Introduction to the Finite Element Method (1)

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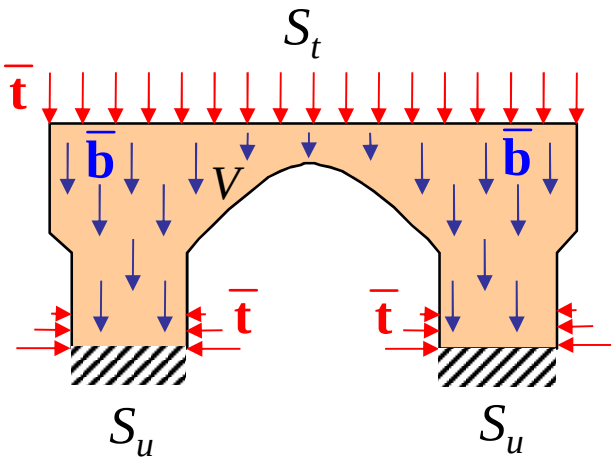
Outline

- ❑ Governing equations of mechanics
- ❑ Variational principles of elasticity
- ❑ Principle of the finite element method
- ❑ Example 1 – axially loaded bar
- ❑ Example 2 – beam bending
- ❑ Generalization

Summary of governing equations of theory of elasticity



displace-ment

$$u, v, w$$

external forces

$$\bar{b}_x, \bar{b}_y, \bar{b}_z$$

kinematic equations

$$\varepsilon_x = \frac{\partial u}{\partial x}$$

...

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

...

$$\tau_{xy} = \frac{E\gamma_{xy}}{2(1+\nu)}$$

...

$$\sigma_x = \frac{E\left[(1-\nu)\varepsilon_x + \nu\varepsilon_y + \nu\varepsilon_z\right]}{(1+\nu)(1-2\nu)}$$

...

equilibrium equations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \bar{b}_x = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \bar{b}_y = 0$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + \bar{b}_z = 0$$

strain

$$\varepsilon_x, \varepsilon_y, \varepsilon_z$$

$$\gamma_{xy}, \gamma_{xz}, \gamma_{yz}$$

stress

$$\sigma_x, \sigma_y, \sigma_z$$

$$\tau_{xy}, \tau_{xz}, \tau_{yz}$$

constitutive equations

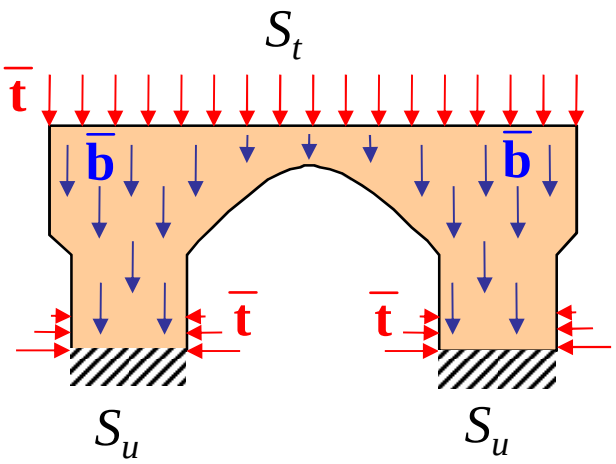
❑ Boundary conditions

Kinematic (essential) B.C.

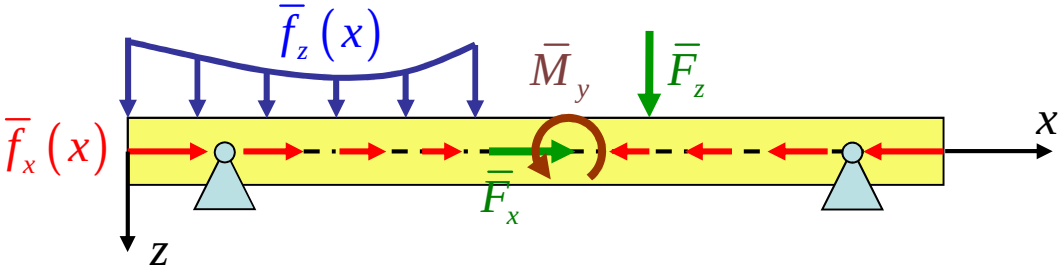
$$\mathbf{u} = \bar{\mathbf{u}} \qquad \text{on } S_u$$

Static (natural) B.C.

