Heat Conduction in 1D

heat conduction:

conductivity (conductance)

$$\begin{tikzcd}
{[K]}\ \boxed{T(x)} \arrow[d, "\frac{d}{dx}"] & & \boxed{Q(x)} \ \text{internal heat source (e.g. hydration)} \ [J m^{-3} s^{-1}] \\
\boxed{\frac{dT(x)}{dx}} \ [K m^{-1}] \ \text{temperature gradient} \arrow[rr, "-\lambda(x)", "\text{Fourier's law}"'] & & \boxed{q_x(x)} \ \text{heat flux} \ [J m^{-2} s^{-1}] \arrow[u, "\frac{d}{dx}"']
\end{tikzcd}$$

elasticity:

Young's modulus

$$\begin{tikzcd}
\boxed{w(x)} \ [m] \arrow[d, "\frac{d}{dx}"] & & \boxed{\bar{b}(x)} \ [N m^{-1}] \\
\boxed{\varepsilon(x)} \ [-] \arrow[rr, "E(x)", "\text{Hooke's law}"'] & & \boxed{\sigma(x)} \ [N m^{-2}] \arrow[u, "\frac{d}{dx}\ A(x)"']
\end{tikzcd}$$

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governing equations: $\frac{d}{dx}\left(-\lambda \frac{dT}{dx}\right) = Q$ $\qquad$ $\frac{d}{dx}\left(EA \frac{dw}{dx}\right) + \bar{b} = 0$

B.C.'s:

a) $T|_{\Gamma_T} = \bar{T}$

b) i) $Q|_{\Gamma_q} = \bar{Q}$

ii) $Q|_{\Gamma_{qc}} = \alpha(x)\left(T_\Gamma - T_0\right)$

iii) $Q|_{\Gamma_{qr}} = \varepsilon(x) \cdot \sigma \cdot \left(T_\Gamma^4 - T_\infty^4\right)$

a) $w|_{\Gamma_w} = \bar{w}$

b) $A\sigma|_{\Gamma_t} = \bar{t}$

Weak form:

$$\int_\Omega \frac{d\delta T}{dx} A\lambda \frac{dT}{dx} d\Omega = -\left[\delta T A \bar{q}\right]_{\Gamma_q} + \int_\Omega \delta T \bar{Q} \, d\Omega$$

$$\int_\Omega \frac{d\delta w}{dx} EA \frac{dw}{dx} d\Omega = \left[\delta w A \bar{t}\right]_\Gamma + \int_\Omega \delta w \bar{b} \, d\Omega$$

Element matrices for heat conduction

conductivity matrix: $\underline{k}^e = \int_{\Omega^e} (\underline{B}^e)^T A^e \lambda^e \underline{B}^e \, dx$

boundary temperature: $\left[(\underline{N}^e)^T A^e \bar{T}\right]_{\Gamma_t^e} = \underline{f}^e_{\Gamma_T}$

internal heat matrix: $\underline{f}^e_\Omega = \int_\Omega (\underline{N}^e)^T \underbrace{\underline{N}^e d\Omega \, \bar{Q}}_{\bar{Q}} = \frac{L^e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} \bar{Q}_1 \\ \bar{Q}_2 \end{bmatrix}$

boundary flux matrix (prescribed flux) : $\underline{f}^e_{\Gamma_{\bar{q}}} = -\begin{bmatrix} \bar{q}_1 \\ \bar{q}_2 \end{bmatrix}$

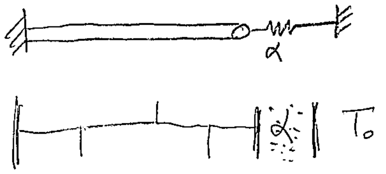
boundary flux matrix (prescribed air temperature) :

$$\underline{f}^e_{\Gamma_{q_c}} = \int_{\Gamma^e} (\underline{N}^e)^T \alpha(x)\, \underline{N}^e \, \bar{\underline{T}}^e_0 \, d\Gamma^e = \underline{K}^e_\Gamma \, \bar{\underline{T}}_0 = \begin{bmatrix} \alpha^e_1 & 0 \\ 0 & \alpha^e_2 \end{bmatrix} \begin{bmatrix} \bar{T}^e_{0,1} \\ \bar{T}^e_{0,1} \end{bmatrix} = \begin{bmatrix} \alpha^e_1 \bar{T}^e_{0,1} \\ \alpha^e_2 \bar{T}^e_{0,2} \end{bmatrix}$$

film coefficient (in standards)

