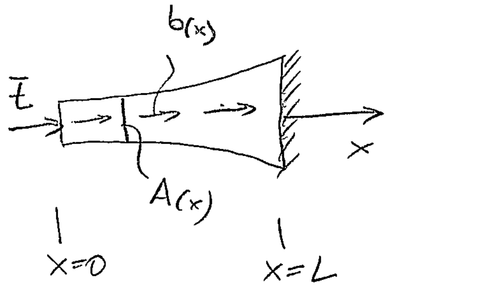
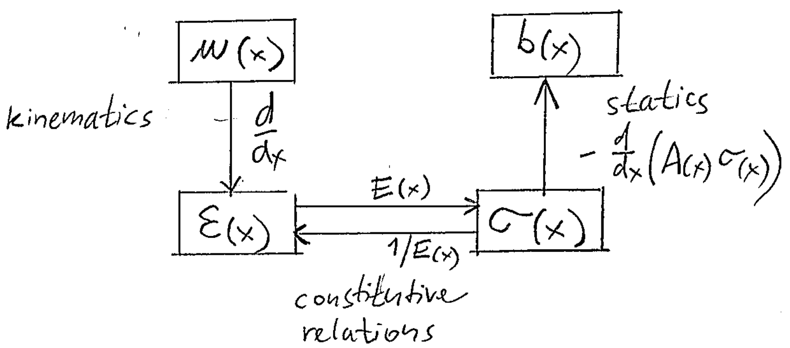
Strong and Weak Forms for 1-D Problems

- strong form:
  - governing equations and BCs for a physical system (usually PDE becoming ODE in 1D)
- weak form:
  - integral form of the strong form
  - by a clever manipulation we can decrease the derivative order ⇒ "weaker continuity requirements"
- 3 ingredients of FEM:
  a) strong formulation
  b) weak form from the strong form
  c) approximation of weak form by approximation functions

Strong form for 1D elasticity





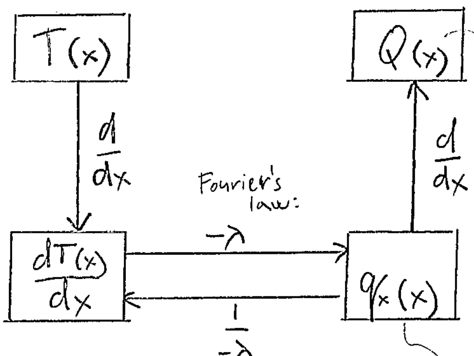
- assuming linear behavior in the strain-displacement and stress-strain laws the governing eq. is $\frac{d}{dx}\left(EA\frac{du}{dx}\right)+\bar{b}=0$ on $0<x<L$

- BC is needed at the free ends:
$$\sigma|_\Gamma = \left[E \frac{dw}{dx}\right]_\Gamma = -\bar{t}$$

Strong form for 1D heat conduction

- decribes heat flow if there is a temperature difference within a body

+ what flows in must flow out, in steady state the time doesn't matter

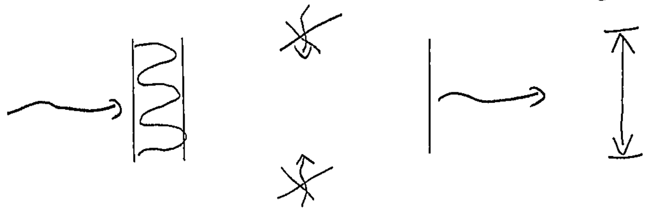


internal heat source (e.g. during hydration) $[J m^{-3} s^{-1}]$

heat flux $[J m^{-2} s^{-1}]$

coefficient of thermal conductivity (rate of energy escape) $[K m^{-1}]$

heat goes against the gradient



long enough $\Rightarrow$ 1D conduction

conduction		elasticity
T	$\leftrightarrow$	w
E	$\leftrightarrow$	$-\lambda$
b	$\leftrightarrow$	Q

- governing equation: $\frac{d}{dx}\left(-\lambda \frac{dT}{dx}\right) = Q$ on $0 < x < L$

- BCs: a) temperature at surface (prescribed), $T|_\Gamma = \bar{T}$
  = Dirichlet b.c., not often used

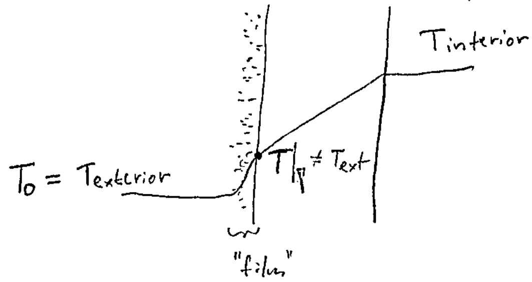
$n(0) = -1$ $n(L) = 1$

b) prescribed heat flux $q_x(0)$ or $q_x(L)$: $q_x(x) \cdot \underbrace{n(x)}_{\{-1,1\}} = \bar{q}_n(x)$

... b) prescribed heat flux

a) constant (Neumann b.c.) – perfectly insulated boundary $Q|_{\Gamma} = \bar{Q}$

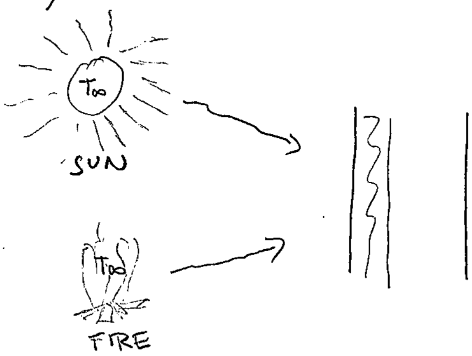
b) temperature dependent (common, used in practice)



$$Q|_{\Gamma} = \alpha_{(x)} \left(T_{\Gamma} - T_0 \right)$$

"film coefficient" $[J\,m^{-2}\,K^{-1}\,s^{-1}]$, in codes for different materials

c) from radiation



Stefan-Boltzmann constant

$$Q|_{\Gamma} = \varepsilon_{(x)} \cdot \sigma \cdot \left(T_{\Gamma}^4 - T_{\infty}^4 \right)$$

surface emmissivity $(0 < \varepsilon < 1)$, in codes

$\Rightarrow$ 4 parts of the boundary: Γ_T – temp. is fixed

$\Gamma_q \begin{cases} \Gamma_{\bar{q}} & \text{– known flux} \\ \Gamma_{qc} & \text{– convective b.c.} \\ \Gamma_{qr} & \text{– radiation} \end{cases}$

