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AN IMPROVED MODEL FOR TORSION OF ELASTIC BEAMS WITH THIN-WALLED CLOSED CROSS SECTIONS

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Abstract

The main purpose of this contribution is to present a newly derived modified torsional model—an elastic beam model for non-uniform torsion of beams with thin-walled closed cross sections. The derivation is based on previous work of the authors [4] on the modified torsional model based on the Hellinger-Reissner variational principle. As the derived model is a refinement of a widely used model of the same phenomena [2], we demonstrate the difference in predictions of both models using an example of a prismatic beam with rectangular thin-walled cross section which is clamped on both sides and its right support is rotated by a prescribed angle. For this particular example, the analytical solution of governing equations of both theories is found and the results are compared. The influence of basic geometrical parameters on the difference in predictions of both models is investigated.

Keywords

non-uniform torsion, warping function, elastic, Hellinger-Reissner variational principle, thin-walled beam, closed cross section

Contents

1 Introduction

In technical literature and engineering practice, non-uniform torsion of beam with closed cross-section is often described by a model which we refer to as classical, based on equations (explained in more detail later in the text)

$$f'''(x) - \beta^2 f'(x) = \frac{\beta^2}{GJ} m_x(x),$$

$$\nu f'(x) - \varphi''(x) = \frac{1}{GI_c} m_x(x),$$

which could be found in [1, chapter 2.5.4, p.247] or [2, Chapter 6, p.130]. The history of this equation dates back to famous publication of Umanskij [3] from 1939, who as first considered problem of non-uniform torsion of beams with closed cross section. We would like to mention on this place that the differential equation describing non-uniform torsion of beams with "I" cross-section was derived by S.P. Timošenko in 1905 and generalized by many other authors such as C.Weber or H.Wagner for beams with arbitrary open cross-section. Maybe the most famous version comes from V.Z. Vlasov from 1940. [2, Chapter 1, p.14]

The purpose of this work is to derive an alternative model (in this work referred as modified torsional model) for torsion of beams with thin-walled closed cross section with restrained warping analogous to the generally known widely used model described in [1, chapter 2.5.4, p.247] or [2, Chapter 6, p.130]. The modified torsional model will be derived utilizing the Hellinger - Reissner variational principle. To achieve this, it wil be used the general format of governing equations of the model given by Hellinger - Reissner variational principle for prescribed approximation of displacement and stress field derived in the bachelor thesis [4] of the author of this work.

Afterwards, our aim is to demonstrate the difference between the classical and the modified model. To perform this, we will pick an example of prismatic beam with rectangular thin-walled cross section which is clamped on both sides while its right support is twisted by the prescribed angle. This particular case allows us to find analytical solution so we can eliminate the influence of numerical error, while being very simple and generally representative from the engineering point of view.

For this particular example, we will investigate effective torsional stiffness k_{eff} (defined for the purpose of easier comparison) predicted by both models, looking for the case where the difference of both models will be of engineering relevance.

2 Derivation of modified torsional model for elastic beams with thin-walled closed cross section

We will now derive the modified torsional model for elastic beams with thin-walled closed cross section. The model will be derived using Hellinger - Reissner variational principle. The general format of governing equations of the model given by Hellinger - Reissner variational principle for prescribed approximation of displacement field and stress field is taken from [4]. For the purpose of convenience of the reader, the general form of governing equations and corresponding notation from [4] is reminded hereinafter. Unknown functions of following equations are incorporated in vector $\mathbf{d}(x)$. This vector contains unknown functions depending only on Cartesian coordinate x [1], which are used to approximate the displacement field.

Requiring zero value of variation of Hellinger-Reissner functional with respect to $\mathbf{d}(x)$ and $\mathbf{c}(x)$ (for more thorough description see [4]), we obtain two sets of differential equations (1) and (2)

$$-E\mathbf{A}_{nn}\mathbf{d}''(x) + \mathbf{A}_{\sigma B}^T\mathbf{c}(x) - \mathbf{A}_{\sigma u}^T\mathbf{c}'(x) = \mathbf{f}(x), \tag{1}$$

$$\mathbf{A}_{\sigma B}\mathbf{d}(x) + \mathbf{A}_{\sigma u}\mathbf{d}'(x) - \frac{1}{G}\mathbf{A}_{\sigma\sigma}\mathbf{c}(x) = 0 \tag{2}$$

We can easily rewrite equation (2) as (assuming existence of $\mathbf{A}_{\sigma\sigma}^{-1}$)

$$\mathbf{c}(x) = G\mathbf{A}_{\sigma\sigma}^{-1}\left(\mathbf{A}_{\sigma B}\mathbf{d}(x) + \mathbf{A}_{\sigma u}\mathbf{d}'(x)\right) \tag{3}$$

When substitung (3) into (1) we reduce our set of differential equations to a compact form (4)

$$-\left(E\mathbf{A}_{nn} + G\mathbf{A}_{\sigma u}^T\mathbf{A}_{\sigma\sigma}^{-1}\mathbf{A}_{\sigma u}\right)\mathbf{d}''(x) + G\left(\mathbf{A}_{\sigma B}^T\mathbf{A}_{\sigma\sigma}^{-1}\mathbf{A}_{\sigma u} - \mathbf{A}_{\sigma u}^T\mathbf{A}_{\sigma\sigma}^{-1}\mathbf{A}_{\sigma B}\right)\mathbf{d}'(x) + \\ + G\mathbf{A}_{\sigma B}^T\mathbf{A}_{\sigma\sigma}^{-1}\mathbf{A}_{\sigma B}\mathbf{d}(x) = \mathbf{f}(x). \tag{4}$$

Where $\mathbf{f}(x)$ is the vector of prescribed internal loads (see more detailed description in [4]). Notation used in previous equations has following meaning

$$\mathbf{y} = \begin{pmatrix} y \\ z \end{pmatrix},$$

[1] x-axis is presumed to coincide with the centerline of the beam.

$$\mathbf{A}_{nn} = \int_{\mathcal{S}} \mathbf{n}_u^T(s)\mathbf{n}_u(s)t(s)\,\mathrm{d}s, \tag{5}$$

$$\mathbf{A}_{\sigma B} = \int_{\mathcal{S}} \mathbf{n}_\tau^T(s)\frac{\mathrm{d}\mathbf{n}_u(s)}{\mathrm{d}s}t(s)\,\mathrm{d}s, \tag{6}$$

$$\mathbf{A}_{\sigma u} = \int_{\mathcal{S}} \mathbf{n}_\tau^T(s)\mathbf{n}_v(s)t(s)\,\mathrm{d}s, \tag{7}$$

$$\mathbf{A}_{\sigma\sigma} = \int_{\mathcal{S}} \mathbf{n}_\tau^T(s)\mathbf{n}_\tau(s)t(s)\,\mathrm{d}s. \tag{8}$$

Where $\mathbf{n}_u(s)$ describes out of plane displacement of cross section in direction of x and $\mathbf{n}_v(s)$ describes in plane displacement of cross section. In terms of this work, local curve coordinate s is defined as the distance from a selected origin measured along the centerline of the section. The centerline is assumed to be a closed curve of total length $s_{\max}$.

