FEM Formulation for Vector Field Problems - Linear Elasticity

- used for industrial stress analysis (under operating conditions most products are not expected to undergo material or geometric non-linearities)

- assumptions: a) deformations are small (max 10^{-2} of body size $\Rightarrow$ error $< 1\%$)
  b) behavior of material is linear
  c) dynamic effects are neglected
  d) no gaps or overlaps occur during deformation

Kinematics (2D)

- displacement vector consists of 2 components:

$$\underline{u} = \begin{bmatrix} u_x \\ u_y \end{bmatrix}$$

- extensions under loading, causing changes in length (= strains) are

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} \quad , \quad \varepsilon_{xy} = \frac{\partial u_y}{\partial y}$$

- shear strain γ_{xy} represents the change in angle between 2 unit vectors:

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$$

- sometimes so called tensorial shear strain is used instead, defined as $\varepsilon_{xy} = \frac{1}{2}\gamma_{xy}$
- the rotations are negligible for infinitesimal displacement fields

- for FEM purposes the strains are usually arranged into a column matrix:

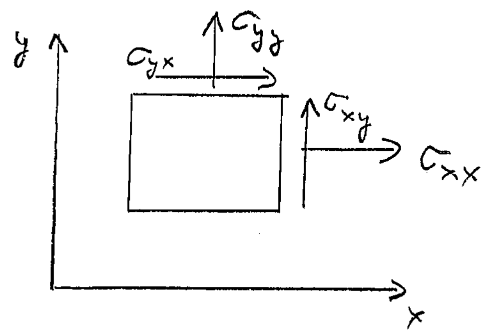
$$\underline{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{xy} \\ \gamma_{xy} \end{bmatrix} = \underline{\nabla}_s \underline{u}$$

where $\underline{\nabla}_s = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}$ is the "symmetric gradient matrix operator"

and $\underline{u} = \begin{bmatrix} u_x \\ u_y \end{bmatrix}$ is the displacement

Stress and traction

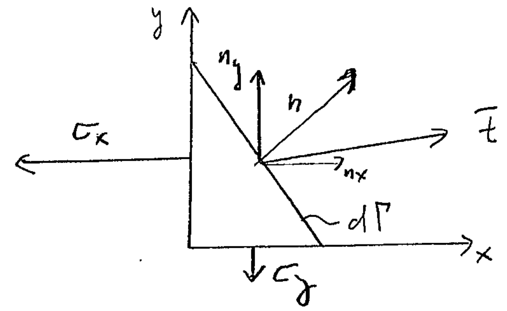
- in 2D the stresses correspond to the forces per unit area acting on planes normal to x- and y-axes + there are shear stresses σ_{xy} and σ_{yx} (they are equal from the moment equilibrium)



- the stresses are arranged in a matrix form similarly to strains:

$$\underline{\sigma} = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix}$$

- tractions are forces per unit area applied on a body surface (× stresses provide only the information about the point in the structure)



- from equilibrium we get $\underline{t} = \underline{\sigma}\, \underline{n}$ ← must be in tensorial arrangement

$$\begin{bmatrix} t_x \\ t_y \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} n_x \\ n_y \end{bmatrix}$$

Equilibrium in the body

- infinitezimal unit cube (square) with body forces included:

$$\rightarrow \quad \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + b_x = 0$$

$$\uparrow \quad \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + b_y = 0$$

- in tensorial form : $\underline{\nabla} \cdot \underline{\sigma} + \underline{b} = \underline{0}$
- in matrix form : $\underline{\nabla_s}^T \underline{\sigma} + \underline{b} = \underline{0}$

↳ transpose of strain-displacement operator (⇒ "self-adjoint" or "symmetric" systems of PDEs)

⇒ symmetry of the discrete equations (stiffness and conductivity matrices)

Constitutive equations

- relations between stresses and strains
- elastic behavior yields linear equations
- linear elasticity is the simplest case: $\sigma = E\varepsilon$ in 1D
- in 2D: $\underline{\sigma} = \underline{D}\,\underline{\varepsilon}$ (= "generalized Hooke's law")

- $\underline{D}$ (= "stiffness matrix") is always symmetric, pos.-def. matrix due to energy consideration
  - different for plane stress (a plate) or plane strain (a tunnel) case ⇒ this is due to the simplification from 3D
  - characterized for isotropic materials by 2 constants (e.g. E and ν), independent of coordinate system

