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# ON FIRST STRAIN-GRADIENT THEORIES IN LINEAR ELASTICITY

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**Abstract**—This paper contains a study of the linear theory of elasticity in which the potential energy-density depends on the gradient of the strain in addition to the strain.

In the first part of the paper, three forms of the theory are compared and the relations connecting the stresses in the three forms and the boundary conditions in the three forms are derived. In the second part of the paper, ambiguities in the form of the moment-equation of equilibrium and the definition of couple-stress are resolved by a derivation based on conservation principles rather than the variational principles employed previously.

## INTRODUCTION

THE purpose of this paper is to clarify certain aspects of the linear theory of elasticity in which the potential energy of deformation is a function of the six components of the strain and the eighteen components of the gradient of the strain.

Stress-equations of equilibrium, constitutive equations and boundary conditions of the "strain-gradient theory" were first given in a general, non-linear form by Toupin [1, §7]. Subsequently [2], linear versions of the theory were given in three forms—distinguished by different groupings of the eighteen additional variables in the potential energy-density: I, the eighteen components of the second gradient of the displacement; II, the eighteen components of the first gradient of the strain; III, the eight components of the gradient of the rotation and the ten components of the fully symmetric part of the second gradient of the displacement (or of the gradient of the strain). The components in the second and third sets are simply linear combinations of those in the first. The third form of the theory is the most convenient one for reduction to the theory in which the potential energy-density is a function of the strain and the gradient of the rotation [1, 3–6].

In [2], the three forms of the strain-gradient theory were shown to lead to the same displacement-equations of motion for isotropic materials. However, the general identity of the stress-equations of motion and the general relations among the stresses in the three forms and among the traction boundary conditions for the three forms were not exhibited. These results are derived in the first part of the present paper.

The second part of the paper is concerned with the moment equation and the couple-stress. In the derivation of the equations of the strain-gradient theory by the variational methods employed previously, the moment equation does not appear explicitly. Although a moment equation can be deduced subsequently from the condition of invariance of the potential energy-density in a rigid rotation of the deformed body [7], the equation can be produced in a variety of forms. As a result, the identification of the couple-stress is uncertain to a constant factor. To clarify the situation, the complete equations of the linear strain-gradient theory are rederived, here, starting from principles of conservation of linear momentum, angular momentum and energy. The moment equation and couple-stress thereby are displayed without ambiguity. A theorem of uniqueness of solutions leaves the spherical part of the couple-stress undetermined just as in the theory in which the potential energy-density depends on the strain and the gradient of the rotation.

## 1. KINEMATIC VARIABLES

The kinematic variables to be employed are defined in terms of derivatives of components of displacement as follows:

$$
\begin{aligned}
\varepsilon_{ij} &= \tfrac{1}{2}(u_{j,i}+u_{i,j}) = u_{(j,i)} = \varepsilon_{ji} = \text{strain},\\
\omega_{ij} &= \tfrac{1}{2}(u_{j,i}-u_{i,j}) = u_{[j,i]} = -\omega_{ji} = \text{rotation},\\
w_i &= \tfrac{1}{2}e_{ijk}u_{k,j} = \text{vector rotation},\\
\tilde{\kappa}_{ijk} &= u_{k,ij} = \tilde{\kappa}_{jik} = \text{second gradient of displacement},\\
\hat{\kappa}_{ijk} &= \tfrac{1}{2}(u_{k,ji}+u_{j,ki}) = u_{(k,j)i} = \hat{\kappa}_{ikj} = \text{gradient of strain},\\
\bar{\kappa}_{ij} &= \tfrac{1}{2}e_{jlk}u_{k,li} = \text{gradient of rotation } (\bar{\kappa}_{ii} = 0),\\
\bar{\bar{\kappa}}_{ijk} &= \tfrac{1}{3}(u_{k,ij}+u_{i,jk}+u_{j,ki}) = u_{(k,ij)}\\
&= \bar{\bar{\kappa}}_{jki} = \bar{\bar{\kappa}}_{kij} = \bar{\bar{\kappa}}_{kji} = \text{symmetric part of } \tilde{\kappa}_{ijk} \text{ or } \hat{\kappa}_{ijk},
\end{aligned}
\tag{1.1}
$$

where $e_{ijk}$ is the alternator.

The following relations among the variables are found by eliminating the displacement from the definitions (1.1):

$$
\begin{aligned}
\omega_{ij} &= e_{ijk}w_k,\\
w_i &= \tfrac{1}{2}e_{ijk}\omega_{jk},
\end{aligned}
\tag{1.2}
$$

$$
\begin{aligned}
\tilde{\kappa}_{ijk} &= \hat{\kappa}_{ijk}+\hat{\kappa}_{jki}-\hat{\kappa}_{kij} = \bar{\bar{\kappa}}_{ijk}+\tfrac{2}{3}\bar{\kappa}_{il}e_{ljk}+\tfrac{2}{3}\bar{\kappa}_{jl}e_{lik},\\
\hat{\kappa}_{ijk} &= \bar{\bar{\kappa}}_{ijk}-\tfrac{1}{3}\bar{\kappa}_{jl}e_{kil}-\tfrac{1}{3}\bar{\kappa}_{kl}e_{jil} = \tfrac{1}{2}(\tilde{\kappa}_{ijk}+\tilde{\kappa}_{ikj}),\\
\bar{\kappa}_{ij} &= w_{j,i} = \tfrac{1}{2}e_{jkl}\omega_{kl,i} = \tfrac{1}{2}\tilde{\kappa}_{ilk}e_{jlk} = \hat{\kappa}_{lik}e_{jlk},\\
\bar{\bar{\kappa}}_{ijk} &= \tfrac{1}{3}(\tilde{\kappa}_{ijk}+\tilde{\kappa}_{jki}+\tilde{\kappa}_{kij}) = \tfrac{1}{3}(\hat{\kappa}_{ijk}+\hat{\kappa}_{jki}+\hat{\kappa}_{kij}).
\end{aligned}
\tag{1.3}
$$

## 2. EULER EQUATIONS AND NATURAL BOUNDARY CONDITIONS

The derivations of the three forms of the strain-gradient theory, given in [2], are summarized, here, with some minor alterations.

Hamilton's principle is written for independent variations $\delta u_i$ between fixed limits of $u_i$ at times $t_0$ and $t_1$:

$$\delta\int_{t_0}^{t_1}(\mathscr{T}-\mathscr{W})\mathrm{d}t+\int_{t_0}^{t_1}\delta\mathscr{W}_1\,\mathrm{d}t = 0, \tag{2.1}$$

where $\mathscr{T}$ and $\mathscr{W}$ are the total kinetic and potential energies in a volume V:

$$\mathscr{T} = \int_{\mathrm{V}}\mathrm{T}\mathrm{d}\mathrm{V}, \qquad \mathscr{W} = \int_{\mathrm{V}}\mathrm{W}\mathrm{d}\mathrm{V}$$

and $\delta\mathscr{W}_1$ is the variation of work done by external forces.

The kinetic energy-density is taken to be

$$\mathrm{T} = \tfrac{1}{2}\rho\dot{u}_i\dot{u}_i. \tag{2.2}$$

In [2], velocity-gradient terms were included in the kinetic energy-density. Such terms are appropriate if the strain-gradient equations are regarded as a low frequency approximation to the equations of a certain elastic material with a deformable micro-structure [2]. However, if the strain-gradient equations are viewed as a moderately long wavelength limit of the finite difference equations of a simple, crystal lattice, the velocity-gradient terms must be omitted. The latter view is adopted, here, in order to avoid carrying along complicated terms which are not germane to the present study.