Weak form in 1D

- equivalent to a strong form, but in integral form
- in stress (elasticity) analysis it is called "principle of virtual work"

- utilizing per-partes: $\frac{d}{dx}(w \cdot f) = w \frac{df}{dx} + \frac{dw}{dx} f$

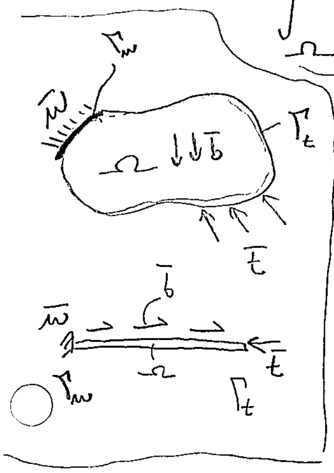
$$\Downarrow$$

$$w \frac{df}{dx} = \frac{d}{dx}(wf) - \frac{dw}{dx} f$$

$\Rightarrow$ after integration over domain $(0,L)$: $\int_0^L w \frac{df}{dx} dx = \underbrace{\int_0^L \frac{d}{dx}(wf)\,dx}_{[wf]_0^L} - \int_0^L f \frac{dw}{dx} dx$

- using per-partes we can express the equivalence of internal and external energies:

$$\underbrace{\int_\Omega \sigma \frac{dw}{dx}\, d\Omega}_{\text{energy from internal stress and strain}} = \underbrace{\int_\Gamma n\sigma w \, d\Gamma - \int_\Omega \frac{d\sigma}{dx} w \, d\Omega}_{\text{energy from external environment}}$$



- there can be two independent states (A and B)
  - → in A there is a deformation $\left(\frac{dw^A}{dx}\right) = \varepsilon^A$ caused by a displacement field w^A
  - → in B there is a stress-state caused by $\bar{t}^B$ and $\bar{b}^B$

$$\int_\Omega \sigma^B \frac{dw^A}{dx}\, d\Omega = \int_\Gamma n\sigma^B w^A \, d\Gamma - \int_\Omega \frac{d\sigma^B}{dx} w^A \, d\Omega$$

$$\underbrace{\int_\Omega \sigma^B \varepsilon^A \, d\Omega}_{\substack{\text{work in volume:} \\ [m^3]\cdot[Nm^{-2}]\cdot[-] = [Nm] = [J] \\ = \text{internal energy}}} = \underbrace{\int_\Gamma \bar{t}^B w^A \, d\Gamma + \int_\Omega \bar{b}^B w^A \, d\Omega}_{= \text{external energy}}$$

BCs: $\bar{t}^B = n\sigma^B$ on Γ → $n\sigma = \bar{t}$ on Γ_t ; $n\sigma = t_R$ on Γ_w = reactions

$$\frac{d\sigma^B}{dx} + \bar{b}^B = 0 \quad \text{in } \Omega$$

- in the principle of virtual displacements the stresses are true and deformations virtual:

$$\int_\Omega \delta\varepsilon\, \sigma \, d\Omega = \int_{\Gamma_t} \delta w\, \bar{t} \, d\Gamma + \int_{\Gamma_w} \underbrace{t_R \delta w}_{0,\ \delta w = 0 \text{ at the boundary}} \, d\Gamma + \int_\Omega \delta w\, \bar{b} \, d\Omega$$

- in FEM δw = "test function", w = "trial solution"
- $\sigma = E\varepsilon$, in 1D the area is constant, Γ - $x=0$ or $x=L$

$$\int_0^L \delta\varepsilon\, EA\, \varepsilon\, dx = \left[\delta w\, \bar{t}\right]_\Gamma + \int_0^L \delta w\, \underset{\uparrow\,[Nm^{-1}]}{\bar{b}}\, dx$$

$$\Downarrow$$

Weak formulation: Find $w(x)$ among the smooth functions that satisfy $w|_\Gamma = \bar{w}$ such that

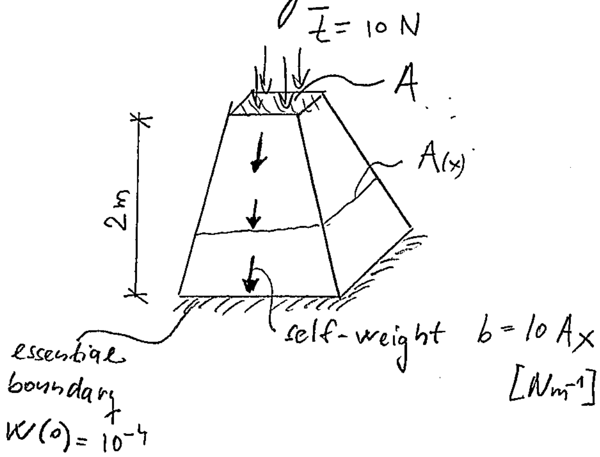
$$\int_0^L \frac{d\delta w}{dx}\, EA\, \frac{dw}{dx}\, dx = \left[\delta w\, A\bar{t}\right]_\Gamma + \int_0^L \delta w\, \bar{b}\, dx$$

only first derivatives
→ <u>weak</u> continuity requirements

$\forall \delta w$ with $\delta w|_\Gamma = 0$
$\parallel$
arbitrary function

EXAMPLE 1

Develop the weak form for the strong form



a) strong form:

i) $\frac{d}{dx}\left(EA\frac{dw}{dx}\right) + 10\,Ax = 0 \qquad 0 < x < 2$

ii) $w|_{x=0} \equiv w(0) = 10^{-4}$

iii) $A\sigma|_{x=L} = \left[AE\frac{dw}{dx}\right]_{x=2} = 10$

b) weak form:

$$\int_0^2 \frac{d\delta w}{dx}\, EA\, \frac{dw}{dx}\, dx = \int_0^2 10\, \delta w\, Ax\, dx + \left[\delta w\, 10\right]_{x=2} = 0 \qquad \forall \delta w \text{ with } \delta w(0) = 0$$

