NUMERICAL ANALYSIS OF CABLED-TRUSS STRUCTURES

Daniel Colmenares
Faculty of Civil Engineering
Czech Technical University

Abstract

This work presents a computational method used in the static analysis of cable structures. The Dynamic Relaxation Method is explained and a sample design is solved to prove its utility for non-linear analysis. Rapid convergence of less than 120 iterations was achieved using kinetic damping. Finally, small numerical errors were obtained in the final comparison of the example solved.

Keywords: cable structures, dynamic relaxation, non-linear analysis, static steady solution

Introduction

The cables, as structural elements, are known for their excellent flexibility, their zero bending moments and, strength. This strength enables them to span very long distances with minimal cross-sectional area [8]. The cables become excellent constructional elements for overarching long-span capabilities. Cabled-truss structures are constructed using two types of elements: (i) cables and (ii) bars. The formers are able to exert tension forces, while the later ones able to exert both tension as well as compression forces.

The analysis of this kind of structures is complex due to their geometrically nonlinear behavior. The aims of this work are to review The Dynamic Relaxation Method as a possible solution for the analysis of these kind of structures. And finally, solve a numerical model example using the afore mentioned method. This example is implemented and tested using MATLAB scripts, in which, the linear and geometric stiffness are considered for each element.

The first part of the work aims to explain the main philosophy of Dynamic Relaxation Method (DRM). Basic theory and equations are introduced, as well as, initial and boundary conditions. After that, a section of the work is dedicated to explain how to set the fictitious masses. Then, the general iteration of the method and kinetic damping technique are presented. Finally, the second part of the work is dedicated to solved a cabled-truss structure and comparison is done between DRM and the non-linear finite element analysis with Newton-Raphson approach used in [6].

Dynamic Relaxation Method

The DRM is a numerical method for solving non-linear analysis of cable-membrane and cabled-truss structures. This numerical method evaluates the response of the structure from the loading stage until the equilibrium position (form finding), achieved by the damping effects. Even though, the method is based on classical dynamics, this technique is used for static structural analysis. It was developed in 1965 by Day [2-5] for the analysis of concrete vessels and Wood [10] has modified it in 2002. The DRM extracts the static solution of the structure by vibration analysis. It traces the temporal response of the structure caused by the excitation of the external forces until it finds the steady state solution.

The importance of this method is related to the computationally efficiency and rapid convergence of the algorithm. The final state, i.e. the geometry that satisfied equilibrium of the structure, is the main objective of the method. One of the most important advantages of this procedure is that no assembled structural stiffness matrix is required. Hence, it is suitable for large complex non-linear problems [9].

The main idea of the method can be explained with an analogy of a simple mass spring system. In the mentioned system, the static steady solution after an excitation, is found when the structure is placed in a position where (i) the kinetic energy is equal to zero and (ii) the sum of external and internal forces in all directions is equal to zero in each node. Finally, this is exactly what the method aims, to find the static solution of a loaded structure.

Basic theory

The method is a direct application of Newton's second law [5]

$$\sum \vec{F} = M * \vec{a} = M * \dot{v}, \tag{1}$$

where,

$$\sum \vec{F} \ \textit{is a sum of all forces in the system},$$

$$M \ \textit{is the mass of the system},$$

$$\vec{a} \ \textit{is the acceleration of the system},$$

$$\dot{v} \ \textit{is a derivative of the velocity respect to time}$$

The motion, the masses, and even the damping are all fictitious. All these parameters are assigned in a way that shortens the transient response, so that the static state of the structure is found efficiently. Mass is assumed to be concentrated at the joints and assigned in each degree of freedom to improve the path through the solution. The motion on each node is analyzed as translation. That is, the motion on each axis [1] is calculated independently from each other. Finally, discrete-time analysis is done sampling uniformly at Δt. During the mentioned time step the acceleration is thus considered as constant, i.e., a linear changed of velocity is assumed [5].

[1] Each axis x, y and z.