Three forms are considered for the potential energy-density:

$$\mathrm{W} = \tilde{\mathrm{W}}(\varepsilon_{ij},\tilde{\kappa}_{ijk}) = \hat{\mathrm{W}}(\varepsilon_{ij},\hat{\kappa}_{ijk}) = \overline{\mathrm{W}}(\varepsilon_{ij},\bar{\kappa}_{ij},\bar{\bar{\kappa}}_{ijk}) \tag{2.3}$$

—all for the same displacement field. In the case of isotropic materials,

$$\begin{aligned}\tilde{\mathrm{W}} &= \tfrac{1}{2}\lambda\varepsilon_{ii}\varepsilon_{jj}+\mu\varepsilon_{ij}+\tilde{a}_1\tilde{\kappa}_{iik}\tilde{\kappa}_{kjj}+\tilde{a}_2\tilde{\kappa}_{ijj}\tilde{\kappa}_{ikk}\\ &\quad+\tilde{a}_3\tilde{\kappa}_{iik}\tilde{\kappa}_{jjk}+\tilde{a}_4\kappa_{ijk}\tilde{\kappa}_{ijk}+\tilde{a}_5\tilde{\kappa}_{ijk}\kappa_{kji},\end{aligned} \tag{2.4}$$

$$\begin{aligned}\hat{\mathrm{W}} &= \tfrac{1}{2}\lambda\varepsilon_{ii}\varepsilon_{jj}+\mu\varepsilon_{ij}\varepsilon_{ij}+\hat{a}_1\hat{\kappa}_{iik}\hat{\kappa}_{kjj}+\hat{a}_2\hat{\kappa}_{ijj}\hat{\kappa}_{ikk}\\ &\quad+\hat{a}_3\hat{\kappa}_{iik}\hat{\kappa}_{jjk}+\hat{a}_4\hat{\kappa}_{ijk}\hat{\kappa}_{ijk}+\hat{a}_5\hat{\kappa}_{ijk}\hat{\kappa}_{kji},\end{aligned} \tag{2.5}$$

where

$$\begin{aligned}\hat{a}_1 &= 2\tilde{a}_1-4\tilde{a}_3, \quad \hat{a}_2 = -\tilde{a}_1+\tilde{a}_2+\tilde{a}_3,\\ \hat{a}_3 &= 4\tilde{a}_3, \quad \hat{a}_4 = 3\tilde{a}_4-\tilde{a}_5, \quad \hat{a}_5 = -2\tilde{a}_4+2\tilde{a}_5,\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}\overline{\mathrm{W}} &= \tfrac{1}{2}\lambda\varepsilon_{ii}\varepsilon_{jj}+\mu\varepsilon_{ij}\varepsilon_{ij}+2\bar{\bar{a}}_1\bar{\kappa}_{ij}\bar{\kappa}_{ij}+2\bar{\bar{a}}_2\bar{\kappa}_{ij}\bar{\kappa}_{ji}\\ &\quad+\tfrac{3}{2}\bar{a}_1\bar{\bar{\kappa}}_{iij}\bar{\bar{\kappa}}_{kkj}+\bar{a}_2\bar{\bar{\kappa}}_{ijk}\bar{\bar{\kappa}}_{ijk}+\bar{f}e_{ijk}\bar{\kappa}_{ij}\bar{\bar{\kappa}}_{kll},\end{aligned} \tag{2.7}$$

where

$$\begin{aligned}18\bar{\bar{a}}_1 &= -2\hat{a}_1+4\hat{a}_2+\hat{a}_3+6\hat{a}_4-3\hat{a}_5, \quad 18\bar{\bar{a}}_2 = 2\hat{a}_1-4\hat{a}_2-\hat{a}_3,\\ 3\bar{a}_1 &= 2(\hat{a}_1+\hat{a}_2+\hat{a}_3), \quad \bar{a}_2 = \hat{a}_4+\hat{a}_5, \quad 3\bar{f} = \hat{a}_1+4\hat{a}_2-2\hat{a}_3,\end{aligned} \tag{2.8}$$

The variation of the work done by external forces is given a separate form for each form of W:

$$\delta\mathscr{W}_1 = \delta\tilde{\mathscr{W}}_1 = \delta\hat{\mathscr{W}}_1 = \delta\overline{\mathscr{W}}_1, \tag{2.9}$$

where, for $\tilde{W}$,

$$\delta\tilde{\mathscr{W}}_1 = \int_V F_k\delta u_k \mathrm{dV} + \int_S (\tilde{P}_k\delta u_k + \tilde{R}_k \mathrm{D}\delta u_k)\mathrm{dS} + \oint_C \tilde{E}_k\delta u_k \mathrm{ds}; \tag{2.10}$$

for $\hat{W}$,

$$\delta\hat{\mathscr{W}}_1 = \int_V F_k\delta u_k \mathrm{dV} + \int_S (\hat{P}_k\delta u_k + \hat{R}_k \mathrm{D}\delta u_k)\mathrm{dS} + \oint_C \hat{E}_k\delta u_k \mathrm{ds}; \tag{2.11}$$

and, for $\overline{W}$,

$$\delta\overline{\mathscr{W}}_1 = \int_V F_k\delta u_k \mathrm{dV} + \int_S [\overline{P}_k\delta u_k + \overline{Q}_k(\delta_{kj} - n_k n_j)\delta w_j + \overline{R}\delta\varepsilon_{nn}]\mathrm{dS} + \oint_C \overline{E}_k\delta u_k \mathrm{ds}; \tag{2.12}$$

where S is the bounding surface, $n_i$ are the components of the outward unit normal to S, C is an edge of S, s is the coordinate along C, D is the normal derivative:

$$\mathrm{D}\varphi = n_i\varphi_{,i}; \tag{2.13}$$

and

$$\varepsilon_{nn} = n_i n_j \varepsilon_{ij}, \tag{2.14}$$

In (2.12), $\bar{Q}_k(\delta_{kj} - n_k n_j)\delta w_j$ replaces $Q_k n_j \delta w_i e_{ijk}$ which was employed inadvertently in [2, (12.12)]. The change constitutes a rotation of ninety degrees about the normal to S.

As shown in [2], the three forms of the variational principle lead to the following three sets of Euler equations (stress-equations of motion) and natural (traction) boundary conditions:

I:

$$\tilde{\tau}_{jk,j} - \tilde{\mu}_{ijk,ij} + F_k = \rho\ddot{u}_k, \tag{2.15}$$

$$\begin{aligned} \tilde{P}_k &= n_j(\tilde{\tau}_{jk} - \tilde{\mu}_{ijk,i}) - \mathrm{D}_j(n_i\tilde{\mu}_{ijk}) + (\mathrm{D}_l n_l)n_i n_j \tilde{\mu}_{ijk}, \\ \tilde{R}_k &= n_i n_j \tilde{\mu}_{ijk}, \\ \tilde{E}_k &= s_p[n_l n_i \tilde{\mu}_{ijk}]e_{plj}, \end{aligned} \tag{2.16}$$

where the components of stress, $\tilde{\tau}_{ij}$, and double stress, $\tilde{\mu}_{ijk}$, are defined by

$$\tilde{\tau}_{ij} = \frac{\partial\tilde{W}}{\partial\varepsilon_{ij}} = \tilde{\tau}_{ji}, \tag{2.17}$$

$$\tilde{\mu}_{ijk} = \frac{\partial\tilde{W}}{\partial\tilde{\kappa}_{ijk}} = \tilde{\mu}_{jik}, \tag{2.18}$$

the $\mathrm{D}_i$ are the components of the surface gradient:

$$\mathrm{D}_i\varphi = \varphi_{,i} - n_i\mathrm{D}\varphi,$$

the $s_p$ are the components of the unit vector tangent to C and the bold face brackets in the third of (2.16) indicate that the enclosed quantity is the difference of its values, at C, on the two portions of S that intersect at C. Equations (2.15)–(2.18), without the acceleration term, are linear forms of Toupin's results [1, §7].

II:
$$\hat{\tau}_{jk,j}-\hat{\mu}_{ijk,ij}+F_k = \rho\ddot{u}_k, \tag{2.19}$$

$$\begin{aligned}\hat{P}_k &= n_j(\hat{\tau}_{jk}-\hat{\mu}_{ijk,i})-D_j(n_i\hat{\mu}_{ijk})+(D_l n_l)n_i n_j\hat{\mu}_{ijk},\\ \hat{R}_k &= n_i n_j\hat{\mu}_{ijk},\\ \hat{E}_k &= s_p[n_l n_i\hat{\mu}_{ijk}]e_{plj},\end{aligned} \tag{2.20}$$

where

$$\hat{\tau}_{ij} = \frac{\partial\hat{W}}{\partial\varepsilon_{ij}} = \hat{\tau}_{ji}, \tag{2.21}$$

$$\hat{\mu}_{ijk} = \frac{\partial\hat{W}}{\partial\hat{\kappa}_{ijk}} = \hat{\mu}_{ikj}, \tag{2.22}$$

