Multidimensional Scalar (Temperature) Field Problems

- weak formulation derived using per-partes in more dimensions

- the same structure of equations as in 1D and 2D is practically the same as 3D

- variables such as heat flux (or displacement) are in vector forms – they have magnitude and direction – and therefore can be expressed in terms of components and base vectors : $\underline{q} = q_x \underline{i} + q_y \underline{j} \Rightarrow \underline{q} = \begin{bmatrix} q_x \\ q_y \end{bmatrix}$

- scalar product : $\underbrace{\underline{q} \cdot \underline{r}}_{\text{vector notation}} = q_x r_x + q_y r_y = \underbrace{\underline{q}^T \underline{r}}_{\text{matrix notation}}$

- gradient : measure of the slope of a field (2D counterpart of a derivative)

  $\Rightarrow$ gradient vector operator : $\underline{\nabla} = \frac{\partial}{\partial x} \underline{i} + \frac{\partial}{\partial y} \underline{j}$

  e.g. $\underbrace{\underline{\nabla} f = \frac{\partial f}{\partial x} \underline{i} + \frac{\partial f}{\partial y} \underline{j}}_{\text{= the steepest descent of } f}$

  - in matrix form the gradient operator is

$$\underline{\nabla} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} \rightarrow \underline{\nabla} f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix}$$

- scalar product of the gradient operator with a vector field gives the divergence of the vector field (e.g. heat flowing from a point):

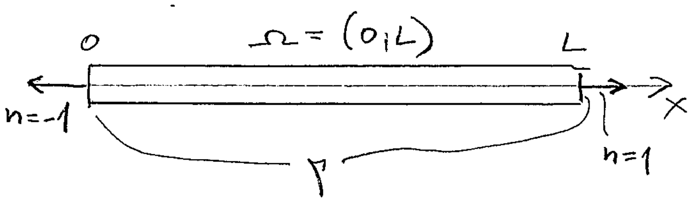
$$\underline{\nabla} \cdot \underline{q} = \left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} \right) \equiv \operatorname{div} \underline{q}$$

scalar quantity.

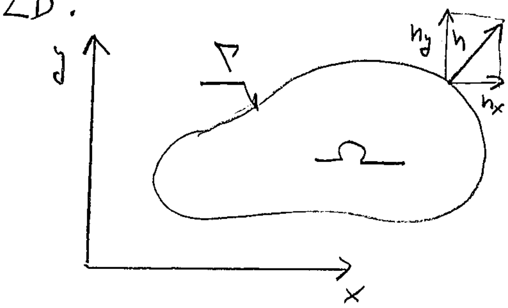
$\Rightarrow$ in matrix notation: $\operatorname{div} \underline{q} = \underline{\nabla}^T \underline{q}$

Divergence theorem and Green's formula

1D:



2D:



$\underline{n} = n_x \underline{i} + n_y \underline{j}$

normal to the surface

$n_x^2 + n_y^2 = 1 = \|\underline{n}\|$

- the integral of a derivative is in 1D:

$$\int_\Omega \frac{df(x)}{dx}\, dx = \left[f n \right]_\Gamma$$

— the value at the beginning minus the value at the end

$n \in \{-1; 1\}$

$\Rightarrow$ in more dimensions generalized by Green's theorem:

$$\int_\Omega \underline{\nabla} f \, d\Omega = \int_\Gamma f \underline{n} \, d\Gamma$$

- divergence theorem relates an area integral of the vector divergence to a contour integral of a vector field: $\int_\Omega \underline{\nabla}^T \underline{q} = \int_\Gamma \underline{q}^T \underline{n} \, d\Gamma$

Green's formula (per-partes in more dimensions)

$$\int_\Omega w \, \underline{\nabla}^T \underline{q} \, d\Omega = \int_\Gamma w \underline{q}^T \underline{n} \, d\Gamma - \int_\Omega (\underline{\nabla} w)^T \underline{q} \, d\Omega$$

w – an arbitrary function

↳ in a rectangular domain with 1D heat flow where $\underline{q} = q_x \underline{i}$ we have $\underline{n} = n \underline{i}$ ($n(0) = -1$ and $n(L) = 1$ and therefore

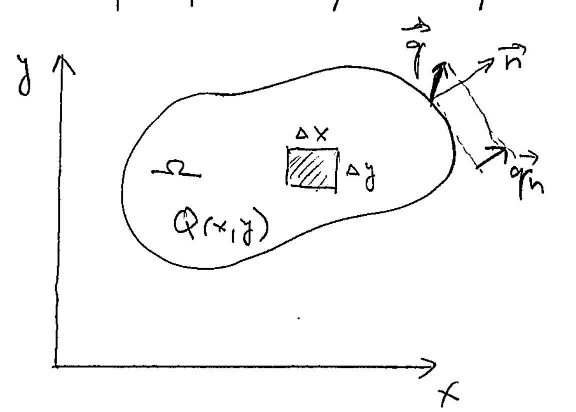
$$\int_\Omega w \frac{\partial q_x}{\partial x} d\Omega = \underbrace{\int_\Gamma w q_x n \, d\Gamma}_{[q_x w]_0^L} - \int_\Omega \frac{\partial w}{\partial x} q_x \, d\Omega$$

