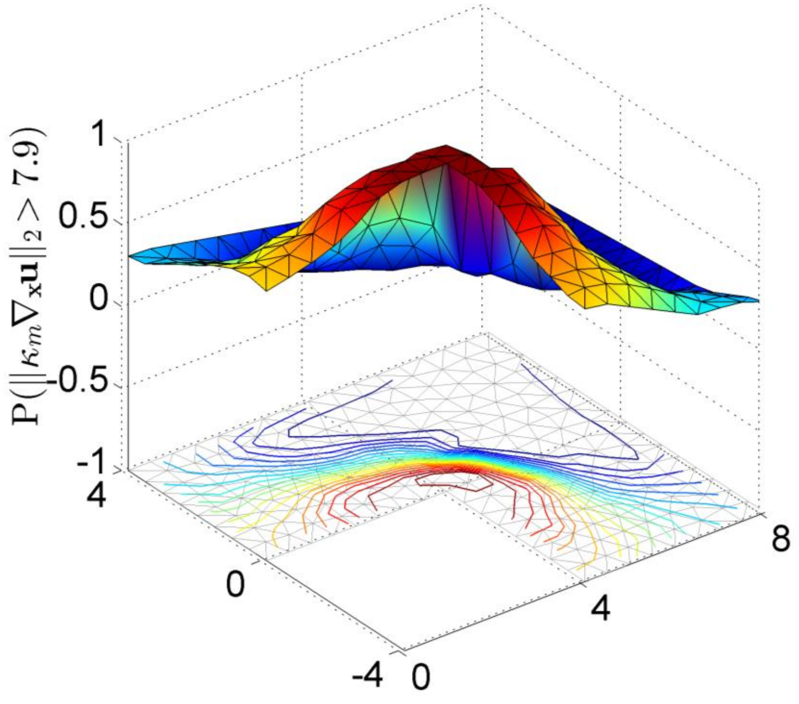
Efficient methods for propagation of uncertainty in description of groundwater flow

Jan Havelka, Jan Sýkora

Motivation

- Evaluation of response statistics
- Modeling of random processes
- Propagation of uncertainties



Extended model

- Stationary flow with uncertain parameter:

$$\begin{aligned} -\nabla_{\mathbf{x}} \cdot (\kappa_m(\mathbf{x},\mathbf{y})\nabla_{\mathbf{x}} u(\mathbf{x},\mathbf{y})) &= \widetilde{f}(\mathbf{x}), && \mathbf{x} \in D,\, \mathbf{y} \in \mathbb{R}^m, \\ \mathbf{n}(\mathbf{x}) \cdot (\kappa_m(\mathbf{x},\mathbf{y})\nabla_{\mathbf{x}} u(\mathbf{x},\mathbf{y})) &= \widetilde{f}_N(\mathbf{x}), && \mathbf{x} \in \partial D_N,\, \mathbf{y} \in \mathbb{R}^m, \\ u(\mathbf{x},\mathbf{y}) &= \widetilde{f}_D(\mathbf{x}), && \mathbf{x} \in \partial D_D,\, \mathbf{y} \in \mathbb{R}^m. \end{aligned}$$