III:
$$\bar{\tau}_{jk,j}-\tfrac{1}{2}\bar{\mu}_{il,ij}e_{ljk}-\bar{\bar{\mu}}_{ijk,ij}+F_k = \rho\ddot{u}_k, \tag{2.23}$$

$$\begin{aligned}\bar{P}_k &= n_j[\bar{\tau}_{jk}+\tfrac{1}{2}(\bar{\mu}_{li,l}-\bar{\mu}_{nn,i})e_{jik}-\bar{\bar{\mu}}_{ijk,i}]-(D_j-n_jD_ln_l)(n_i\bar{\bar{\mu}}_{ijk}+n_pn_qn_k\bar{\bar{\mu}}_{pqj}),\\ \bar{Q}_k &= n_i\bar{\mu}_{ij}(\delta_{jk}-n_jn_k)+2n_qn_in_j\bar{\bar{\mu}}_{ijp}e_{qpk},\\ \bar{R} &= n_in_jn_k\bar{\bar{\mu}}_{ijk},\\ \bar{E}_k &= s_p[\tfrac{1}{2}\delta_{pk}\bar{\mu}_{nn}+n_qn_i(\bar{\bar{\mu}}_{ijk}+n_kn_l\bar{\bar{\mu}}_{lij})e_{jpq}],\end{aligned} \tag{2.24}$$

where

$$\bar{\tau}_{ij} = \frac{\partial\bar{W}}{\partial\varepsilon_{ij}} = \bar{\tau}_{ji}, \tag{2.25}$$

$$\bar{\mu}_{ij} = \frac{\partial\bar{W}}{\partial\bar{\kappa}_{ij}},\quad \bar{\mu}_{ii} = 0, \tag{2.26}$$

$$\bar{\bar{\mu}}_{ijk} = \frac{\partial\bar{W}}{\partial\bar{\bar{\kappa}}_{ijk}} = \bar{\bar{\mu}}_{jki} = \bar{\bar{\mu}}_{kij} = \bar{\bar{\mu}}_{kji} \tag{2.27}$$

and $\bar{\mu}_{nn} = n_in_j\bar{\mu}_{ij}$.

In [2], $\bar{\mu}_{ij}$ was designated the "deviator of the couple-stress". If $u_{(k,ij)}$ is set equal to zero in $\bar{W}$, $\bar{\mu}_{ij}$ does, in fact, reduce to the deviator of the couple-stress tensor $\mu_{ij}$ which was defined in [6] by means of the angular momentum principle and appears in the angular momentum equation [6, (1.9)]. That $\bar{\mu}_{ij}$ remains the deviator of the couple-stress when $\bar{W}$ depends on $u_{(k,ij)}$ is verified in Section 5.

## 3. RELATIONS AMONG THE THREE FORMS

From (2.3) and the definitions (2.17), (2.21) and (2.25), it follows that the stresses, in the three forms of the theory, are the same:

$$\tilde{\tau}_{ij} = \hat{\tau}_{ij} = \bar{\tau}_{ij}. \tag{3.1}$$

To find the relations among the double stresses, we first form the relations

$$
\begin{aligned}
\tilde{\mu}_{ijk} &= \frac{\partial \hat{W}}{\partial \hat{\kappa}_{pqr}} \frac{\partial \hat{\kappa}_{pqr}}{\partial \tilde{\kappa}_{ijk}} = \frac{\partial \bar{W}}{\partial \bar{\bar{\kappa}}_{pqr}} \frac{\partial \bar{\bar{\kappa}}_{pqr}}{\partial \tilde{\kappa}_{ijk}} + \frac{\partial \bar{W}}{\partial \bar{\kappa}_{pq}} \frac{\partial \bar{\kappa}_{pq}}{\partial \tilde{\kappa}_{ijk}}, \\
\hat{\mu}_{ijk} &= \frac{\partial \tilde{W}}{\partial \tilde{\kappa}_{pqr}} \frac{\partial \tilde{\kappa}_{pqr}}{\partial \hat{\kappa}_{ijk}} = \frac{\partial \bar{W}}{\partial \bar{\bar{\kappa}}_{pqr}} \frac{\partial \bar{\bar{\kappa}}_{pqr}}{\partial \hat{\kappa}_{ijk}} + \frac{\partial \bar{W}}{\partial \bar{\kappa}_{pq}} \frac{\partial \bar{\kappa}_{pq}}{\partial \hat{\kappa}_{ijk}}, \\
\bar{\mu}_{ij} &= \frac{\partial \tilde{W}}{\partial \tilde{\kappa}_{pqr}} \frac{\partial \tilde{\kappa}_{pqr}}{\partial \bar{\kappa}_{ij}} = \frac{\partial \hat{W}}{\partial \hat{\kappa}_{pqr}} \frac{\partial \hat{\kappa}_{pqr}}{\partial \bar{\kappa}_{ij}}, \\
\bar{\bar{\mu}}_{ijk} &= \frac{\partial \tilde{W}}{\partial \tilde{\kappa}_{pqr}} \frac{\partial \tilde{\kappa}_{pqr}}{\partial \bar{\bar{\kappa}}_{ijk}} = \frac{\partial \hat{W}}{\partial \hat{\kappa}_{pqr}} \frac{\partial \hat{\kappa}_{pqr}}{\partial \bar{\bar{\kappa}}_{ijk}}.
\end{aligned}
\tag{3.2}
$$

Then we replace the derivatives of the energy, in (3.2), with the definitions (2.18), (2.22), (2.26) and (2.27) and we use the relations (1.3) in evaluating the derivatives of the kinematic quantities. The results are

$$\tilde{\mu}_{ijk} = \tfrac{1}{2}(\hat{\mu}_{ijk} + \hat{\mu}_{jik}) = \bar{\bar{\mu}}_{ijk} + \tfrac{1}{4}\bar{\mu}_{il}e_{ljk} + \tfrac{1}{4}\bar{\mu}_{jl}e_{lik}, \tag{3.3}$$

$$\hat{\mu}_{ijk} = \tilde{\mu}_{ijk} + \tilde{\mu}_{kij} - \tilde{\mu}_{jki} = \bar{\bar{\mu}}_{ijk} + \tfrac{1}{2}\bar{\mu}_{jl}e_{lik} + \tfrac{1}{2}\bar{\mu}_{kl}e_{lij}, \tag{3.4}$$

$$\bar{\mu}_{ij} = \tfrac{4}{3}\tilde{\mu}_{ipq}e_{jpq} = \tfrac{2}{3}(\hat{\mu}_{ipq} + \hat{\mu}_{piq})e_{jpq}, \tag{3.5}$$

$$\bar{\bar{\mu}}_{ijk} = \tfrac{1}{3}(\tilde{\mu}_{ijk} + \tilde{\mu}_{jki} + \tilde{\mu}_{kij}) = \tfrac{1}{3}(\hat{\mu}_{ijk} + \hat{\mu}_{jki} + \hat{\mu}_{kij}). \tag{3.6}$$

A useful, alternative form of the second of (3.3) is

$$\tilde{\mu}_{ijk} = \bar{\bar{\mu}}_{ijk} + \tfrac{1}{2}\bar{\mu}_{il}e_{ljk} + \tfrac{1}{4}\bar{\mu}_{kl}e_{lij}. \tag{3.7}$$

From (3.7) and the first of (3.3), we find

$$\tilde{\mu}_{ijk,ij} = \hat{\mu}_{ijk,ij} = \tfrac{1}{2}\bar{\mu}_{il,ij}e_{ljk} + \bar{\bar{\mu}}_{ijk,ij}. \tag{3.8}$$

Accordingly, the three stress-equations of motion, (2.15), (2.19) and (2.23), are the same and they lead to the same displacement-equations of motion—as shown in [2] for isotropic materials.