= per-partes in 1D

Strong form for 2D heat conduction

- derived from an energy balance in a control volume and Fourier law relating heat flux to a temperature gradient
- in steady-state conduction the temperature is not a function of time
- heat flux $\underline{q}$ flowing through the boundaries of a control volume equals to a generated heat Q

- heat going through the boundary:
  $q_n = \underline{q} \cdot \underline{n} = \underline{q}^T \underline{n} = q_x n_x + q_y n_y$



- <u>energy balance</u> : "what goes in must go out"

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} - \bar{Q} = 0$$

$$\left(\operatorname{div} \underline{q} - \bar{Q} = 0 \quad \text{or} \quad \underbrace{\underline{\nabla}^T \underline{q}}_{\text{heat flowing from a point}} - \underbrace{\bar{Q}}_{\text{generated heat}} = 0 \right)$$

- <u>Fourier's law</u> : $\underline{q} = -\lambda \, \underline{\nabla} T$

  ↑ heat goes against the temperature gradient

$$\Downarrow$$

$\lambda \underline{\nabla}^2 T + \bar{Q} = 0$, where $\underline{\nabla}^2$ is the Laplacian operator defined as $\underline{\nabla}^2 = \underline{\nabla} \cdot \underline{\nabla} = \underline{\nabla}^T \underline{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

- <u>generalized Fourier's law</u> :

  - $\underline{q} = -\underline{D} \, \underline{\nabla} T$

    ↑ conductivity matrix, always pos.-def.

$$\underbrace{\begin{bmatrix} q_x \\ q_y \end{bmatrix}}_{\underline{q}} = - \underbrace{\begin{bmatrix} \lambda_{xx} & \lambda_{xy} \\ \lambda_{yx} & \lambda_{yy} \end{bmatrix}}_{\underline{D}} \underbrace{\begin{bmatrix} \frac{\partial T}{\partial x} \\ \frac{\partial T}{\partial y} \end{bmatrix}}_{\underline{\nabla} T}$$

$$\Downarrow$$

$$\underline{\nabla}^T \left(\underline{D} \, \underline{\nabla} T \right) + \bar{Q} = 0$$

- for an isotropic material (having the same properties in any coordinate system, e.g. metals, concrete...) :

$$\underline{D} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \lambda \underline{I}$$

<u>1D:</u>

$$\begin{tikzcd} T(x) \arrow[d, "\frac{d}{dx}"'] & \bar{Q}(x) \\ \frac{d}{dx} T(x) \arrow[r, "-\lambda"] & q_x(x) \arrow[u, "\frac{d}{dx}"'] \end{tikzcd}$$

<u>2D:</u>

$$\begin{tikzcd} T(\underline{x}) \arrow[d, "\underline{\nabla}"] & \underline{x} = (x, y) & Q(\underline{x}) \\ \underline{\nabla} T(\underline{x}) \arrow[rr, "-\underline{D}"] & & \underline{q}(\underline{x}) \arrow[u, "-\underline{\nabla}"'] \end{tikzcd} \qquad \underline{q} = \begin{bmatrix} q_x \\ q_y \end{bmatrix}$$

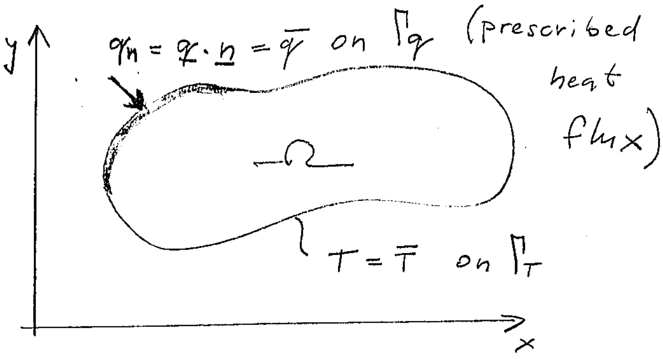
Boundary conditions

a) essential (= Dirichlet) BCs:

$$T(x,y) = \bar{T}(x,y) \quad \text{on } \Gamma_T$$

b) natural (= Neumann) BCs:

$$q_n = \underline{q} \cdot \underline{n} = \bar{q} \quad \text{on } \Gamma_q$$



Weak Formulation

- energy balance equation is premultiplied, together with natural BCs, by a weight function δT and integrated over the domain Ω and boundary Γ_q

a) $$\int_\Omega \delta T \left(\underline{\nabla} \cdot \underline{q} - \bar{Q} \right) d\Omega = 0 \qquad \forall \delta T \text{ smooth enough vanishing at } \Gamma_T$$