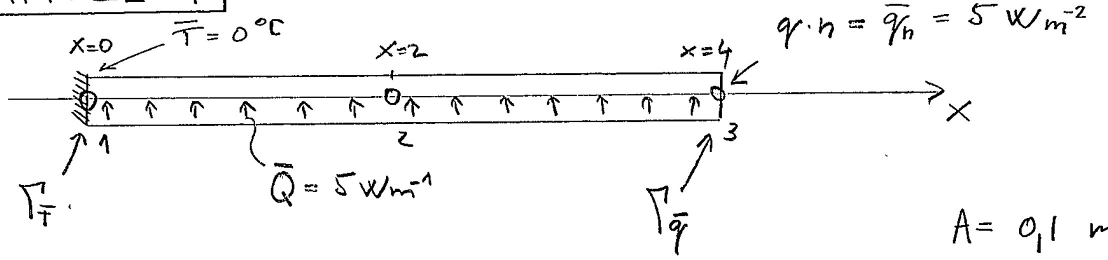
surrounding air temperature

conductivity matrix : $\underline{k}^e_\Omega + \underline{k}^e_\Gamma$



- in mechanics $\underline{k}^e_\Gamma$ would be an additional stiffness of a node
- contribution to conductivity matrix

$\boxed{\text{EXAMPLE 1}}$



$A = 0{,}1\ m^2$

$\lambda = 2\ W K^{-1} m^{-1}$

→ 2 linear 2-node elements

- conductance matrix $\underline{k}^e = \int_{\Omega^e} (\underline{B}^e)^T A^e \lambda^e \underline{B}^e \, dx = \dfrac{A^e \lambda^e}{L^e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$$\underline{k}^{(1)} = \frac{0{,}2}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{array}{cc} \begin{bmatrix} 0{,}1 & -0{,}1 \\ -0{,}1 & 0{,}1 \end{bmatrix} & \begin{array}{c} ① \\ ② \end{array} \\ \begin{array}{cc} ① & ② \end{array} & \end{array}$$

$$\underline{k}^{(2)} = \begin{array}{cc} \begin{bmatrix} 0{,}1 & -0{,}1 \\ -0{,}1 & 0{,}1 \end{bmatrix} & \begin{array}{c} ② \\ ③ \end{array} \\ \begin{array}{cc} ② & ③ \end{array} & \end{array}$$

$$\underline{k} = \begin{array}{cc} \begin{bmatrix} 0.1 & -0.1 & 0 \\ -0.1 & 0.2 & -0.1 \\ 0 & -0.1 & 0.1 \end{bmatrix} & \begin{array}{c} ① \\ ② \\ ③ \end{array} \\ \begin{array}{ccc} ① & ② & ③ \end{array} & \end{array}$$

$[W K^{-1}]$

- boundary flux matrix: $\underline{f}^e_{\Gamma_q} = -\left[(\underline{N}^e)^T A^e \bar{q} \right]_{\Gamma^e_{\bar{q}}}$ $\quad \bar{q}$ [Wm^{-2}]

at the element boundary with prescribed heat flux is $(\underline{N}^e)^T = 1$

$$\underline{f}_\Gamma = \begin{bmatrix} 0 \\ 0 \\ -0{,}5 \end{bmatrix} \begin{array}{c} ① \\ ② \\ ③ \end{array} \quad [Wm^{-2}]$$

- element source flux matrix: $\underline{f}^e_\Omega = \int_{x_1^e}^{x_2^e} (\underline{N}^e)^T \underline{N}^e \, dx \, \bar{Q}$

$$\underline{f}^e_\Omega = \frac{L^e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \bar{Q}_1 \\ \bar{Q}_2 \end{bmatrix}, \text{ since } \bar{Q}_1 = Q_2 \quad \underline{f}^e_\Omega = \frac{L^e \bar{Q}}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\underline{f}^{(1)}_\Omega = \begin{bmatrix} 5 \\ 5 \end{bmatrix} \begin{array}{c} ① \\ ② \end{array}, \quad \underline{f}^{(2)}_\Omega = \begin{bmatrix} 5 \\ 5 \end{bmatrix} \begin{array}{c} ② \\ ③ \end{array} \Rightarrow \underline{f}_\Omega = \begin{bmatrix} 5 \\ 10 \\ 5 \end{bmatrix}$$

[Wm^{-2}]

Partition and solution

$$\underset{\text{conductivity}}{\underline{K}} \; \underset{\text{temperature}}{\underline{T}} = \underset{\text{flux}}{\underline{f}}$$