To derive the model with the prepared aparatus, we therefore need to find suitable approximation of displacement field and stress field. For displacement field, we will adopt the same approximation as given in [1, chapter 2.5.4, p.247] or [2, Chapter 6, p.130] except for convenience we consider the warping function $\chi(x)$ with the opposite sign than the authors does for their warping function $f(x)$ in [1, chapter 2.5.4, p.247] or [2, Chapter 6, p.130]. Therefore it holds $f(x) = -\chi(x)$. Displacement field approximation is so given as:

$$\begin{aligned} u(x,s) &= \chi(x)\psi(s) \\ v_s(x,s) &= \phi_x(x)\rho(s) \end{aligned} \tag{9}$$

where u is displacement in direction of Cartesian x-axis and v_s is displacement tangential to the centerline of the cross section. $\varrho(s)$ is the perpendicular distance from shear centre to the point of centerline specified by coordinate s (it could aquire both non-positive and non-negative value - see [4]). $\psi(s)$ represents presumed warping function of the cross section describing out of plane displacement of cross section in direction of Cartesian x-axis.

$$\psi(s) = \psi_0 + \omega(s) - \frac{\Omega}{\Pi}\pi(s) \tag{10}$$

Here, ω is the sectorial coordinate defined by the usual expression

$$\omega(s) = \int_0^s \rho(\bar{s})\,\mathrm{d}\bar{s} \tag{11}$$

Constant ψ_0 will be later selected to decouple axial effects from torsion. The correction term in (10) contains function

$$\pi(s) = \int_0^s \frac{\mathrm{d}\bar{s}}{t(\bar{s})} \tag{12}$$

where t denotes the thickness of the section. Quantities Ω and Π are simply the values obtained by integrating ρ or $1/t$ over the entire centerline:

$$\Omega = \omega(s_{\max}) = \int_0^{s_{\max}} \rho(\bar{s})\,\mathrm{d}\bar{s} \tag{13}$$

$$\Pi = \pi(s_{\max}) = \int_0^{s_{\max}} \frac{\mathrm{d}\bar{s}}{t(\bar{s})} \tag{14}$$

Note that

$$\psi(s_{\max}) = \psi_0 + \omega(s_{\max}) - \frac{\Omega}{\Pi}\pi(s_{\max}) = \psi_0 + \Omega - \frac{\Omega}{\Pi}\Pi = \psi_0 = \psi(0) \tag{15}$$

which is necessary for continuity of function ψ, since the points where $s = 0$ and $s = s_{\max}$ physically coincide. It is good to select constant ψ_0 such that

$$\int_0^{s_{\max}} \psi(\bar{s})t(\bar{s})\,\mathrm{d}\bar{s} = 0 \tag{16}$$

because then the axial effects are decoupled from torsion. This can be achieved by setting

$$\psi_0 = \frac{1}{A}\left(\frac{\Omega}{\Pi}\int_0^{s_{\max}} \pi(\bar{s})t(\bar{s})\,\mathrm{d}\bar{s} - \int_0^{s_{\max}} \omega(\bar{s})t(\bar{s})\,\mathrm{d}\bar{s}\right) \tag{17}$$

where

$$A = \int_0^{s_{\max}} t(\bar{s})\,\mathrm{d}\bar{s} \tag{18}$$

is the sectional area.

Now we will try to find suitable approximation for the stress field. For this purpose we will inspire ourselves by the equations of static equilibrium of our thin-walled beam.

In the absence of body and surface forces, the equilibrium equation integrated along the thickness can be written as

$$\frac{\partial}{\partial x}\left(t(s)\sigma(x,s)\right) + \frac{\partial}{\partial s}\left(t(s)\tau_{xs}(x,s)\right) = 0 \tag{19}$$

We consider linear elastic behaviour of the material desribed by Young's modulus E and shear modulus G. Substituting

$$\sigma(x,s) = E\frac{\partial u(x,s)}{\partial x} = E\chi'(x)\psi(s) \tag{20}$$

we obtain

$$\frac{\partial}{\partial s}\left(t(s)\tau_{xs}(x,s)\right) = -t(s)E\chi''(x)\psi(s) \tag{21}$$

from which

$$\tau_{xs}(x,s)=\frac{t(0)\tau_{xs}(0)}{t(s)}-\frac{E\chi''(x)}{t(s)}\int_0^s t(\bar{s})\psi(\bar{s})\,\mathrm{d}\bar{s} \tag{22}$$

This motivates the shear stress approximation in the form

$$\tau_{xs}(x,s)=C_1(x)f_1(s)+C_2(x)f_2(s) \tag{23}$$

where

$$\begin{aligned} f_1(s) &= \frac{1}{t(s)} \\ f_2(s) &= -\frac{1}{t(s)}\int_0^s t(\bar{s})\psi(\bar{s})\,\mathrm{d}\bar{s}\equiv\frac{\bar{S}_\psi(s)}{t(s)} \end{aligned} \tag{24}$$

Note that the definition of quantity $\bar{S}_\psi$ includes the negative sign:

$$\bar{S}_\psi(s)=-\int_0^s t(\bar{s})\psi(\bar{s})\,\mathrm{d}\bar{s}=\int_s^{s_{\max}} t(\bar{s})\psi(\bar{s})\,\mathrm{d}\bar{s} \tag{25}$$

In terms of the general notation introduced in [4], our present approximations are described by

$$\mathbf{d}=\begin{pmatrix}\phi_x\\ \chi\end{pmatrix},\qquad \mathbf{n}_u=\begin{pmatrix}0\\ \psi\end{pmatrix},\qquad \mathbf{n}_v=\begin{pmatrix}\rho\\ 0\end{pmatrix},\qquad \mathbf{n}_\tau=\begin{pmatrix}1/t\\ \bar{S}_\psi/t\end{pmatrix},\qquad \mathbf{c}=\begin{pmatrix}C_1\\ C_2\end{pmatrix}$$