b) $$\int_{\Gamma_q} \delta T \left(\bar{q} - \underline{q} \cdot \underline{n} \right) d\Gamma = 0 \qquad \forall \delta T \text{ ——— '' ———}$$

applying Green's formula to the first term:

$$\int_\Omega \delta T \, \underline{\nabla} \cdot \underline{q} \, d\Omega = \int_\Gamma \delta T \, \underline{q} \cdot \underline{n} \, d\Gamma - \int_\Omega \underline{\nabla} \delta T \cdot \underline{q} \, d\Omega$$

therefore a) can be written as:

$$\int_\Omega \underline{\nabla} \delta T \cdot \underline{q} = \int_\Gamma \delta T \, \underline{q} \cdot \underline{n} \, d\Gamma - \int_\Omega \delta T \bar{Q} \, d\Omega = \int_{\Gamma_q} \delta T \, \underbrace{\underline{q} \cdot \underline{n}}_{\text{from b) it is } \delta T \bar{q}} \, d\Gamma + \int_{\Gamma_T} \delta T \, \underline{q} \cdot \underline{n} \, d\Gamma - \int_\Omega \delta T \bar{Q} \, d\Omega$$

- after we set $\delta T = 0$ on Γ_T, the weak form is given by:

$$\int_\Omega \underline{\nabla} \delta T \cdot \underline{q} \, d\Omega = \int_{\Gamma_q} \delta T \, \bar{q} \, d\Gamma - \int_\Omega \delta T \, \bar{Q} \, d\Omega$$

- in the matrix form and after the substitution of Furier's law $\underline{q} = -\underline{D}\,\underline{\nabla} T$ into the first term we obtain the weak form for heat conduction:

find T such that:

$$\int_\Omega (\underline{\nabla}\,\delta T)^T \underline{D}\,\underline{\nabla} T \, d\Omega = -\int_{\Gamma_q} \delta T \, \bar{q} \, d\Gamma + \int_\Omega \delta T \, \bar{Q} \, d\Omega$$

MULTIDIMENSIONAL FEM APPROXIMATION

- in more dimensions (2D, 3D) the power of FEM becomes apparent (we can solve on arbitrary shape of bodies)
- meshing is the challenge (i.e. division into elements) together with the construction of shape functions

- shape functions are polynomials given by Pascal's triangle (giving monomials that must be included in FEM approximation so that the solution converges with the mesh refinement):

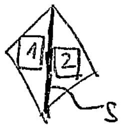
$$\begin{array}{ccccccc} & & & 1 & & & \\ & & x & & y & & \\ & x^2 & & xy & & y^2 & \\ x^3 & & x^2y & & xy^2 & & y^3 \end{array}$$

linear must consist of all these terms:

$$\Theta^e(x,y) = \alpha_1^e + \alpha_2^e x + \alpha_3^e y$$

- there must be continuity between elements:

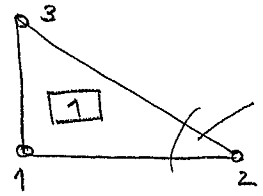
$$\Theta^{(1)}(s) = \Theta^{(2)}(s)$$



s (boundary between elements)

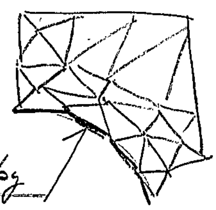
Three-node triangular element

- easy to mesh any shape, relatively inaccurate
- nodes in the corners of the elements, numbered counterclockwise ! :



none of the angles should be smaller than 45°

- introduced an error in geometry in case of rounded geometries :



approximated by straight segments

- element coordinates : (x_i^e, y_i^e) where i = node number

Linear approximation of trial solution

$$\theta^e(x,y) = \alpha_0^e + \alpha_1^e x + \alpha_2^e y = \underbrace{\begin{bmatrix} 1 & x & y \end{bmatrix}}_{\underline{P}(x)} \underbrace{\begin{bmatrix} \alpha_0^e \\ \alpha_1^e \\ \alpha_2^e \end{bmatrix}}_{\underline{\alpha}^e} = \underline{P}(x,y)\,\underline{\alpha}^e$$

desired quantity (e.g. Temperature)

- nodal values :

$$\theta_1^e = \alpha_0^e + \alpha_1^e x_1^e + \alpha_2^e y_1^e \quad \text{(at point with coordinates } (x_1^e, y_1^e))$$
$$\theta_2^e = \alpha_0^e + \alpha_1^e x_2^e + \alpha_2^e y_2^e$$
$$\theta_3^e = \alpha_0^e + \alpha_1^e x_3^e + \alpha_2^e y_3^e$$

$$\Downarrow$$

$$\underbrace{\begin{bmatrix} \theta_1^e \\ \theta_2^e \\ \theta_3^e \end{bmatrix}}_{\underline{d}^e \text{ (or eg. } \underline{T}^e)} = \overbrace{\begin{bmatrix} 1 & x_1^e & y_1^e \\ 1 & x_2^e & y_2^e \\ 1 & x_3^e & y_3^e \end{bmatrix}}^{\underline{M}^e} \underbrace{\begin{bmatrix} \alpha_0^e \\ \alpha_1^e \\ \alpha_2^e \end{bmatrix}}_{\underline{\alpha}^e} \Rightarrow \underline{\alpha}^e = (\underline{M}^e)^{-1}\underline{d}^e$$