The relations among the boundary conditions may be obtained by equating coefficients of like variations in (2.10), (2.11) and (2.12). From (2.10) and (2.11), it follows immediately that

$$\tilde{P}_k = \hat{P}_k, \quad \tilde{R}_k = \hat{R}_k, \quad \tilde{E}_k = \hat{E}_k. \tag{3.9}$$

To find the corresponding relations between the first and third forms, we can proceed by noting that, in (2.10),

$$
\begin{aligned}
\tilde{R}_k D\delta u_k &= 2\tilde{R}_k n_j \delta w_l e_{ljk} + \tilde{R}_k n_j D_k \delta u_j + \tilde{R}_k n_k \delta \varepsilon_{nn}, \\
2\tilde{R}_k n_j \delta w_l e_{ljk} &= 2n_j \tilde{R}_k(\delta_{li} - n_l n_i)\delta w_i e_{ljk},
\end{aligned}
$$

and

$$
\begin{aligned}
\int_S \tilde{R}_k n_j D_k \delta u_j dS &= \int_S [D_k(\tilde{R}_k n_j \delta u_j) - D_k(\tilde{R}_k n_j)\delta u_j] dS \\
&= \int_S [(D_l n_l) n_k \tilde{R}_k n_j - D_k(\tilde{R}_k n_j)]\delta u_j dS + \oint_C s_l[n_i \tilde{R}_j n_k] e_{lij} \delta u_k ds
\end{aligned}
$$

by the surface divergence theorem. Thus, (2.10) takes the form

$$\delta\tilde{\mathscr{W}}_1 = \int_V F_k\delta u_k \mathrm{dV} + \int_S [\tilde{P}_k - (\mathrm{D}_j - n_j \mathrm{D}_l n_l)(\tilde{R}_j n_k)]\delta u_k \mathrm{dS}$$
$$+ \int_S 2n_j\tilde{R}_k e_{ljk}(\delta_{li} - n_l n_i)\delta w_i \mathrm{dS} + \int_S \tilde{R}_k n_k \delta\varepsilon_{nn}\mathrm{dS}$$
$$+ \oint_C (\tilde{E}_k + s_l[n_i\tilde{R}_j n_k]e_{lij})\delta u_k \mathrm{ds}. \tag{3.10}$$

Equating coefficients of like variations in (3.10) and (2.12), we have

$$\begin{aligned} \bar{P}_k &= \tilde{P}_k - (\mathrm{D}_j - n_j\mathrm{D}_i n_i)(\tilde{R}_j n_k), \\ \bar{Q}_k &= 2n_i\tilde{R}_j e_{ijk}, \\ \bar{R} &= n_i\tilde{R}_i, \\ \bar{E}_k &= \tilde{E}_k + s_l[n_i\tilde{R}_j n_k]e_{lij} \end{aligned} \tag{3.11}$$

or, in vector notation,

$$\begin{aligned} \bar{\mathbf{P}} &= \tilde{\mathbf{P}} - \mathbf{n}\cdot\nabla\times(\mathbf{n}\times\tilde{\mathbf{R}}\mathbf{n}), \\ \bar{\mathbf{Q}} &= 2\mathbf{n}\times\mathbf{R}, \\ \bar{\mathbf{R}} &= \tilde{\mathbf{R}}\cdot\mathbf{n}, \\ \bar{\mathbf{E}} &= \tilde{\mathbf{E}} + \mathbf{s}\cdot[\mathbf{n}\times\tilde{\mathbf{R}}\mathbf{n}]. \end{aligned} \tag{3.12}$$

Conversely,

$$\begin{aligned} \tilde{\mathbf{P}} &= \bar{\mathbf{P}} + \tfrac{1}{2}\mathbf{n}\cdot\nabla\times(\bar{\mathbf{Q}}\mathbf{n}) = \hat{\mathbf{P}}, \\ \tilde{\mathbf{R}}(&= \mathbf{n}\times\tilde{\mathbf{R}}\times\mathbf{n} + \mathbf{n}\cdot\tilde{\mathbf{R}}\mathbf{n}) = \tfrac{1}{2}\bar{\mathbf{Q}}\times\mathbf{n} + \bar{\mathbf{R}}\mathbf{n} = \hat{\mathbf{R}}, \\ \tilde{\mathbf{E}} &= \bar{\mathbf{E}} - \tfrac{1}{2}\mathbf{s}\cdot[\bar{\mathbf{Q}}\mathbf{n}] = \hat{\mathbf{E}}. \end{aligned} \tag{3.13}$$

The formulas (3.12) and (3.13) show how the same traction boundary conditions can be set in each of the three forms of the theory. The same results may also be found by substituting (3.3)–(3.8) in (2.16), (2.20) and (2.24).

## 4. MOMENT EQUATIONS

The assumption that the first gradient of the displacement enters the potential energy-density only in the symmetric form $\varepsilon_{ij}$ makes W *ab initio* invariant in a rigid rotation of the deformed body and, in the variational derivation, precludes the display of a differential equation expressing the equilibrium of moments. Toupin [7] has shown how a moment equation can be found by assuming an energy function

$$\mathrm{W} = \mathrm{W}'(u_{j,i}, \tilde{\kappa}_{ijk}) \tag{4.1}$$

and applying the requirement of rotational invariance subsequently. It is shown, here, how such a procedure can lead to a variety of equivalent forms of the moment equation. To exhibit this result in a simple context, we shall omit time dependence and edges.

Hamilton's principle then reduces to

$$\int_V \delta W' \, dV = \int_V F_k \delta u_k \, dV + \int_S (\tilde{P}'_k \delta u_k + \tilde{R}'_k D\delta u_k) \, dS. \tag{4.2}$$

Starting with

$$\delta W' = \frac{\partial W'}{\partial u_{j,i}} \delta u_{j,i} + \frac{\partial W'}{\partial \tilde{\kappa}_{ijk}} \delta \tilde{\kappa}_{ijk},$$

we find

$$\begin{aligned}\int_V \delta W' \, dV = &-\int_V \left[\left(\frac{\partial W'}{\partial u_{k,i}}\right)_{,i} - \left(\frac{\partial W'}{\partial \tilde{\kappa}_{ijk}}\right)_{,ij}\right] \delta u_k \, dV \\ &+ \int_V n_j \left[\frac{\partial W'}{\partial u_{k,j}} - \left(\frac{\partial W'}{\partial \tilde{\kappa}_{ijk}}\right)_{,i}\right] \delta u_k \, dS + \int_S n_i \frac{\partial W'}{\partial \tilde{\kappa}_{ijk}} \delta u_{k,j} \, dS\end{aligned} \tag{4.3}$$

after application of the chain rule and the divergence theorem.

Define

$$\tilde{\mu}'_{ijk} = \frac{\partial W'}{\partial \tilde{\kappa}_{ijk}} = \tilde{\mu}'_{jik}, \tag{4.4}$$

$$\tau'_{jk} = \frac{\partial W'}{\partial u_{k,j}} - \tilde{\mu}'_{ijk,i}. \tag{4.5}$$

Then

$$\int_V \delta W' \, dV = -\int_V \tau'_{jk,j} \delta u_k \, dV + \int_S n_j \tau'_{jk} \delta u_k \, dS + \int_S n_i \tilde{\mu}'_{ijk} \delta u_{k,j} \, dS.$$

Now,

$$\int_S n_i \tilde{\mu}'_{ijk} \delta u_{k,j} \, dS = \int_S n_i \tilde{\mu}'_{ijk} D_j \delta u_k \, dS + \int_S n_i n_j \tilde{\mu}'_{ijk} D\delta u_k \, dS$$

and

$$\int_S n_i \tilde{\mu}'_{ijk} D_j \delta u_k \, dS = \int_S D_j(n_i \tilde{\mu}'_{ijk} \delta u_k) \, dS - \int_S D_j(n_i \tilde{\mu}'_{ijk}) \delta u_k \, dS.$$

Also,

$$\int_S D_j(n_i \tilde{\mu}'_{ijk} \delta u_k) \, dS = \int_S (D_l n_l) n_i n_j \tilde{\mu}'_{ijk} \delta u_k \, dS$$

by the surface divergence theorem for a smooth surface. Assembling these results, we have

$$\begin{aligned}\int_V \delta W' \, dV = &-\int_V \tau'_{jk,j} \delta u_k \, dV + \int_S n_i n_j \tilde{\mu}'_{ijk} D\delta u_k \, dS \\ &+ \int_S [n_j \tau'_{jk} - D_j(n_i \tilde{\mu}_{ijk}) + (D_l n_l) n_i n_j \tilde{\mu}'_{ijk}] \delta u_k \, dS.\end{aligned} \tag{4.6}$$

Upon equating coefficients of like variations in (4.6) and (4.2), we find

$$\tau'_{jk,j} + F_k = 0, \tag{4.7}$$

$$\begin{aligned} P'_k &= n_j\tau'_{jk} - \mathrm{D}_j(n_i\tilde{\mu}'_{ijk}) + (\mathrm{D}_l n_l)n_i n_j \tilde{\mu}'_{ijk}, \\ R'_k &= n_i n_j \tilde{\mu}'_{ijk}, \end{aligned} \tag{4.8}$$

which are equivalent to Toupin's results (10.13) and (10.14) in [7].