Basic Equations

Newton's law describes at any time step Δt the out of balance force in x coordinate direction (same for $y\, and\, z$ direction) at any joint i. Adding an additional viscous damping term proportional to the velocity of the joint the formulation follows as

$$R_{ix}^{t} = M_{ix} * \dot{v}_{ix}^{t} + C_{ix} * v_{ix}^{t}, \quad (2)$$

where,

R_{ix}^{t} *is a residual force at time t at a joint i in the x direction,*

M_{ix} *is a fictitious mass assigned to the joint i in the x direction,*

C_{ix} *is a viscous damping factor assigned tothe joint i in the x direction,*

$\dot{v}_{ix}^{t}$, v_{ix}^{t} *are acceleration and velocity at time t at the joint i in the x direction.*

As the iterative process goes, each iteration updates the coordinates of the joints of the structure, i.e., the geometry is changed. Leapgrog and Velocity Verlet integration are used [11] as follows: the residuals are calculated at the end of the time intervals of 0,Δt, $2\Delta t$ … and velocities are calculated at the mid point of time intervals as $\Delta t/2$, $3\Delta t/2$, $5\Delta t/2$… The final expression of the velocity at each node in each direction is

$$v_{ix}^{t+\Delta t/2} = A * v_{ix}^{t-\frac{\Delta t}{2}} + B_{ix} * R_{ix}^{t}, \qquad (3)$$

$$A = \left(\frac{M_{ix}/\Delta t - C_{ix}/2}{M_{ix}/\Delta t + C_{ix}/2}\right),$$

$$B_{ix} = \left(\frac{1}{M_{ix}/\Delta t + C_{ix}/2}\right).$$

The current coordinates of the joint i at the time $t + \Delta t$ in the x direction[2] can be updated as follows

$$x_{i}^{t+\Delta t} = x_{i}^{t} + \Delta t * v_{ix}^{t+\frac{\Delta t}{2}}. \qquad (4)$$

Once the current coordinates are determined, the contribution of each element connected to the node i is summed with the external force, to obtain the residual force at time $(t + \Delta t)$. The residuals are recalculated as

$$R_{ix}^{t+\Delta t} = F_{ix} + \sum_{j=1}^{m} T_{ixj}^{t+\Delta t}, \qquad (5)$$

$$T_{ixj}^{t+\Delta t} = T_{j}^{t+\Delta t} * \frac{(x_{k}^{t+\Delta t} - x_{i}^{t+\Delta t})}{l_{j}^{t+\Delta t}}, \qquad (6)$$

$$T_{j}^{t+\Delta t} = \frac{E_j A_j}{l_j^0}\left(l_j^{t+\Delta t} - l_j^0\right) + T_j^0, \qquad (7)$$

where,

F_{ix} *is an applied force at the joint i in the x direction,*

$T_{ixj}^{t+\Delta t}$ *is a current internal force of the element j linked to the joint i in the x direction,*

$T_{j}^{t+\Delta t}$ *is a current internal force of the element j,*

E_j *is a elastic modulus of the element j,*

[2] Also for the y and z direction.

A_j *is a cross sectional area of the element* j,

l_j^0 *is a initial lenght of the linking element* j,

$l_j^{t+\Delta t}$ *is the current lenght of the element* j *at the time* $t + \Delta t$,

T_j^0 *is the initial prestress in the linking element* j,

m *represents the total elements connected to the joint* i,

$\frac{\left(x_k^{t+\Delta t} - x_i^{t+\Delta t}\right)}{l_j^{t+\Delta t}}$ *will give the direction of the force at joint* i, *where the element is linking the joints* k *and* i.

Note that all the previous expressions can be formulated for y and z directions. If the linking element is a cable and the internal force $T_m^{t+\Delta t}$ is less than zero then $T_m^{t+\Delta t}$ must be set equal to zero [9].