- the value of a function $\theta^e(x,y)$ is given by:

$$\theta^e(x,y) = \mathbf{p}(x,y)\,\underline{\alpha}^e = \underbrace{\mathbf{p}(x,y)\left(\underline{M}^e\right)^{-1}}_{\text{shape functions } \underline{N}^e(x,y)} \underline{d}^e$$

- as in 1D the desired quantity $\theta^e(x,y)$ can be expressed by means of shape functions and nodal values: $\theta^e(x,y) = \underline{N}^e(x,y)\,\underline{d}^e$, where

$$\underline{N}^e(x,y) = \mathbf{p}(x,y)\left(\underline{M}^e\right)^{-1} = \left[N_1^e(x,y) \quad N_2^e(x,y) \quad N_3^e(x,y)\right]$$

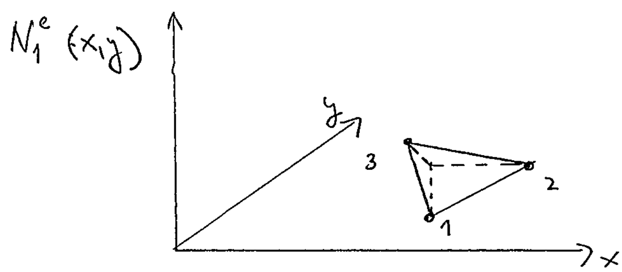
and

$$\left(\underline{M}^e\right)^{-1} = \frac{1}{2A^e}\begin{bmatrix} y_2^e - y_3^e & y_3^e - y_1^e & y_1^e - y_2^e \\ x_3^e - x_2^e & x_1^e - x_3^e & x_2^e - x_1^e \\ x_2^e y_3^e - x_3^e y_2^e & x_3^e y_1^e - x_1^e y_2^e & x_1^e y_2^e - x_2^e y_1^e \end{bmatrix}$$

where A^e is the area of the element e, and if the nodes are numbered counterclockwise it can be expressed as

$$2A^e = \det\left(\underline{M}^e\right) = \left(x_2^e y_3^e - x_3^e y_2^e\right) - \left(x_1^e y_3^e - x_3^e y_1^e\right) + \left(x_1^e y_2^e - x_2^e y_1^e\right)$$

- note: $\underline{N}^e(x,y)$ consists of linear functions giving 1 at corresponding node and vanishing at the other nodes:



→ kronecker delta property:

$$N_i^e(x_j^e, y_j^e) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$$N_1^e = \frac{1}{2A^e}\left(x_2^e y_3^e - x_3^e y_2^e + (y_2^e - y_3^e)x + (x_3^e - x_2^e)y\right)$$

$$N_2^e = \frac{1}{2A^e}\left(x_3^e y_1^e - x_1^e y_3^e + (y_3^e - y_1^e)x + (x_1^e - x_3^e)y\right)$$

$$N_3^e = \frac{1}{2A}\left(x_1^e y_2^e - x_2^e y_1^e + (y_1^e - y_2^e)x + (x_2^e - x_1^e)y\right)$$

global approximation and continuity

- global shape functions:

$$\underline{N}^T = \sum_{e=1}^{hel} (\underline{L}^e)^T (\underline{N}^e)^T$$

"gather operator" localizing the local matrix to the global one, consisting of 0's and 1's

- trial solution:

$$\theta = \underline{N}\,\underline{d}$$

derivatives of shape functions (gradient of $\underline{N}^e$)

$$\underline{\nabla}\,\theta^e = \begin{bmatrix} \frac{\partial \theta^e}{\partial x} \\ \frac{\partial \theta^e}{\partial y} \end{bmatrix} = \underbrace{\underline{\nabla}\,\underline{N}^e}_{\underline{B}^e}\,\underline{d}^e = \underbrace{\begin{bmatrix} \frac{\partial N_1^e}{\partial x} & \frac{\partial N_2^e}{\partial x} & \frac{\partial N_3^e}{\partial x} \\ \frac{\partial N_1^e}{\partial y} & \frac{\partial N_2^e}{\partial y} & \frac{\partial N_3^e}{\partial y} \end{bmatrix}}_{\underline{B}^e} \underbrace{\begin{bmatrix} \theta_1^e \\ \theta_2^e \\ \theta_3^e \end{bmatrix}}_{\underline{d}^e}$$

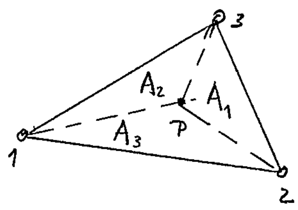
- in case of the 3-node triangular element:

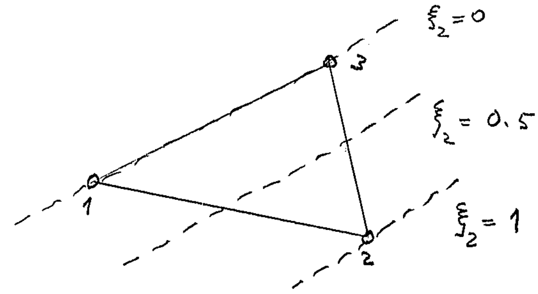
$$\underline{B}^e = \frac{1}{2A^e}\begin{bmatrix} y_2^e - y_3^e & y_3^e - y_1^e & y_1^e - y_2^e \\ x_3^e - x_2^e & x_1^e - x_3^e & x_2^e - x_1^e \end{bmatrix}$$

constant in each element (independent of x and y)
→ constant gradients in the elements (like 1D linear elements)

Derivation of base functions using triangular coordinates (= parent element coordinate approach)

- necessary for more complex shapes than triangles and higher order functions





- the triangular coordinates of a point P are given by:

$$\xi_i = \frac{A_i}{A}$$ ← area of a triangle generated by connecting the two other nodes than i with the point P

  - if P goes to the node i the corresponding triangular coordinate ξ_i is becoming equal to one

$$\Downarrow$$

$\xi_i(x_j^e, y_j^e) = \delta_{ij}$ and because $\xi_i = \frac{A_i}{A}$, the function is linear $\Rightarrow$ identical to the linear shape functions

- the relationship between the triangular and physical coordinates is

$$x = \sum_{i=1}^{3} x_i^e \xi_i \quad , \quad y = \sum_{i=1}^{3} y_i^e \xi_i$$

and therefore $\Theta^e = \sum_{i=1}^{3} \theta_i^e \xi_i = \theta_1^e \xi_1 + \theta_2^e \xi_2 + \theta_3^e \xi_3$

- it is obvious that $\xi_1 + \xi_2 + \xi_3 = 1$ which in combination with $x = x_1^e \xi_1 + x_2^e \xi_2 + x_3^e \xi_3$ and $y = y_1^e \xi_1 + y_2^e \xi_2 + y_3^e \xi_3$ gives

$$\begin{bmatrix} 1 \\ x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ x_1^e & x_2^e & x_3^e \\ y_1^e & y_2^e & y_3^e \end{bmatrix}}_{\text{corresponds to } (\underline{M}^e)^T} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}$$

and by inversion we get:

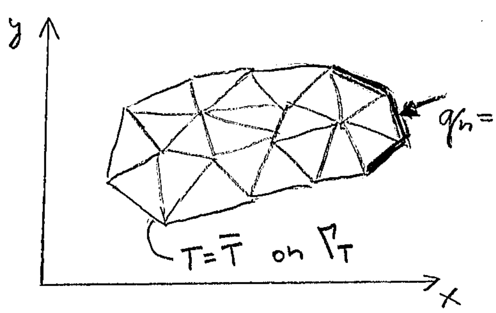
$$\begin{bmatrix} \xi_1(x) \\ \xi_2(x) \\ \xi_3(x) \end{bmatrix} = \underline{N}^T = \frac{1}{2A^e} \underbrace{\begin{bmatrix} x_1^e y_2^e - x_3^e y_2^e & y_2^e - y_3^e & x_3^e - x_2^e \\ x_3^e y_1^e - x_1^e y_3^e & y_3^e - y_1^e & x_1^e - x_3^e \\ x_1^e y_2^e - x_2^e y_1^e & y_1^e - y_2^e & x_2^e - x_1^e \end{bmatrix}}_{(\underline{M}^{-1})^T} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix}$$

FEM Formulation for 2D Heat Conduction

- the weak form is in matrix form written as:

$$\int_\Omega (\underline{\nabla}\delta T)^T \underline{D}\, \underline{\nabla} T \, d\Omega = -\int_{\Gamma_q} (\delta T)^T \bar{q} \, d\Gamma + \int_\Omega (\delta T)^T \bar{Q} \, d\Omega$$

where $\underline{\nabla} T = \begin{bmatrix} \frac{\partial T}{\partial x} \\ \frac{\partial T}{\partial y} \end{bmatrix}$, $\underline{D} = \begin{bmatrix} \lambda_{xx} & \lambda_{xy} \\ \lambda_{yx} & \lambda_{yy} \end{bmatrix}$



- the integrals in the weak solution can be replaced by a sum over all elements

$$\sum_{e=1}^{n_{el}} \left(\int_{\Omega^e} (\underline{\nabla} \delta T^e)^T \underline{D}^e (\underline{\nabla} T^e) \, d\Omega + \int_{\Gamma_q^e} (\delta T^e)^T \bar{q} \, d\Gamma - \int_{\Omega^e} (\delta T^e)^T \bar{Q} \, d\Omega \right) = 0$$