- plane stress (thin plate, no stress on z-plane surface):

$$\underline{D} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

- plane strain (long structure, constrained z-deformation):

$$\underline{D} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

- if $\nu = 0.5$ → incompressible material (special elements must be used in plane strain)

Strong form of 2D linear elasticity

- equilibrium: $\underline{\nabla}_s^T \underline{\sigma} + \underline{b} = \underline{0}$

kinematics: $\underline{\varepsilon} = \underline{\nabla}_s \underline{u}$

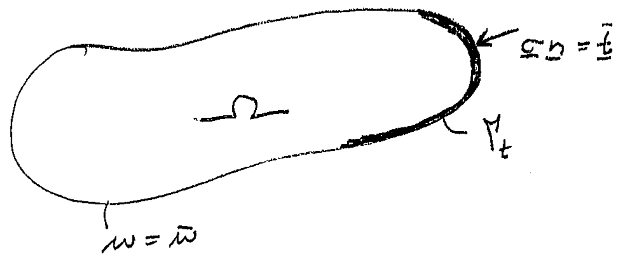
constitutive equations: $\underline{\sigma} = \underline{D}\,\underline{\varepsilon}$

Boundary conditions:

a) prescribed traction on Γ_t: $\underline{\underline{\sigma}}\,\underline{n} = \underline{\bar{t}}$

$$\underbrace{\begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}}_{N/m^2} \begin{bmatrix} \cdot \\ \cdot \end{bmatrix} = \underbrace{\begin{bmatrix} \cdot \\ \cdot \end{bmatrix}}_{N/m^2}$$

b) displacement on Γ_u: $\underline{u} = \underline{\bar{u}}$
= essential BCs, must be satisfied by the displacement field



$$\Gamma_u \cap \Gamma_t = \emptyset$$
$$\Gamma_u \cap \Gamma_t = \Gamma$$

Weak form for 2D linear elasticity

- using the Gauss theorem, the weak form can be derived using the Principle of Virtual Work as in case of 1D elasticity:

$$\underbrace{\int_\Omega \delta\underline{\underline{\varepsilon}} : \underline{\underline{\sigma}} \, d\Omega}_{\substack{\text{work of true}\\ \text{stresses on virtual}\\ \text{deformations}\\ \text{(internal energy)}}} = \int_{\Gamma_t} \delta\underline{u} \cdot \underline{\bar{t}} \, d\Gamma + \cancel{\int_{\Gamma_u} \delta\underline{u} \cdot \underline{t}_R \, d\Gamma} + \int_\Omega \delta\underline{u} \cdot \underline{b} \, d\Omega$$

$\emptyset$, vanishes because $\delta\underline{u}$ vanishes at Γ_u

$\Rightarrow$ weak form: "Find $\underline{u} \in V$ such that

$$\int_\Omega (\underline{\nabla}_s \delta\underline{u})^T \underline{D}\, \underline{\nabla}_s \underline{u} \, d\Omega = \int_{\Gamma_t} (\delta\underline{u})^T \underline{\bar{t}} \, d\Gamma + \int_\Omega (\delta\underline{u})^T \underline{b} \, d\Omega$$

$\forall\, \delta\underline{u} \in V_0$ (smooth functions with $\delta\underline{u} = 0$ on Γ_u)

FEM discretization

- $\underline{w} = \begin{bmatrix} w_x \\ w_y \end{bmatrix}$ ... usually both approximated by the same shape function

  ↳ 2 degrees of freedom per node $\underline{d} = \begin{bmatrix} w_{x1} \\ w_{y1} \\ w_{x2} \\ \vdots \end{bmatrix}$

- approximation of trial solutions and weight functions:

$$\underline{w}(x,y) \approx \underline{w}^e(x,y) = \underline{N}(x,y)\,\underline{d}^e \qquad (x,y) \in \Omega^e$$
$$\delta\underline{w}^T(x,y) \approx (\delta\underline{w}^e)^T(x,y) = (\delta\underline{d}^e)^T\,(\underline{N}^e(x,y))^T \qquad (x,y) \in \Omega^e$$

where the element shape function is:

$$\underline{N}^e = \begin{bmatrix} N_1^e & 0 & N_2^e & 0 & \dots \\ 0 & N_1^e & 0 & N_2^e & \dots \end{bmatrix}$$

approximates w_{x2}

approximates w_{y1}

and the nodal displacements:

$$\underline{d}^e = \begin{bmatrix} w_{x1}^e & w_{y1}^e & w_{x2}^e & \dots \end{bmatrix}$$
$$\delta\underline{d}^e = \begin{bmatrix} \delta w_{x1}^e & \delta w_{y1}^e & \dots \end{bmatrix}$$