- essential B.C.s: $\underline{E} = [1]$ (prescribed temperature at Γ_T)

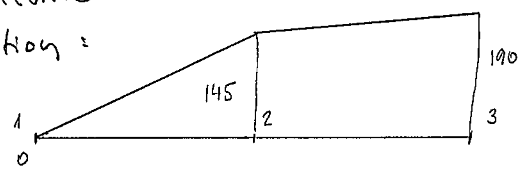
- no convective BCs $\Rightarrow$ no $\underline{K}_\Gamma$ (boundary conductivity matrix) and no convective boundary heat flux $\underline{f}_{\Gamma_{qc}}$

- free nodes: $\underline{F} = [2, 3]$

$$\begin{bmatrix} 0.1 & -0.1 & 0 \\ -0.1 & 0.2 & -0.1 \\ 0 & -0.1 & 0.1 \end{bmatrix} \begin{bmatrix} 0 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 5 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -0.5 \end{bmatrix} + \begin{bmatrix} r_1 \\ 0 \\ 0 \end{bmatrix}$$

$$\underline{T}_{(F)} = \underline{K}^{-1}_{(F)} \left(\underline{f}_{(F)} - \underline{K}_{(F,E)} \underbrace{\underline{\bar{T}}_{(E)}}_{0} \right) = \begin{bmatrix} 0.2 & -0.1 \\ -0.1 & 0.1 \end{bmatrix}^{-1} \begin{bmatrix} 10 \\ 4{,}5 \end{bmatrix} = \begin{bmatrix} 145 \\ 190 \end{bmatrix} \quad [°C]$$

temperature distribution:



[°C]

temperature gradient in the elements:

$$\frac{dT^{(1)}}{dx} = \underline{B}^{(1)} \underline{T}^{(1)} = \underbrace{\frac{1}{L^e}\begin{bmatrix}-1 & 1\end{bmatrix}}_{\underline{B}^e} \cdot \begin{bmatrix}0 \\ 145\end{bmatrix} = \frac{1}{2}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{bmatrix}0 \\ 145\end{bmatrix} = 72{,}5 \quad \mathrm{Km}^{-1}$$

$$\frac{d\underline{T}^{(2)}}{dx} = \underline{B}^{(2)} \underline{T}^{(2)} = \frac{1}{2}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{bmatrix}145 \\ 190\end{bmatrix} = 22{,}5 \quad \mathrm{Km}^{-1}$$

exact solution (posible to find in 1D)

$$\frac{d}{dx}\left(A\lambda \frac{dT}{dx}\right) + \bar{Q} = 0 \qquad 0 < x < L$$

$$\frac{d}{dx}\left(0.2 \frac{dT}{dx}\right) + 5 = 0 \quad \Rightarrow \quad \frac{d^2T}{dx^2} = -25$$

$BC^{(1)}$: $T(0) = 0$

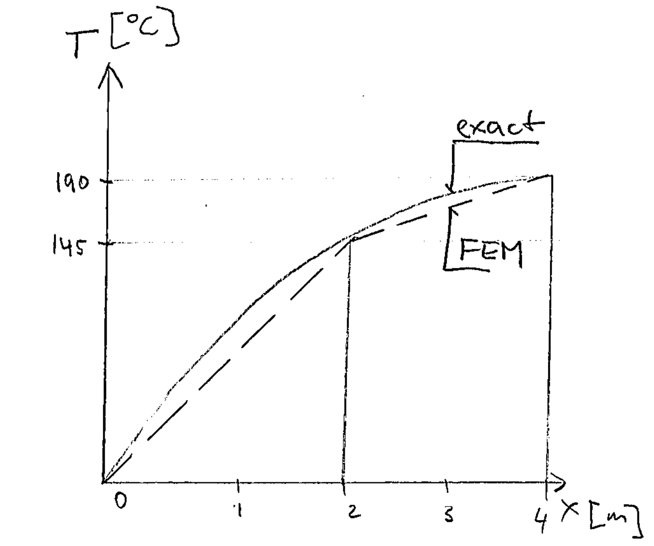
$BC^{(2)}$: $\bar{q}(4) = -\lambda \frac{dT}{dx} n \Big|_{x=4} = 5 \Rightarrow \frac{dT}{dx}(4) = \frac{5}{-2} = -2{,}5$