The conditions of invariance of the potential energy-density in a rigid rotation of the deformed body are, in the present linear case,

$$\frac{\partial \mathrm{W}'}{\partial u_{[j,i]}} = 0. \tag{4.9}$$

With (4.9), $\tilde{\mu}'_{ijk}$ becomes $\tilde{\mu}_{ijk}$ and the definition (4.5) yields

$$\tau'_{[jk]} + \tilde{\mu}_{i[jk],i} = 0, \tag{4.10}$$

which, allowing for differences in notation, is Toupin's moment equation (10.20) in [7].

Now, return to the definition (4.5) and replace it with

$$\tau''_{jk} = \frac{\partial \mathrm{W}'}{\partial u_{k,j}} - \tilde{\mu}'_{ijk,i} + 2A\tilde{\mu}'_{k[ij],i}, \tag{4.11}$$

where $A$ is a constant, and note that $\tilde{\mu}_{k[ij],ij} = 0$. Then, instead of (4.7) and (4.8), we have

$$\tau''_{jk,j} + F_k = 0, \tag{4.12}$$

$$\begin{aligned} \tilde{P}'_k &= n_j(\tau''_{jk} - 2A\tilde{\mu}'_{k[ij],i}) - \mathrm{D}_j(n_i\tilde{\mu}'_{ijk}) + (\mathrm{D}_l n_l)n_i n_j \tilde{\mu}'_{ijk}, \\ \tilde{R}'_k &= n_i n_j \tilde{\mu}'_{ijk}. \end{aligned} \tag{4.13}$$

In terms of W′, (4.12) and (4.13) are the same as (4.7) and (4.8). The invariance conditions (4.9), applied to the definition (4.11), produce

$$\tau''_{[jk]} + (1+A)\tilde{\mu}_{i[jk],i} = 0. \tag{4.14}$$

Again, (4.14) and (4.10) are equivalent equations; but derivatives of the potential energy-density are distributed differently between the two terms in each equation.

By the first of (3.5),

$$\tilde{\mu}_{i[jk]} = \tfrac{3}{8}\bar{\mu}_{il}e_{jkl}. \tag{4.15}$$

Hence (4.14) can be written as

$$\tau''_{[jk]} + \tfrac{3}{8}(1+A)\bar{\mu}_{il,i}e_{jkl} = 0. \tag{4.16}$$

This is to be compared with the moment equation derived from the principle of angular momentum [6, (1.9)]:

$$\tau_{[jk]} + \tfrac{1}{2}\mu_{il,i}e_{jkl} = 0. \tag{4.17}$$

Noting that $A$ is arbitrary and that $\bar{\mu}_{ij}$ is a deviator whereas $\mu_{ij}$ is not, it is apparent that the invariance conditions (4.9) do not contribute to the identification of the couple-stress.

## 5. DERIVATION BASED ON CONSERVATION PRINCIPLES

In this section, principles of conservation of linear and angular momentum and of energy are employed in the derivation of the equations of strain-gradient theory. For completeness, body double forces, with and without moment, are included along with body forces, edges and time dependence.

Let $t_i$ and $m_i$ be the components of force and couple, per unit area, acting on the surface S of a body occupying a volume V; and let $F_i$ and $C_i$ be the components of force and couple, per unit volume, in V. Then principles of linear and angular momentum are expressed by

$$\int_S t_i \,\mathrm{dS} + \int_V F_i \,\mathrm{dV} = \int_V \rho \ddot{u}_i \,\mathrm{dV}, \tag{5.1}$$

$$\int_S (x_j t_k e_{ijk} + m_i)\,\mathrm{dS} + \int_V (x_j F_k e_{ijk} + C_i)\,\mathrm{dV} = \int_V \rho x_j \ddot{u}_k e_{ijk}\,\mathrm{dV}. \tag{5.2}$$

Application of (5.1) to an elementary tetrahedron leads, in the limit as the tetrahedron shrinks to zero, to the definition of stress, $\boldsymbol{\tau}$, such that

$$t_j = n_i \tau_{ij}. \tag{5.3}$$

Substitution of (5.3) into (5.1) and application of the divergence theorem lead to the stress-equations of motion:

$$\tau_{jk,j} + F_k = \rho \ddot{u}_k. \tag{5.4}$$

Similarly, application of (5.2) to an elementary tetrahedron leads to the definition of couple-stress, $\boldsymbol{\mu}$, such that

$$m_j = n_i \mu_{ij} \tag{5.5}$$

which, with (5.2), yields the moment-equation

$$\mu_{ij,i} + \tau_{ki} e_{ijk} + C_j = 0 \tag{5.6}$$

or

$$\tau_{[jk]} + \tfrac{1}{2}\mu_{il,i} e_{jkl} + \tfrac{1}{2} C_l e_{jkl} = 0. \tag{5.7}$$

Now, in (5.4), write $\tau_{jk} = \tau_{(jk)} + \tau_{[jk]}$. Then, with (5.7), (5.4) becomes

$$\tau_{(jk),j} - \tfrac{1}{2}\mu_{il,ij} e_{jkl} + F_k - \tfrac{1}{2} C_{l,j} e_{jkl} = \rho \ddot{u}_k. \tag{5.8}$$

In (5.8), separate $\mu_{ij}$ into its deviatoric and spherical parts:

$$\mu_{ij} = \mu^{\mathrm{D}}_{ij} + \tfrac{1}{3}\delta_{ij}\mu_{kk}. \tag{5.9}$$

But the curl of the divergence of a spherical tensor vanishes. Hence (5.8) is

$$\tau_{(jk),j} - \tfrac{1}{2}\mu^{\mathrm{D}}_{il,ij} e_{jkl} + F_k - \tfrac{1}{2} C_{l,j} e_{jkl} = \rho \ddot{u}_k. \tag{5.10}$$

As for energy densities, we suppose, again, that the kinetic energy-density is

$$T = \tfrac{1}{2}\rho \dot{u}_i \dot{u}_i. \tag{5.11}$$

Then

$$\dot{T} = \rho \ddot{u}_i \dot{u}_i. \tag{5.12}$$

For the potential energy-density, we assume, again,

$$\mathrm{W} = \overline{\mathrm{W}}(\varepsilon_{ij}, \bar{\kappa}_{ij}, \bar{\bar{\kappa}}_{ijk});$$

so that

$$\dot{\mathrm{W}} = \bar{\tau}_{ij}\dot{\varepsilon}_{ij} + \bar{\mu}_{ij}\dot{\bar{\kappa}}_{ij} + \bar{\bar{\mu}}_{ijk}\dot{\bar{\bar{\kappa}}}_{ijk}, \tag{5.13}$$

where $\bar{\tau}_{ij}$, $\bar{\mu}_{ij}$ and $\bar{\bar{\mu}}_{ijk}$ are defined, as before, by (2.25), (2.26) and (2.27).

We now adopt the following principle of conservation of energy:

$$\int_{\mathrm{V}} (\dot{\mathrm{T}} + \dot{\mathrm{W}})\,\mathrm{dV} = \int_{\mathrm{V}} (F_j\dot{u}_j + C_j\dot{w}_j + \Phi_{(ij)}\dot{\varepsilon}_{ij})\,\mathrm{dV} + \int_{\mathrm{S}} (t_j\dot{u}_j + m_j\dot{w}_j + n_i\bar{\bar{\mu}}_{ijk}\dot{\varepsilon}_{jk})\,\mathrm{dS}, \tag{5.14}$$

where the symmetric components $\Phi_{(ij)}$ are the densities of body double forces without moment. The antisymmetric part $\Phi_{[ij]}$ is the body couple per unit volume: $\frac{1}{2}e_{ijk}C_k$.