- the integral over the domain Ω is calculated as a sum of integrals over the element domains Ω^e:

$$\sum_{e=1}^{n_{el}} \left(\int_{\Omega^e} (\underline{\nabla}_s\,\delta\underline{w}^e)^T\,\underline{D}^e\,\underline{\nabla}_s\,\underline{w}^e\,d\Omega - \int_{\Gamma_t^e} (\delta\underline{w}^e)^T\,\underline{\bar{t}}\,d\Gamma - \int_{\Omega^e} (\delta\underline{w}^e)^T\,\underline{b}\,d\Omega \right) = 0$$

- the strains in Ω^e: $\underline{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{bmatrix} \approx \underline{\varepsilon}^e = \underline{\nabla}_s\,\underline{w}^e = \underline{\nabla}_s\,\underline{N}^e\underline{d}^e = \underline{B}^e\underline{d}^e$

where $\underline{B}^e \equiv \underline{\nabla}_s\,\underline{N}^e = \begin{bmatrix} \frac{\partial N_1^e}{\partial x} & 0 & \frac{\partial N_2^e}{\partial x} & 0 & \frac{\partial N_3^e}{\partial x} \\ 0 & \frac{\partial N_1^e}{\partial y} & \ddots & & \dots \\ \frac{\partial N_1^e}{\partial y} & \frac{\partial N_1^e}{\partial x} & & & \end{bmatrix}$

and therefore $\left(\nabla_s \delta \underline{d}^e\right)^T = \left(\underline{B}^e \delta \underline{d}^e\right)^T = \left(\delta \underline{d}^e\right)^T \left(\underline{B}^e\right)^T$

- since $\underline{d}^e = \underline{L}^e \underline{d}$ and $\left(\delta \underline{d}^e\right)^T = \left(\delta \underline{d}\right)^T \left(\underline{L}^e\right)^T$, we obtain:

$$(\delta \underline{d})^T \left(\sum_{e=1}^{n_{el}} (\underline{L}^e)^T \left(\int_{\Omega^e} (\underline{B}^e)^T \underline{D}^e \underline{B}^e \, d\Omega \, \underline{L}^e \underline{d} - \int_{\Gamma_t^e} (\underline{N}^e)^T \underline{\bar{t}} \, d\Gamma - \int_{\Omega^e} (\underline{N}^e)^T \underline{\bar{b}} \, d\Omega \right) \right) = 0$$

$\forall\, \delta \underline{d}$ on Γ_t (and vanishing on the essential boundary)

- element stiffness matrix : $\underline{k}^e = \int_{\Omega^e} (\underline{B}^e)^T \underline{D}^e \underline{B}^e \, d\Omega$

- element external force matrix : $\underline{f}^e = \underbrace{\int_{\Omega^e} (\underline{N}^e)^T \underline{\bar{b}} \, d\Omega}_{\underline{f}^e_\Omega} + \underbrace{\int_{\Gamma_t^e} (\underline{N}^e)^T \underline{\bar{t}} \, d\Gamma}_{\underline{f}^e_\Gamma}$

body force matrix ↗ boundary force matrix ↗

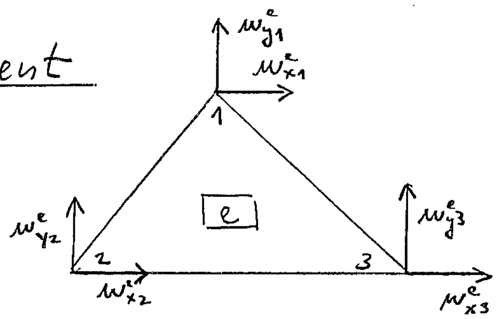
→ discretized weak form:

$$(\delta \underline{d})^T \left(\left(\sum_{e=1}^{n_{el}} (\underline{L}^e)^T \underline{k}^e \underline{L}^e \right) \underline{d} - \sum_{e=1}^{n_{el}} (\underline{L}^e)^T \underline{f}^e \right) = 0 \qquad \forall\, \delta \underline{d} \text{ vanishing on } \Gamma_u$$

$$\underline{k}\, \underline{d} - \underline{f} = \underline{0}$$

Three-node triangular element

- linear approximation of the displacement field → strains are constant in the element