↳ find $w(x)$ such that $w(0) = 10^{-4}$ and equation b) holds for all smooth $\delta w(x)$ with $\delta w(0) = 0$

Weak form of heat conduction in 1D

$$\int_\Omega \frac{d\delta T}{dx} A\lambda \frac{dT}{dx}\, d\Omega = -\left[\delta T A \bar{q}\right]_{\Gamma_q} + \int_\Omega \delta T \bar{Q}\, d\Omega$$

$\forall \delta T \in V_0$

$\uparrow$

admissible set of functions

FEM Approximation in 1D

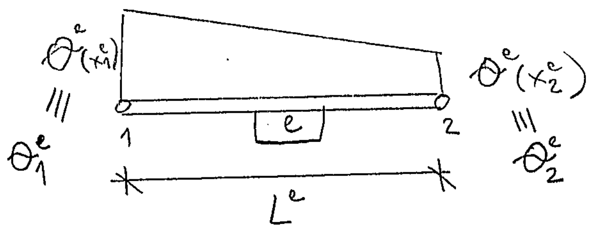
- domain divided into segments creating FE mesh
- usually the physical quantity is approximated by a polynomial
- with mesh refinement we improve the result accuracy

- approximating function (polynomial) is in form:

$$\theta^e = \alpha_0^e + \alpha_1^e x + \alpha_2^e x^2 + \dots$$

- for compatibility requirement there must be continuity between elements:
$\theta^{(1)}(x_2^{(1)}) = \theta^{(2)}(x_1^{(2)})$

- to reach more accurate results with refinement (convergence requirement) not a single member of the approximating polynomial can miss (e.g. $\alpha_0^e + \alpha_2^e x^2$ is not correct)



$$\theta^e_{(x)} = \alpha_0^e + \alpha_1^e x$$

$$\Downarrow$$

$$\theta^e_{(x)} = \underbrace{\begin{bmatrix} 1 & x \end{bmatrix}}_{\underline{p}(x)} \underbrace{\begin{bmatrix} \alpha_0^e \\ \alpha_1^e \end{bmatrix}}_{\underline{\alpha}^e} = \underline{p}(x)\, \underline{\alpha}^e$$

polynomial matrix $\underline{p}(x)$, $\underline{\alpha}^e$ coefficients

- we have a function value for each node:

$$\theta_1^e = \alpha_0^e + \alpha_1^e x_1^e$$
$$\theta_2^e = \alpha_0^e + \alpha_1 x_2^e$$

$$\Rightarrow \begin{bmatrix} \theta_1^e \\ \theta_2^e \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & x_1^e \\ 1 & x_2^e \end{bmatrix}}_{\underline{M}^e} \underbrace{\begin{bmatrix} \alpha_0^e \\ \alpha_1^e \end{bmatrix}}_{\underline{\alpha}^e}$$

matrix of nodal values $\underline{d}^e$ — matrix of positions $\underline{M}^e$

$$\Downarrow$$

$$\underline{d}^e = \underline{M}^e \underline{\alpha}^e \Rightarrow \underline{\alpha}^e = (\underline{M}^e)^{-1} \underline{d}^e$$

- expression of the continuous function using nodal values:

$$\theta^e_{(x)} = \underline{p}(x)\underline{\alpha}^e = \underbrace{\underline{p}(x)(\underline{M}^e)^{-1}}\underline{d}^e$$

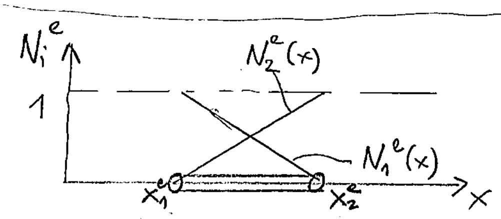
$$\underline{N}^e_{(x)} = \begin{bmatrix} N_1^e(x) & N_2^e(x) \end{bmatrix} = \underline{p}(x)\,(\underline{M}^e)^{-1}$$

element shape functions (= interpolant of nodal values)

A) Linear 1D Element

- for $\theta^e = \alpha_0^e + \alpha_1^e x^2$ the matrix of positions is

$$(\underline{M}^e)^{-1} = \underbrace{\frac{1}{x_2^e - x_1^e}}_{\text{element length } L^e} \begin{bmatrix} x_2^e & -x_1^e \\ -1 & 1 \end{bmatrix}$$



$$\left.\begin{matrix} N_1^e(x_1^e) = 1, & N_1^e(x_2^e) = 0 \\ N_2^e(x_1^e) = 0, & N_2^e(x_2^e) = 1 \end{matrix}\right\} N_i^e(x_j^e) = \delta_{ij}$$

kronecker delta is a fundamental property of shape functions

- shape functions:

$$\underline{N}^e = \begin{bmatrix} N_1^e & N_2^e \end{bmatrix} = \underline{p}(x)(\underline{M}^e)^{-1} = \begin{bmatrix} 1 & x \end{bmatrix} \begin{bmatrix} x_2^e & -x_1^e \\ -1 & 1 \end{bmatrix} \cdot \frac{1}{L^e} =$$

$$= \frac{1}{L^e}\begin{bmatrix} x_2^e - x & x - x_1^e \end{bmatrix}$$

- we also need the derivatives of the shape functions:

$$\frac{d\theta^e}{dx} = \left[\frac{dN_1^e}{dx} \quad \frac{dN_2^e}{dx}\right]\begin{bmatrix}\theta_1^e \\ \theta_2^e\end{bmatrix} = \underline{B}^e \underline{d}^e$$

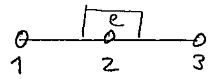
$$\downarrow$$

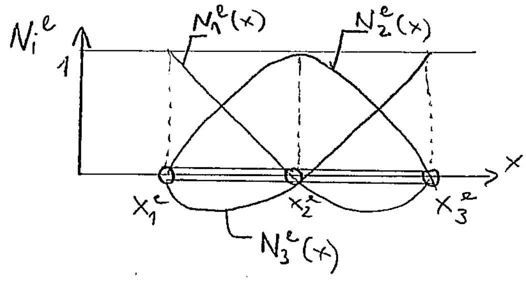
$$\underline{B}^e = \frac{1}{L^e}\left[-1 \quad 1\right]$$