$$\mathbf{n}\boldsymbol{\sigma} = \bar{\mathbf{t}} \qquad \text{on } S_t$$



Bernoulli-Euler theory: summary of governing equations (in plane)



Axial stretching/compressing

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(EA\frac{\mathrm{d}u}{\mathrm{d}x}\right) = -\bar{f}_x$$

$$\begin{tikzcd}
\boxed{u(x)} \arrow[rr] \arrow[d, "\varepsilon = \frac{\mathrm{d}u}{\mathrm{d}x}"'] & & \boxed{\bar{f}_x(x)} \\
\boxed{\varepsilon(x)} \arrow[rr, "N = EA\varepsilon"] & & \boxed{N(x)} \arrow[u, "\frac{\mathrm{d}N}{\mathrm{d}x} = -\bar{f}_x"']
\end{tikzcd}$$

Bernoulli-Euler theory: summary of governing equations (in plane)

Bending

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI\,\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right) = \overline{f_z}$$

$$\boxed{w(x)} \longrightarrow \boxed{\overline{f_z}(x)}$$

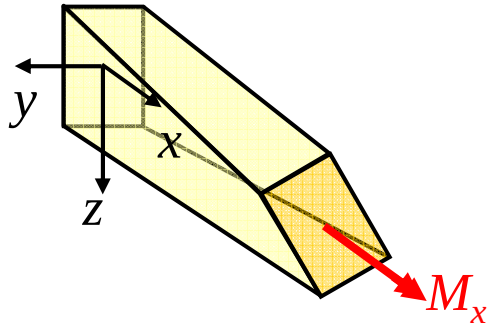
$$\kappa = -\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} \qquad\qquad -\frac{\mathrm{d}^2 M}{\mathrm{d}x^2} = \overline{f_z}$$

$$\boxed{\kappa(x)} \xrightarrow{\quad M = EI\,\kappa \quad} \boxed{M(x)}$$

❑ Boundary conditions

	Kinematic (essential) B.C.		Static (natural) B.C.
axial stretching/ compression	$u = \bar{u}$	or	$EA\dfrac{du}{dx} = \bar{N}$
bending	$w = \bar{w}$	or	$-\dfrac{d}{dx}EI\dfrac{d^2w}{dx^2} = \bar{V}$
	$\dfrac{dw}{dx} = \bar{w}'$	or	$-EI\dfrac{d^2w}{dx^2} = \bar{M}$

St. Venant torsion of member with arbitrary cross-section: summary of governing equations



$$GI_k \frac{d^2\varphi_x}{dx^2} = 0$$

$$\varphi_x \longrightarrow \bar{m}_x = 0$$

$$\theta = \frac{d\varphi_x}{dx} \qquad \frac{dM_x}{dx} = 0$$

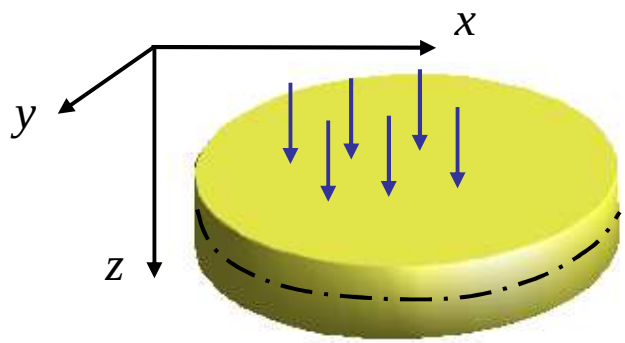
$$M_x = GI_k\theta$$

$$\theta \longrightarrow M_x$$

- Boundary conditions

Kinematic (essential) B.C.		Static (natural) B.C.
$\varphi_x = \bar{\varphi}_x$	or	$GI_k \dfrac{d\varphi_x}{dx} = \bar{M}_x$