- FEM approximation:
  a) trial solution: $T(x,y) \approx T^e(x,y) = \underline{N}^e(x,y) \, \underline{T}^e$
  b) weight function: $\delta T(x,y) \approx \delta T^e(x,y) = \underline{N}^e(x,y) \, \delta \underline{T}^e$

$$\underline{T}^e = \begin{bmatrix} T_1^e \\ T_2^e \\ \vdots \\ T_n^e \end{bmatrix}, \quad \delta \underline{T}^e = \begin{bmatrix} \delta T_1^e \\ \delta T_2^e \\ \vdots \\ \delta T_n^e \end{bmatrix}$$

n number of nodes

- element nodal temperatures are related to the global temperature matrix by scatter matrix $\underline{L}^e$:

$$\underline{T}^e = \underline{L}^e \underline{T}$$

therefore we get:

$$T^e(x,y) = \underline{N}^e(x,y) \, \underline{L}^e \, \underline{T}$$
$$(\delta T^e)^T(x,y) = \left(\underline{N}^e(x,y) \, \delta \underline{T}^e\right)^T = (\delta \underline{T})^T (\underline{L}^e)^T (\underline{N}^e)^T(x,y)$$

$$\underline{\nabla} T^e(x,y) = \underbrace{\left(\underline{\nabla} \underline{N}^e(x,y)\right)}_{\underline{B}^e} \underline{T}^e = \underline{B}^e(x,y) \, \underline{T}^e = \underline{B}^e(x,y) \, \underline{L}^e \, \underline{T}$$

$$(\underline{\nabla} \delta T^e)^T = (\underline{B}^e \delta \underline{T}^e)^T = (\delta \underline{T}^e)^T (\underline{B}^e)^T = (\underline{L}^e \delta \underline{T})^T (\underline{B}^e)^T = (\delta \underline{T})^T (\underline{L}^e)^T (\underline{B}^e)^T$$

- the global matrices can be partitioned as:

$$\underline{T} = \begin{bmatrix} \bar{\underline{T}}_E \\ \underline{T}_F \end{bmatrix} \qquad \delta \underline{T} = \begin{bmatrix} \underline{0} \\ \delta \underline{T}_F \end{bmatrix}$$

δT vanishes here, $\bar{\underline{T}}_E$ are prescribed

→ substituting the trial solution and weight function approximations into weak solution yields:

$$(\delta \underline{I})^T \left(\sum_{e=1}^{n_{el}} (\underline{L}^e)^T \left(\int_{\Omega^e} (\underline{B}^e)^T \underline{D}^e \underline{B}^e \, d\Omega \, \underline{L}^e \underline{I} + \int_{\Gamma_q^e} (\underline{N}^e)^T \bar{q} \, d\Gamma - \int_{\Omega^e} (\underline{N}^e)^T \bar{Q} \, d\Omega \right) \right) = 0$$

$$\forall \, \delta \underline{I}_F$$

↑ integration only on the level of elements

- element conductivity matrix: $\underline{k}^e = \int_{\Omega^e} (\underline{B}^e)^T \underline{D}^e \underline{B}^e \, d\Omega$

- element flux matrix: $\underline{f}^e = \underbrace{-\int_{\Gamma_q^e} (\underline{N}^e)^T \bar{q} \, d\Gamma}_{\underline{f}_\Gamma^e} + \underbrace{\int_{\Omega^e} (\underline{N}^e)^T \bar{Q} \, d\Omega}_{\underline{f}_\Omega^e}$

$\underline{f}_\Gamma^e$ ↗ element boundary flux

$\underline{f}_\Omega^e$ ↑ source heat flux

⇓

discretized weak solution:

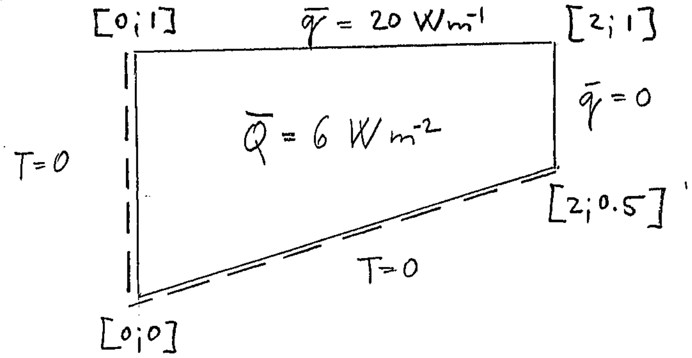
$$\underline{w}^T \left[\underbrace{\left(\sum_{e=1}^{n_{el}} (\underline{L}^e)^T \underline{k}^e \underline{L}^e \right)}_{\underline{K}} \underline{d} - \underbrace{\sum_{e=1}^{n_{el}} (\underline{L}^e)^T \underline{f}^e}_{\underline{f}} \right] = 0 \qquad \forall \, \delta \underline{I}_F$$