B) <u>Quadratic 1D Element</u>

$$\theta^e(x) = \alpha_0^e + \alpha_1^e x + \alpha_2^e x^2 = \underbrace{\left[1 \quad x \quad x^2\right]}_{\underline{p}(x)} \underbrace{\begin{bmatrix}\alpha_0^e \\ \alpha_1^e \\ \alpha_2^e\end{bmatrix}}_{\underline{\alpha}^e} = \underline{p}(x)\,\underline{\alpha}^e$$

- we need 3 nodes for 1 element (we have 3 nodal values α_j^e)





$$\begin{aligned}\theta_1^e &= \alpha_0^e + \alpha_1^e x_1^e + \alpha_2^e x_1^{e\,2} \\ \theta_2^e &= \alpha_0^e + \alpha_1^e x_2^e + \alpha_2^e x_2^{e\,2} \\ \theta_3^e &= \alpha_0^e + \alpha_1^e x_3^e + \alpha_2^e x_3^{e\,2}\end{aligned} \Rightarrow \underbrace{\begin{bmatrix}\theta_1^e \\ \theta_2^e \\ \theta_3^e\end{bmatrix}}_{\underline{d}^e} = \underbrace{\begin{bmatrix}1 & x_1^e & (x_1^e)^2 \\ 1 & x_2^e & (x_2^e)^2 \\ 1 & x_3^e & (x_3^e)^2\end{bmatrix}}_{\underline{M}^e} \underbrace{\begin{bmatrix}\alpha_0^e \\ \alpha_1^e \\ \alpha_2^e\end{bmatrix}}_{\underline{\alpha}^e}$$

nodal values → $\underline{d}^e$

$$\theta^e = \underbrace{\underline{p}(x)\,(\underline{M}^e)^{-1}}\,\underline{d}^e$$

$$\underline{N}^e = \frac{2}{(L^e)^2}\left[(x-x_2^e)(x-x_3^e) \quad -2(x-x_1^e)(x-x_3^e) \quad (x-x_1^e)(x-x_2^e)\right]$$

$$\underline{B}^e = \frac{d\underline{N}^e}{dx}$$

Direct derivation of shape functions

$$N_1^e(x) = \frac{(x-x_2^e)(x-x_3^e)}{(x_1^e-x_2^e)(x_1^e-x_3^e)}$$

we want $N_1^e(x)$ to vanish at x_2^e and x_3^e

at x_1^e we must have $N_1^e(x) = 1$

$$L^e = (x_3^e - x_1^e)$$

- in Galerkin FEM the trial solutions (w) are approximated in the same way as the weight (test) functions (δw):

$$w^e(x) = \underline{N}^e(x)\, \underline{d}^e$$
$$\delta w^e(x) = \underline{N}^e(x)\, \delta \underline{d}^e$$

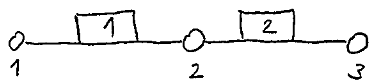
Global approximation

- global approximation of w (trial function) and δw (test function) is obtained by gathering the contributions from individual elements
- gathering matrices, containing zeros and ones, are used for a correct location of local contributions to a global matrix

$$w = \left(\sum_{e=1}^{n_{el}} \underline{N}^e \underline{L}^e\right) \underline{d} = \underline{N}\, \underline{d} \quad \rightarrow \quad \underline{N}^T = \sum_{e=1}^{n_{el}} (\underline{L}^e)^T (\underline{N}^e)^T$$

$$\delta w = \left(\sum_{e=1}^{n_{el}} \underline{N}^e \underline{L}^e\right) \delta \underline{d} = \underline{N}\, \delta \underline{d}$$

- $\underline{L}^e$ = gathering matrices : e.g.

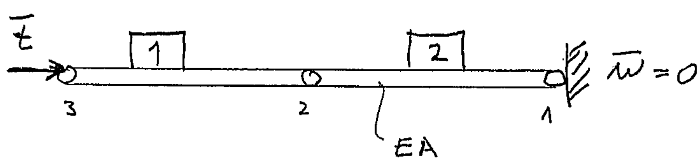


$$\underline{d}^{(1)} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{\underline{L}^{(1)}} \underbrace{\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}}_{\underline{d}} = \underline{L}^{(1)} \underline{d}$$

$$\underline{d}^{(2)} = \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\underline{L}^{(2)}} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}}_{\underline{d}} = \underline{L}^{(2)} \underline{d}$$

$$\underline{N} = \sum_{e=1}^{n_{el}} \underline{N}^e \underline{L}^e = \underline{N}^{(1)} \underline{L}^{(1)} + \underline{N}^{(2)} \underline{L}^{(2)} = \left[\underbrace{N_1^{(1)}}_{N_1} \quad \underbrace{N_2^{(1)} + N_1^{(2)}}_{N_2} \quad \underbrace{N_2^{(2)}}_{N_3} \right]$$

FE formulation for 1D problems



weak solution : $\displaystyle\int_0^L \left(\frac{d\delta u}{dx} \right)^T EA \left(\frac{du}{dx} \right) dx - \int_0^L (\delta u)^T \bar{b} \, dx - \left[(\delta u)^T \bar{t} A \right]_{x=0} = 0$

$\forall \, \delta u(x)$ with $\delta u(l) = 0$

- approximation of the weight (test) function : $\delta u(x) = \underline{N}(x) \, \delta \underline{d}$
- approximation of the trial solution : $u(x) = \underline{N}(x) \, \underline{d}$

↳ it must satisfy $u_1 = \bar{u} = 0$

⇓

FEM approximation = linear combination of the shape functions

- we integrate over element domain (we sum up the element contributions)

$$\sum_{e=1}^{n_{el}} \left(\int_{x_1^e}^{x_2^e} \left(\frac{d\delta u^e}{dx} \right)^T E^e A^e \left(\frac{du^e}{dx} \right) dx - \int_{x_1^e}^{x_2^e} (\delta u^e)^T \bar{b} \, dx - \left[(\delta u^e)^T A^e \bar{t} \right]_{x=0} \right) = 0$$

$(\delta u^e)^T = (\delta \underline{d}^e)^T (\underline{N}^e)^T$

$\left(\frac{d\delta u^e}{dx} \right)^T = (\delta \underline{d}^e)^T (\underline{B}^e)^T$