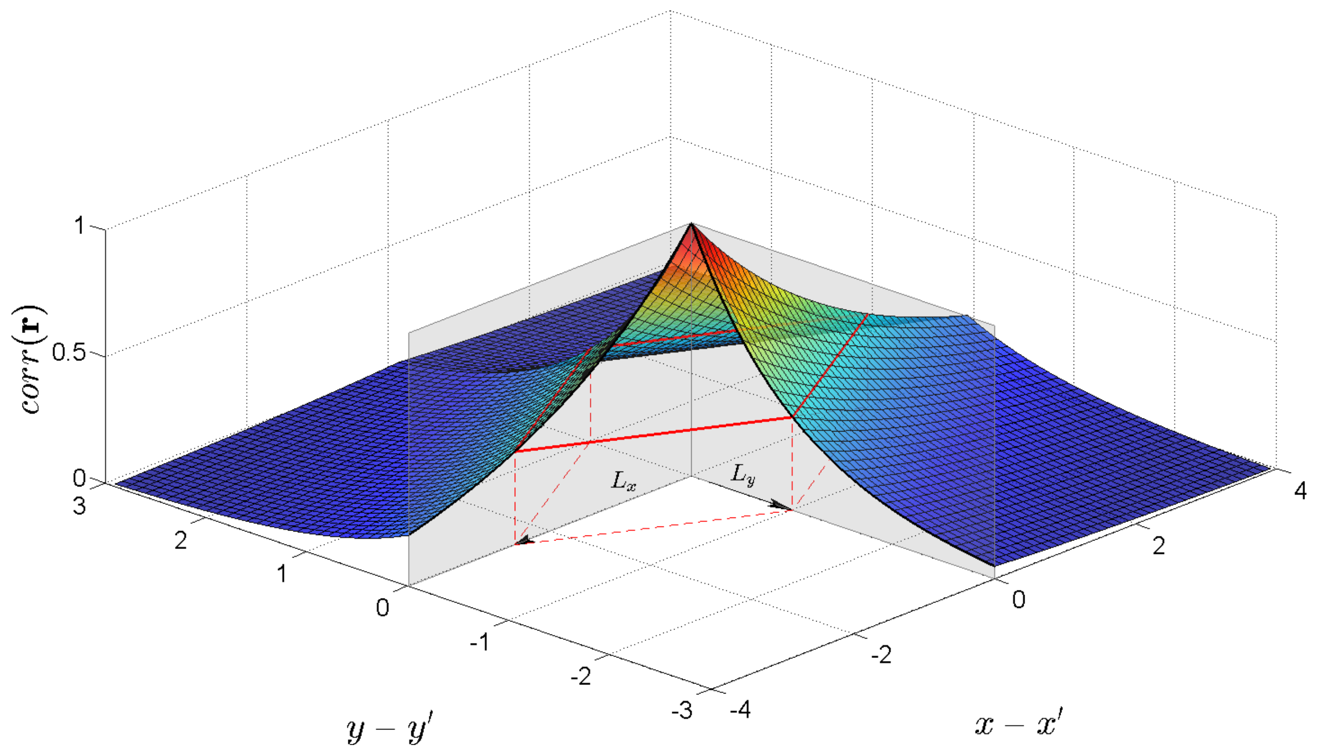
- Aproximation of material parameter:
  - Statistics μ_κ, σ
  - Covariance function (exp.,gauss.,...) C_κ
  - Karhunen-Loève expansion

$$\kappa_m(\mathbf{x},\mathbf{y}) = \mu_\kappa(\mathbf{x}) + \sum_{i=1}^{m} \sqrt{\lambda_i}\phi_i(\mathbf{x}) y_i$$

Covariance function

- Exponencial

$$C_\kappa = \sigma^2 \exp\left(-\left|\frac{x-x'}{l_x}\right| - \left|\frac{y-y'}{l_y}\right|\right) \qquad (l_x = 2.5,\ l_y = 1)$$

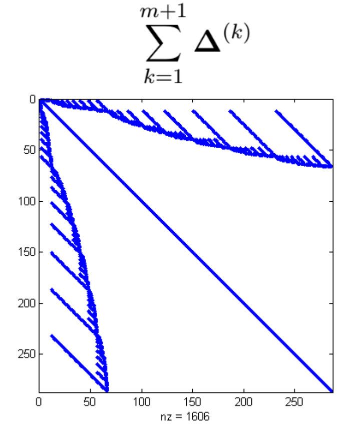


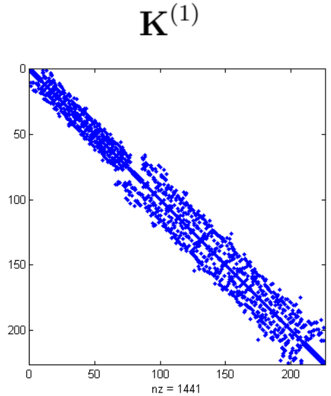
Methods

- Spatial coordinations
  - Galerkin method
- Stochastic
  - Monte Carlo method
  - Stochastic collocation method
  - Stochastic Galerkin method

$$u(\mathbf{x},\mathbf{y}) \approx \sum_{j=1}^{M}\sum_{r=1}^{n} \widetilde{u}_{jr} N_r(\mathbf{x}) H_j(\mathbf{y}).$$

$$\mathbf{K} = \sum_{k=1}^{m+1} \left[\boldsymbol{\Delta}^{(k)} \otimes \mathbf{K}^{(k)} \right]$$