- nodes must be numbered counterclockwise
- each nodes has 2 DOFs : $\underline{d}^e = \left[u_{x1} \; u_{y1} \; u_{x2} \; u_{y2} \; u_{x3} \; u_{y3} \right]^T$

- the displacement field is then expressed in the form of :

$$\begin{bmatrix} w_x \\ w_y \end{bmatrix}^e = \begin{bmatrix} N_1^e & 0 & N_2^e & 0 & N_3^e & 0 \\ 0 & N_1^e & 0 & N_2^e & 0 & N_3^e \end{bmatrix} \underline{d}^e$$

- applying the symmetric gradient operator $\underline{\nabla}_s$ gives

$$\underline{\varepsilon}^e = \underbrace{\underline{\nabla}_s \, \underline{N}^e}_{\underline{B}^e} \underline{d}$$

where $$\underline{B}^e = \frac{1}{2A^e} \begin{bmatrix} y_2^e - y_3^e & 0 & y_3^e - y_1^e & 0 & y_1^e - y_2^e & 0 \\ 0 & x_3^e - x_2^e & 0 & x_1^e - x_3^e & 0 & x_2^e - x_1^e \\ x_3^e - x_2^e & y_2^e - y_3^e & x_1^e - x_3^e & y_3^e - y_1^e & x_2^e - x_1^e & y_1^e - y_2^e \end{bmatrix}$$

not a function of x and y → constant in the element

- <u>element stiffness</u> is given by $\underline{K}^e = \int_{\Omega^e} (\underline{B}^e)^T \underline{D}^e \, \underline{B}^e \, d\Omega$ and

if the material properties are constant the stiffness for a unit thishness is $\underline{K}^e = A^e (\underline{B}^e)^T \underline{D}^e \, \underline{B}^e$

- <u>element body force matrix</u> $\underline{f}_\Omega^e = \int_\Omega (\underline{N}^e)^T \, \underline{b} \, d\Omega$ can be evaluated by a :

i) direct numerical integration, or
ii) interpolating $\underline{b}$ usually with a linear function and integrating in a closed form

- in the triangular element we interpolate the body force in the element by the linear shape functions :

$$\underline{b} = \begin{bmatrix} b_x \\ b_y \end{bmatrix} = \sum_{\underset{\uparrow \text{nodes}}{i=1}}^{3} (N_i)^T \begin{bmatrix} b_{xi} \\ b_{yi} \end{bmatrix}$$

x- and y-component of body forces at node i and after the substitution into $\underline{f}_\Omega^e = \int_\Omega (\underline{N}^e)^T \bar{\underline{b}}\, d\Omega$ we obtain:

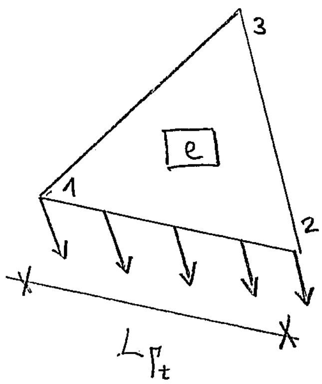
$$\underline{f}_\Omega^e = \frac{A^e}{12}\begin{bmatrix} 2b_{x1} + b_{x2} + b_{x3} \\ 2b_{y1} + b_{y2} + b_{y3} \\ b_{x1} + 2b_{x2} + b_{x3} \\ b_{y1} + 2b_{y2} + b_{y3} \\ b_{x1} + b_{x2} + 2b_{x3} \\ b_{y1} + b_{y2} + 2b_{y3} \end{bmatrix}$$

- if b_x and b_y are the same in all nodes, we obtain:

$$\underline{f}_\Omega^e = \frac{A^e}{3}\begin{bmatrix} b_x \\ b_y \\ b_x \\ b_y \\ b_x \\ b_y \end{bmatrix}$$ ← all nodes get the same portion