Based on these column matrices, the square matrices defined in [4] can be evaluated:

$$\mathbf{A}_{nn}=\begin{pmatrix}0 & 0\\ 0 & I_\psi\end{pmatrix},\qquad I_\psi=\int_0^{s_{\max}}\psi^2(s)t(s)\,\mathrm{d}s \tag{26}$$

$$\mathbf{A}_{\sigma B}=\begin{pmatrix}0 & 0\\ 0 & I_A\end{pmatrix},\qquad I_A=\int_0^{s_{\max}}\bar{S}_\psi(s)\frac{\mathrm{d}\psi(s)}{\mathrm{d}s}\,\mathrm{d}s \tag{27}$$

$$\mathbf{A}_{\sigma u}=\begin{pmatrix}\Omega & 0\\ I_B & 0\end{pmatrix},\qquad I_B=\int_0^{s_{\max}}\bar{S}_\psi(s)\varrho(s)\,\mathrm{d}s \tag{28}$$

$$\mathbf{A}_{\sigma\sigma}=\begin{pmatrix}\Pi & I_D\\ I_D & I_C\end{pmatrix},\qquad I_D=\int_0^{s_{\max}}\frac{\bar{S}_\psi(s)}{t(s)}\,\mathrm{d}s,\qquad I_C=\int_0^{s_{\max}}\frac{\bar{S}_\psi^2(s)}{t(s)}\,\mathrm{d}s \tag{29}$$

Knowing general form of governing equations from [4] - equations (4), we can now write the governing equations of the modified model as the set of ODEs

$$K_1\varphi''(x)-K_3\chi'(x)=m_x(x) \tag{30}$$

$$K_2\chi''(x)+K_3\varphi'(x)+K_4\chi(x)=b(x) \tag{31}$$

where,

$$K_1=-\frac{G}{\Pi I_c-I_d^2}\left(\Omega\left(-I_bI_d+I_c\Omega\right)+I_b\left(-I_d\Omega+I_b\Pi\right)\right) \tag{32}$$

$$K_2 = -EI_\psi \tag{33}$$

$$K_3 = \frac{G}{\Pi I_c - I_d^2}\left(-I_a I_d \Omega + I_a I_b \Pi\right) \tag{34}$$

$$K_4 = \frac{G}{\Pi I_c - I_d^2} I_a^2 \Pi \tag{35}$$

We can rewrite equation (31) as

$$\varphi'(x) = \frac{1}{K_3}\left(b(x) - K_2\chi''(x) - K_4\chi(x)\right) \tag{36}$$

When substituting (36) into (30), we can reduce this system of two second-order ODEs to one third-order ODE which reads

$$K_1\frac{1}{K_3}\left(b'(x) - K_2\chi'''(x) - K_4\chi'(x)\right) - K_3\chi'(x) = m_x(x) \tag{37}$$

or in the rewritten form

$$-\frac{K_1K_2}{K_3}\chi'''(x) - \left(\frac{K_1K_4}{K_3} + K_3\right)\chi'(x) = m_x(x) - \frac{K_1}{K_3}b'(x) \tag{38}$$

We can now also derive expression for total torsional moment, which is defined by

$$M_k(x) = \int_0^{s_{\max}} \tau_{xs}(x,s)\rho(s)t(s)\,\mathrm{d}s. \tag{39}$$

Now because we presumed the distribution of shear stress described by

$$\tau_{xs}(x,s) = C_1(x)\frac{1}{t(s)} + C_2(x)\frac{\bar{S}_\psi(s)}{t(s)}, \tag{40}$$

we obtain

$$\begin{aligned} M_k(x) &= \int_0^{s_{\max}} \tau_{xs}(x,s)\rho(s)t(s)\,\mathrm{d}s = \\ &= \int_0^{s_{\max}} \left(C_1(x)f_1(s) + C_2(x)f_2(s)\right)\rho(s)t(s)\,\mathrm{d}s = \\ &= \int_0^{s_{\max}} \left(C_1(x)\frac{1}{t(s)} + C_2(x)\frac{\bar{S}_\psi(s)}{t(s)}\right)\rho(s)t(s)\,\mathrm{d}s = \\ &= C_1(x)\int_0^{s_{\max}} \rho(s)\,\mathrm{d}s + C_2(x)\int_0^{s_{\max}} \bar{S}_\psi(s)\rho(s)t(s)\,\mathrm{d}s = \\ &= C_1(x)\Omega + C_2(x)I_B \end{aligned} \tag{41}$$

Based on the theory from [4] we can also express $C_1(x)$ and $C_2(x)$ as

$$C_1(x) = \frac{G(\chi(x)I_AI_D + \varphi'(x)I_BI_D - \varphi'(x)I_C\Omega)}{I_D^2 - I_C\Pi}, \tag{42}$$

$$C_2(x) = \frac{G(\chi(x) I_A \Pi + \varphi'(x) I_B \Pi - \varphi'(x) I_D \Omega)}{-I_D^2 + I_C \Pi}, \tag{43}$$

and we obtain the formula for total torsional moment

$$\begin{aligned} M_k(x) &= G\chi(x)\left(\frac{I_A I_B \Pi}{I_C \Pi - I_D^2} - \frac{I_A I_D \Omega}{I_C \Pi - I_D^2}\right) + \\ &+ G\varphi'(x)\left(\Omega\left(\frac{I_C \Omega}{I_C \Pi - I_D^2} - \frac{I_B I_D}{I_C \Pi - I_D^2}\right) + I_B\left(\frac{I_B \Pi}{I_C \Pi - I_D^2} - \frac{I_D \Omega}{I_C \Pi - I_D^2}\right)\right). \end{aligned} \tag{44}$$