With (5.3), (5.5), the divergence theorem and the chain rule, the surface integral in (5.14) becomes

$$\int_{\mathrm{V}} (\tau_{ij,i}\dot{u}_j + \mu_{ij,i}\dot{w}_j + \bar{\bar{\mu}}_{ijk,i}\dot{\varepsilon}_{jk} + \tau_{ij}\dot{u}_{j,i} + \mu_{ij}\dot{w}_{j,i} + \bar{\bar{\mu}}_{ijk}\dot{\varepsilon}_{jk,i})\,\mathrm{dV}. \tag{5.15}$$

But

$$\tau_{ij}\dot{u}_{j,i} = \tau_{(ij)}\dot{u}_{(j,i)} + \tau_{[ij]}\dot{u}_{[j,i]} = \tau_{(ij)}\dot{\varepsilon}_{ij} + \tau_{ki}e_{ijk}\dot{w}_j$$

and

$$\mu_{ij}\dot{w}_{j,i} = \mu^{\mathrm{D}}_{ij}\dot{\bar{\kappa}}_{ij}, \qquad \bar{\bar{\mu}}_{ijk}\dot{\varepsilon}_{jk,i} = \bar{\bar{\mu}}_{ijk}\dot{\bar{\bar{\kappa}}}_{ijk};$$

so that (5.15) becomes

$$\int_{\mathrm{V}} [\tau_{ij,i}\dot{u}_j + (\mu_{ij,i} + \tau_{ki}e_{ijk})\dot{w}_j + (\tau_{(jk)} + \bar{\bar{\mu}}_{ijk,i})\dot{\varepsilon}_{jk} + \mu^{\mathrm{D}}_{ij}\dot{\bar{\kappa}}_{ij} + \bar{\bar{\mu}}_{ijk}\dot{\bar{\bar{\kappa}}}_{ijk}]\,\mathrm{dV}.$$

Hence, with (5.4), (5.6) and (5.12), the principle of conservation of energy (5.14) is converted to

$$\int_{\mathrm{V}} \dot{\mathrm{W}}\,\mathrm{dV} = \int_{\mathrm{V}} [(\tau_{(jk)} + \bar{\bar{\mu}}_{ijk,i} + \Phi_{(jk)})\dot{\varepsilon}_{jk} + \mu^{\mathrm{D}}_{ij}\dot{\bar{\kappa}}_{ij} + \bar{\bar{\mu}}_{ijk}\dot{\bar{\bar{\kappa}}}_{ijk}]\,\mathrm{dV}. \tag{5.16}$$

Finally, inserting (5.13) in the left hand side of (5.16) and equating coefficients of like kinematic variables on both sides of the equation, we find

$$\tau_{(jk)} = \bar{\tau}_{jk} - \bar{\bar{\mu}}_{ijk,i} - \Phi_{(jk)} \tag{5.17}$$

$$\mu^{\mathrm{D}}_{ij} = \bar{\mu}_{ij}. \tag{5.18}$$

Thus, $\bar{\mu}_{ij}$ is indeed the deviator of the couple-stress.

Upon substituting (5.17) and (5.18) into (5.10), we recover (with the addition of body double forces) the stress-equations of motion (2.23) which were obtained from Hamilton's principle:

$$\bar{\tau}_{jk,j} - \tfrac{1}{2}\mu^{\mathrm{D}}_{il,ij}e_{jkl} - \bar{\bar{\mu}}_{ijk,ij} + F_k - \tfrac{1}{2}C_{l,j}e_{jkl} - \Phi_{(jk),j} = \rho\ddot{u}_k. \tag{5.19}$$

In the case of an isotropic material, $\overline{W}$ is given by (2.7). Then, from (2.25), (2.26), (2.27) and (5.18),

$$\bar{\tau}_{pq} = \lambda\delta_{pq}\varepsilon_{ii} + 2\mu\varepsilon_{pq}, \tag{5.20}$$

$$\mu^{D}_{pq} = 4\bar{d}_1\bar{\kappa}_{pq} + 4\bar{d}_2\bar{\kappa}_{qp} + \bar{f}e_{pqi}\bar{\bar{\kappa}}_{ijj}, \tag{5.21}$$

$$\begin{aligned}\bar{\bar{\mu}}_{pqr} = {} & \bar{a}_1(\bar{\bar{\kappa}}_{iir}\delta_{pq} + \bar{\bar{\kappa}}_{iip}\delta_{qr} + \bar{\bar{\kappa}}_{iiq}\delta_{rp}) + 2\bar{a}_2\bar{\bar{\kappa}}_{pqr} \\ & + \tfrac{1}{3}\bar{f}\bar{\kappa}_{ij}(\delta_{pq}e_{ijr} + \delta_{qr}e_{ijp} + \delta_{rp}e_{ijq}).\end{aligned} \tag{5.22}$$

[Note that, in (5.21), $\bar{\bar{\kappa}}_{ijk}$ contributes to the couple-stress—contrary to the statement in [2] following (12.1)].

When (5.20), (5.21) and (5.22) are inserted in (5.19) and $\varepsilon_{ij}$, $\bar{\kappa}_{ij}$ and $\bar{\bar{\kappa}}_{ijk}$ are replaced by their expressions in terms of $u_i$, we find the displacement-equation of motion

$$(\lambda+2\mu)(1-l_1^2\nabla^2)\nabla\nabla\cdot\mathbf{u} - \mu(1-l_2^2\nabla^2)\nabla\times\nabla\times\mathbf{u} + \mathbf{F} + \tfrac{1}{2}\nabla\times\mathbf{C} - \nabla\cdot\boldsymbol{\Phi} = \rho\ddot{\mathbf{u}}, \tag{5.23}$$

where

$$l_1^2 = (3\bar{a}_1 + 2\bar{a}_2)/(\lambda+2\mu), \quad l_2^2 = (3\bar{d}_1 + \bar{a}_1 + 2\bar{a}_2 - \bar{f})/3\mu. \tag{5.24}$$

Necessary and sufficient conditions for positive definiteness of W are

$$\begin{gathered}\mu > 0, \quad 3\lambda+2\mu > 0, \quad -\bar{d}_1 < \bar{d}_2 < \bar{d}_1, \\ \bar{a}_2 > 0, \quad 5\bar{a}_1 + 2\bar{a}_2 > 0, \quad 5\bar{f}^2 < 6(\bar{d}_1 - \bar{d}_2)(5\bar{a}_1 + 2\bar{a}_2),\end{gathered} \tag{5.25}$$

which replace (12.18) of [2]. From (5.25), it follows immediately that $l_1^2 > 0$. To show that $l_2^2 > 0$, note first that

$$\begin{aligned}&[2(\bar{a}_1 + 2\bar{a}_2) - 3\bar{d}_1 - \tfrac{1}{2}\bar{f}]^2 + \tfrac{9}{20}[6(\bar{d}_1 - \bar{d}_2)(5\bar{a}_1 + 2\bar{a}_2) - 5\bar{f}^2] + \tfrac{216}{5}\bar{d}_1\bar{a}_2 \\ &+ \tfrac{27}{10}(d_1 + d_2)(5\bar{a}_1 + 2\bar{a}_2) = (3\bar{d}_1 + \bar{a}_1 + 2\bar{a}_2 - \bar{f})(3\bar{d}_1 + 4\bar{a}_1 + 8\bar{a}_2 + 2\bar{f}).\end{aligned} \tag{5.26}$$

Since, by (5.25), the left hand side of (5.26) is the sum of positive terms, the right hand side is positive. The two factors on the right must both be positive because, if they were negative, the sum of the second factor and twice the first would be negative—in violation of (5.25). Hence

$$3\bar{d}_1 + \bar{a}_1 + 2\bar{a}_2 - \bar{f} > 0$$

and, consequently, $l_2^2 > 0$.

## 6. UNIQUENESS OF SOLUTIONS

As deduced from conservation principles in the preceding section, the equations of the strain-gradient theory for isotropic materials are sixty-three in number:

{3} $$\tau_{ij,i} + F_j = \rho\ddot{u}_j, \tag{6.1}$$

{3} $$\mu_{ij,i} + \tau_{ki}e_{ijk} + C_j = 0, \tag{6.2}$$

{6} $$\tau_{(jk)} = \bar{\tau}_{jk} - \bar{\bar{\mu}}_{ijk,i} - \Phi_{(jk)}, \tag{6.3}$$

{6} $$\varepsilon_{ij} = u_{(j,i)}, \tag{6.4}$$

{3} $$w_j = \tfrac{1}{2}u_{l,k}e_{jkl}, \tag{6.5}$$

$\{8\}$ $$\bar{\kappa}_{ij} = w_{j,i}, \tag{6.6}$$

$\{10\}$ $$\bar{\bar{\kappa}}_{ijk} = u_{(k,ij)}, \tag{6.7}$$

$\{6\}$ $$\bar{\tau}_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij}, \tag{6.8}$$

$\{8\}$ $$\mu^{\mathrm{D}}_{ij} = 4\bar{d}_1\bar{\kappa}_{ij} + 4\bar{d}_2\bar{\kappa}_{ji} + \bar{f}e_{ijp}\bar{\bar{\kappa}}_{pqq}, \tag{6.9}$$

$\{10\}$ $$\bar{\bar{\mu}}_{ijk} = \bar{a}_1(\bar{\bar{\kappa}}_{ppk}\delta_{ij} + \bar{\bar{\kappa}}_{ppi}\delta_{jk} + \bar{\bar{\kappa}}_{ppj}\delta_{ki}) + 2\bar{a}_2\bar{\bar{\kappa}}_{ijk} + \tfrac{1}{3}\bar{f}\bar{\kappa}_{pq}(\delta_{ij}e_{pqk} + \delta_{jk}e_{pqi} + \delta_{ki}e_{pqj}). \tag{6.10}$$

Whereas there are sixty-three equations, there are sixty-four dependent variables: 3 of $u_i$, 3 of $w_i$, 6 of $\varepsilon_{ij}$, 8 of $\bar{\kappa}_{ij}$, 10 of $\bar{\bar{\kappa}}_{ijk}$, 9 of $\tau_{ij}$, 9 of $\mu_{ij}$, 6 of $\bar{\tau}_{ij}$ and 10 of $\bar{\bar{\mu}}_{ijk}$. The additional variable is the spherical part of the couple-stress which, because it does not contribute to the change of potential energy-density (5.13), is indeterminate within the framework of the theory as represented by the sixty-three equations. With this understanding, we proceed to the proof of a theorem of uniqueness of solutions in the usual manner.

