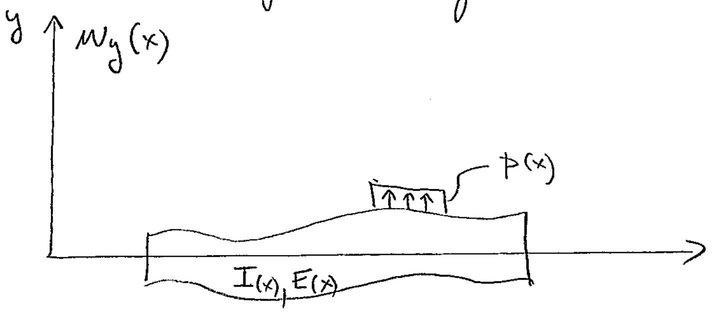
BEAMS

- loads do not act in the local x-direction, but perpendicular to it $\Rightarrow$ unknown is the displacement in the perpendicular direction carried by bending



- linear elastic material $\Rightarrow \sigma_x(x,y) = E(x)\,\varepsilon(x,y)$, where
$$\varepsilon_x(x,y) = \frac{dw_x}{dx} - y\frac{d^2w_y}{dx^2}$$

- $$M(x) = \int_A y\,\sigma_x\,dA = E(x)\frac{dw_x}{dx}\underbrace{\int_A dA}_{S(x)=0} - E(x)\frac{d^2w_y}{dx^2}\underbrace{\int_A y^2\,dA}_{I_y(x)} =$$
$$= -EI\frac{d^2w_y}{dx^2}$$

- from equilibrium of an infinitezimal beam slice we get:
$$\frac{d^2}{dx^2}\left(EI\frac{d^2w_y}{dx^2}\right) = p$$

Strong formulation

- given: distributed load $p(x)$
  displacements w_y
  rotations θ
  boundary moments $\bar{m}$
  boundary shear forces $\bar{S}$

governing equation: $\frac{d^2}{dx^2}\left(EI\frac{d^2 w_y}{dx^2}\right) - \bar{p} = 0$ in Ω.

BCs: $w_y = \bar{w}_y$ on Γ_w ... prescribed deflection

$\frac{dw_y}{dx} = \bar{\Theta}$ on Γ_Θ ... rotation

$-EI\frac{d^2 w_y}{dx^2} = \bar{m}$ on Γ_m ... end-moment

$-\frac{d}{dx}\left(EI\frac{d^2 w_y}{dx^2}\right) = \bar{s}$ on Γ_s ... end-shear force

Weak formulation

- energetically equivalent to the strong formulation
- by multiplication by an arbitrary function δw_y and integration over Ω we get:

$$\int_0^L \delta w_y \left(\frac{d^2}{dx^2}\left(EI\frac{d^2 w_y}{dx^2}\right) - \bar{p}\right) dx = 0$$

- after per-partes integration of the first term we obtain:

$$\left[\delta w_y \frac{d}{dx}\left(EI\frac{d^2 w_y}{dx^2}\right)\right]_0^L - \int \frac{d\delta w_y}{dx}\frac{d}{dx}\left(EI\frac{d^2 w_y}{dx^2}\right) dx -$$
$$- \int_0^L \delta w_y \bar{p}\, dx = 0$$

- after 2nd per-partes integration of the first term:

$$\left[\delta w_y \frac{d}{dx}\left(EI\frac{d^2 w_y}{dx}\right)\right]_0^L - \left[\frac{d\delta w_y}{dx} EI \frac{d^2 w_y}{dx^2}\right]_0^L + \int_0^L \frac{d^2 \delta w_y}{dx^2} EI \frac{d^2 w_y}{dx^2} dx -$$
$$- \int_0^L \delta w_y \bar{p}\, dx = 0$$