$u^e = \underline{N}^e \underline{d}^e$

$\frac{du^e}{dx} = \underline{B}^e \underline{d}^e$

$$\sum_{e=1}^{n_{el}} (\delta \underline{d}^e)^T \Bigg[\underbrace{\int_{x_1^e}^{x_2^e} (\underline{B}^e)^T E^e A^e \underline{B}^e \, dx \; \underline{d}^e}_{\underline{K}^e} - \underbrace{\int_{x_1^e}^{x_2^e} (\underline{N}^e)^T \bar{b} \, dx}_{\underline{f}_\Omega^e} - \underbrace{\left[(\underline{N}^e)^T A^e \bar{t} \right]_{x=0}}_{f_\Gamma^e} \Bigg] = 0$$

doesn't depend on x, arbitrary
only fulfils B.Cs

- element stiffness matrix : $\underline{k}^e = \int_{\Omega^e} (\underline{B}^e)^T E^e A^e \underline{B}^e \, dx$

element portion of natural boundary

- element external force matrix : $\underline{f}^e = \underbrace{\int_{\Omega^e} (\underline{N}^e)^T \bar{b} \, dx + \left[(\underline{N}^e)^T A^e \bar{t} \right]_{\Gamma_t^e}}_{\text{element body and boundary force matrices}}$

- element matrices play the same role as in case of the Direct Stiffness Method

- using the gathering matrices : $\delta \underline{d}^e = \underline{L}^e \underline{d}^e$ and $\underline{d}^e = \underline{L}^e \underline{d}$ we get (by gathering of local contributions)

$$\delta \underline{d}^T \left(\sum_{e=1}^{n_{el}} (\underline{L}^e)^T \underline{k}^e \underline{L}^e \underline{d} - \sum_{e=1}^{n_{el}} (\underline{L}^e)^T \underline{f}^e \right) = 0$$

system (=global) stiffness matrix $\underline{K} = \sum_{e=1}^{n_{el}} (\underline{L}^e)^T \underline{k}^e \underline{L}^e$

$$\underline{f} = \sum_{e=1}^{n_{el}} (\underline{L}^e)^T \underline{f}^e$$

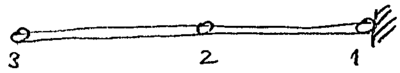
- in practice direct assembly is used

$\Downarrow$

in compact form we get : $\delta \underline{d}^T \left(\underline{K} \underline{d} - \underline{f} \right) = 0 \qquad \forall \delta \underline{d}$ except $\delta \underline{d}(\Gamma_u) = 0$

arbitrary nodal values restricted only by $\delta d_1 = 0$

- if we introduce the residual $\underline{r} = \underline{k}\,\underline{d} - \underline{f}$, then $\delta\underline{d}^T \underline{r} = 0$

↳ for 

we have $\delta\underline{d}^T \underline{r} = \delta w_2 r_2 + \delta w_3 r_3 = 0$

and since δw_2 and δw_3 are arbitrary, we get $r_2 = r_3 = 0$ but r_1 is unknown (δw_1 must be zero)

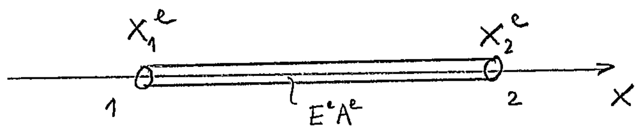
unbalanced force :

$$\underline{r} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} & & \\ & \underline{k} & \\ & & \end{bmatrix} \begin{bmatrix} \bar{w}_1 \\ w_2 \\ w_3 \end{bmatrix} - \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} f_1 + r_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} & & \\ & \underline{k} & \\ & & \end{bmatrix} \begin{bmatrix} \bar{w}_1 \\ w_2 \\ w_3 \end{bmatrix}$$

ELEMENT MATRICES FOR 2-NODE ELEMENT



- $\underline{N}^e = \frac{1}{L^e}\left[x_2^e - x \qquad x - x_1^e\right] \quad \rightarrow \quad \underline{B}^e = \frac{d}{dx}\underline{N}^e = \frac{1}{L^e}\left[-1 \quad 1\right]$

- $\underline{K}^e = \int_{x_1^e}^{x_2^e} (\underline{B}^e)^T E^e A^e \underline{B}^e \, dx = \int_{x_1^e}^{x_2^e} \frac{1}{L^e}\begin{bmatrix} -1 \\ 1 \end{bmatrix} E^e A^e \frac{1}{L^e}\left[-1 \quad 1\right] dx = \frac{E^eA^e}{(L^e)^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \int_{x_1^e}^{x_2^e} dx$

$$= \frac{E^eA^e}{L^e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

... for element with constant cross-section it is equivalent to the Direct Stiffness Method which is based on physical arguments

• body force matrix : $\underline{f}^e_\Omega = \int_{x_1^e}^{x_2^e} (\underline{N}^e)^T \, \bar{b}(x) \, dx$

- as the body force distribution is linearly varying through an element, it can be expressed in terms of linear shape functions as $\bar{b}(x) = \underline{N}^e(x) \, \underline{\bar{b}}$, where $\bar{b}$ contains nodal values of the body force $\underline{\bar{b}} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

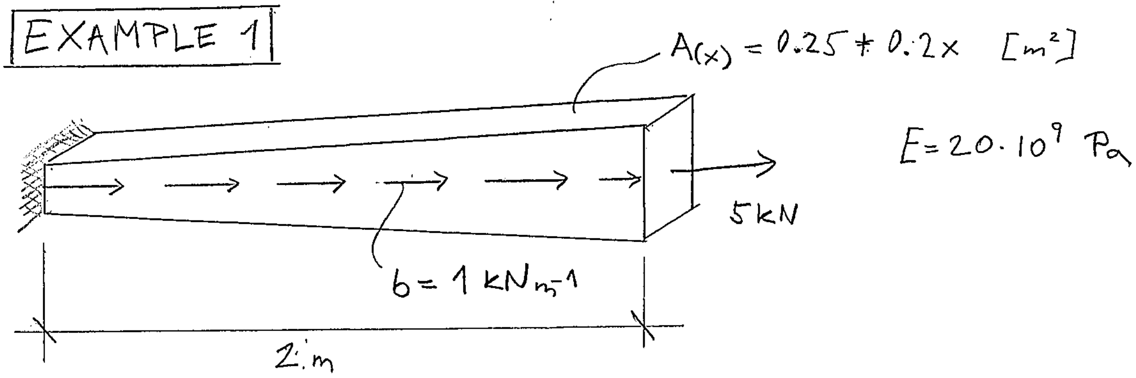
$\Downarrow$

$$\underline{f}^e_\Omega = \int_{x_1^e}^{x_2^e} (\underline{N}^e)^T \underline{N}^e \, dx \; \underline{\bar{b}} = \frac{1}{(L^e)^2} \int_{x_1^e}^{x_2^e} \begin{bmatrix} (x_2^e - x)^2 & (x_2^e - x)(x - x_1^e) \\ (x_2^e - x)(x - x_1^e) & (x - x_1^e)^2 \end{bmatrix} dx \; \underline{\bar{b}} =$$