Finally, the stopping criteria is set up by the residuals and the kinetic energy of the structure

$$U_k = \sum_{i=1}^{n} \sum_{j=\{x,y,z\}} M_{ij} * v_{ij}^2, \qquad (8)$$

where,

U_k *is the kinetic energy of the structure*

n *corresponds to the numer of nodes*,

M_{ij} *is a mass of the joint* i *in the* j *direction and*

v_{ij} *is the velocity of the joint* i *in the* j *direction*.

Initial and Boundary Conditions

To ensure that at the time zero (i) the velocity is equal to zero $v_{ix}^0 = 0$ and (ii) the residuals are equal to the external forces applied $R_{ix}^0 = F_{ix}$, the initial velocity at the time $\Delta t/2$ must be calculated as

$$v_{ix}^{\Delta t/2} = \frac{\Delta t}{2M_{ix}} * F_{ix}. \qquad (9)$$

Boundary conditions are imposed assigning large masses to the fixed joints, in the direction on the degree of freedom of interest, as follows

$$M_{in} = 10^{48},$$

where,

M_{in} *is the mass of the joint* i *in the* n *direction*.

Also, it is possible to set to zero the residuals forces, velocities or displacements of the fixed degrees of freedom of the particular joint i [9].

Fictitious Masses

It was shown by Barnes [1] that, for any time step Δt, the convergence is ensured using the following equation for the fictitious masses

$$M_{ix} = \frac{\Delta t}{2} * S_{ix}, \qquad (10)$$

$$S_m = \left(\frac{E_m * A_m}{l_m^0} + \frac{T_m^{t+\Delta t}}{l_m^{t+\Delta t}}\right), \qquad (11)$$

where,

$$S_{ix} \text{ is the largest direct stiffness of an element linked to the joint } i \text{ in the } x \text{ direction and}$$

$$S_m \text{ is the elastic and geometric stiffness of the linking cable or truss element } m.$$

General Iteration

The general procedure can be shown in Figure 1 reproduced from [9]

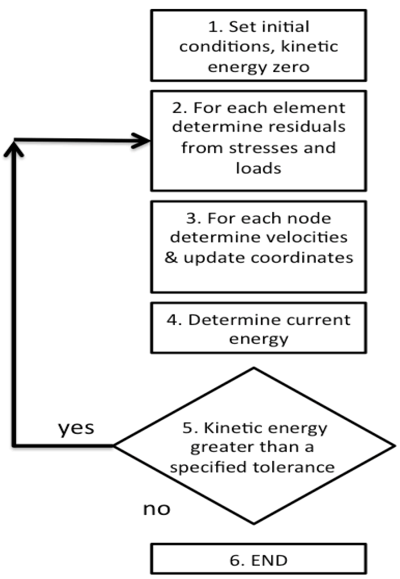

Figure 1: General Procedure

Kinetic Damping

The kinetic damping is an alternative to viscous damping. The whole structure is analyzed while undamped oscillations are done. The objective is to measure the kinetic energy in each iteration and look for a peak of energy. When an energy peak is detected the static equilibrium condition has just been passed. Then, all the velocities are reset to zero and the current coordinates at the time $(t + \Delta t/2)$ of the nodes are set to the position in $(t - \Delta t/2)$ to achieve convergence [9]. Further iterations must be done through further falls of the kinetic energy, till stopping criteria is fulfilled. Then, the equation (3) is changed as

$$v_{ix}^{t+\Delta t/2} = A * v_{ix}^{t-\frac{\Delta t}{2}} + B_{ix} * R_{ix}^{t}, \quad (12)$$

where,

$$A = 1,$$

$$B_{ix} = \frac{\Delta t}{M_{ix}}.$$

To return to the position of an equilibrium, the coordinates[3] must be updated as follows

$$x_i^{t-\Delta t/2} = x_i^{t+\Delta t/2} - \frac{3\Delta t * v_{ix}^{t+\frac{\Delta t}{2}}}{2} + \frac{\Delta t^2 R_{ix}^{t}}{2M_{ix}}. \quad (13)$$

When restarting the analysis, the velocities in the fist time step are calculated as

$$v_{ix}^{\frac{\Delta t}{2}} = \frac{\Delta t}{2M_{ix}} * R_{ix}^{t*}, \quad (14)$$

where,

$$R_{ix}^{t*} \text{ are the residuals calculated from the } x_i^{t-\Delta t/2} \text{ positions.}$$

Control Criteria

The Dynamic Relaxation Method has two main control criteria: (i) kinetic energy level and (ii) the size

[3] In x, y and z direction.

of the residuals, i.e., the difference between applied external forces and internal reaction forces. With the mentioned criteria, it can be guaranteed that the solution found corresponds to the steady static solution.