Theory of thin plates: summary of governing equations



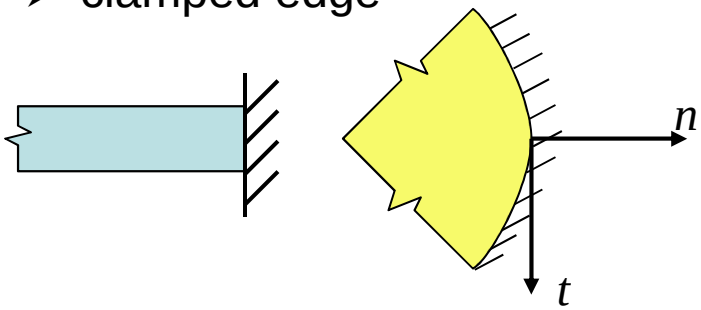
$$w(x,y) \longrightarrow D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) - \bar{f}_z = 0 \longrightarrow \bar{f}_z(x,y)$$

$$\frac{\partial^2 m_x}{\partial x^2} + 2\frac{\partial^2 m_{xy}}{\partial x \partial y} + \frac{\partial^2 m_y}{\partial y^2} + \bar{f}_z = 0$$

$$\boldsymbol{\varepsilon} = \left\{-\frac{\partial^2 w}{\partial x^2}, -\frac{\partial^2 w}{\partial y^2}, -2\frac{\partial^2 w}{\partial x \partial y}\right\}^T \longrightarrow \boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon} \longrightarrow \boldsymbol{\sigma} = \left\{m_x, m_y, m_{xy}\right\}^T$$

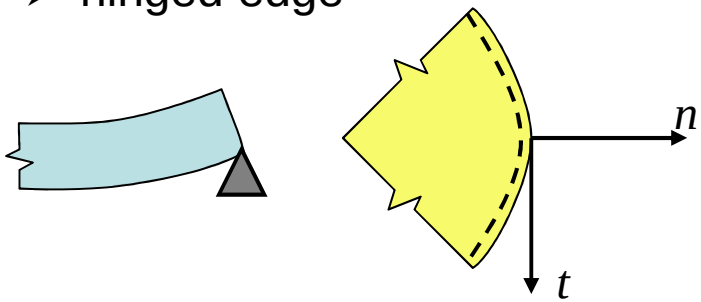
- Boundary conditions for various plate edge arrangements

  - clamped edge



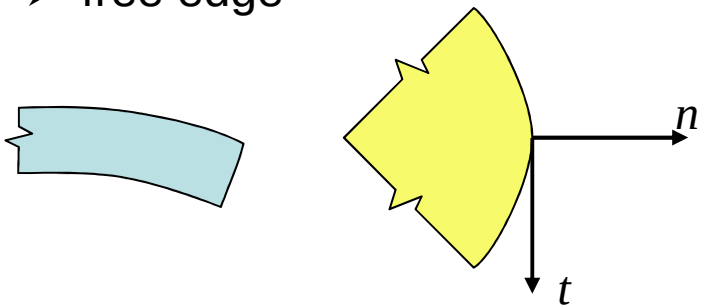
$$\frac{\partial w}{\partial n} = 0 \quad \wedge \quad w = 0$$

  - hinged edge



$$m_n = 0 \quad \wedge \quad w = 0$$

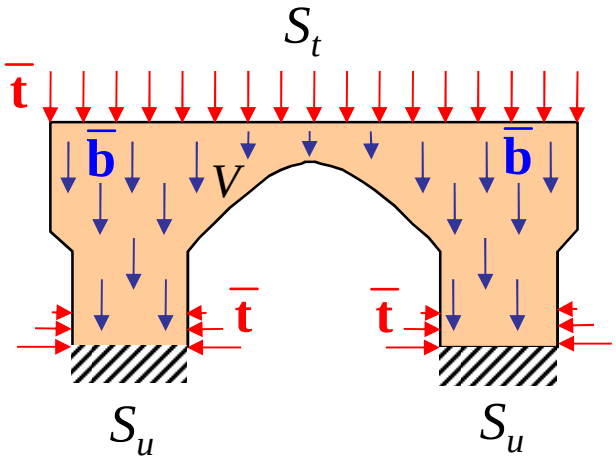
  - free edge



$$m_n = 0 \quad \wedge \quad \frac{\partial m_{nt}}{\partial t} + v_n = 0$$

Variational principles of elasticity

- Consider a deformable body under the action of prescribed body forces $\bar{\mathbf{b}}$ in domain V, surface tractions $\bar{\mathbf{t}}$ on boundary S_t and displacement prescribed $\bar{\mathbf{u}}$ on boundary S_u.