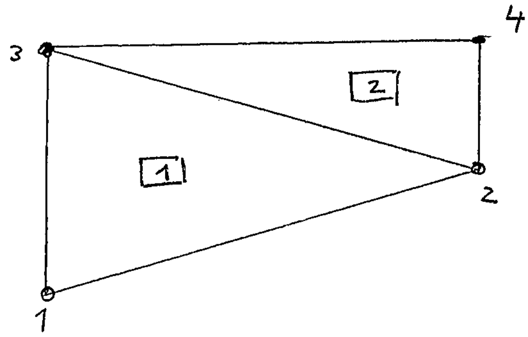
$\underline{K}$ - global matrix, in practise the direct assembly is used instead

- again as in 1D case we get the system:

$$\begin{bmatrix} \underline{K}_E & \underline{K}_{EF} \\ \underline{K}_{EF}^T & \underline{K}_F \end{bmatrix} \begin{bmatrix} \underline{d}_E \\ \underline{d}_F \end{bmatrix} = \begin{bmatrix} \underline{f}_E + \underline{r}_E \\ \underline{f}_F \end{bmatrix}$$

$\boxed{\text{EXAMPLE 1}}$





essential BCs at nodes :

$$\underline{E} = [1, 2, 3]$$

free node :

$$\underline{F} = [4]$$

- isotropic conductivity $\lambda = 5 W k^{-1}$

$$\Rightarrow \underline{D}^e = \lambda \underline{I} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \quad [W k^{-1}]$$

- $$\underline{B}^e = \frac{1}{2A^e} \begin{bmatrix} y_2^e - y_3^e & y_3^e - y_1^e & y_1^e - y_2^e \\ x_3^e - x_2^e & x_1^e - x_3^e & x_2^e - x_1^e \end{bmatrix} \quad \text{where}$$

$$2A^e = (x_2^e y_3^e - x_3^e y_2^e) - (x_1^e y_3^e - x_3^e y_1^e) - (x_1^e y_2^e - x_2^e y_1^e)$$

- since $\underline{B}^e$ and $\underline{D}^e$ are constants :

$$\underline{k}^e = \int_{\Omega^e} (\underline{B}^e)^T \underline{D}^e \underline{B}^e \, d\Omega = \int_{\Omega^e} (\underline{B}^e)^T \underline{B}^e \lambda \, d\Omega = (\underline{B}^e)^T \underline{B}^e \lambda \underbrace{\int_{\Omega^e} d\Omega}_{= A^e}$$

- element 1: $A^{(1)} = 1$

$$\underline{B}^{(1)} = 0.5 \begin{bmatrix} -0.5 & 1 & -0.5 \\ -2 & 0 & 2 \end{bmatrix}$$

$$\underline{k}^{(1)} = \lambda A^{(1)} (\underline{B}^{(1)})^T \underline{B}^{(1)} = \begin{array}{c} \begin{bmatrix} 5.3125 & -0.625 & -4.6875 \\ -0.625 & 1.25 & -0.625 \\ -4.6875 & -0.625 & 5.3125 \end{bmatrix} \\ \begin{array}{ccc} \quad 1 \quad & \quad 2 \quad & \quad 3 \quad \end{array} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array}$$

- element 2: $A^{(2)} = 0.5$

$$\underline{B}^{(2)} = \begin{bmatrix} 0 & 0.5 & -0.5 \\ -2 & 2 & 0 \end{bmatrix}$$

$$\underline{k}^{(2)} = \lambda A^{(2)} (\underline{B}^{(2)})^T \underline{B}^{(2)} = \begin{array}{cc} \begin{bmatrix} 10 & -10 & 0 \\ -10 & 10.625 & -0.625 \\ 0 & -0.625 & 0.625 \end{bmatrix} & \begin{array}{c} 2 \\ 4 \\ 3 \end{array} \\ \begin{array}{ccc} 2 & \quad 4 & \quad 3 \end{array} & \end{array}$$

→ global conductivity matrix assembly:

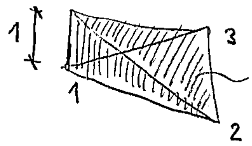
$$\underline{k} = \begin{array}{cc} \begin{bmatrix} 5.3125 & -0.625 & -4.6875 & 0 \\ -0.625 & 11.25 & -0.625 & -10 \\ -4.6875 & -0.625 & 5.9375 & -0.625 \\ 0 & -10 & -0.625 & 10.625 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \\ \begin{array}{cccc} 1 & \quad 2 & \quad 3 & \quad 4 \end{array} & \end{array}$$