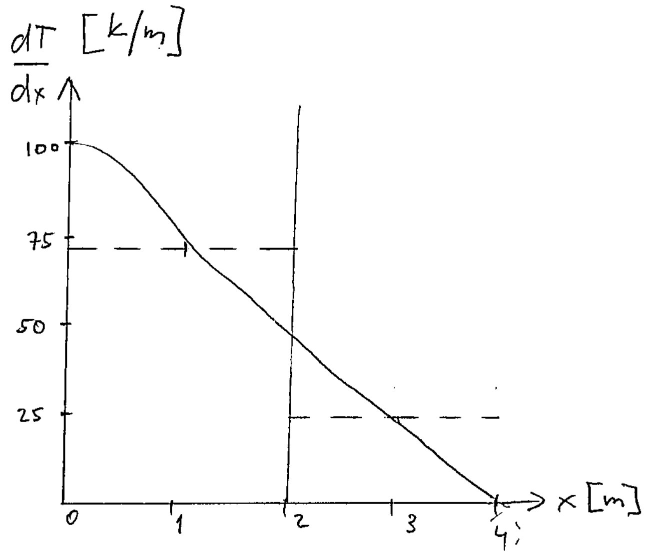
$$\left.\begin{aligned} \frac{d^2T}{dx} = -25 \Rightarrow \frac{dT}{dx} &= -25x + c_1 \\ \frac{dT}{dx}(4) &= -2{,}5 \end{aligned}\right\} \begin{aligned} -25 \cdot 4 + c_1 &= -2{,}5 \\ c_1 &= 97{,}5 \end{aligned}$$

$$\left.\begin{aligned} \frac{dT}{dx} = -25x + 97{,}5 \Rightarrow T &= -12{,}5x^2 + 97{,}5x + c_2 \\ T(0) &= 0 \end{aligned}\right\} c_2 = 0$$

$$T^{exact} = -12{,}5\, x^2 + 97.5x$$

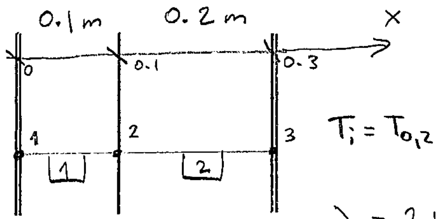
flux $\dfrac{dT^{exact}}{dx} = -25x + 97.5$





↳ nodal values in FEM are exact in 1D (only)

EXAMPLE 2



$T_e = T_{0,1} = 10°C$

$T_i = T_{0,2} = 20°C$

$\lambda_1 = 0.5\ W m^{-1} K^{-1}$

$\lambda_2 = 2\ W m^{-1} K^{-1}$

$\alpha = 20\ W m^{-2} K^{-1}$

$A = 1 m^2$ (→ 1D element, wall must be high)

no essential BCs → no need for partitioning $\underline{E} = [0]$

- conductivity matrix

$$\underline{k}^{(1)}_{\Omega} = \int_0^{0.1} \left(\underline{B}^{(1)}\right)^T \lambda^{(1)} \cancel{A^{(1)}}^{\,1 m^2} \underline{B}^{(1)}\, dx = \frac{\lambda^{(1)}}{L^{(1)}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{0.5}{0.1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix}$$

$$\underline{k}^{(2)}_{\Omega} = \frac{2}{0.2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 10 & -10 \\ -10 & 10 \end{bmatrix}$$

$[W K^{-1} m^{-2}]$ ← per unit meter

- boundary conductivity matrix

$$\underline{k}^{(1)}_{\Gamma} = \Big(\underbrace{\underline{N}^{(1)}(x_1^{(1)})}_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}\Big)^T \underbrace{\alpha^{(1)}(x_1^{(1)})}_{20}\ \underbrace{\underline{N}^{(1)}(x_1^{(1)})}_{[1\ \ 0]} + \Big(\underbrace{\underline{N}^{(1)}(x_2^{(1)})}_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}\Big)^T \underbrace{\alpha^{(1)}(x_2^{(1)})}_{0}\ \underbrace{\underline{N}^{(1)}(x_2^{(1)})}_{[0\ \ 1]} = \begin{bmatrix} 20 & 0 \\ 0 & 0 \end{bmatrix}$$

$[W K^{-1} m^{-2}]$

$$\underline{k}_{\Gamma}^{(2)} = \begin{bmatrix} 0 & 0 \\ 0 & 20 \end{bmatrix}$$

- global conductivity matrix assembly

$$\underline{k} = \underline{k}_{\Omega} + \underline{k}_{\Gamma} = \begin{bmatrix} 5 & -5 & 0 \\ -5 & 15 & -10 \\ 0 & -10 & 10 \end{bmatrix} + \begin{bmatrix} 20 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 20 \end{bmatrix} = \begin{bmatrix} 25 & -5 & 0 \\ -5 & 15 & -10 \\ 0 & -10 & 30 \end{bmatrix}$$