- <u>the boundary force matrix</u> $\underline{f}_\Gamma^e = \int_{\Gamma_t^e} (\underline{N}^e)^T \bar{\underline{t}}\, d\Gamma$ is

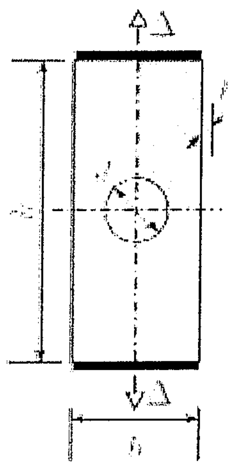
in case of the constant traction per element:



$$\bar{\underline{t}} = \begin{bmatrix} \bar{t}_x \\ \bar{t}_y \end{bmatrix}$$

$$\underline{f}_\Gamma^e = \frac{L_{\Gamma_t}}{2}\begin{bmatrix} \bar{t}_x \\ \bar{t}_y \\ \bar{t}_x \\ \bar{t}_y \\ 0 \\ 0 \end{bmatrix}$$

Example 1: displacement field in the perforated plate subjected to prescribed displacement at the boundary



$b = 100$ mm
$h = 600$ mm
$t = 1$ mm
$\Delta = 0.028$ mm
$E = 70$ GPa
$\nu = 0.25$

MATLAB code: (what follows after the % symbol is just a comment)

```
%% Load nodal data
data=load('exercise05_3_nodes.dat');
xy=data(:,1:2);
num_nodes=length(data(:,1));
```

- loads the data from a file ....dat containing the information about the geometry :

# of node	x	y
...	...	...

```
%% Load element data
data=load('exercise05_3_elements.dat');
nodes=data(:,1:3);
E=data(:,4);
nu=data(:,5);
t=data(:,6);
num_elems=length(data(:,1));
```

- loads the data about the elements

# of element	node 1	node 2	node 3	E	ν	t
...	...	...	...	...	...	...

```
%% Load boundary conditions
data=load('exercise05_3_dirichlet.dat');
r=zeros(2*num_nodes,1);

p=data(:,1); % Dofs with prescribed displacement
r(p)=data(:,2);
u=setdiff(1:length(r),p);
```

essential displacements $\underline{d}(E)$

- loads the information about nodes – if there is a prescribed displacement

- direct global assembly of $\underline{K}$

```
%% Assemble stiffness matrix and RHS vector
K=sparse(2*num_nodes,2*num_nodes);

for e=1:num_elems
    id=2*nodes(e,[1,1,2,2,3,3])-[1,0,1,0,1,0];
    Ke=T3_stiffness_2D(xy(nodes(e,:),:), E(e), nu(e), t(e));
    K(id,id)=K(id,id)+Ke;
end
```

function for computation of element stiffness matrix

shows non-zero entries in $\underline{K}$

```
spy(K); % Show the sparsity pattern
disp('Sparsity pattern of stiffness matrix...');
pause;

r(u)=-K(u,u)\(K(u,p)*r(p));
```

→ solves the system $\underline{d}_{(F)} = \underline{K}^{-1}_{(F,F)}\left(-\underline{K}_{(F,E)}\,\underline{d}_{(E)}\right)$

```
T3show(xy,nodes,r,100);
```

↳ plot the results using the function T3show

```
%% Functions:
  % Calculates the entries needed for the evaluation of B-matrix
    function [b,c,A]=AreaCoordinates(xy)
    A = .5*det([1,1,1; xy']);
    b = [xy(2,2)-xy(3,2), xy(3,2)-xy(1,2), xy(1,2)-xy(2,2)];
    c = [xy(3,1)-xy(2,1), xy(1,1)-xy(3,1), xy(2,1)-xy(1,1)];
    end
```

$$\underline{B}^e = \begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix}$$

```
  % Evalueates the stiffness D-matrix
    function D=PlaneStressMatrix(E,nu)
      D=[1,nu,0;
         nu,1,0;
         0,0,.5*(1-nu)]*E/(1-nu*nu);
    end
```