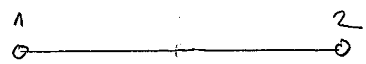
$$= \frac{L^e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \bar{b}_1 \\ \bar{b}_2 \end{bmatrix}$$

• assembly of global matrices as in the Direct Stiffness Method

$\boxed{\text{EXAMPLE 1}}$



→ 2 nodes, 1 element with linear basis functions



$$\underline{N}^e_{(x)} = \frac{1}{L^e} \left[x_2^e - x \quad x - x_1^e \right] \;\rightarrow\; \underline{B}^e = \frac{d\underline{N}^e_{(x)}}{dx} = \frac{1}{L^e} \left[-1 \quad 1 \right]$$

a) develop element stiffness matrix

$$\underline{K}^e = \int_{x_1^e}^{x_2^e} (\underline{B}^e)^T E^e A^e \underline{B}^e \, dx = \int_0^2 \frac{1}{2}\begin{bmatrix}-1\\1\end{bmatrix} 20\cdot10^9 \cdot (0.25+0.2x)\,\frac{1}{2}\,[-1 \quad 1]\,dx =$$

$$= 5\cdot10^9 \begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} \left[0.25x + 0.1x^2\right]_0^2 = 5\cdot10^9 \begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}(0.5+0.4) =$$

$$= 4{,}5\cdot10^9 \cdot \begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} \quad [Nm^{-1}]$$

b) external force matrix

$$\underline{f}_\Gamma^e = \begin{bmatrix}R_1\\5\cdot10^3\end{bmatrix} \quad [N]$$

c) body force matrix

$$\underline{f}_\Omega^e = \frac{L^e}{6}\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}b_1\\b_2\end{bmatrix}, \quad \text{since } b_1 = b_2 = b \text{ then}$$

$$\underline{f}_\Omega^e = \frac{L^e b}{2}\begin{bmatrix}1\\1\end{bmatrix} = \frac{2\cdot10^3}{2}\begin{bmatrix}1\\1\end{bmatrix} = 10^3\begin{bmatrix}1\\1\end{bmatrix} \quad [N]$$

d) displacement matrix

$$\underline{d}^e = \begin{bmatrix}0\\d_2\end{bmatrix} \quad [m]$$

→ unknown DOFs (= free) $F = [2]$

→ prescribed DOFs (= essential BCs) $E = [1]$

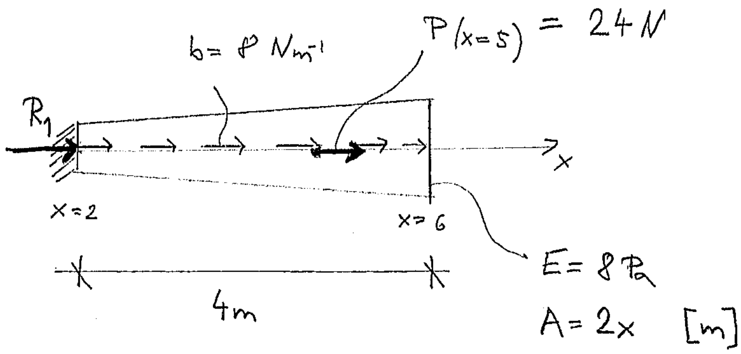
$$\underline{d}_{(F)} = \underline{K}_{(F,F)}^{-1}\left(\underline{f}_{(F)} - \underline{K}_{(F,E)}\underbrace{\underline{d}_{(E)}}_{0}\right) \Rightarrow d_2 = \frac{1}{4{,}5\cdot10^9}\cdot 6\cdot10^3 = \underline{1.\overline{33}\cdot10^{-6}\,m}$$

$$\underline{f}_{(E)} = \underline{K}_{(E,F)}\,\underline{d}_{(F)} + \underline{K}_{(E,E)}\underbrace{\underline{d}_{(E)}}_{0} \Rightarrow R_1 + 10^3 = -4{,}5\cdot10^9 \cdot 1{,}\overline{33}\cdot10^{-6}$$

$$R_1 = \underline{-7\cdot10^3\,N}$$

EXAMPLE 2

- tapered elastic bar :



⇓

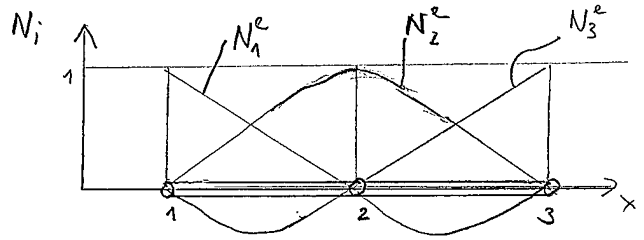
modelled by a single three-node element:

a) <u>element shape functions for the 3-node quadratic element</u>

$$N_1^{(1)} = \frac{(x-x_2^{(1)})(x-x_3^{(1)})}{(x_1^{(1)}-x_2^{(1)})(x_1^{(1)}-x_3^{(1)})} = \frac{(x-4)(x-6)}{(-2)(-4)} = \frac{1}{8}(x-4)(x-6)$$

$$N_2^{(1)} = \frac{(x-x_1^{(1)})(x-x_3^{(1)})}{(x_2^{(1)}-x_1^{(1)})(x_2^{(1)}-x_3^{(1)})} = \frac{(x-2)(x-6)}{2(-2)} = -\frac{1}{4}(x-2)(x-6)$$