Virtual work principle

- Virtual work is work done by all forces acting on a deformable body as this body is given a small hypothetical displacement – virtual displacement $\delta\mathbf{u}$, which is consistent with constraints present. The acting forces are constant during the virtual displacement.

$$\delta W_{virt} = \int_V \bar{\mathbf{b}}^T \delta\mathbf{u}\, dV + \int_{S_t} \bar{\mathbf{t}}^T \delta\mathbf{u}\, dS$$

- The necessary and sufficient condition for the body to be in equilibrium is that the virtual work of external statically compatible forces ($\bar{\mathbf{t}}, \bar{\mathbf{b}}$) is equal to the virtual work of internal forces (stresses) for any kinematically compatible admissible virtual displacement and deformation field ($\delta\mathbf{u}$, $\delta\boldsymbol{\varepsilon}$).

$$\forall \delta\mathbf{u}, \delta\boldsymbol{\varepsilon} = \partial\delta\mathbf{u} :$$

$$\underbrace{\int_V \boldsymbol{\sigma}^T \delta\boldsymbol{\varepsilon}\, dV}_{\text{virt. work on internal forces}} = \underbrace{\int_{S_t} \bar{\mathbf{t}}^T \delta\mathbf{u}\, dS + \int_V \bar{\mathbf{b}}^T \delta\mathbf{u}\, dV}_{\text{virt. work of external forces}}$$

Note: statically compatible forces $\bar{\mathbf{t}}, \bar{\mathbf{b}}$ - there is overall equilibrium for the body from the viewpoint of rigid body mechanics.

Principle of total potential energy

- Define potential energy of applied loads: $E_{ext} = -\int_V \mathbf{b}^T \mathbf{u}\, dV - \int_{S_t} \mathbf{t}^T \mathbf{u}\, dS$
- Postulate existence of positive definite strain energy density function Φ : $\boldsymbol{\sigma} = \dfrac{\partial \Phi}{\partial \boldsymbol{\varepsilon}}$
- Define strain energy of the body: $E_{int} = \int_V \Phi dV = \int_V \left(\int \boldsymbol{\sigma}^T \, d\boldsymbol{\varepsilon} \right) dV$
- Define the total potential energy of the body: $E_{pot} = E_{int} + E_{ext}$
- For the body to be in equilibrium, the first variation of E_{pot} must be zero:

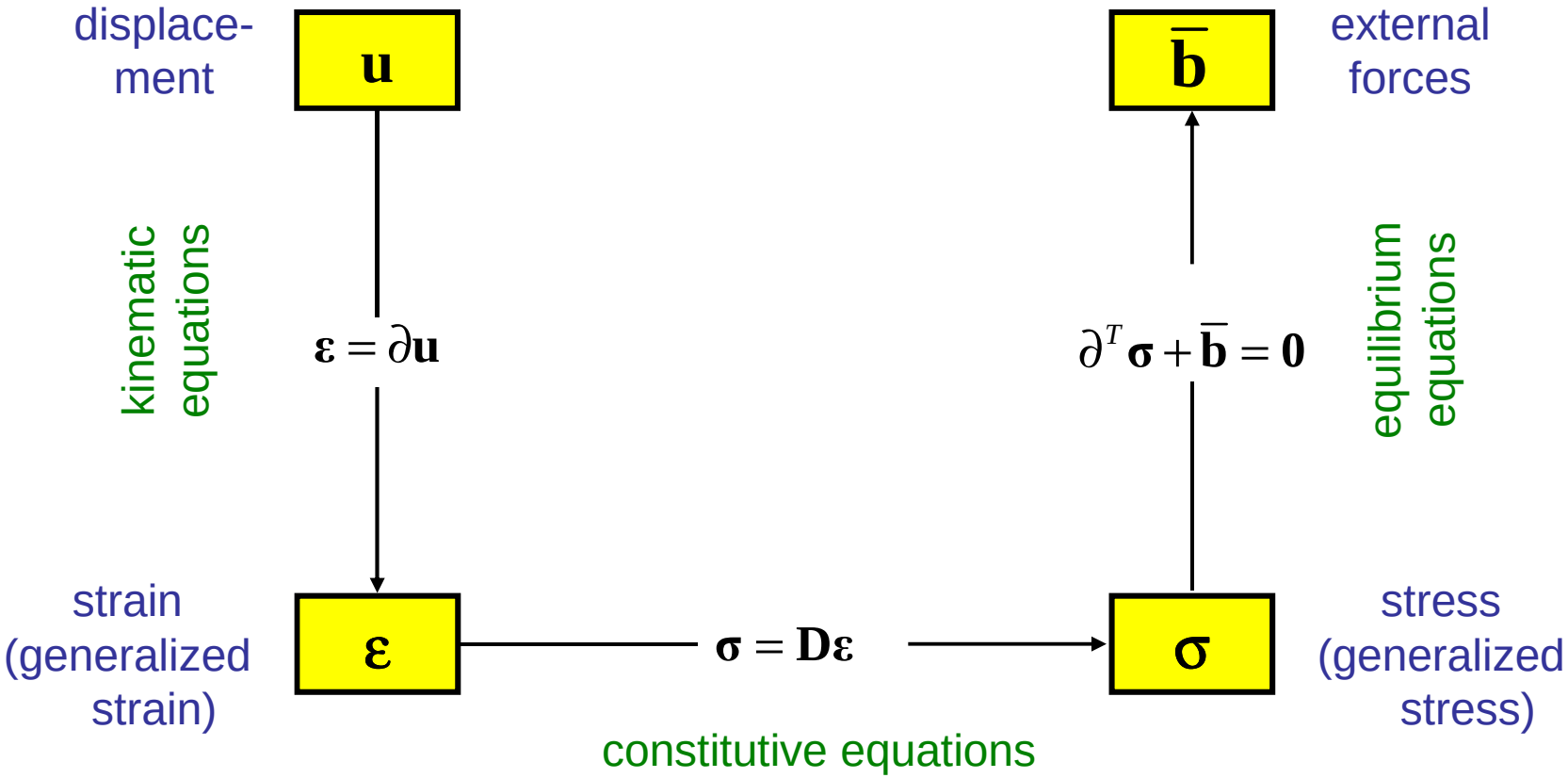
$$\delta E_{pot} = \delta \left(E_{int} + E_{ext} \right) = 0$$