$[W K^{-1} m^{-2}]$

- convective flux matrix

$$\underline{f}_{\Gamma_c}^{e} = \underline{k}_{\Gamma}^{e} \cdot \begin{bmatrix} T_{0,1}^{e} \\ T_{0,2}^{e} \end{bmatrix} = \begin{bmatrix} \alpha_1^e & 0 \\ 0 & \alpha_2^e \end{bmatrix} \begin{bmatrix} T_{0,1}^{e} \\ T_{0,2}^{e} \end{bmatrix} = \begin{bmatrix} \alpha_1^e \, T_{0,1}^{e} \\ \alpha_2^e \, T_{0,2}^{e} \end{bmatrix}$$

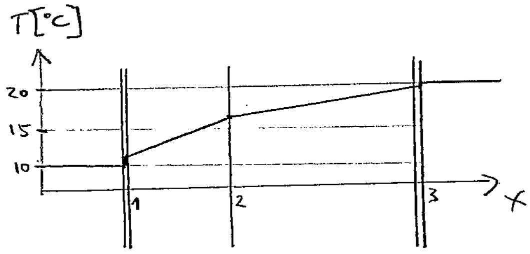
$$\underline{f}_{\Gamma_c}^{(1)} = \begin{bmatrix} 20 \cdot 10 \\ 0 \end{bmatrix} = \begin{bmatrix} 200 \\ 0 \end{bmatrix}$$

$$\underline{f}_{\Gamma_c}^{(2)} = \begin{bmatrix} 0 \\ 20 \cdot 20 \end{bmatrix} = \begin{bmatrix} 0 \\ 400 \end{bmatrix}$$

$$\underline{f} = \begin{bmatrix} 200 \\ 0 \\ 400 \end{bmatrix} \quad [W m^{-2}]$$

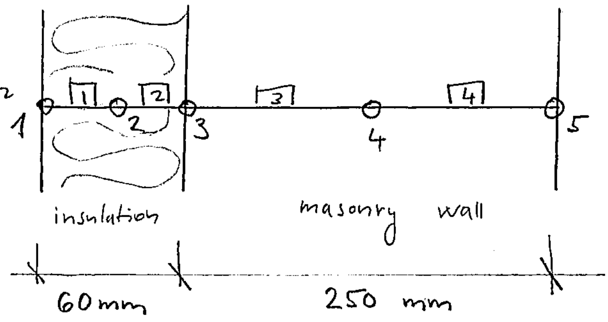
solution of $\underline{k}\,\underline{T} = \underline{f}$

$$\underline{T} = \underline{k}^{-1}\underline{f} = \begin{bmatrix} 11,25 \\ 16,25 \\ 18,75 \end{bmatrix}$$



HOMEWORK 3

$T_e = -17°C$
$\alpha_e = 30\ Wm^{-2}k^{-2}$



$T_i = 22°C$
$\alpha_i = 15\ Wm^{-2}k^{-1}$

- linear basis function: $\underline{N}^e_{(x)} = \frac{1}{L^e}\left[x_2^e - x \qquad x - x_1^e\right]$
- $\underline{T}^e_{(x)} = \underline{N}^e_{(x)}\, \underline{d}^e$
- $\frac{d\underline{T}^e_{(x)}}{dx} = \underline{B}^e \underline{d}^e$, $\underline{B}^e = \frac{1}{L^e}\left[-1 \quad 1\right]$

a) conductivity matrix: $\underline{k}^e_\Omega = \int_\Omega \underline{B}^T \cancelto{1m^2}{A}\, \lambda\, \underline{B}\, d\Omega = \frac{\lambda^e}{L^e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

b) boundary conductivity matrix: $\underline{k}^e_\Gamma = \int_{\Gamma^e} \left(\underline{N}^e\right)^T \alpha^e\, \underline{N}^e dx = \alpha^e_{(x_1)}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} +$

$$+ \alpha^e_{(x_2)}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \alpha^e_{(x_1^e)} & 0 \\ 0 & \alpha^e_{(x_2^e)} \end{bmatrix}$$

c) boundary flux matrix (convection): $\underline{k}^e_\Gamma \underline{T}^e = \begin{bmatrix} \alpha_1^e\, T_{0,1}^e \\ \alpha_2^e\, T_{0,2}^e \end{bmatrix} = \underline{f}^e_{\Gamma qc}$