- element source matrix: $\underline{f}^e_\Omega = \int_{\Omega^e} (\underline{N}^e)^T \bar{Q} \, d\Omega$ where the triangular shape functions are:

$$N_1^e = \frac{1}{2A^e} \left(x_2^e y_3^e - x_3^e y_2^e + (y_2^e - y_3^e) x + (x_3^e - x_2^e) y \right)$$
$$N_2^e = \frac{1}{2A^e} \left(x_3^e y_1^e - x_1^e y_3^e + (y_3^e - y_1^e) x + (x_1^e - x_3^e) y \right)$$
$$N_3^e = \frac{1}{2A} \left(x_1^e y_2^e - x_2^e y_1^e + (y_1^e - y_2^e) x + (x_2^e - x_1^e) y \right)$$

and $\underline{N}^e = [N_1^e \;\; N_2^e \;\; N_3^e]$

- if $\bar{Q}$ is constant, we get $\int_{\Omega^e} (\underline{N}^e)^T \bar{Q} \, d\Omega = \frac{\bar{Q} A^e}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$



Volume = $\frac{A^e}{3}$

- element source matrices are given by:

$$\underline{f}^{(1)}_\Omega = \frac{\bar{Q} A^{(1)}}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{array}{cc} \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \end{array}$$

$$\underline{f}_{\Omega}^{(2)} = \frac{\bar{Q} A^{(2)}}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{array}{c} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \end{array} \begin{array}{c} (2) \\ (4) \\ (3) \end{array}$$

$\Rightarrow$ direct assembly $\rightarrow$ global source matrix

$$\underline{f}_{\Omega} = \begin{bmatrix} 2 \\ 2+1 \\ 2+1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 3 \\ 1 \end{bmatrix} \begin{array}{c} (1) \\ (2) \\ (3) \\ (4) \end{array}$$

- element boundary flux matrix : $\underline{f}_{\Gamma}^{e} = -\int_{\Gamma_q^e} (\underline{N}^e)^T \bar{q} \, d\Gamma$

  - element 1 has 2 edges on the essential boundary (where $\bar{T}$ is prescribed) and 1 interior edge $\Rightarrow \Gamma_q^{(1)} = \emptyset$

  - element 2 : $\bar{q} = 20 \, Wm^{-1}$ on one edge contributes to the boundary flux matrix :

$$\underline{N}^{(2)}\Big|_{y=1} = \begin{bmatrix} 0 \\ 0.5x \\ -0.5x + 1.0 \end{bmatrix}$$

$$\underline{f}_{\Gamma}^{(2)} = -20 \int_{x=0}^{x=2} \begin{bmatrix} 0 \\ 0.5x \\ -0.5x + 1.0 \end{bmatrix} dx = \begin{bmatrix} 0 \\ -20 \\ -20 \end{bmatrix} \begin{array}{c} (2) \\ (4) \\ (3) \end{array}$$

total heat energy is distributed equally between nodes 3 and 4

$\Rightarrow$ global flux matrix by direct assembly : $\underline{f}_{\Gamma} = \begin{bmatrix} 0 \\ 0 \\ -20 \\ -20 \end{bmatrix} \begin{array}{c} (1) \\ (2) \\ (3) \\ (4) \end{array}$

- $\underline{f} = \underline{f}_{\Omega} + \underline{f}_{\Gamma} + \underline{r} = \begin{bmatrix} 2 \\ 3 \\ -17 \\ -19 \end{bmatrix} + \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ 0 \end{bmatrix}$

- the resulting system of equations is given by:

$$\underbrace{\left[\begin{array}{ccc|c} 5.3125 & -0.625 & -4.6875 & 0 \\ -0.625 & 11.25 & -0.625 & -10 \\ -4.6875 & -0.625 & 5.9375 & -0.625 \\ \hline 0 & -10 & -0.625 & 10.625 \end{array}\right]}_{\underline{K}} \underbrace{\left[\begin{array}{c} 0 \\ 0 \\ 0 \\ \hline T_4 \end{array}\right]}_{\underline{T}} = \underbrace{\left[\begin{array}{c} r_1+2 \\ r_2+3 \\ r_3-17 \\ \hline -19 \end{array}\right]}_{\underline{f}}$$

$$\underline{T}(F) = \underline{K}^{-1}(F,F)\,\underline{f}(F): \quad T_4 = -\frac{19}{10.625} = -1.788\ ^\circ C$$

- element temperature matrices:

$$\underline{T}^{(1)} = \begin{array}{c} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \qquad \underline{T}^{(2)} = \begin{bmatrix} 0 \\ -1.788 \\ 0 \end{bmatrix} \begin{array}{c} 2 \\ 4 \\ 3 \end{array}$$

$$\Downarrow$$

the flux matrices are (from $\underline{q} = -\underline{D}\,\underline{\nabla}\,T$):

$$\underline{q}^{(1)} = -\lambda \underline{B}^{(1)} \underline{T}^{(1)} = -5 \cdot 0.5 \cdot \begin{bmatrix} -0.5 & 1 & -0.5 \\ -2 & 0 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\underline{q}^{(2)} = -\lambda \underline{B}^{(2)} \underline{T}^{(2)} = 5 \cdot \begin{bmatrix} 0 & 0.5 & -0.5 \\ -2 & 2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -1.788 \\ 0 \end{bmatrix} = \begin{bmatrix} 4.47 \\ 17.88 \end{bmatrix}$$