3 Simplification of various relations for modified torsional model

Some of the integrals defined above can be simplified by using the definition of $\bar{S}_\psi$ and then changing the order of integration variables. Let us show this procedure in detail for I_B defined in (28):

$$\begin{aligned} I_B &= \int_0^{s_{\max}} \bar{S}_\psi(s)\varrho(s)\,\mathrm{d}s = -\int_0^{s_{\max}}\int_0^{s} \psi(\bar{s})t(\bar{s})\,\mathrm{d}\bar{s}\,\varrho(s)\,\mathrm{d}s = \\ &= -\int_0^{s_{\max}}\int_{\bar{s}}^{s_{\max}} \varrho(s)\,\mathrm{d}s\,\psi(\bar{s})t(\bar{s})\,\mathrm{d}\bar{s} = -\int_0^{s_{\max}} (\omega(s_{\max}) - \omega(\bar{s}))\,\psi(\bar{s})t(\bar{s})\,\mathrm{d}\bar{s} = \\ &= -\Omega\int_0^{s_{\max}} \psi(\bar{s})t(\bar{s})\,\mathrm{d}\bar{s} + \int_0^{s_{\max}} \omega(\bar{s})\psi(\bar{s})t(\bar{s})\,\mathrm{d}\bar{s} = \int_A \omega\psi\,\mathrm{d}A \end{aligned} \tag{45}$$

Here we have exploited the fact that $\int_A \psi\,\mathrm{d}A = 0$. In a similar fashion, it can be shown that

$$I_A = \int_0^{s_{\max}} \bar{S}_\psi(s)\frac{\mathrm{d}\psi(s)}{\mathrm{d}s}\,\mathrm{d}s = \int_A \psi^2\,\mathrm{d}A = I_\psi \tag{46}$$

$$I_D = \int_0^{s_{\max}} \bar{S}_\psi(s)\frac{1}{t(s)}\,\mathrm{d}s = \int_A \pi\psi\,\mathrm{d}A \tag{47}$$

Therefore, one could write I_ψ instead of I_A and replace I_B and I_D by $I_{\omega\psi}$ and $I_{\pi\psi}$.

4 Classical solution of non-uniform torsion of beam with closed cross-section

We will now remind the governing equations of model of non-uniform torsion of beam with closed cross section as found in [1, chapter 2.5.4, p.247] or [2, Chapter 6, p.130]. We will refer to this model as to classical torsional model. In following equations, $f(x)$ has the same meaning as $\chi(x)$ in equations (30) and (31).

$$f'''(x) - \beta^2 f'(x) = \frac{\beta^2}{GJ} m_x(x) \tag{48}$$

$$\nu f'(x) - \varphi''(x) = \frac{1}{GI_c} m_x(x) \tag{49}$$

$$\beta^2 = \nu \frac{GJ}{EI_\psi}, \qquad \nu = 1 - \frac{J}{I_c} \tag{50}$$

$$I_\psi = \int_0^{s_{\max}} \psi^2(s) t(s)\,\mathrm{d}s, \qquad I_c = \int_0^{s_{\max}} \rho^2(s)\,\mathrm{d}s, \qquad J = \frac{\Omega^2}{\Pi} \tag{51}$$

Total torsional moment can be calculated as (see [2, Chapter 6, p.128])

$$M_k^{VL}(x) = G\chi(x)(J - I_c) + G\varphi'(x) I_c. \tag{52}$$

5 Stiffness parameters for rectangular thin-walled cross section

After the derivation of the modified torsional was done in previous sections and classical torsional model was reminded, our aim is now to demonstrate the difference between classical and modified model. For this purpose, we will take an example of prismatic beam with rectangular thin-walled cross section (of constant thickness) which is clamped on both sides while its right support is twisted by the prescribed angle. More precisely written, we suppose that the centerline of the beam is represented by a line segment $\langle 0, L\rangle$, $\varphi(0) = 0$, $\varphi(L) = \varphi_L$, $\chi(0) = 0$, $\chi(L) = 0$, $m(x) = 0$ and $b(x) = 0$. We will presume that functions $\varphi(x)$ and $\chi(x)$ are sufficiently smooth to enable following operations.

To be able to compute numerical results, we therefore need firstly to calculate various stiffness parameters presented in previous chapters both for modified and classical torsional theory.

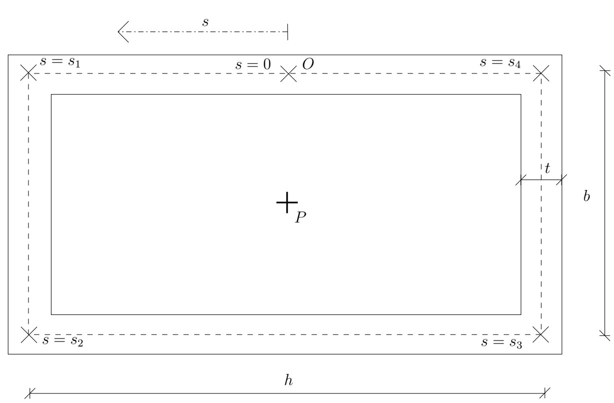

Figure 1: Considered rectangular thin-walled cross section

For the chosen example function $\omega(s)$ is piecewise linear and we can write it as follows

On the interval $\left(0, \frac{b}{2}\right)$ $\omega(s) = \frac{h}{2}s$,

On the interval $\left(\frac{b}{2}, \frac{b}{2}+h\right)$ $\omega(s) = \frac{b}{2}s + \frac{hb}{4} - \frac{b^2}{4}$,

On the interval $\left(\frac{b}{2}+h, \frac{3b}{2}+h\right)$ $\omega(s) = \frac{h}{2}s + \frac{3hb}{4} - \frac{h}{2}\left(\frac{b}{2}+h\right)$,

On the interval $\left(\frac{3b}{2}+h, \frac{3b}{2}+2h\right)$ $\omega(s) = \frac{b}{2}s + \frac{5hb}{4} - \frac{b}{2}\left(\frac{3b}{2}+h\right)$,

On the interval $\left(\frac{3b}{2}+2h, 2b+2h\right)$ $\omega(s) = \frac{h}{2}s + \frac{7hb}{4} - \frac{h}{2}\left(\frac{3b}{2}+2h\right)$.

Function $\pi(s)$ is linear and $\pi(s) = \frac{1}{t}s$.