Consider two sets of the sixty-four dependent variables $u'_i \ldots$ and $u''_i \ldots$ (with $\mu'_{ii}$ and $\mu''_{ii}$ arbitrary) and their differences $u_i = u'_i - u''_i \ldots$ Similarly, define body force and body double force differences: $F_i = F'_i - F''_i \ldots$ If each set of variables and body forces is a solution of (6.1)–(6.10), so is the difference set and, from (6.1), we may form the equation

$$\int_{t_0}^{t} \mathrm{d}t \int_{\mathrm{V}} (\tau_{ij,i} + F_j - \rho\ddot{u}_j)\dot{u}_j \,\mathrm{dV} = 0, \tag{6.11}$$

where $t_0$ is an initial time and V is a volume bounded by a surface S with an edge C. Now

$$\tau_{ij,i}\dot{u}_j = (\tau_{ij}\dot{u}_j)_{,i} - \tau_{ij}\dot{u}_{j,i} = (\tau_{ij}\dot{u}_j)_{,i} - \tau_{[ij]}\dot{u}_{[j,i]} - \tau_{(ij)}\dot{u}_{(j,i)}.$$

Hence, with (6.2) and (6.3),

$$\tau_{ij,i}\dot{u}_j = (\tau_{ij}\dot{u}_j)_{,i} + \tfrac{1}{2}e_{ijk}(\mu_{lk,l} + C_k)\dot{u}_{[j,i]} - (\bar{\tau}_{ij} - \bar{\bar{\mu}}_{kij,k} - \Phi_{(ij)})\dot{u}_{(j,i)};$$

or, with (6.4) and (6.5),

$$\tau_{ij,i}\dot{u}_j = (\tau_{ij}\dot{u}_j)_{,i} + (\mu_{lk,l} + C_k)\dot{w}_k - (\bar{\tau}_{ij} - \bar{\bar{\mu}}_{kij,k} - \Phi_{(ij)})\dot{\varepsilon}_{ij}.$$

Further,

$$\mu_{lk,l}\dot{w}_k + \bar{\bar{\mu}}_{kij,k}\dot{\varepsilon}_{ij} = (\mu_{ij}\dot{w}_j + \bar{\bar{\mu}}_{ijk}\dot{\varepsilon}_{jk})_{,i} - \mu_{ij}\dot{w}_{j,i} - \bar{\bar{\mu}}_{ijk}\dot{\varepsilon}_{jk,i};$$

or, with (6.6) and (6.7),

$$\mu_{lk,l}\dot{w}_k + \bar{\bar{\mu}}_{kij,k}\dot{\varepsilon}_{ij} = (\mu_{ij}\dot{w}_j + \bar{\bar{\mu}}_{ijk}\dot{\varepsilon}_{jk})_{,i} - \mu^{\mathrm{D}}_{ij}\dot{\bar{\kappa}}_{ij} - \bar{\bar{\mu}}_{ijk}\dot{\bar{\bar{\kappa}}}_{ijk},$$

where we have used $\bar{\kappa}_{ii} = 0$ and the symmetry of $\bar{\bar{\mu}}_{ijk}$. Also, by (6.8), (6.9) and (6.10),

$$\bar{\tau}_{ij}\dot{\varepsilon}_{ij} + \mu^{\mathrm{D}}_{ij}\dot{\bar{\kappa}}_{ij} + \bar{\bar{\mu}}_{ijk}\dot{\bar{\bar{\kappa}}}_{ijk} = \dot{\mathrm{W}},$$

where W is defined by the same form as (2.7) but in terms of the difference variables. Thus, the integrand of (6.11) becomes

$$F_j\dot{u}_j + C_j\dot{w}_j + \Phi_{(ij)}\dot{\varepsilon}_{ij} + (\tau_{ij}\dot{u}_j + \mu_{ij}\dot{w}_i + \bar{\bar{\mu}}_{ijk}\dot{\varepsilon}_{jk})_{,i} - \dot{\mathrm{T}} - \dot{\mathrm{W}},$$

where T is given by (2.2). Accordingly, application of the divergence theorem and an integration with respect to time convert (6.11) to

$$\int_{\mathrm{V}} [\mathrm{T}+\mathrm{W}]_{t_0}^{t}\, \mathrm{dV} = \int_{t_0}^{t} \mathrm{d}t \int_{\mathrm{V}} (F_j\dot{u}_j + C_j\dot{w}_j + \Phi_{(ij)}\dot{\varepsilon}_{ij})\, \mathrm{dV} + \int_{t_0}^{t} \mathrm{d}t \int_{\mathrm{S}} n_i(\tau_{ij}\dot{u}_j + \mu_{ij}\dot{w}_j + \bar{\bar{\mu}}_{ijk}\dot{\varepsilon}_{jk})\, \mathrm{dS}. \tag{6.12}$$

Thus, starting with the system of sixty-three equations (6.1)–(6.10), we have recovered the principle of conservation of energy for the difference system. We have now to reduce the twelve variables $\dot{u}_j$, $\dot{w}_j$, $\dot{\varepsilon}_{jk}$, in the surface integral of (6.12), to six as only six are independent of each other on S. For example, if the $u_j$ are known on S, so is the normal component, $n_jw_j$, of $w_j$; and, if the $u_j$ and the tangential component, $(\delta_{ij}-n_in_j)w_j$, of $w_j$ are known on S, so are all the components of $\varepsilon_{ij}$ except the normal component $n_in_j\varepsilon_{ij}$. It is also convenient to express the coefficients of $\dot{u}_j$, $(\delta_{ij}-n_in_j)\dot{w}_j$ and $n_in_j\dot{\varepsilon}_{ij}$ in forms that are independent of $\tau_{[ij]}$ and $\mu_{ii}$ so that the coefficients can be computed directly from the constitutive equations (6.8)–(6.10).

From (6.2) and (6.3),

$$\tau_{jk} = \bar{\tau}_{jk} - \tfrac{1}{2}\mu_{li,l}e_{ijk} - \bar{\bar{\mu}}_{ijk,i} - \tfrac{1}{2}C_ie_{ijk} - \Phi_{(jk)}. \tag{6.13}$$

Also,

$$\begin{aligned} n_i\mu_{ij}\dot{w}_j &= n_i\mu_{ij}n_jn_k\dot{w}_k + n_i\mu_{ij}(\delta_{jk} - n_jn_k)\dot{w}_k \\ &= \tfrac{1}{2}\mu_{nn}n_ke_{kij}\dot{u}_{j,i} + n_i\mu_{ij}(\delta_{jk} - n_jn_k)\dot{w}_k \\ &= \tfrac{1}{2}n_ke_{kij}(\mu_{nn}\dot{u}_j)_{,i} - \tfrac{1}{2}n_ke_{kij}\mu_{nn,i}\dot{u}_j + n_i\mu_{ij}(\delta_{jk} - n_jn_k)\dot{w}_k \end{aligned}$$

and

$$\int_{\mathrm{S}} n_ke_{kij}(\mu_{nn}\dot{u}_j)_{,i}\, \mathrm{dS} = \oint_{\mathrm{C}} [\mu_{nn}^{\mathrm{D}}]s_j\dot{u}_j\, \mathrm{d}s.$$