with $\delta w_y = 0$ on Γ_w and $\frac{d\delta w_y}{dx} = 0$ on Γ_Θ

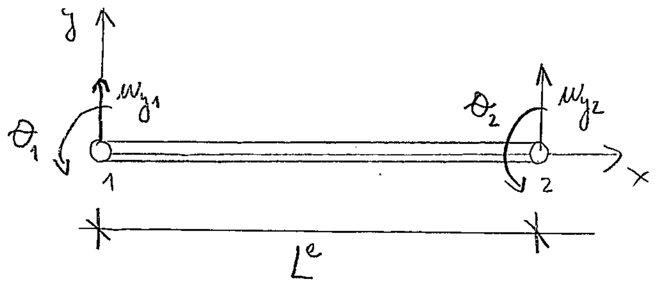
$\Rightarrow$ weak form: given $\bar{p}, \bar{s}, \bar{m}, \bar{w}_y, \bar{\theta}$; find $w_y(x)$ such that:

$$\underbrace{\int_0^L \frac{d^2 \delta w_y}{dx^2} EI \frac{d^2 w_y}{dx^2} dx}_{\text{internal energy}} = \underbrace{\int_0^L \delta w_y \bar{p}\, dx + \delta w_y \bar{s}\Big|_{\Gamma_s} - \frac{d \delta w_y}{dx} \bar{m}\Big|_{\Gamma_m}}_{\text{work of external forces}}$$

BCs: $w_y = \bar{w}_y$ at Γ_w

$\frac{dw_y}{dx} = \bar{\theta}$ at Γ_θ

FEM approximation



nodal displacements:

$\underline{d}^e = [w_{y1}, \theta_1, w_{y2}, \theta_2]^T$

nodal forces (conjugate to displacements):

$\underline{f}^e = [f_{y1}, m_1, f_{y2}, m_2]$

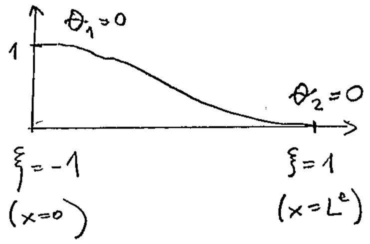
- we need C^1 continuity (second derivatives in the weak formulation) $\rightarrow$ "Hermite polynomials" are used, derivatives of displacements can be seen as rotations

- the Hermite polynomials for an element of length L^e are given by:

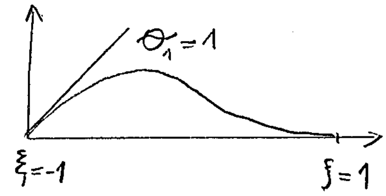
$$N_{u1} = \frac{1}{4}(1-\xi)^2(2+\xi)$$
$$N_{\theta 1} = \frac{L^e}{8}(1-\xi)^2(1+\xi)$$
$$N_{u2} = \frac{1}{4}(1+\xi)^2(2-\xi)$$
$$N_{\theta 2} = \frac{L^e}{8}(1+\xi)^2(\xi-1)$$

where $\xi = \frac{2x}{L^e} - 1$, therefore $-1 \leq \xi \leq 1$

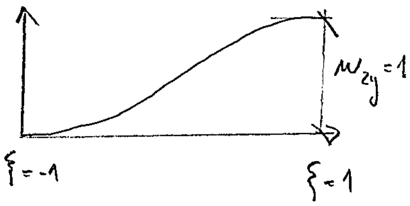
N_{u1}:



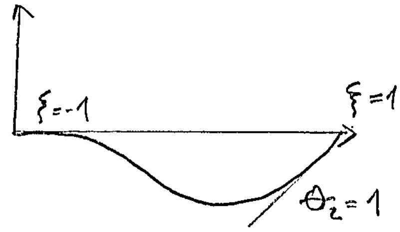
$N_{\theta 1}$:



N_{u2}:



$N_{\theta 2}$:



- the weight (δu_y) and trial solutions (u_y) are interpolated with the same weight functions:

$$u_y^e = \underline{N}^e \underline{d}^e \quad , \quad \delta u_y^e = \underline{N}^e \delta \underline{d}^e$$