Example

The example solved in this work is taken from the optimized cable truss structure from GS10 published in [6]. Figure 2 shows the topology of the cable-truss structure (dash line represents a cable) and the load condition. We will analyze the response of the structure in two different cases: (i) with prestress and (ii) without prestress. In both cases, we will make the comparison of stresses and deflections between DRM and the non-linear finite element analysis with Newton-Raphson approach used in [6]. Table 1 shows the materials and properties used. Table 2 shows the cross-sectioned areas of each element.

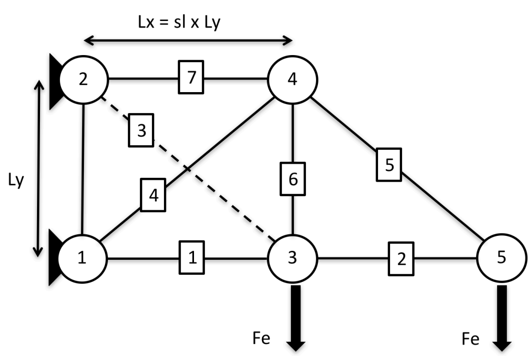

Figure 2: Optimized cabled-truss obtained from GS10

Table 1: Geometric and material parameters

Parameter	Value
Structural Height (L_y); $sl = 1$	9.144 m
Loading force (F_e)	448.2 kN
Initial cable strain $(istrn)$	8x10^{-4}
Admissible areas (A_{ad})	From 645.16 to 19359.00 with increment 645.16 mm^2
Elasticity modulus (E_{Al}^{a})	6.895x10^4 MPa
Material density (ρ_{al})	2.768 g/cm^3
Tensile Modulus (τ_{cf}^{b})	20.685x10^{4}MPa
Material density (ρ_{cf})	0.4613 g/cm^3
Max. Allowable stress (σ_j^{max})	1.724x10^{2}MPa
Max. Allowable displacement (q_w^{max})	50.8 mm

[a] Aluminum
[b] Carbon – nanotube fibers

Table 2: Areas obtained for the optimized solution using S10

A_i (mm^2)	Cable-truss
A1	13548.36
A2	10322.56
A3	12258.04
A4	12903.20
A5	13548.36
A6	2580.64
A7	18064.48

The load condition is a force $F_e = 448.2\ kN$ applied on the nodes 3 and 5 as depicted in Figure 2.

Results and Discussion

Case (i) Structure with prestress

The typical behavior of the DRM used with kinetic damping is obtained. It is shown in Figure 3 and Figure 4 with the logarithmic scale. It is appreciable one large peak of energy, related with high frequency modes caused by large out-of-balance forces and, a fast convergence with slight and rapid energy peaks near the equilibrium state.