From the previous we obtain

$$\Omega = \omega(s_{\max}) = \int_0^{s_{\max}} \rho(\bar{s})\,\mathrm{d}\bar{s} = 2bh, \tag{53}$$

$$\Pi = \pi(s_{\max}) = \int_0^{s_{\max}} \frac{\mathrm{d}\bar{s}}{t(\bar{s})} = \frac{1}{t}\left(2b+2h\right), \tag{54}$$

sectional area

$$A = \int_0^{s_{\max}} t(\bar{s})\,\mathrm{d}\bar{s} = 2t(b+h), \tag{55}$$

$$\begin{aligned} \psi_0 &= \frac{1}{A}\left(\frac{\Omega}{\Pi}\int_0^{s_{\max}} \pi(\bar{s})t(\bar{s})\,\mathrm{d}\bar{s} - \int_0^{s_{\max}} \omega(\bar{s})t(\bar{s})\,\mathrm{d}\bar{s}\right) = \\ &= \frac{bht(2b+2h) - t\left(2b^2h+2bh^2\right)}{2t(b+h)} = 0, \end{aligned} \tag{56}$$

$$I_A = \int_A \psi^2\,\mathrm{d}A = I_\psi = \frac{b^2h^2t(b-h)^2}{24(b+h)}, \tag{57}$$

$$I_B = \int_A \omega\psi\,\mathrm{d}A = \frac{b^2h^2t(b-h)(2b+h)}{24(b+h)}, \tag{58}$$

$$I_C = \int_0^{s_{\max}} \frac{\bar{S}_\psi^2(s)}{t(s)}\,\mathrm{d}s = \frac{b^2h^2t(b-h)^2\left(3b^2+12bh+8h^2\right)}{1920(b+h)}, \tag{59}$$

$$I_D = \int_A \pi\psi\,\mathrm{d}A = \frac{1}{24}bh(b-h)(b+2h). \tag{60}$$

With the computed cross-sectional parametres I_A - I_D we can express K_1 - K_4 according to the expressions (32)-(35).

$$K_1 = -G\frac{3b^2h^2t\left(3b^2+2bh+3h^2\right)}{2(b+h)\left(b^2+4bh+h^2\right)}, \tag{61}$$

$$K_2 = -E\frac{b^2\ h^2 t(b-h)^2}{24(b+h)}, \tag{62}$$

$$K_3 = G\frac{5b^2h^2t(b-h)^2}{2(b+h)\left(b^2+4bh+h^2\right)}, \tag{63}$$

$$K_4 = G\frac{5b^2h^2t(b-h)^2}{2(b+h)\left(b^2+4bh+h^2\right)}. \tag{64}$$

We see that for our specific example $K_3 = K_4$.

For classical theory as described previously holds

$$I_c = \frac{1}{2}bht(b+h), \tag{65}$$

$$I_\psi = \frac{b^2h^2t(b-h)^2}{24(b+h)}, \tag{66}$$

$$J = \frac{2b^2h^2t}{b+h}, \tag{67}$$

$$\nu = \frac{(b-h)^2}{(b+h)^2}, \tag{68}$$

$$\beta^2 = \frac{48G}{E(b+h)^2}, \tag{69}$$

$$\beta = \sqrt{\frac{48G}{E(b+h)^2}}. \tag{70}$$

6 Analytical solution according to modified theory for chosen example

We will now try to solve equations (30) and (31) analytically for the special case of beam clamped on both ends, while its right support is rotated by an angle of φ_L. We still consider hollow rectangular cross section depicted in Fig.1. In other words we presume that centerline of the beam is represented by an interval by line segment $\langle 0, L\rangle$, $\varphi(0) = 0$, $\varphi(L) = \varphi_L$, $\chi(0) = 0$, $\chi(L) = 0$, $m(x) = 0$ and $b(x) = 0$. We will presume that functions $\varphi(x)$ and $\chi(x)$ are sufficiently smooth to enable following operations.

In order to make comparison with solution of classical model (equations (48) and (49)), we will follow very similar procedure as later for the classical torsional model.

As we showed previously, we can obtain equation (38) when combining equations (30) and (31). Equation (38) reads

$$-\frac{K_1K_2}{K_3}\chi'''(x) - \left(\frac{K_1K_4}{K_3} + K_3\right)\chi'(x) = m_x(x) - \frac{K_1}{K_3}b'(x)$$

We can rewrite this equation as

$$\chi'''(x) + \frac{\dfrac{K_1K_4}{K_3} + K_3}{\dfrac{K_1K_2}{K_3}}\chi'(x) = -\frac{1}{\dfrac{K_1K_2}{K_3}}m_x(x) + \frac{1}{\dfrac{K_1K_2}{K_3}}\frac{K_1}{K_3}b'(x) \tag{71}$$

If we introduce a new parameter

$$\tilde{\beta}^2 = -\frac{\dfrac{K_1K_4}{K_3} + K_3}{\dfrac{K_1K_2}{K_3}}, \tag{72}$$

which is analogical to β^2 in classical model, we can write (71) as

$$\chi'''(x) - \tilde{\beta}^2\chi'(x) = -\frac{1}{\dfrac{K_1K_2}{K_3}}m_x(x) + \frac{1}{\dfrac{K_1K_2}{K_3}}\frac{K_1}{K_3}b'(x). \tag{73}$$

We will now take this equation together with equation (30), which reads

$$K_1\varphi''(x) - K_3\chi'(x) = m_x(x)$$

and find unknown functions $\chi(x)$ and $\varphi(x)$ analytically. We will represent the unknown functions $\chi(x)$ and $\varphi(x)$ as a linear combination of functions $e^{\lambda x}$. Substituting $e^{\lambda x}$ to equation (71), we obtain the characteristic equation

$$\lambda^3 - \tilde{\beta}^2\lambda = \lambda(\lambda^2 - \tilde{\beta}^2) = 0. \tag{74}$$

We now see that we can write $\chi(x)$ as

$$\chi(x) = C_1 + C_2e^{\tilde{\beta}x} + C_3e^{-\tilde{\beta}x}. \tag{75}$$

Utilizing equation (30), we can also write $\varphi(x)$ as

$$\varphi(x) = C_4 + C_1x + C_2\frac{K_3}{K_1}\frac{1}{\tilde{\beta}}e^{\tilde{\beta}x} - C_3\frac{K_3}{K_1}\frac{1}{\tilde{\beta}}e^{-\tilde{\beta}x}. \tag{76}$$

Yet unknown coefficients C_1 - C_4 can be specified by applying boundary conditions $\varphi(0) = 0$, $\varphi(L) = \varphi_L$, $\chi(0) = 0$, $\chi(L) = 0$.