- to evaluate the domain integral in the weak form we need to evaluate $\frac{d^2 u_y^e}{dx^2} = \frac{d^2 \underline{N}^e}{dx^2} \underline{d}^e$ for construction of the stiffness matrix

$$\frac{d^2 \underline{N}^e}{dx^2} = \frac{1}{L^e} \left[\frac{6\xi}{L^e} \quad 3\xi - 1 \quad -\frac{6\xi}{L^e} \quad 3\xi + 1 \right] = \underline{B}^e$$

Discretization

- discrete equation $\underline{k}\,\underline{d} = \underline{f} + \underline{r}$
- the element matrices:

a) stiffness matrix: $\underline{K}^e = \int_{\Omega^e} (\underline{B}^e)^T EI \, \underline{B}^e \, dx$

- if the flexural stiffness EI is constant over the element, the element stiffness matrix is given by:

$$\underline{K}^e = \frac{EI}{(L^e)^3} \begin{bmatrix} 12 & 6L^e & -12 & 6L^e \\ & (4L^e)^2 & -6L^e & (2L^e)^2 \\ & & 12 & -6L^e \\ & & & 4(L^e)^2 \end{bmatrix}$$

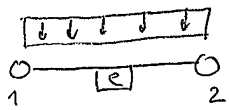
b) external force matrix:

$$\underline{f}^e = \underbrace{\int_{\Omega^e} (\underline{N}^e)^T \, \bar{p} \, dx}_{\underline{f}^e_\Omega} + \underbrace{\left[(\underline{N}^e)^T \bar{s} \right]_{\Gamma_s} + \left[\frac{d(\underline{N}^e)^T}{dx} \, \bar{m} \right]_{\Gamma_m}}_{\underline{f}^e_\Gamma}$$

$\bar{s}$ ← prescribed shear force at the boundary

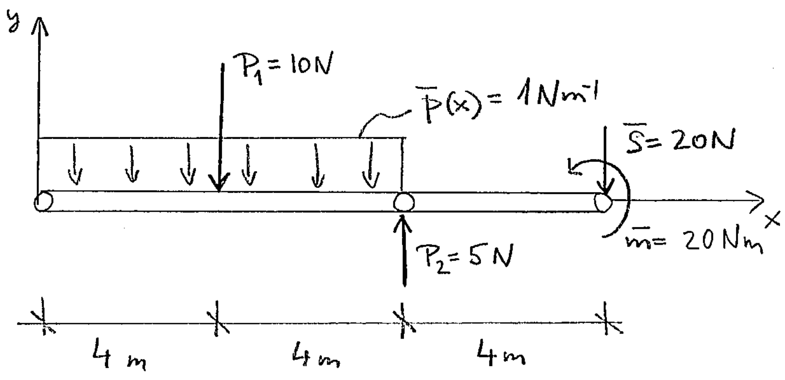
$\underline{f}^e_\Gamma$ ... element boundary force matrix

$\underline{f}^e_\Omega$ ... element body force matrix

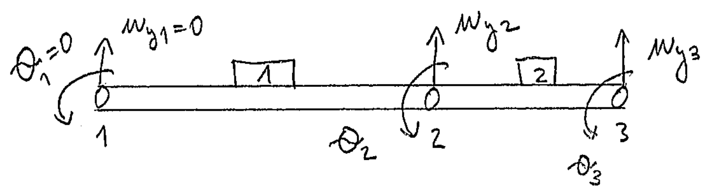
- for a constant load  the nodal forces can be calculated as

$$\underline{f}^e_\Omega = \int_{\Omega^e} (\underline{N}^e)^T \bar{p} \, dx = \int_0^{L^e} \begin{bmatrix} N_{u1} \\ N_{\theta 1} \\ N_{u2} \\ N_{\theta 2} \end{bmatrix} \bar{p} \, dx = \frac{\bar{p} L^e}{2} \begin{bmatrix} 1 \\ L^e/6 \\ 1 \\ -L^e/6 \end{bmatrix}$$