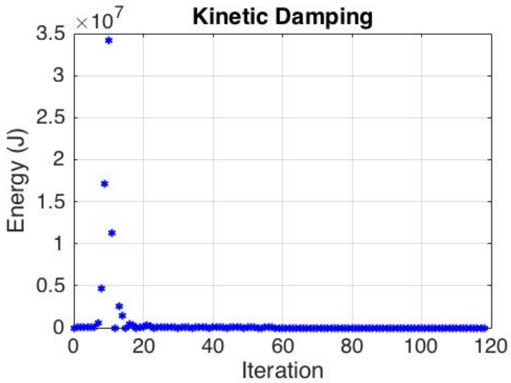

Figure 3: Kinetic Damping

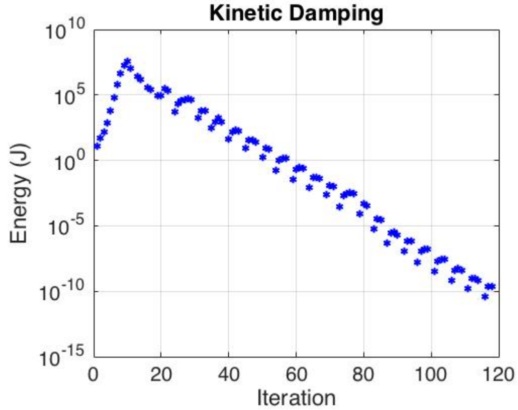

Figure 4 Kinetic Damping (Logarithmic Scale)

The final displacements of the nodes 3, 4 and 5 are depicted in Figures 5 and 6. Numerical values are shown in Table 3. All of the values are inside the allowable limits. The signs follow the convention: (i) to the right direction is positive (+) and (ii) to top direction is positive (+).

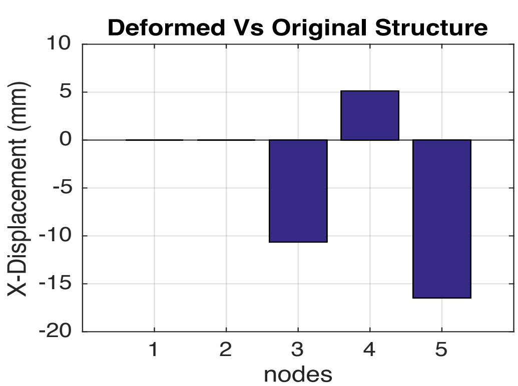

Figure 5: Displacement in the x direction

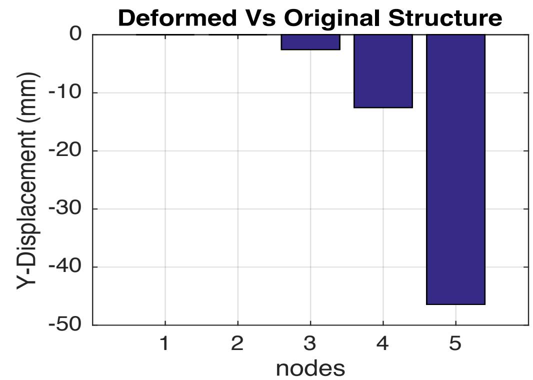

Figure 6: Displacement in the y direction

Table 3: Final displacements per node

Node	x (mm)	y (mm)
1	0	0
2	0	0
3	-10.6386	-2.5497
4	5.1313	-12.5353
5	-16.4674	-46.4213

As nodes 1 and 2 are fixed in the x and y direction, no displacements were expected.

The forces and stresses of each member are shown in Figures 6 and 7. Tension in the member is positive (+). Cable member 3 is in tension as expected. Numerical values are shown in the Tables 4 and 5.