The necessary and sufficient conditions for the body to be in equilibrium are:

1. $\bar{\mathbf{t}}, \bar{\mathbf{b}}$ are statically compatible
2. The deformation field which is related to stress through elastic constitutive law minimizes the total potential energy E_{pot} with respect to all other kinematically compatible, admissible deformation fields.

Principle of the finite element method

- Recall the governing equations of elasticity problems



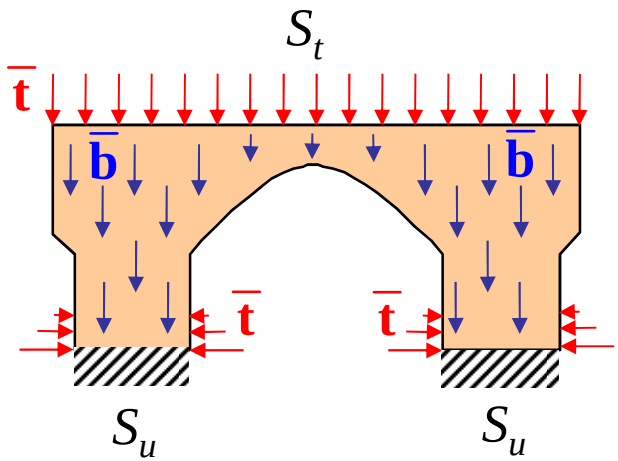
- The boundary value problem (BVP)

$$\mathbf{u} \longleftarrow \boxed{\partial^T \mathbf{D}\, \partial \mathbf{u} + \bar{\mathbf{b}} = \mathbf{0}} \longrightarrow \bar{\mathbf{b}}$$

strong form

$$\boxed{\begin{aligned} \mathbf{u} &= \bar{\mathbf{u}} \quad && \text{on } S_u \\ \mathbf{n}\boldsymbol{\sigma} &= \bar{\mathbf{t}} \quad && \text{on } S_t \end{aligned}}$$

boundary conditions

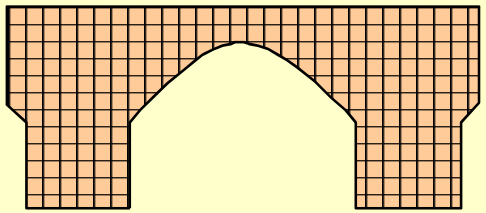


- the solution of BVP is a function (or functions) defined over the problem domain, e.g. $u(x)$, $v(x)$, $w(x)$
- the finite element method allows us to obtain approximate numerical solution of BVPs

