^e_{1y} = -u^e_{1x} \sin\phi^e + u^e_{1y} \cos\phi^e$$

$$\Downarrow$$

in matrix notation: $\underline{d}'^e = \underline{T}^e \underline{d}^e$

$$\begin{bmatrix} u'^e_{1x} \\ u'^e_{1y} \\ u'^e_{2x} \\ u'^e_{2y} \end{bmatrix} = \begin{bmatrix} \cos\phi^e & \sin\phi^e & 0 & 0 \\ -\sin\phi^e & \cos\phi^e & 0 & 0 \\ 0 & 0 & \cos\phi^e & \sin\phi^e \\ 0 & 0 & -\sin\phi^e & \cos\phi^e \end{bmatrix} \begin{bmatrix} u^e_{1x} \\ u^e_{1y} \\ u^e_{2x} \\ u^e_{2y} \end{bmatrix}$$

- $\underline{T}^e$ is orthogonal $\Rightarrow (\underline{T}^e)^T = (\underline{T}^e)^{-1} \Rightarrow \underline{T}^e (\underline{T}^e)^T = I$

- in local coordinates (x', y') the element stiffness equation is: $\begin{bmatrix} k^e & -k^e \\ -k^e & k^e \end{bmatrix} \begin{bmatrix} u'^e_{1x} \\ u'^e_{2x} \end{bmatrix} = \begin{bmatrix} F'^e_{1x} \\ F'^e_{2x} \end{bmatrix}$

$$\Downarrow$$

it is expanded by adding y-components:

$$\underbrace{\begin{bmatrix} F'^e_{1x} \\ F'^e_{1y} \\ F'^e_{2x} \\ F'^e_{2y} \end{bmatrix}}_{\underline{F}'^e} = \underbrace{k^e \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\underline{k}'^e} \underbrace{\begin{bmatrix} u'^e_{1x} \\ u'^e_{1y} \\ u'^e_{2x} \\ u'^e_{2y} \end{bmatrix}}_{\underline{d}'^e}$$

$$\Rightarrow \quad \underline{d}^e = (\underline{T}^e)^T \underline{d}'^e \; , \quad \underline{F}^e = (\underline{T}^e)^T \underline{F}'^e$$

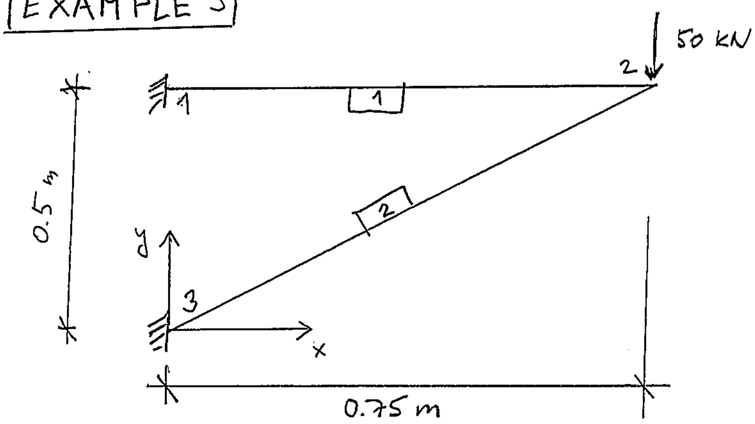
- relation between $\underline{F}^e$ and $\underline{d}^e$:
(global nodal values)

$$\underline{F}^e = (\underline{T}^e)^T \underline{F}'^e = (\underline{T}^e)^T \underline{k}'^e \underline{d}'^e = \underbrace{(\underline{T}^e)^T \underline{K}'^e \underline{T}^e}_{\underline{K}^e \text{ - global stiffness matrix, symmetric}} \underline{d}^e$$

global stiffness of each element: $\underline{k}^e = (\underline{T}^e)^T \underline{K}'^e \underline{T}^e = \frac{EA}{L} \begin{bmatrix} c^2 & cs & -c^2 & -cs \\ & s^2 & -cs & -s^2 \\ & & c^2 & cs \\ & & & s^2 \end{bmatrix}$

where $c^e = \dfrac{x_2^e - x_1^e}{L^e}$, $s^e = \dfrac{y_2^e - y_1^e}{L^e}$, and $L^e = \sqrt{(x_2^e - x_1^e)^2 + (y_2^e - y_1^e)^2}$