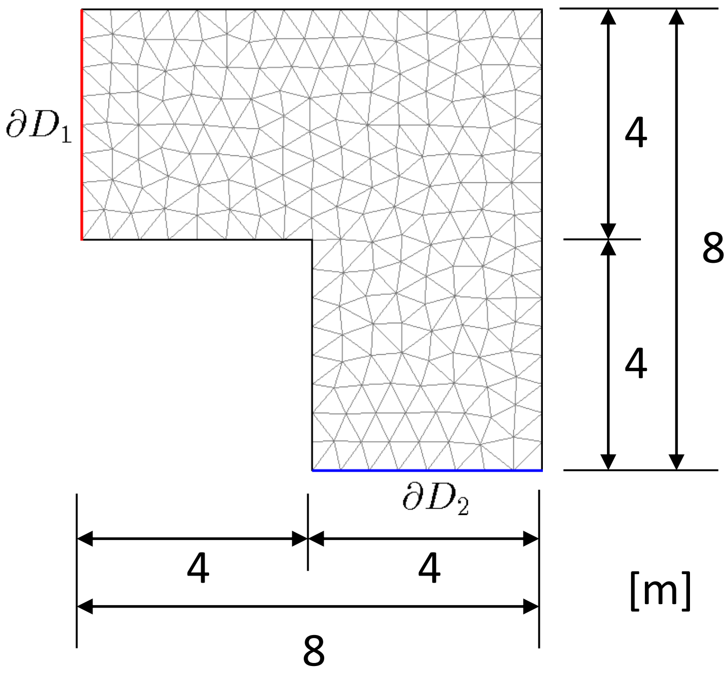


Example

- Input data – FEM (geometry, boundary conditions, discretization)

$$u(\mathbf{x}) = 0, \quad \mathbf{x} \in \partial D_1,$$
$$u_p(\mathbf{x}) = 0, \quad \mathbf{x} \in \partial D_2.$$