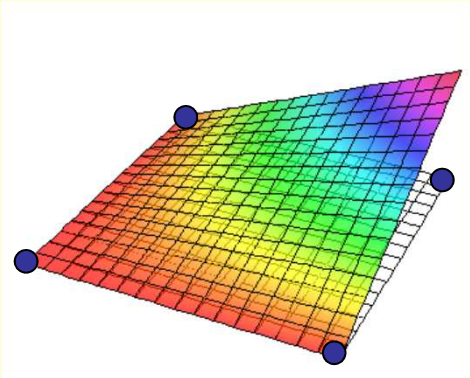
- Solution of the boundary value problem by the finite element method (FEM)

$$\mathbf{u}(\mathbf{x}) \approx \mathbf{N}(\mathbf{x})\mathbf{d}$$

- The problem domain is divided into elements of finite size (discretized)



- On each element, displacement is approximated by function of suitable type (e.g. linear, quadratic,...) ... shape function $\mathbf{N}(\mathbf{x})$
- The coefficients of these functions $\mathbf{d}$ (usually values of this function in certain element points – nodes) become the primary unknowns of the problem



$$\mathbf{u}(\mathbf{x}) \approx \mathbf{N}(\mathbf{x})\mathbf{d}$$

$$\boldsymbol{\varepsilon} = \partial \mathbf{u}$$

$$\boldsymbol{\varepsilon}(\mathbf{x}) \approx \mathbf{B}(\mathbf{x})\mathbf{d}$$

- Strain is calculated from the approximated displacement by kinematic equations
- Since only the shape functions depend on $\mathbf{x}$, the derivatives are applied only to them (and not $\mathbf{d}$)

$$\mathbf{u}(\mathbf{x}) \approx \mathbf{N}(\mathbf{x})\mathbf{d}$$

$$\boldsymbol{\varepsilon} = \partial \mathbf{u}$$

- Constitutive equations are used to calculate stress using the approximated displacement

$$\boldsymbol{\varepsilon}(\mathbf{x}) \approx \mathbf{B}(\mathbf{x})\mathbf{d} \xrightarrow{\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}} \boldsymbol{\sigma}(\mathbf{x}) \approx \mathbf{D}\mathbf{B}(\mathbf{x})\mathbf{d}$$

$$\mathbf{u}(\mathbf{x}) \approx \mathbf{N}(\mathbf{x})\mathbf{d}$$

$$\bar{\mathbf{b}}(\mathbf{x})$$

- The equilibrium equations are not directly used; instead an energy or variational principle is used (e.g. principle of virtual work, principle of total potential energy)

$$\boldsymbol{\varepsilon} = \boldsymbol{\partial}\mathbf{u}$$

$$\cancel{\boldsymbol{\partial}^T\boldsymbol{\sigma} + \bar{\mathbf{b}} = \mathbf{0}}$$

$$\boldsymbol{\varepsilon}(\mathbf{x}) \approx \mathbf{B}(\mathbf{x})\mathbf{d} \xrightarrow{\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}} \boldsymbol{\sigma}(\mathbf{x}) \approx \mathbf{D}\mathbf{B}(\mathbf{x})\mathbf{d}$$

- This leads to the weak form of the problem

$$\bar{\mathbf{b}}(\mathbf{x})$$

$$\forall \delta\mathbf{u}, \delta\boldsymbol{\varepsilon} = \partial\delta\mathbf{u}:$$

$$\int_V \boldsymbol{\sigma}^T \delta\boldsymbol{\varepsilon}\,\mathrm{d}V = \int_{S_t} \bar{\mathbf{t}}^T \delta\mathbf{u}\,\mathrm{d}S + \int_V \bar{\mathbf{b}}^T \delta\mathbf{u}\,\mathrm{d}V$$

$$\boldsymbol{\varepsilon} = \partial\mathbf{u}$$

$$\boldsymbol{\varepsilon}(\mathbf{x}) \approx \mathbf{B}(\mathbf{x})\mathbf{d} \xrightarrow{\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}} \boldsymbol{\sigma}(\mathbf{x}) \approx \mathbf{D}\mathbf{B}(\mathbf{x})\mathbf{d}$$