EXAMPLE 3



$E = 200 \text{ GPa} = 200 \cdot 10^9 \text{ Pa}$

$A = 1000 \text{ mm}^2 = 10^{-3} \text{ m}^2$

a) stiffness matrices (local):

$L^{(1)} = 0{,}75$

$L^{(2)} = \sqrt{0.75^2 + 0.5^2} = 0.901388$

$$\underline{K}^{(1)} = 266{,}67 \cdot 10^6 \begin{array}{c} \begin{array}{cc} ① & ② \end{array} \\ \left[\begin{array}{cc|cc} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{c} ① \\ \\ ② \end{array}$$

$c^{(1)} = 1$, $s^{(1)} = 0$

$c^{(2)} = \frac{3}{\sqrt{13}}$, $s^{(2)} = \frac{2}{\sqrt{13}}$

$$\underline{k}^{(2)} = 17{,}07 \cdot 10^6 \cdot \begin{array}{c} \left[\begin{array}{cc|cc} 9 & 6 & -9 & -6 \\ 6 & 4 & -6 & -4 \\ \hline -9 & -6 & 9 & 6 \\ -6 & -4 & 6 & 4 \end{array}\right] \\ \begin{array}{cc} ② & ③ \end{array} \end{array} \begin{array}{c} ② \\ \\ ③ \end{array}$$

$[\text{Nm}^{-1}]$

b) Assembly of global stiffness matrix

$$\underline{K} = 10^6 \begin{array}{c} \begin{array}{cccccc} \hphantom{-266,67} & ① & \hphantom{-266,67} & ② & \hphantom{-153,63} & ③ \end{array} \\ \left[\begin{array}{cc:cc:cc} 266,67 & 0 & -266,67 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hdashline -266,67 & 0 & 420,3 & 102,42 & -153,63 & -102,42 \\ 0 & 0 & 102,42 & 68,28 & -102,42 & -68,28 \\ \hdashline 0 & 0 & -153,63 & -102,42 & 153,63 & 102,42 \\ 0 & 0 & -102,42 & -68,28 & 102,42 & 68,28 \end{array}\right] \end{array} [Nm^{-1}]$$

$$\underline{d} = \begin{bmatrix} 0 \\ 0 \\ w_{2x} \\ w_{2y} \\ 0 \\ 0 \end{bmatrix} [m] \qquad \underline{f} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -50\,000 \\ 0 \\ 0 \end{bmatrix} [N]$$

- free DOFs : 3, 4

$$\hookrightarrow \underline{K}_F = \underline{K}(3{:}4, 3{:}4) = \begin{bmatrix} 420,3 & 102,42 \\ 102,42 & 68,28 \end{bmatrix} \cdot 10^6 \quad [Nm^{-1}]$$

$$\underline{d}_F = \underline{d}(3{:}4) = \begin{bmatrix} w_{2x} \\ w_{2y} \end{bmatrix} [m] \qquad \underline{f}_F = \underline{f}(3{:}4) = \begin{bmatrix} 0 \\ -50\,000 \end{bmatrix} [N]$$

- essential (constrained) BCs : 1, 2, 5, 6

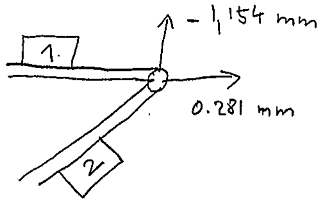
$$\hookrightarrow \underline{K}_E = \underline{K}(\{1{:}2, 5{:}6\}, \{1{:}2, 5{:}6\}) = \begin{bmatrix} 266,67 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 153,63 & 102,42 \\ 0 & 0 & 102,42 & 68,28 \end{bmatrix} \cdot 10^6 \quad [Nm^{-1}]$$

$$\bar{\underline{d}}_E = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} [m] \qquad \underline{r}_E = \begin{bmatrix} r_{1x} \\ r_{1y} \\ r_{3x} \\ r_{3y} \end{bmatrix} [N]$$

$$\underline{K}_{EF} = \underline{k}(\{1{:}2,5{:}6\},3{:}4) = 10^6 \begin{bmatrix} -266{,}67 & 0 \\ 0 & 0 \\ -153{,}63 & -102{,}42 \\ -102{,}42 & -68{,}28 \end{bmatrix} [Nm^{-1}]$$

c) solve the equations:

$$\underline{d}_F = \underline{K}_F^{-1}\left(\underline{f}_F - \cancelto{0}{\underline{K}_{EF}^T \underline{d}_E}\right) = \begin{bmatrix} 0.000\,281 \\ -0.001\,154 \end{bmatrix} [m]$$