Hence

$$\begin{aligned} \int_{\mathrm{S}} (n_j\tau_{jk}\dot{u}_k + n_i\mu_{ij}\dot{w}_j)\, \mathrm{dS} &= \int_{\mathrm{S}} n_j[\bar{\tau}_{jk} + \tfrac{1}{2}(\mu_{li,l}^{\mathrm{D}} - \mu_{nn,i}^{\mathrm{D}})e_{jik} - \bar{\bar{\mu}}_{ijk,i}]\dot{u}_k\, \mathrm{dS} \\ &+ \int_{\mathrm{S}} n_j(\tfrac{1}{2}C_ie_{jik} - \Phi_{(jk)})\dot{u}_k\, \mathrm{dS} \\ &+ \int_{\mathrm{S}} n_i\mu_{ij}^{\mathrm{D}}(\delta_{jk} - n_jn_k)\dot{w}_k\, \mathrm{dS} + \oint_{\mathrm{C}} \tfrac{1}{2}[\mu_{nn}^{\mathrm{D}}]s_k\dot{u}_k\, \mathrm{d}s, \end{aligned} \tag{6.14}$$

in which we have used

$$\mu_{ij}(\delta_{jk}-n_jn_k) = \mu^{D}_{ij}(\delta_{jk}-n_jn_k), \quad \mu_{li,l}-\mu_{nn,i} = \mu^{D}_{li,l}-\mu^{D}_{nn,i}.$$

As for the last term in (6.12), we write

$$n_i\bar{\bar{\mu}}_{ijk}\dot{\varepsilon}_{jk} = n_i\bar{\bar{\mu}}_{ijk}\dot{u}_{k,j} = n_i\bar{\bar{\mu}}_{ijk}\mathrm{D}_j\dot{u}_k+n_in_j\bar{\bar{\mu}}_{ijk}\mathrm{D}\dot{u}_k.$$

But

$$\mathrm{D}\dot{u}_k = 2\dot{w}_ln_je_{ljk}+n_j\mathrm{D}_k\dot{u}_j+n_k\dot{\varepsilon}_{nn}$$

Hence

$$\begin{aligned} n_i\bar{\bar{\mu}}_{ijk}\dot{\varepsilon}_{jk} &= (n_i\bar{\bar{\mu}}_{ijk}+n_pn_q\bar{\bar{\mu}}_{pqj}n_k)\mathrm{D}_j\dot{u}_k+2n_qn_in_j\bar{\bar{\mu}}_{ijp}e_{qpk}\dot{w}_k+n_in_jn_k\bar{\bar{\mu}}_{ijk}\dot{\varepsilon}_{nn} \\ &= \mathrm{D}_j[(n_i\bar{\bar{\mu}}_{ijk}+n_pn_q\bar{\bar{\mu}}_{pqj}n_k)\dot{u}_k]-\mathrm{D}_j(n_i\bar{\bar{\mu}}_{ijk}+n_pn_q\bar{\bar{\mu}}_{pqj}n_k)\dot{u}_k \\ &\quad +2n_qn_in_j\bar{\bar{\mu}}_{ijp}e_{qpk}\dot{w}_k+n_in_jn_k\bar{\bar{\mu}}_{ijk}\dot{\varepsilon}_{nn}. \end{aligned}$$

Then, with the surface divergence theorem,

$$\begin{aligned} \int_S n_i\bar{\bar{\mu}}_{ijk}\dot{\varepsilon}_{jk}\,\mathrm{dS} &= \int_S [(n_j\mathrm{D}_ln_l-\mathrm{D}_j)(n_i\bar{\bar{\mu}}_{ijk}+n_pn_q\bar{\bar{\mu}}_{pqj}n_k)]\dot{u}_k\,\mathrm{dS} \\ &\quad +\int_S n_in_j(2n_q\bar{\bar{\mu}}_{ijp}e_{qpk}\dot{w}_k+\bar{\bar{\mu}}_{ijk}n_k\dot{\varepsilon}_{nn})\,\mathrm{dS} \\ &\quad +\oint_C s_p[n_qe_{jpq}(n_i\bar{\bar{\mu}}_{ijk}+n_ln_in_k\bar{\bar{\mu}}_{lij})]\dot{u}_k\,\mathrm{ds}. \end{aligned} \tag{6.15}$$

Upon inserting (6.14) and (6.15) in (6.12), we obtain, finally,

$$\begin{aligned} \int_V [\mathrm{T}+\mathrm{W}]_{t_0}^{t}\,\mathrm{dV} &= \int_{t_0}^{t}\mathrm{d}t\int_V (F_j\dot{u}_j+C_j\dot{w}_j+\Phi_{(ij)}\dot{\varepsilon}_{ij})\,\mathrm{dV} \\ &\quad +\int_{t_0}^{t}\mathrm{d}t\int_S [\bar{P}_k\dot{u}_k+\bar{Q}_k(\delta_{kj}-n_kn_j)\dot{w}_j+\bar{R}\dot{\varepsilon}_{nn}]\,\mathrm{dS} \\ &\quad +\int_{t_0}^{t}\mathrm{d}t\int_C \bar{E}_k\dot{u}_k\,\mathrm{ds}, \end{aligned} \tag{6.16}$$

where

$$\begin{aligned} \bar{P}_k &= n_j[\bar{\tau}_{jk}+\tfrac{1}{2}(\mu^{D}_{li,l}-\mu^{D}_{nn,i})e_{jik}-\bar{\bar{\mu}}_{ijk,i}]-(\mathrm{D}_j-n_j\mathrm{D}_ln_l)(n_i\bar{\bar{\mu}}_{ijk}+n_pn_qn_k\bar{\bar{\mu}}_{pqj})+n_j(\tfrac{1}{2}C_ie_{jik}-\Phi_{(jk)}), \\ \bar{Q}_k &= n_i\mu^{D}_{ij}(\delta_{jk}-n_jn_k)+2n_in_jn_q\bar{\bar{\mu}}_{ijp}e_{qpk}, \\ \bar{R} &= n_in_jn_k\bar{\bar{\mu}}_{ijk}, \\ \bar{E}_k &= s_p[\tfrac{1}{2}\delta_{pk}\mu^{D}_{nn}+n_qn_i(\bar{\bar{\mu}}_{ijk}+n_kn_l\bar{\bar{\mu}}_{lij})e_{jpq}]. \end{aligned} \tag{6.17}$$

It will be observed that (6.17) have the same form as (2.24) with the addition of the body double forces. Note that the body couple **C** has been placed in the expression for **P**, following Koiter [8].

Conditions sufficient for a unique solution of (6.1)–(6.10) are now obtained from (6.16) in the usual manner—based on the assumption of positive definiteness of T and W:

1. At each point in V: **F**, **C**, **Φ** and initial values of **u** and $\dot{\mathbf{u}}$.

2. At each point on S: (a) a component of **u** (or $\bar{\mathbf{P}}$), in any direction, and the resultant of $\bar{\mathbf{P}}$ (or **u**) in the plane at right angles; (b) in the tangent plane, a component of **w** (or $\bar{\mathbf{Q}}$) and the component of $\bar{\mathbf{Q}}$ (or **w**) at right angles; (c) $\varepsilon_{nn}$ or $\bar{R}$.

3. At each point on C: a component of **u** (or $\bar{\mathbf{E}}$), in any direction, and the resultant of $\bar{\mathbf{E}}$ (or **u**) in the plane at right angles.

Note that, according to the linear momentum principle, $n_j\tau_{jk}$ and not $\bar{P}_k$ are the components of force per unit area. Also, by the angular momentum principle, $n_i\mu_{ij}(\delta_{jk}-n_jn_k)$ and not $\bar{Q}_k$ are the components of tangential couple per unit area.

As in classical elasticity, the uniqueness theorem is subject to the regularity restrictions implied by the forms of the divergence theorems employed in the proof. In particular, if a singularity is present, an additional condition, generally obtained through a limit process, is required. Sternberg and Muki [9] have shown how failure to observe such a requirement can lead to physically irrelevant "pseudo-solutions".

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**Абстракт**—В работе исследуется линейная теория упругости, в которой плотность потенциальной энергии зависит от градиента дефоямации и от дефоямации.

В первой части работы сравниваются три вида теории и выводятся зависимости, касающиеся напряжений и граничных условий в этих трех видах.

Во второй части решаются неоднозначности выражения для моментного уравнения и определяется моментное напряжение, используя вывод основывающийся в большей степени на принципах сохранения, чем на вариационных принципах, которые использованы ранее.