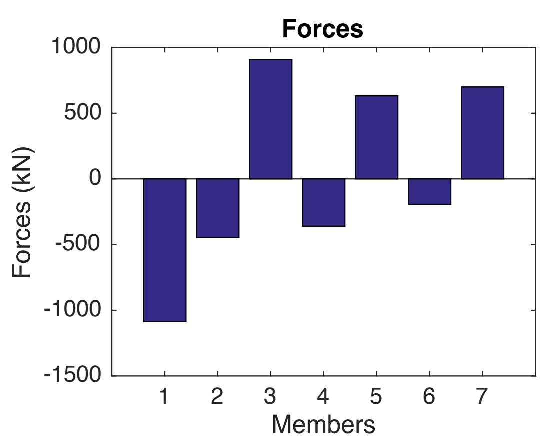

Figure 7: Forces in each Member

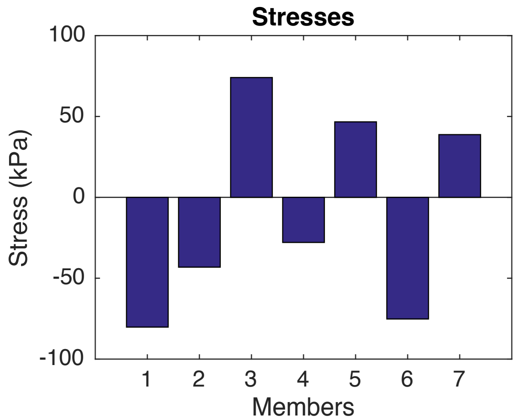

Figure 8: Stresses in each Member

Table 4: Forces in each member

Member	Force (kN)
1	-1.0868×10^3
2	-0.4455×10^3
3	0.9076×10^3
4	-0.3598×10^3
5	0.6319×10^3
6	-0.1940×10^3
7	0.7001×10^3

Table 5: Stresses in each member

Member	Stress (MPa)
1	-80.2172
2	-43.1575
3	74.0428
4	-27.8825
5	46.6431
6	-75.1936
7	38.7575

Finally, in Table 6 we see the comparison between DRM and the non-linear finite element analysis with Newton-Raphson approach used in [6].

Table 6: Comparison of Stresses

	Stresses (MPa)		
Member	**ANSYS**	**Dynamic Relaxation Method**	**Ect (%)**
1	-80.5650	-80.2172	0.4336
2	-43.0920	-43.1575	0.1517
3	74.6110	74.0428	0.7674
4	-26.6260	-27.8825	4.5063
5	46.4310	46.6431	0.4546
6	-78.2320	-75.1936	4.0408
7	38.0720	38.7575	1.7687

Only in members 4 and 6 it can be seen an error near 4.50%

Case (ii) Structure without prestress

The typical behavior of the DRM used with kinetic damping is obtained. It is shown in Figure 9 and Figure 10; note the logarithmic scale. Also, it is appreciable one large peak of energy, and rapid energy peaks near the equilibrium state.

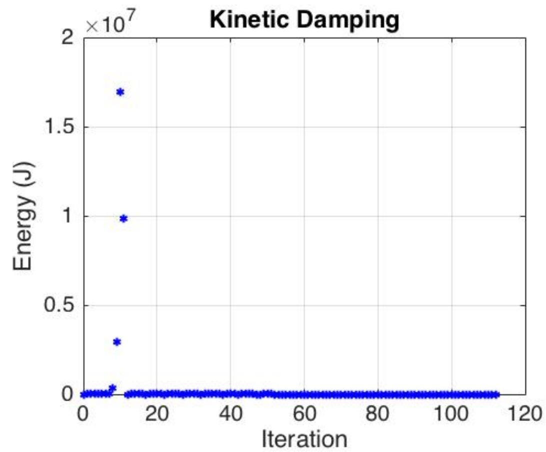

Figure 9: Kinetic Damping

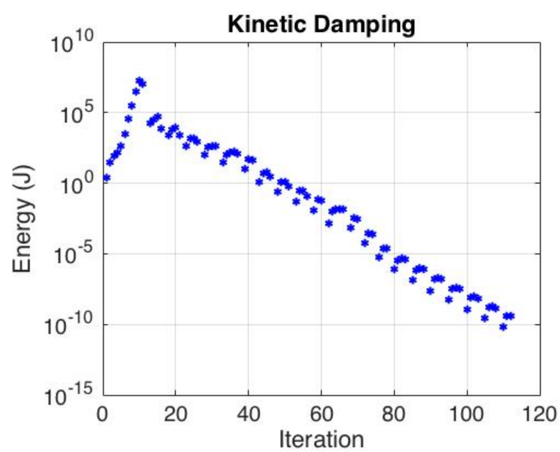

Figure 10: Kinetic Damping (Logarithmic Scale)

The final displacements of the nodes 3, 4 and 5 are depicted in Figures 11 and 12. Numerical values are shown in Table 6. All of the values are inside the allowable limits. The signs follow the convention: (i) to the right direction is positive (+) and (ii) to the top direction is positive (+).