- in local coordinates:

$$\underline{d}'^{(1)} = \begin{bmatrix} 0 \\ 0 \\ 0.000281 \\ -0.001154 \end{bmatrix} [m]$$

$$d_x'^{(1)} = \begin{bmatrix} 0 \\ 0.000\,281 \end{bmatrix} [m]$$

$$\underline{d}'^{(2)} = \underline{T}^{(2)}\,\underline{d}^{(2)} = \frac{1}{\sqrt{13}}\begin{bmatrix} 3 & 2 & 0 & 0 \\ -2 & 3 & 0 & 0 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & -2 & 3 \end{bmatrix}\begin{bmatrix} 0.000\,281 \\ -0.001\,154 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.000\,406 \\ -0.001\,116 \\ 0 \\ 0 \end{bmatrix} [m]$$

$$d_x'^{(2)} = \begin{bmatrix} -0.000\,406 \\ 0 \end{bmatrix} [m]$$

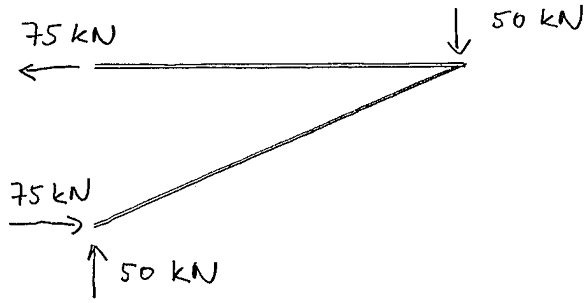
d) postprocessing:

$$\sigma^{(1)} = \frac{u_{2x}'^{(1)} - u_{1x}'^{(1)}}{L^{(1)}} \cdot E^{(1)} = \frac{0.281 \cdot 10^{-3}}{0.75} \cdot 200 \cdot 10^9 = 74{,}9 \cdot 10^6 \; Pa$$

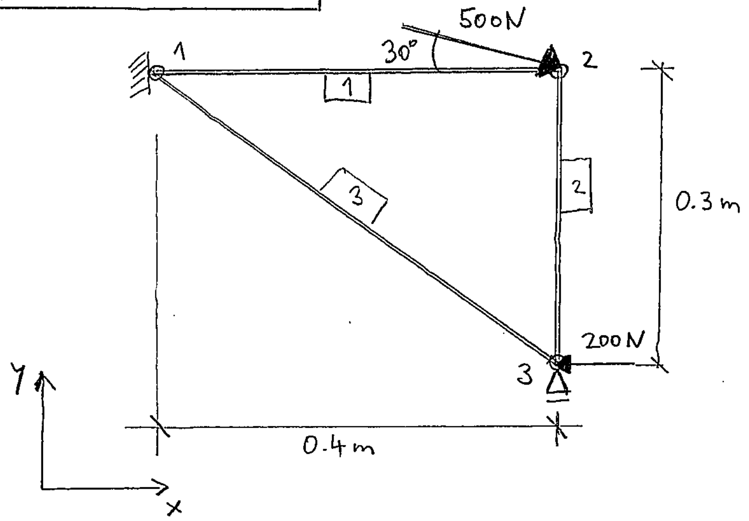
$\sigma^{(2)} = \ldots$ accordingly

e) reactions :

$$\underline{r}_E = \cancel{\underline{k}_E \underline{d}_E}^{\,0} + \underline{k}_{EF}\,\underline{d}_F = \begin{bmatrix} -75\,000 \\ 0 \\ 75\,000 \\ 50\,000 \end{bmatrix} \quad [N]$$



HOMEWORK 1



$E = 100 \cdot 10^9$ Pa

$A = 0{,}01 \ m^2$

- no of DOFs, $\underline{d}$, $\underline{f}$
- element stiffness matrices in global coordinates, displacements, reactions, stresses in members