7 Analytical solution according to classical theory for chosen example

We will now solve equations (48) and (49) analytically with the same assumptions as in previous chapter. We remind that we consider special case of beam clamped on both ends, while its right support is rotated by an angle of φ_L. We also still consider hollow rectangular cross section depicted in Fig.1. The centerline of the beam is represented by an interval $\langle 0, L\rangle$, $\varphi(0) = 0$, $\varphi(L) = \varphi_L$, $f(0) = 0$, $f(L) = 0$, $m(x) = 0$ and $b(x) = 0$. We presume that functions $\varphi(x)$ and $\chi(x)$ are sufficiently smooth to enable following operations as well.

We will now find solution of equations describing the classical model (48) and (49) in the similar manner as we did when solving equations of the modified model (30) and (31) previously. Let us remind that equations (48) and (49) take form[2]

$$f'''(x) - \beta^2 f'(x) = \frac{\beta^2}{GJ} m_x(x)$$

$$\nu f'(x) - \varphi''(x) = \frac{1}{GI_c} m_x(x)$$

We will represent the unknown functions $f(x)$ and $\varphi(x)$ as a linear combination of functions $e^{\lambda x}$. Substituting $e^{\lambda x}$ to equation (48), we obtain the characteristic equation

$$\lambda^3 - \beta^2\lambda = \lambda(\lambda^2 - \beta^2) = 0. \tag{77}$$

We now see that we can write $f(x)$ as

$$f(x) = C_1 + C_2 e^{\beta x} + C_3 e^{-\beta x}. \tag{78}$$

Utilizing equation (49), we can also write $\varphi(x)$ as

$$\varphi(x) = C_4 + C_1 x + C_2 \frac{\nu}{\beta} e^{\beta x} - C_3 \frac{\nu}{\beta} e^{-\beta x}. \tag{79}$$

Value of coefficients C_1 - C_4 can be found by applying boundary conditions $\varphi(0) = 0$, $\varphi(L) = \varphi_L$, $\chi(0) = 0$, $\chi(L) = 0$.

[2]We would like to remind on this place again that $f(x) = -\chi(x)$.

8 Comparison of analytical solutions of modified and classical model - brief review of theoretical background

We will now try to compare solutions obtained by classical model of torsion of beams with thin-walled closed cross section and modified model of the same phenomena proposed in this text. Comparison will be presented for the specific case of prismatic beam with rectangular thin-walled cross section (of constant thickness) which is clamped on both sides while its right support is twisted by an prescribed angle. This particular case was chosen because it allows us to find analytical solution so we can eliminate the influence of numerical error in the comparison and at the same time is very simple, clear and meaningful from the engineering point of view. Because for the sake of clarity we will try to compare both theories mostly graphically, we will firstly review the most important results, which we derived in previous sections. The following expressions form a theoretical background for the subsequent graphical comparison.

1. **Classical model**[3]

   For $m(x) = 0$ and $b(x) = 0$ we get

   $$f(x) = C_1 + C_2 e^{\beta x} + C_3 e^{-\beta x}.$$

   $$\varphi(x) = C_4 + C_1 x + C_2 \frac{\nu}{\beta} e^{\beta x} - C_3 \frac{\nu}{\beta} e^{-\beta x}.$$

   Unknown coefficients C_1 - C_4 can be specified by applying boundary conditions. In our considered example of twisted beam we require $\varphi(0) = 0$, $\varphi(L) = \varphi_L$, $\chi(0) = 0$, $\chi(L) = 0$.

   Moreover, for rectangular cross section of arbitrary positive sizes b and h and constant thickness t we get

   $$\beta^2 = \frac{48G}{E(b+h)^2},$$

   $$\nu = \frac{(b-h)^2}{(b+h)^2},$$

   $$M_k^{VL}(x) = G\varphi'(x)\frac{1}{2}bht(b+h) - G\chi(x)\frac{bht(b-h)^2}{2(b+h)}.$$

2. **Modified model**

   For $m(x) = 0$ and $b(x) = 0$ we get

   $$\chi(x) = C_1 + C_2 e^{\tilde{\beta} x} + C_3 e^{-\tilde{\beta} x}.$$

[3] We would like to remind on this place that $f(x) = -\chi(x)$.

$$\varphi(x) = C_4 + C_1 x + C_2 \frac{K_3}{K_1}\frac{1}{\tilde{\beta}} e^{\tilde{\beta}x} - C_3 \frac{K_3}{K_1}\frac{1}{\tilde{\beta}} e^{-\tilde{\beta}x}.$$

Again, unknown coefficients C_1 - C_4 can be specified by applying boundary conditions $\varphi(0) = 0$, $\varphi(L) = \varphi_L$, $\chi(0) = 0$, $\chi(L) = 0$.

Moreover, for rectangular cross section of arbitrary positive sizes b and h and constant thickness t we get

$$\tilde{\beta}^2 = \frac{80G}{E(3b^2 + 2bh + 3h^2)},$$

$$\frac{K_3}{K_1} = -\frac{5(b-h)^2}{9b^2 + 6bh + 9h^2},$$

$$M_k(x) = G\varphi'(x)\frac{3b^2h^2t\left(3b^2 + 2bh + 3h^2\right)}{2(b+h)\left(b^2 + 4bh + h^2\right)} + G\chi(x)\frac{5b^2h^2t(b-h)^2}{2(b+h)\left(b^2 + 4bh + h^2\right)}.$$

9 Comparison of analytical solutions of modified and classical model

We will now try to compare results for above considered example obtained when using classical model of torsion of beams with thin-walled closed cross section (of constant thickness) and its modified counterpart presented in the chapter 1 of this text. As mentioned before, this will be performed by comparing effective torsional stiffness k_{eff}(defined hereinafter).

Effective torsional stiffness k_{eff} is defined on the basis of following expression

$$M_k(L) = k_{eff}\overline{\Theta}_k, \tag{80}$$

so in other words

$$k_{eff} = \frac{M_k(L)}{\overline{\Theta}_k}, \tag{81}$$

where $M_k(L)$ is the torsional moment at $x = L$ (right support) and

$$\overline{\Theta}_k = \frac{\varphi_L}{L}, \tag{82}$$

is an average twist angle. If the cross section does not have tendency to warp, k_{eff} reduces to free warping torsinal stiffness, which is

$$k_{eff} = G\frac{\Omega^2}{\Pi} = G\frac{2b^2h^2t}{b+h}. \tag{83}$$

We now see that the concept of effective torsional stiffness k_{eff} enables us to easily compare the stiffness in case of free warping to stiffness obtained by classical or modified torsional model, which enables us to consider engineering relevance of obtained results.