EXAMPLE 1



$EI = 10^4 \ Nm^2$



global displacement matrix:

$$\underline{d} = \begin{bmatrix} w_{y1} = 0 \\ \theta_1 = 0 \\ w_{y2} \\ \theta_2 \\ w_{y3} \\ \theta_3 \end{bmatrix} \rightarrow \text{each node has 2 DOFs}$$

essential BC: $E = [1, 2]$

free DOFs: $F = [3, 4, 5, 6]$

- element stiffness matrices are

$$\underline{k}^{(1)} = \frac{EI}{(L^{(1)})^3}\begin{bmatrix} 12 & 6L^{(1)} & -12 & 6L^{(1)} \\ 6L^{(1)} & 4(L^{(1)})^2 & -6L^{(1)} & 2(L^{(1)})^2 \\ -12 & -6L^{(1)} & 12 & -6L^{(1)} \\ 6L^{(1)} & 2(L^{(1)})^2 & -6L^{(1)} & 4(L^{(1)})^2 \end{bmatrix} \begin{array}{c} ① \\ \\ ② \\ \end{array} = 10^3 \begin{array}{c} \begin{bmatrix} 0.23 & 0.94 & -0.23 & 0.94 \\ 0.94 & 5.0 & -0.94 & 2.5 \\ -0.23 & -0.94 & 0.23 & -0.94 \\ 0.94 & 2.5 & -0.94 & 5.0 \end{bmatrix} \\ \quad ① \qquad\qquad ② \end{array}$$

$$\underline{k}^{(2)} = 10^3 \begin{array}{cc} \begin{array}{c} ② \qquad\qquad\qquad ③ \\ \begin{bmatrix} 1.88 & 3.75 & -1.88 & 3.75 \\ 3.75 & 10.0 & -3.75 & 5.0 \\ -1.88 & -3.75 & 1.88 & -3.75 \\ 3.75 & 5.0 & -3.75 & 10.0 \end{bmatrix} \end{array} & \begin{array}{c} \\ ② \\ \\ ③ \\ \end{array} \end{array}$$

- the global stiffness can be directly assembled as:

$$\underline{K} = 10^3 \cdot \begin{array}{cc} \begin{bmatrix} 0.23 & 0.94 & -0.23 & 0.94 & 0 & 0 \\ 0.94 & 5.0 & -0.94 & 2.5 & 0 & 0 \\ -0.23 & -0.94 & 2.11 & 2.81 & -1.88 & 3.75 \\ 0.94 & 2.5 & 2.81 & 15.0 & -3.75 & 5.0 \\ 0 & 0 & -1.88 & -3.75 & 1.88 & -3.75 \\ 0 & 0 & 3.75 & 5.0 & -3.75 & 10.0 \end{bmatrix} & \begin{array}{c} ① \\ \\ ② \\ \\ ③ \\ \end{array} \\ \quad ① \qquad\qquad\qquad ② \qquad\qquad\qquad ③ & \end{array}$$

- boundary force matrix: $\underline{f}_{\Gamma}^{e} = \left[(\underline{N}^e)^T \bar{s} \right]_{\Gamma_s} + \left[\left(\frac{d\underline{N}^e}{dx}\right)^T \bar{m} \right]_{\Gamma_m}$

$$\underline{f}_{\Gamma}^{(1)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{array}{c} ① \\ \\ ② \\ \end{array} \qquad \underline{f}_{\Gamma}^{(2)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \bar{m} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \bar{s} = \begin{bmatrix} 0 \\ 0 \\ -20 \\ 20 \end{bmatrix} \begin{array}{c} ② \\ \\ ③ \\ \end{array}$$

no load on the boundary of the 1st element

$$\Rightarrow \text{by direct assembly} \quad \underline{f}_{\Gamma} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -20 \\ 20 \end{bmatrix} \begin{array}{c} ① \\ \\ ② \\ \\ ③ \\ \end{array}$$

- body force (distributed load) matrix: $\underline{f}^e_\Omega = \int_{\Omega^e} (\underline{N}^e)^T \bar{p}\, dx$