$$(u = u_p + u_g)$$



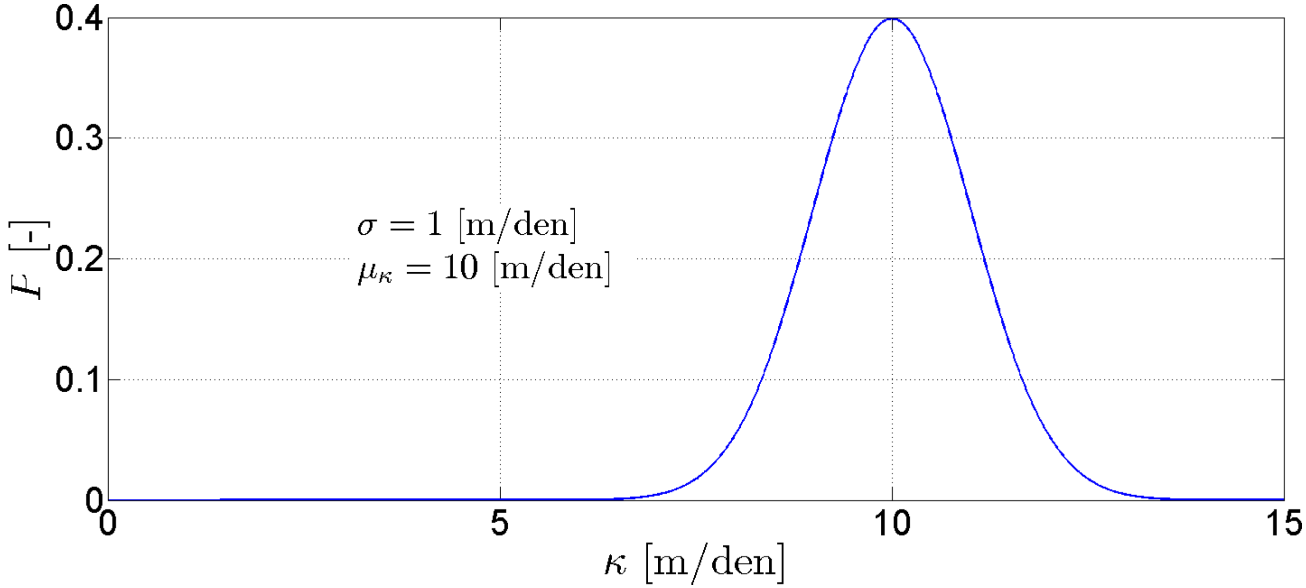
Elements: 384
Nodes: 225

Example

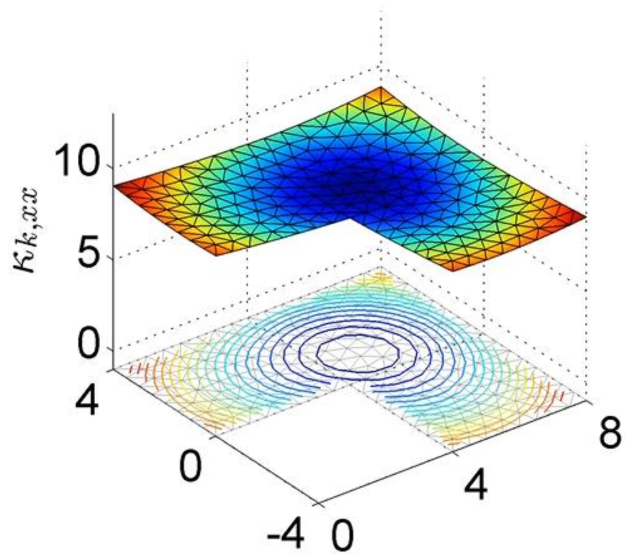
- Input data – stochastics
  1. Covariance function

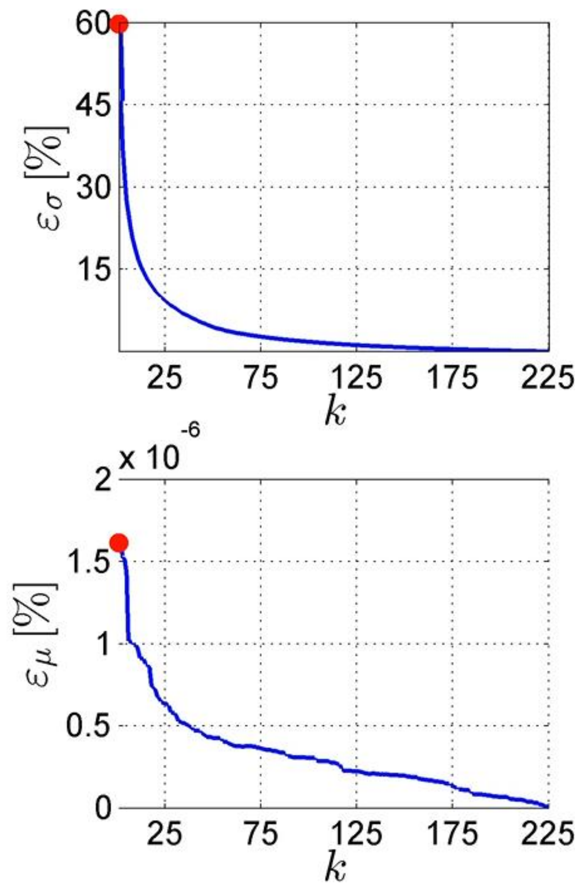
$$C_\kappa = \sigma^2 \exp\left(-\left|\frac{x-x'}{l_x}\right| - \left|\frac{y-y'}{l_y}\right|\right) \qquad (l_x = l_y = 5m)$$