$$N_3^{(1)} = \frac{(x-x_1^{(1)})(x-x_2^{(1)})}{(x_3^{(1)}-x_1^{(1)})(x_3^{(1)}-x_2^{(1)})} = \frac{(x-2)(x-4)}{4\cdot 2} = \frac{1}{8}(x-2)(x-4)$$



b)
$$\left.\begin{aligned} B_1^{(1)} &= \frac{dN_1^{(1)}}{dx} = \frac{1}{4}(x-5) \\ B_2^{(1)} &= \frac{dN_2^{(1)}}{dx} = \frac{1}{2}(4-x) \\ B_3^{(1)} &= \frac{dN_3^{(1)}}{dx} = \frac{1}{4}(x-3) \end{aligned}\right\} \quad \underline{B}^{(1)} = \frac{1}{4}\left[(x-5) \quad (8-2x) \quad (x-3)\right]$$

c) <u>element (= global) stiffness matrix</u>

$$\underline{k}^{(1)} = \underline{k} = \int_{x_1^e}^{x_3^e} (\underline{B}^{(1)})^T E^{(1)} A^{(1)} \underline{B}^{(1)}\, dx = \int_2^6 \frac{1}{4}\begin{bmatrix} x-5 \\ 8-2x \\ x-3 \end{bmatrix} 8\cdot 2x \cdot \frac{1}{4}\Big[(x-5) \ldots$$

$$\ldots (8-2x) \quad (x-3)\Big]\, dx = \begin{bmatrix} 26.67 & -32 & 5.33 \\ & 85.33 & -53.33 \\ & & 48 \end{bmatrix}$$

d) <u>Body force matrix</u>

- contribution from the distributed loading $\bar{b}$ and the point load P

→ distributed into nodes using shape functions

$$\underline{f}_\Omega = \underline{f}_\Omega^{(1)} = \int_{x_1^e}^{x_3^e} (\underline{N}^e)^T \bar{b}\, dx + (\underline{N}^e)^T P \big|_{x=5}$$

element shape function evaluated at $x=5$ (if $\underline{N}$ is linear and force P is in the middle then half of P goes to each node)

$$\underline{f}_\Omega = \underbrace{\int_0^6 \begin{bmatrix} 0.125\,(x-4)(x-6) \\ -0.25\,(x-2)(x-6) \\ 0.125\,(x-2)(x-4) \end{bmatrix} \cdot 8\, dx}_{sum\,=\,32} + \underbrace{\begin{bmatrix} 0.125\,(x-4)(x-6) \\ -0.25\,(x-2)(x-6) \\ 0.125\,(x-2)(x-4) \end{bmatrix}_{x=5} \cdot 24}_{sum\,=\,24} =$$

$$= \begin{bmatrix} 2.33 \\ 39.33 \\ 14.33 \end{bmatrix}$$

$$\underline{f}_\Gamma = \begin{bmatrix} R_1 \\ 0 \\ 0 \end{bmatrix} \longrightarrow \underline{f} = \begin{bmatrix} 2{,}33 + R_1 \\ 39{,}33 \\ 14{,}33 \end{bmatrix}$$

e) <u>solve the system $\underline{K}\,\underline{d} = \underline{f}$</u>

(part of the loading going directly to node 1)

$$\begin{bmatrix} 26{,}67 & -32 & 5.33 \\ -32 & 85.33 & -53.33 \\ 5{,}33 & -53.33 & 48 \end{bmatrix} \cdot \begin{bmatrix} w_1 = 0 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} R_1 + 2{,}33 \\ 39{,}33 \\ 14{,}33 \end{bmatrix}$$

- essential B.C. $= [1] = E$

free DOFs $= [2,3] = F$

$$\underline{d}_{(F)} = \underline{K}^{-1}_{(F,F)} \left(\underline{f}_{(F)} - \underline{K}_{(F,E)} \underbrace{\underline{d}_{(E)}}_{0} \right) = \begin{bmatrix} 85,33 & -53,33 \\ -53,33 & 48 \end{bmatrix}^{-1} \begin{bmatrix} 39,33 \\ 14,33 \end{bmatrix} =$$

$$= \begin{bmatrix} 2,12 \\ 2,65 \end{bmatrix} \; [m]$$

$$\underline{f}_{(E)} = \left[R_1 + 2,33 \right] = \underline{K}_{(E)} \underbrace{\underline{d}_{(E)}}_{0} + \underline{K}_{(E,F)} \underline{d}_{(F)} = \begin{bmatrix} -32 & 5,33 \end{bmatrix} \begin{bmatrix} 2,12 \\ 2,65 \end{bmatrix} =$$

$$= -53,66 \quad \Rightarrow \quad R_1 = 56 \text{ N} \qquad (= 8 \cdot 4 + 24) \checkmark$$

f) <u>postprocessing</u>

- if we know the nodal displacements we can calculate the displacement field:

$$u = N_1^{(1)} u_1 + N_2^{(1)} u_2 + N_3^{(1)} u_3 \,, \quad \underline{d} = \underline{d}^{(1)} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2,12 \\ 2,65 \end{bmatrix} \; [m]$$

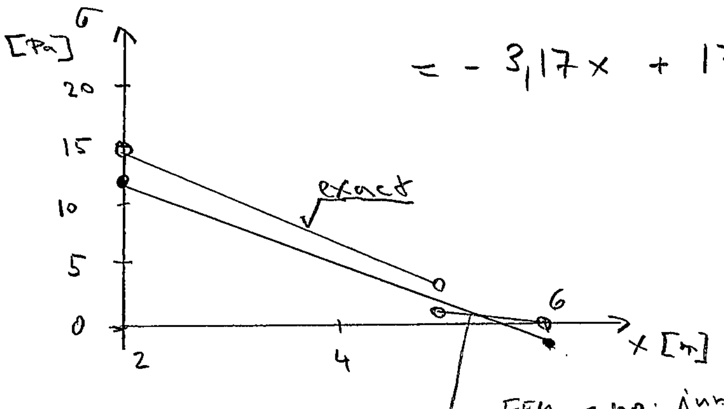
$$u(x) = \frac{1}{8}(x-4)(x-6)\cdot(0) + \frac{(-1)}{4}(x-2)(x-6)(2,12) +$$
$$+ \frac{1}{8}(x-2)(x-4)(2,65) = \underline{-0,198x^2 + 2,249x - 3,705}$$