  - it must also include the concentrated force P_1 (it is not in a node), which has to be distributed to nodes using shape functions (of $\xi \in (-1,1)$):

$$\underline{f}^e_\Omega = (\underline{N}^e)^T(\xi_A)\, P_A$$

point of force action

element 1:

$$\underline{f}^{(1)}_{\Omega_{\bar{p}}} = \int_0^{L^{(1)}} \begin{bmatrix} N_{u1} \\ N_{\theta 1} \\ N_{u2} \\ N_{\theta 2} \end{bmatrix} \bar{p}\, dx = \frac{\bar{p} L^{(1)}}{2} \begin{bmatrix} 1 \\ L^{(1)}/6 \\ 1 \\ -L^{(1)}/6 \end{bmatrix} = \begin{bmatrix} -4 \\ -5{,}33 \\ -4 \\ 5{,}33 \end{bmatrix}$$

$$\underline{f}^{(1)}_{\Omega_{P_1}} = \begin{bmatrix} N_{w1} \\ N_{\theta 1} \\ N_{w2} \\ N_{\theta 2} \end{bmatrix}_{\xi=0} P_1 = \begin{bmatrix} -5 \\ -10 \\ -5 \\ 10 \end{bmatrix} \longrightarrow \underline{f}^{(1)}_\Omega = \begin{bmatrix} -9 \\ -15{,}33 \\ -9 \\ 15{,}33 \end{bmatrix} \begin{matrix} (1) \\ \\ (2) \end{matrix}$$

element 2: the force $P_2 = 5\,N$ acts in the first node, no need to distribute it by the shape functions ($\xi = -1$):

$$\underline{f}^{(2)}_\Omega = \begin{bmatrix} N_{w1} \\ N_{\theta 1} \\ N_{w2} \\ N_{\theta 2} \end{bmatrix}_{\xi=-1} P_2 = \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} (2) \\ \\ (3) \end{matrix}$$

from definition only $N_{u1} = 1$ at $\xi = -1$

$\Downarrow$

by the direct assembly: $\underline{f}_\Omega = \begin{bmatrix} -9 \\ -15.3 \\ -4 \\ 15.3 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} (1) \\ \\ (2) \\ \\ (3) \end{matrix}$

- Solution of the system is then:

$$\underline{d}_{(F)} = \underline{k}^{-1}_{(F,F)} \underline{f}_{(F)} - \cancel{\underline{k}_{(F,E)} \underline{d}_{(E)}}^{0} = \left(\begin{bmatrix} 2.11 & 2.81 & -1.88 & 3.75 \\ 2.81 & 15.0 & -3.75 & 5.0 \\ -1.88 & 3.75 & 1.88 & -3.75 \\ 3.75 & 5.0 & -3.75 & 10.0 \end{bmatrix} \cdot 10^3 \right)^{-1} \begin{bmatrix} -4 \\ 15.3 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -0.55 \\ -0.11 \\ -1.03 \\ -0.12 \end{bmatrix} = \begin{bmatrix} w_{y2} \\ \theta_2 \\ w_{y3} \\ \theta_3 \end{bmatrix}$$

$$\underline{f}_{(E)} = \begin{bmatrix} R_{y1} - 9 \\ R_{\theta 1} - 15.3 \end{bmatrix} = \cancel{\underline{k}_{(E,E)} \underline{d}_{(E)}}^{0} + \underline{k}_{(E,F)} d_{(F)} = 10^3 \begin{bmatrix} -0.23 & 0.94 & 0 & 0 \\ -0.94 & 2.5 & 0 & 0 \end{bmatrix} \begin{bmatrix} -0.55 \\ -0.11 \\ -1.03 \\ -0.12 \end{bmatrix}$$

$$\Rightarrow R_{y1} = 33 \text{ N}$$
$$R_{\theta 1} = 252 \text{ Nm}$$

- post processing: moments and shear forces in elements

- $$m^{(1)} = EI \frac{d^2 w^{(1)}}{dx^2} = EI \begin{bmatrix} \frac{d^2 N_{w1}}{dx^2} & \frac{d^2 N_{\theta 1}}{dx^2} & \frac{d^2 N_{w2}}{dx^2} & \frac{d^2 N_{\theta 2}}{dx^2} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ w_{y2} \\ \theta_2 \end{bmatrix} =$$

$$= -240{,}64 + 25{,}785x \quad [\text{Nm}]$$

- $$m^{(2)} = EI \frac{d^2 \underline{N}}{dx^2} \underline{d}^{(2)} = -104.5 + 39{,}75x \quad [\text{Nm}]$$