  2. Elementary statistics and distribution



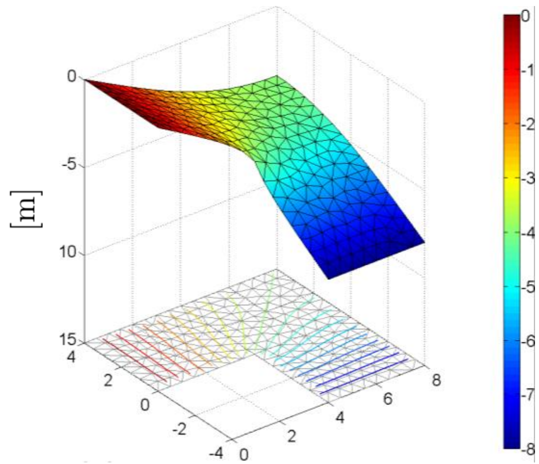
Randomfield realisation



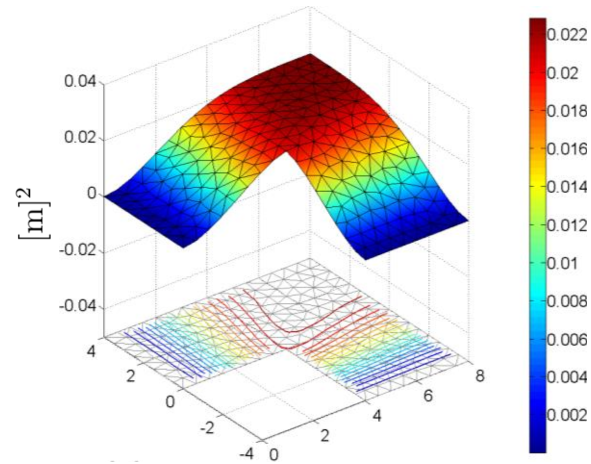


$$\kappa_{k,xx} = \kappa_{k,yy} = \mu_\kappa(\mathbf{x}) + \sum_{i=1}^{k} \sqrt{\lambda_i}\phi_i(\mathbf{x})y_i, \quad k = 1, \ldots, m$$