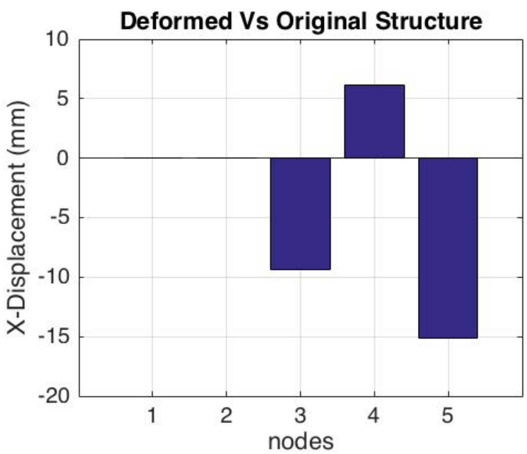

Figure 11: Displacement in the x direction

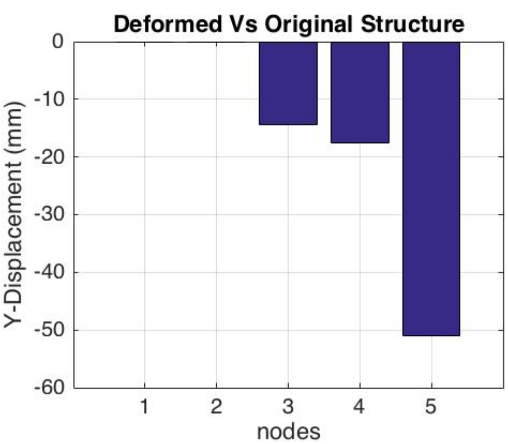

Figure 12 Displacement in the y direction

Table 7: Final displacements per node

Node	x (mm)	y (mm)
1	0	0
2	0	0
3	-9.3190	-14.4649
4	6.1227	-17.4824
5	-15.1160	-51.0110

The forces and stresses of each member are shown in Figures 13 and 14. Tension in the member is positive (+). Cable member 3 is in tension as expected. Numerical values are shown in the Tables 7 and 8.

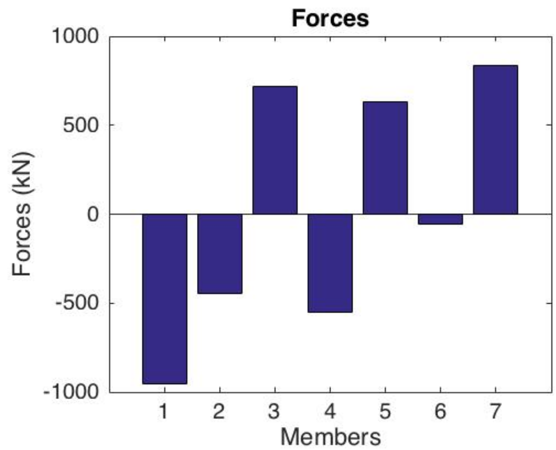

Figure 13: Forces in each member

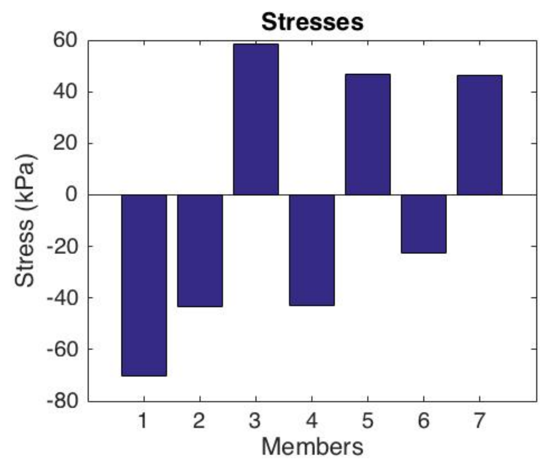

Figure 14: Stresses in each member

Table 8: Forces in each member

Member	Force (kN)
1	-950.8715
2	-445.5277
3	715.5994
4	-551.8836
5	631.9612
6	-58.4631
7	836.2706

Table 9: Stresses in each member

Member	Stress (MPa)
1	-70.1835
2	-43.1606
3	58.3780
4	-42.7711
5	46.6449
6	-22.6545
7	46.2946

Comparing case (i) and (ii), it is appreciable that the use of prestress made a redistribution of forces and stresses in the whole structure. This is even more evident looking into the displacements between these two cases.