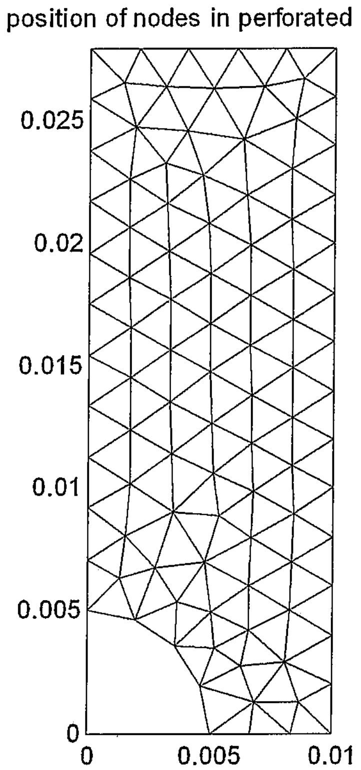
$$\underline{D} = \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{(1-\nu)}{2} \end{bmatrix} \cdot \frac{E}{(1-\nu^2)}$$

```
  % Computes stiffness matrix of CST element
    function [Ke]=T3_stiffness_2D(xye,Ee,nue,te)
      [b,c,A]=AreaCoordinates(xye);
      De=PlaneStressMatrix(Ee,nue);
      Be=[b(1),0,b(2),0,b(3),0;
          0,c(1),0,c(2),0,c(3);
          c(1),b(1),c(2),b(2),c(3),b(3)];
      Ke=A*te*(Be')*De*Be;
    end
```

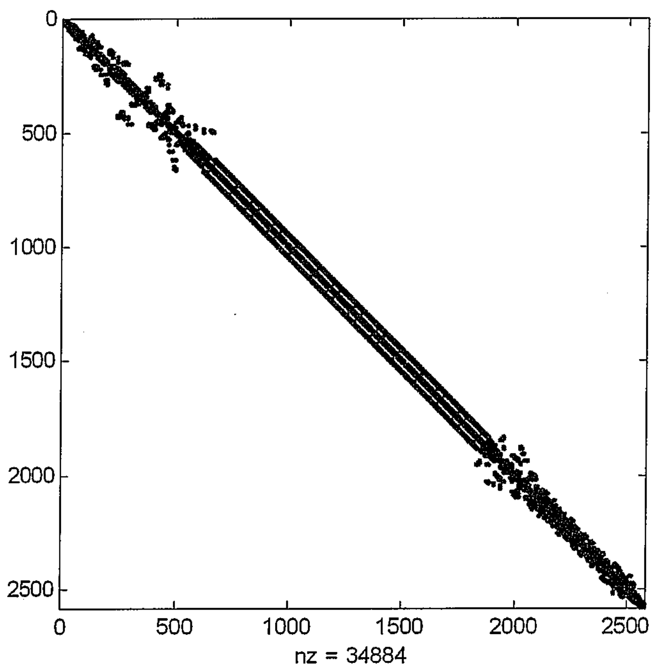
$$\underline{k}^e = \int_{\Omega^e} (\underline{B}^e)^T\,\underline{D}\,\underline{B}^e\,d\Omega^e = A^e t^e (\underline{B}^e)^T \underline{D}^e \underline{B}^e$$

↑ thickness of the element

Undeformed mesh:

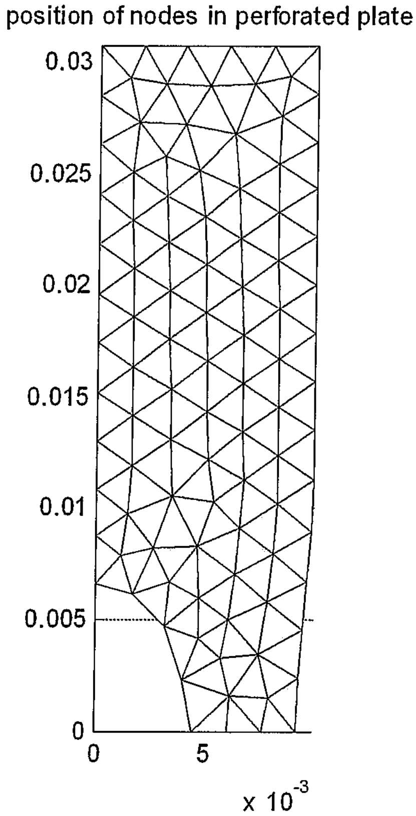


Sparsity pattern of the global stiffness matrix:



Deformed meshes:

a) "big" elements



b) refined mesh