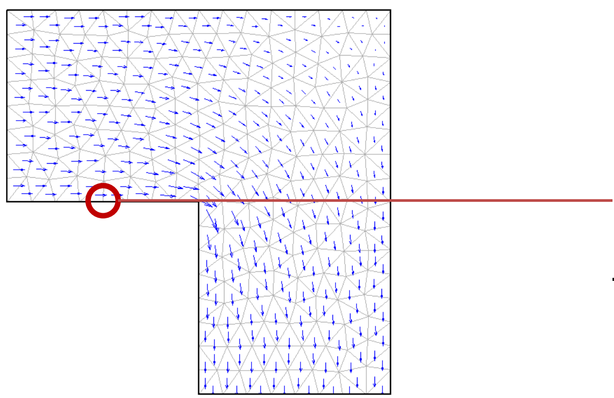
Model response



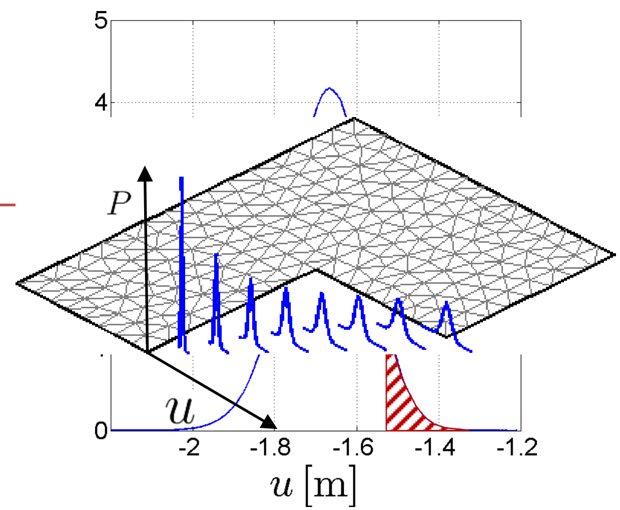
$\mathbb{E}\left[u\right]$



$\mathrm{var}\left[u\right]$

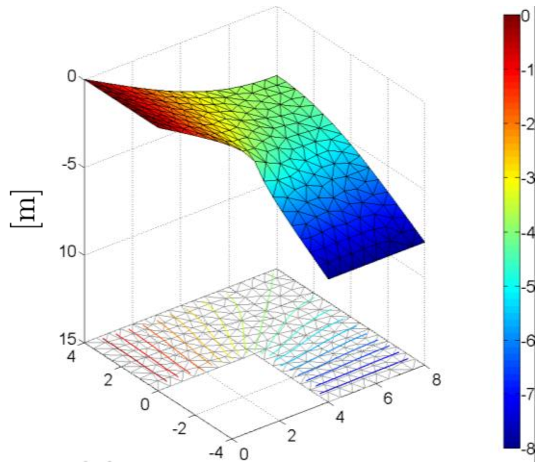


$\nabla_{\mathbf{x}}\mathbb{E}(u(\mathbf{x},\mathbf{y}))$

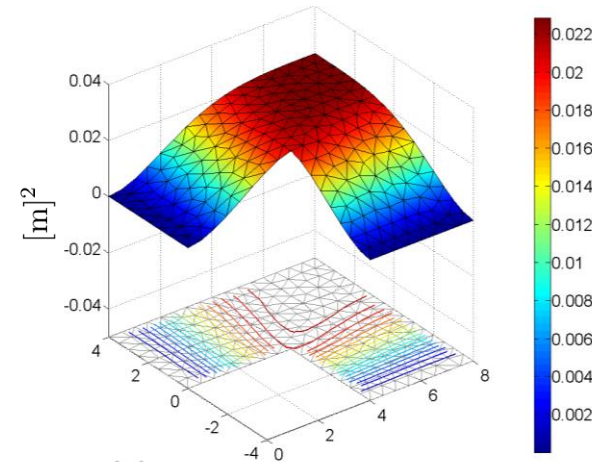


$P(u(\mathbf{x}_5,\mathbf{y}) > -1.5) = 0.032$

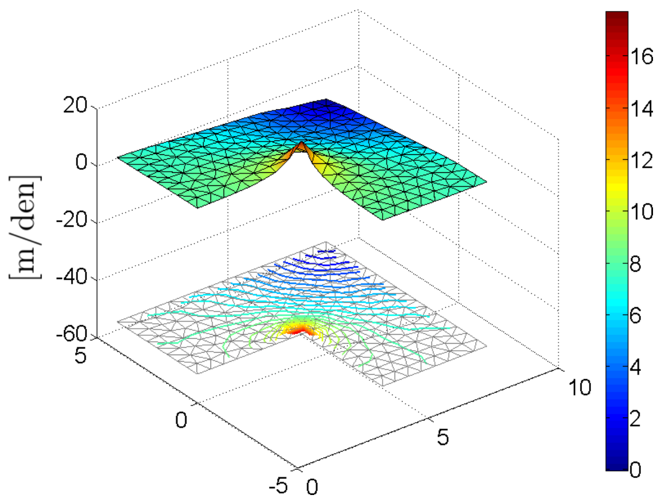
Model response



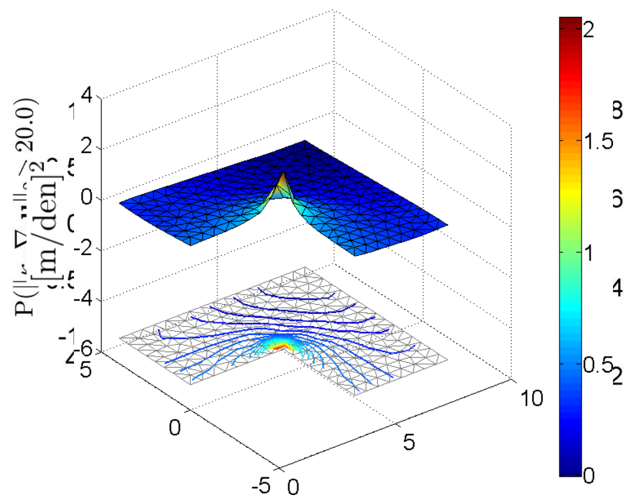
$\mathbb{E}[u]$



$\mathrm{var}[u]$



$\|\mathbb{E}(\kappa \nabla_{\mathbf{x}} u)\|_2$



$\|\mathrm{var}(\kappa \nabla_{\mathbf{x}} u)\|_2$

Conclusion

- General method
- Arbitrary transformation of response
- Current (~98mil. DoF, ~60GB RAM)