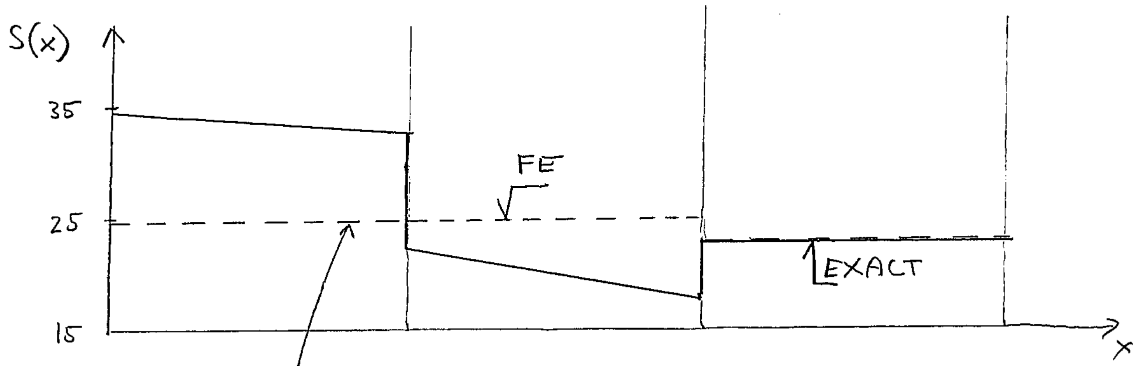
- $$s^{(1)} = -EI \frac{d^3 w^{(1)}}{dx^3} = -EI \begin{bmatrix} \frac{d^3 N_{w1}}{dx^3} & \frac{d^3 N_{\theta 1}}{dx^3} & \frac{d^3 N_{w2}}{dx^3} & \frac{d^3 N_{\theta 2}}{dx^3} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ w_{y2} \\ \theta_2 \end{bmatrix} =$$

$$= -25{,}785 \text{ N}$$

- $$s^{(2)} = -EI \frac{d^3 \underline{N}}{dx^3} \underline{d}^{(2)} = -39{,}75 \text{ N}$$

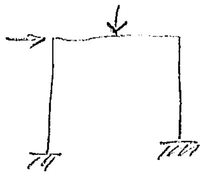
Comparison of analytical and FE solution

- displacements practically identical
- moments in a good agreement
- shear forces :



not good → node should have been placed at the concentrated force point of action

- **note** : in frames we have 6 DoFs per element

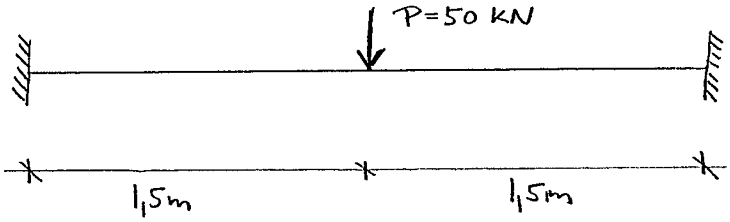


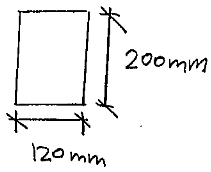
$$\underline{d}^e = \begin{bmatrix} w_{1x} \\ w_{1y} \\ \theta_1 \\ w_{2x} \\ w_{2y} \\ \theta_2 \end{bmatrix}$$

→ the transformation into local coordinates is done as in the Direct Stiffness Method

$$\underline{d}^{e'} = \begin{bmatrix} \cos\phi & \sin\phi & 0 & 0 & 0 & 0 \\ -\sin\phi & \cos\phi & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\phi & \sin\phi & 0 \\ 0 & 0 & 0 & -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \underline{d}^e$$

HOMEWORK 4





– midpoint deflection ?

$E = 200 \cdot 10^9$ Pa