Finally, the comparison of displacements between Dynamic Relaxation Method and the ones obtained in [6] without prestress, are depicted in Table 10 and 11.

Table 10: Comparison of displacements in horizontal direction

Displacement x (mm)			
Node	**ANSYS**	**Dynamic Relaxation Method**	**Ect (%)**
3	-9.2724	-9.3190	0.5001
4	6.1077	6.1227	0.2450
5	-14.9874	-15.1160	0.8508

Table 11: Comparison of displacements in vertical direction

Displacement y (mm)			
Node	**ANSYS**	**Dynamic Relaxation Method**	**Ect (%)**
3	-14.3981	-14.4649	0.4618
4	-17.3621	-17.4824	0.6881
5	-50.7732	-51.0110	0.4662

The numerical error is less than 0.90%, validating the results and the methodology used.

Conclusions

In this work, the DRM was implemented and tested. The method reviewed is based on the application of Newton's second law and, a fictitious mass-damping model. Kinetic damping was used to achieve a stable and rapid convergence, with less than 120 iterations in each case studied.

Finally, the method was evaluated by a typical example, with and without prestress. The numerical error obtained in the first case, was negligible in comparison to the non-linear finite element analysis using the Newton-Raphson approach proposed in [6]. The Dynamic Relaxation Method, had only and error near 4.50% in two of the members of the structure. On the other hand, in the second case, the numerical error was less than 0.90%, proving the validity and applicability of Dynamic Relaxation Method to cabled-truss structures.

Acknowledgement

The author wishes to express his appreciation for the support of Professor Matej Lepš. Czech Technical University, Prague, Czech Republic.

References

[1] Barnes, M. (1977). *"Form-finding and analysis of tension space structures by dynamic relxation",PhD Thesis.* London, England: City University.

[2] Day, A. (1965, January). "An introuction to dynamic relaxation". *The Engineer* .

[3] Day, A., & Bunce, J. (1970). "Analysis of cable networks by dynamic relazation". *Civil Engineering and Public Works Review* , 383-386.

[4] Day, A., & Bunce, J. (1980). "Form finding, control and modification for tension structures". *ARUP Journal* , 19-20.

[5] Day, A., & Bunce, J. (no date). "The analysis of hanging roofs". *ARUP Journal* , 30-31.

[6] Finotto, V., da Silva, W., Valasek, M., & Stemberk, P. (2013, May 14). "Hybrid fuzzy-genetic system for optimising cabled-truss structures". *Advances in Engineering Software*.

[8] Hüttner, M., Máca , J., & Fajman , P. (2012). "Numerical Analysis of Cable Structures". (B. Topping, Ed.) *Proceedings of the Eleventh International Conference on Computational Structures Technology* .

[9] Topping , B., & Ivanyi, P. (2005). "Computer Aided Design of Cable Membrane Structures". In B. Topping, & P. Ivanyi, *"Computer Aided Design of Cable Membrane Structures"* (pp. 39-40). SAXE-COBURG PUBLICATIONS.

[10] Wood, R. (2002). 'A simple technique for controlling element distorsion in dynamic relaxation form-finding of tension membranes". *Computers & Structures , 80*, 2115-2120.

[11] Young, P. (2014, April 21). The leapfrog method and other “symplectic” algorithms for integrating Newton’s laws of motion. *115/242* . Retrieved April 4, 2016, from http://young.physics.ucsc.edu/115/leapfrog.pdf