Further work:

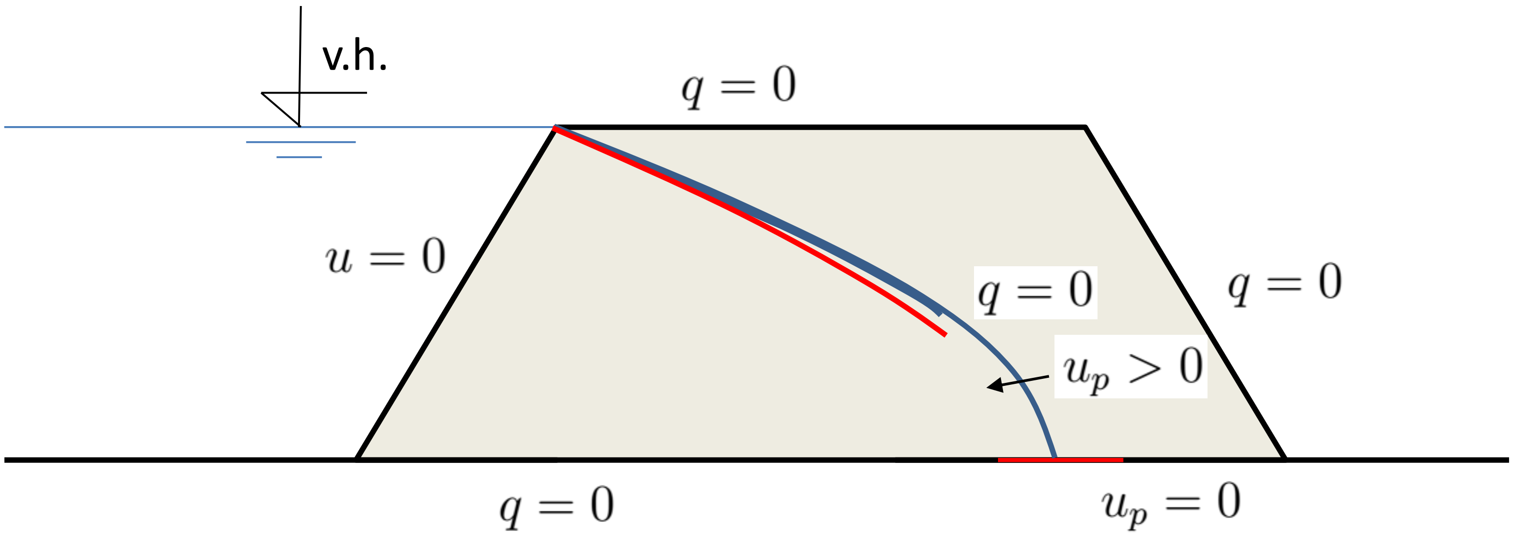
- Probability of exceeding some value (in general)
- Covariance function
- Time dependent flow, coupled tasks
- Log-normal distribution
- Paralelisation
- Further approximation
- Development of solver for this kind of tasks

Thank you for attention

Otázky k diskuzi

1. Průsaková podmínka

$$(u = u_p + u_g)$$



$$\bar{q}_n = k_v(u - u_{ext}) = k_v(u - u_g) = k_v u_p$$

$$\bar{q}_n = \begin{cases} k_v u_p, & \text{pro } u_p > 0, \\ 0, & \text{pro } u_p < 0. \end{cases}$$

Otázky k diskuzi

2. Stochastický proces vs. Náhodné pole

Otázky k diskuzi

3. Konvergence