- the stress field is given by:

$$\sigma(x) = E\frac{du}{dx} = E\frac{d}{dx}\left(\underline{N}^{(1)}\underline{d}^{(1)}\right) = E\,\underline{B}^{(1)}\underline{d}^{(1)} =$$

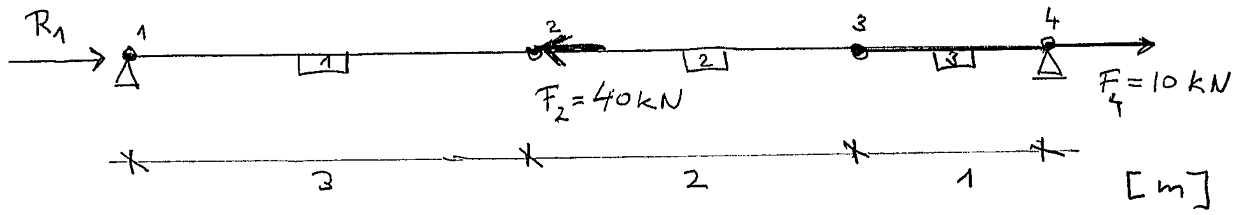
$$= 8 \cdot \frac{1}{4} \cdot \begin{bmatrix} (x-5) & (8-2x) & (x-3) \end{bmatrix} \begin{bmatrix} 0 \\ 2,12 \\ 2,65 \end{bmatrix} =$$

$$= -3,17x + 17,99 \quad [Pa]$$



FEM - no jump $\Rightarrow$ node should be always placed where a point force is applied

EXAMPLE 3



$E^{(1)} = E^{(2)} = 200 \cdot 10^9$ Pa $\qquad A^{(1)} = A^{(2)} = A^{(3)} = 10^{-4}$ m²

$E^{(3)} = 400 \cdot 10^9$ Pa

1) selection of base functions : linear approximation

$$\left.\begin{aligned} N_1^e(x) &= \frac{1}{L^e}(x_2^e - x) \\ N_2^e(x) &= \frac{1}{L^e}(x - x_1^e) \end{aligned}\right\} \underline{N}^e = [N_1^e \quad N_2^e]$$

2) element stiffness matrix

$$\underline{B}^e = \frac{1}{L^e}[-1 \quad 1]$$

$$\underline{K}^e = \int_{x_1^e}^{x_2^e} (\underline{B}^e)^T E^e A^e \underline{B}^e \, dx = \frac{E^e A^e}{L^e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

3) boundary traction contribution (→ load to nodes)

$$\underline{f}_\Gamma^e = (\underline{N}^e)^T_{x=x_1^e} F_1^e + (\underline{N}^e)^T_{x=x_2^e} F_2^e = \begin{bmatrix} F_1^e \\ F_2^e \end{bmatrix}$$

$$\underline{f} = \begin{bmatrix} R_1 \\ -40 \cdot 10^3 \\ 0 \\ 10 \cdot 10^3 \end{bmatrix} [N]$$

4) global stiffness matrix assembly

$$\underline{K}^{(1)} = \frac{200 \cdot 10^5}{3}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad \underline{K}^{(2)} = \underline{K}^{(1)}, \quad \underline{K}^{(3)} = 400 \cdot 10^5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\Rightarrow \quad \underline{k} = \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & 0 & 0 \\ & \frac{5}{3} & -1 & 0 \\ & & 5 & -4 \\ & & & 4 \end{bmatrix} \cdot 10^{7} \qquad [Nm^{-1}]$$

5) <u>solve the system $\underline{k}\,\underline{d} = \underline{f}$</u>

- essential BC prescribed at $\underline{E} = [1]$
- free DOFs: $\underline{F} = [2, 3, 4]$

a) <u>unknown displacements</u>

$$\underline{d}_{(F)} = k^{-1}_{(FF)} \left(\underline{f}_{(F)} + \underline{k}_{(F,E)} \underbrace{\cancel{\underline{d}_{(E)}}}_{0} \right)$$

$$\begin{bmatrix} w_2 \\ w_3 \\ w_4 \end{bmatrix} = 10^{-7} \begin{bmatrix} \frac{5}{3} & -1 & 0 \\ -1 & 5 & -4 \\ 0 & -4 & 4 \end{bmatrix}^{-1} \begin{bmatrix} -40 \cdot 10^{3} \\ 0 \\ 10 \cdot 10^{3} \end{bmatrix} = \begin{bmatrix} -4{,}5 \\ -3{,}5 \\ -3{,}25 \end{bmatrix} \cdot 10^{-3} \qquad [m]$$

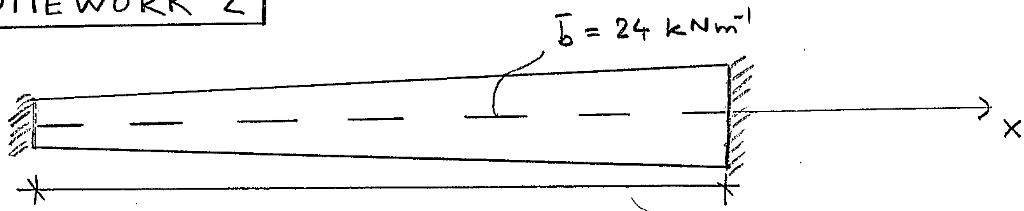
b) <u>reactions</u>

$$\underline{f}_{(E)} = \underline{k}_{(E,F)}\, \underline{d}_{(F)} + \cancel{k_{(E,E)}\, \underline{d}_{(E)}} \to 0$$

$$[R_1] = \left[-\frac{2}{3} \quad 0 \quad 0 \right] \cdot 10^{7} \begin{bmatrix} -4{,}5 \\ -3{,}5 \\ -3{,}25 \end{bmatrix} \cdot 10^{-3} = \underline{30 \text{ kN}}$$

$(30 - 40 + 10 = 0 \ \checkmark)$

$\boxed{\text{HOMEWORK 2}}$



$E = 20 \cdot 10^{9}$ Pa

$A(x) = 0{,}04 + 0{,}02x + 0{,}0025x^2$

1) 2 elements with linear basis functions
2) 3 elements with linear basis functions
3) single element with quadratic basis function
   + plot displacements on x-position