We will of course obtain different results comparing both models when choosing different geometrical setup, which is specified by the length of the beam L, width of the hollow rectangular cross section b, its height h and thickness t (geometrical meaning of these parameters is depicted in Fig.1). Therefore, our goal is to observe how the change of these parameters influences the value of effective torsional stiffness k_{eff} predicted by both models. As the changes of b [4] and L are predictably the most influential on the value of effective stiffness, we will particularly focus on the influence of these two parameters.

[4] Which is on our case of twisted beam equivalent to the change of height.

9.1 Influence of the length of the beam

For this comparison we considered the following values of geometrical and material parametres. Elastic Young's modulus E and elastic shear modulus G are chosen to reflect the real values of corresponding material parameters of steel.

$$E = 210 \cdot 10^3 \text{ MPa}, \; G = 81 \cdot 10^3 \text{ MPa}$$

$$\varphi_L = \frac{\pi}{180} \text{ rad}$$

$$b = 700 \text{ mm}, \; h = 100 \text{ mm}, \; t = 10 \text{ mm}$$

We can observe that the ratio of b to h is 7. Value of ratio b/h could have been chosen arbitrarily except $b/h = 1$. This ratio implies that the cross section is hollow square of constant thickness, which does not have tendency to warp, so the total torsional stiffness is given just by the free warping torsional stiffness. Generally we can say that as b/h approaches 1, the difference of prediction of k_{eff} by both models diminishes. From analysis in subsequent section we will also see that the greater b/h is, the predicted stiffness differs more significantly.

The Fig.2 shows the dependence of effective torsional stiffness k_{eff} predicted by both models on the length of the beam. We can observe that with the growing length the difference of prediction of both models diminishes. On the contrary, we can observe that the stiffness difference is increasing with decreasing lengtht.

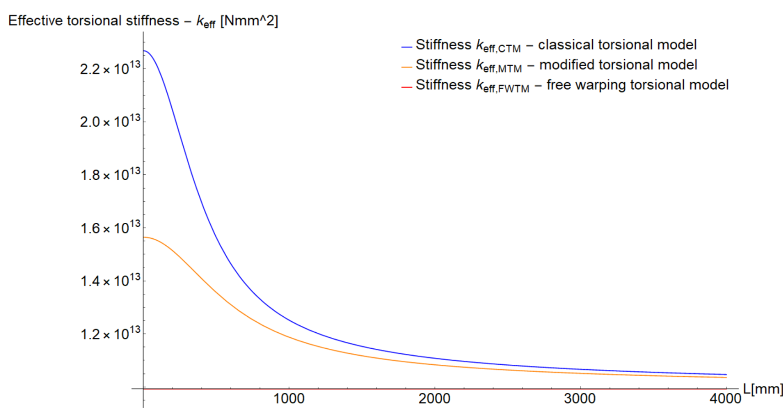

Figure 2: Dependence of effective torsional stiffness k_{eff} on the length of the beam

Fig.3 relates to Fig.2 and depicts the relative stiffness difference of modified and classical model. The relative stiffness difference relates to the difference of k_{eff} predicted by both models divided by k_{eff} for case of free warping.

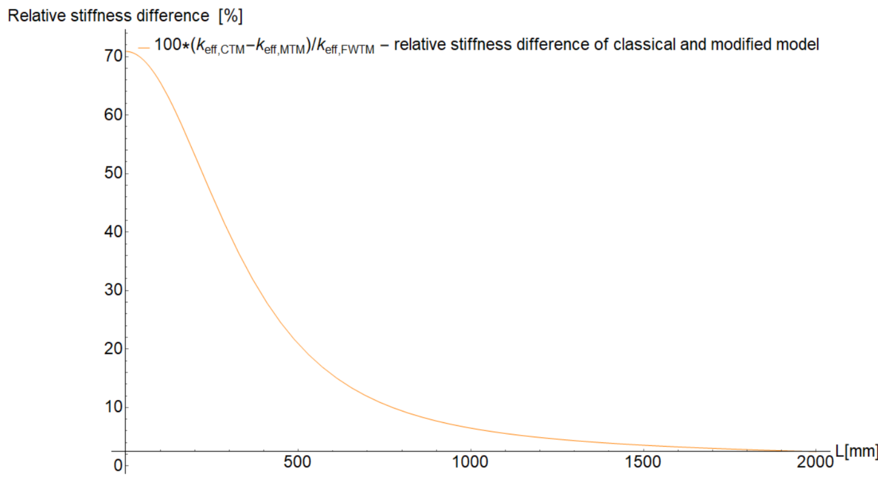

Figure 3: Dependence of relative stiffness difference on the length of the beam

9.2 Influence of the b/h ratio of the rectangular cross section

We will now follow the dependence of k_{eff} on the ratio b/h - width to height of the rectangular cross section. Le us remind that we consider the case of constant thickness of the cross section. Before we proceed to the graphical comparison, let us make brief remark on how we generated sizes b and h for ratio b/h. It is important to realize that with different b and h we get different k_{eff} for free warping torsional stiffness. For the sake of comparability of results we therefore considered such values b and h that the stiffness k_{eff} for free warping torsion remains constant for all considered cross sections.

Fig.4 depicts the dependence of h on b, so that the stiffness k_{eff} for free warping torsion is the same for every b and corresponding $h(b)$ pair. The value of free warping torsional stiffness was chosen as $1.7 \cdot 10^{13}$ Nmm^2.

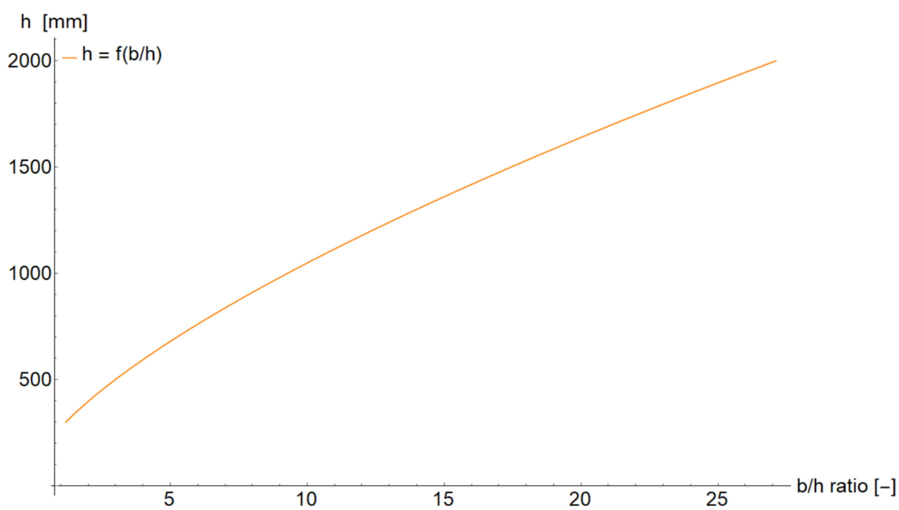

Figure 4: Dependence of the height h on the b/h ratio of the cross section to obtain the same free warping torsional stiffness $1.7 \cdot 10^{13}$ Nmm^2