- Upon substituting the approximations, the discretized weak form is obtained

$$\bar{\mathbf{b}}(\mathbf{x})$$

$$\forall \delta\mathbf{d}:$$

$$\mathbf{d}^T \int_V \mathbf{B}^T \mathbf{D}^T \mathbf{B}\, \mathrm{d}V\, \delta\mathbf{d} = \int_{S_t} \bar{\mathbf{t}}^T \mathbf{N}\, \mathrm{d}S\, \delta\mathbf{d} + \int_V \bar{\mathbf{b}}^T \mathbf{N}\, \mathrm{d}V\, \delta\mathbf{d}$$

$$\boldsymbol{\varepsilon}(\mathbf{x}) \approx \mathbf{B}(\mathbf{x})\mathbf{d} \longrightarrow \boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon} \longrightarrow \boldsymbol{\sigma}(\mathbf{x}) \approx \mathbf{D}\mathbf{B}(\mathbf{x})\mathbf{d}$$

structural stiffness matrix $\mathbf{K}^T$

external force vector $\mathbf{f}_{ext}^T$

$\bar{\mathbf{b}}(\mathbf{x})$

$$\forall \delta\mathbf{d}:$$

$$\mathbf{d}^T \int_V \mathbf{B}^T \mathbf{D}^T \mathbf{B}\, \mathrm{d}V\, \delta\mathbf{d} = \int_{S_t} \bar{\mathbf{t}}^T \mathbf{N}\, \mathrm{d}S\, \delta\mathbf{d} + \int_V \bar{\mathbf{b}}^T \mathbf{N}\, \mathrm{d}V\, \delta\mathbf{d}$$

$$\boldsymbol{\varepsilon}(\mathbf{x}) \approx \mathbf{B}(\mathbf{x})\mathbf{d} \longrightarrow \boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon} \longrightarrow \boldsymbol{\sigma}(\mathbf{x}) \approx \mathbf{D}\mathbf{B}(\mathbf{x})\mathbf{d}$$

$$\mathbf{u}(\mathbf{x}) \approx \mathbf{N}(\mathbf{x})\mathbf{d} \longleftarrow \mathbf{K}\,\mathbf{d} = \mathbf{f}_{ext} \longleftarrow \bar{\mathbf{b}}(\mathbf{x})$$

$$\forall\, \delta\mathbf{d}:$$

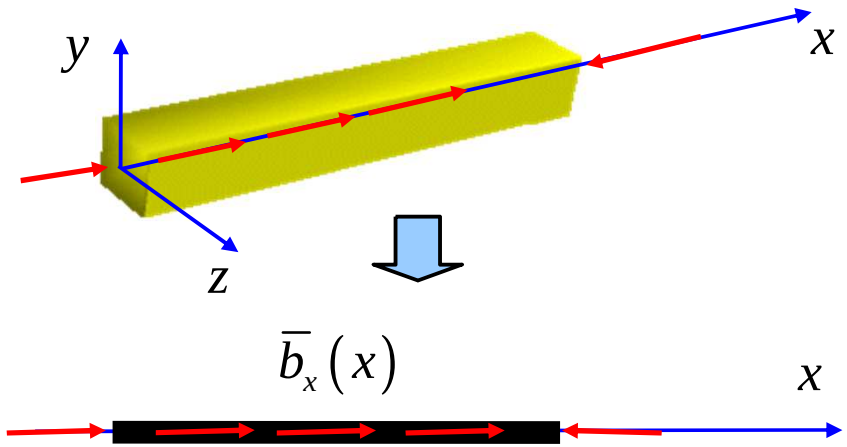
$$\mathbf{d}^T \int_V \mathbf{B} \qquad \mathrm{d}V\, \delta\mathbf{d}$$

- The discretized weak form leads to a system of simultaneous linear algebraic equations ... the primary solution are discrete nodal values $\mathbf{d}$

$$\boldsymbol{\varepsilon}(\mathbf{x}) \approx \mathbf{B}(\mathbf{x})\mathbf{d} \longrightarrow \boldsymbol{\sigma} = \mathbf{D}\,\boldsymbol{\varepsilon} \longrightarrow \boldsymbol{\sigma}(\mathbf{x}) \approx \mathbf{D}\mathbf{B}(\mathbf{x})\mathbf{d}$$

Example 1 – axially loaded bar

Boundary value problem



for simplicity, introduce notation

$$\sigma = \sigma_x, \varepsilon = \varepsilon_x, \bar{b} = \bar{b}_x, \bar{t} = \bar{t}_x$$