For this comparison we considered the following values of geometrical and material parameters. Elastic Young's modulus E and elastic shear modulus G are chosen to reflect the real values of corresponding material parameters of steel.

$$E = 210 \cdot 10^3 \text{ MPa}, \; G = 81 \cdot 10^3 \text{ MPa}$$

$$\varphi_L = \frac{\pi}{180} \text{ rad}$$

$$L = 1000 \text{ mm}, \; t = 10 \text{ mm}$$

Fig. 5 shows the dependence of effective torsional stiffness k_{eff} predicted by both models on the b/h ratio of the cross section. We can observe that with the growing b/h ratio the difference of k_{eff} predicted by corresponding models increases.

Let us denote that the minimal value of b is $b = 300$ mm. That is, because for chosen value of free warping torsional stiffness[5] the size of square with this free warping torsional stiffness is 275.838 mm.

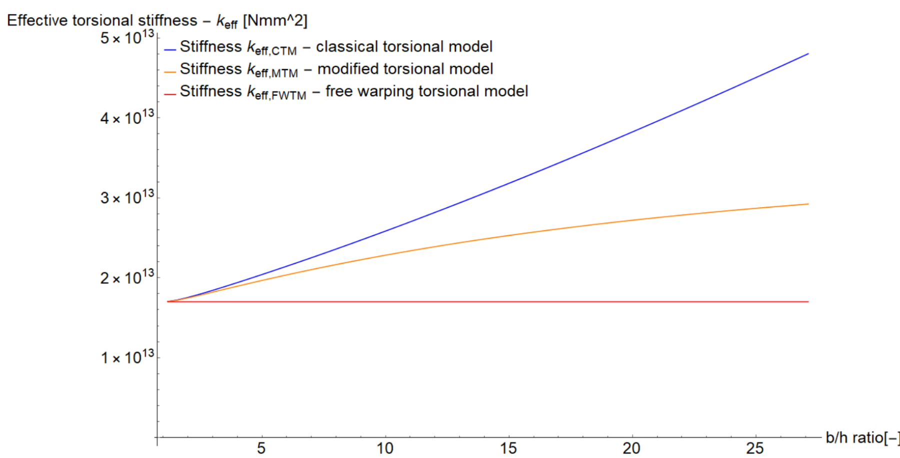

Figure 5: Dependence of effective torsional stiffness k_{eff} on the b/h ratio of the cross section

[5]Which was chosen as $1.7 \cdot 10^{13}$ Nmm^2 as mentioned before in this chapter.

Fig.6 relates to Fig.5 and depicts the relative stiffness difference of modified and classical model. The relative stiffness difference relates to the difference of k_{eff} predicted by both models divided by k_{eff} for case of free warping.

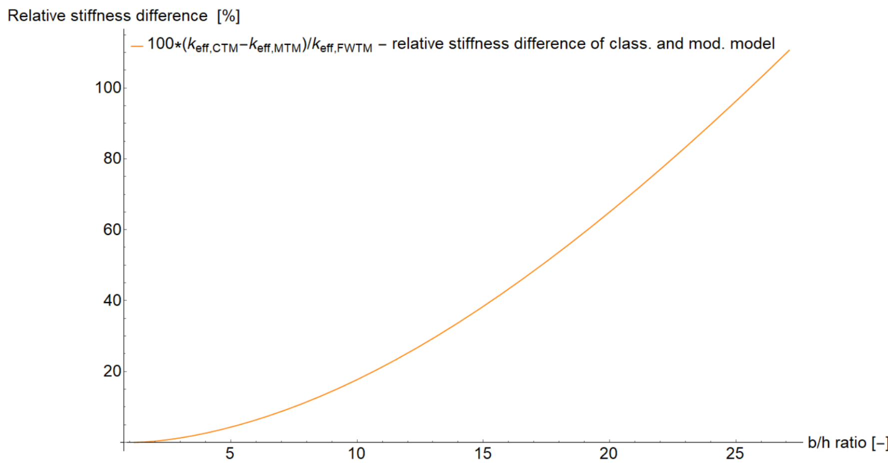

Figure 6: Dependence of relative stiffness difference on the b/h ratio of the cross section

10 Conclusion

In this work we derived model describing non-uniform torsion of beams with thin-walled closed cross section. For this purpose, we utilized Hellinger - Reissner variational principle and the general format of governing equations of the model given by Hellinger - Reissner variational principle for prescribed approximation of displacement (9) and stress (24) field derived in the bachelor thesis [4] of the author of this work.

The derived model is referred to as the modified torsional model in this work and it is shown that its governing equations (30) and (31) are analogous to the widely known model describing non-uniform torsion of beams with thin-walled closed cross section, which we adopted in the form found in [1, chapter 2.5.4, p.247] or [2, Chapter 6, p.130]. This model was selected as the reference model and is referred to as the classical torsional model in this work.

Afterwards, to demonstrate the difference between classical and modified model, we analyze the case of prismatic beam with rectangular thin-walled cross section (of constant thickness) which is clamped on both sides while its right support is twisted by the prescribed angle. This particular case was chosen because it allows us to find analytical solution so we can eliminate the influence of numerical error in the comparison and at the same time is very simple, clear and meaningful from the engineering point of view.

Lastly, we graphically demonstrate, comparing effective torsional stiffness k_{eff} (defined for the purpose of easier comparison), that in case of short beams (see Fig.2 and Fig.3) and cross sections with large b/h ratio (see Fig.4, Fig.5 and Fig.6), the difference of k_{eff} of classical and modified model can be significant from the engineering point of view and in the latter case could reach tens of percents of torsional stiffness of cross section with free warping.

We would like to extend this work in future, namely we would like to compare the results of both theories for our considered example to the results gained by three-dimensional finite element method.

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