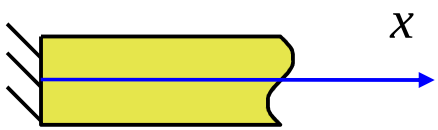
Governing equations

$$\frac{d\sigma(x)}{dx} + \bar{b}(x) = 0 \qquad (1)$$

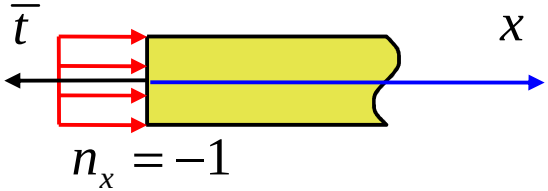
$$\varepsilon(x) = \frac{du(x)}{dx} \qquad (2)$$

$$\sigma(x) = E\varepsilon(x) \qquad (3)$$

boundary conditions, e.g.



$$u(0) = \bar{u} \qquad (4)$$



$$-1 \cdot \sigma(0) = \bar{t} \qquad (5)$$

... this we call the strong form of BVP

Virtual work principle – weak form of BVP

$$\underbrace{\int_V \sigma\,\delta\varepsilon\,dV}_{\text{virt. work on internal forces}} \underbrace{- \int_V \bar{b}\,\delta u\,dV - \int_{S_t} \bar{t}\,\delta u\,dS}_{\text{virt. work of external forces}}\Bigg|_{x=0} = 0$$

... must be satisfied for any virtual displacement and compatible strain δu, $\delta\varepsilon$ that satisfy the governing equations and boundary conditions, e.g. $u(0) = 0, \delta u(0) = 0$ for fixed end

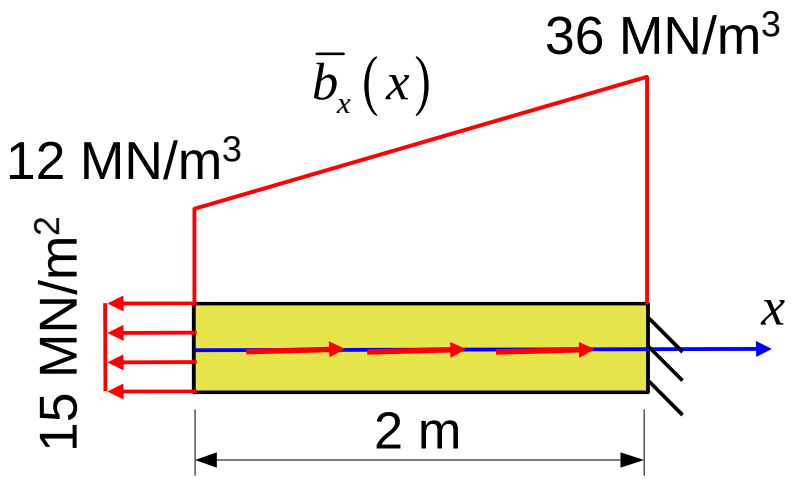
upon substituting the governing stress-strain and strain-displacement relations, we obtain:

$$\int_{x=0}^{L}\left(EA\frac{du(x)}{dx}\frac{d\delta u(x)}{dx}-A\bar{b}(x)\delta u(x)\right)dx-\left.A\bar{t}\,\delta u(x)\right|_{x=0}=0$$

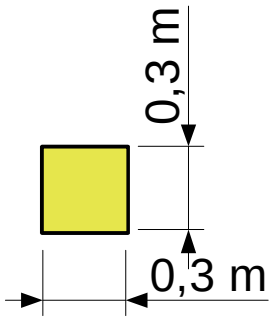
which, together with prescribed boundary conditions, is called the weak form of the BVP

Problem description

Use the FEM to determine the displacement, strain, and stress fields of a bar considering 1-D stress state.

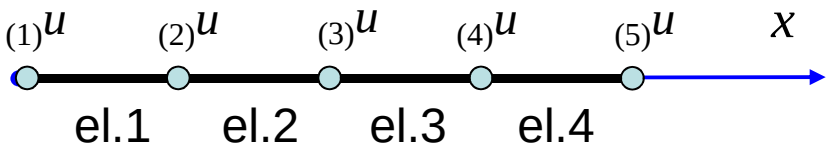


$E = const. = 2000$ MPa



Approximation of displacement

- Discretization of the domain into 4 elements, global vector of nodal displacements



Notation:

(global node) global DOF

element element, el. DOF element, (el. node)

DOF ... d ... degree of freedom (displacement component etc.)

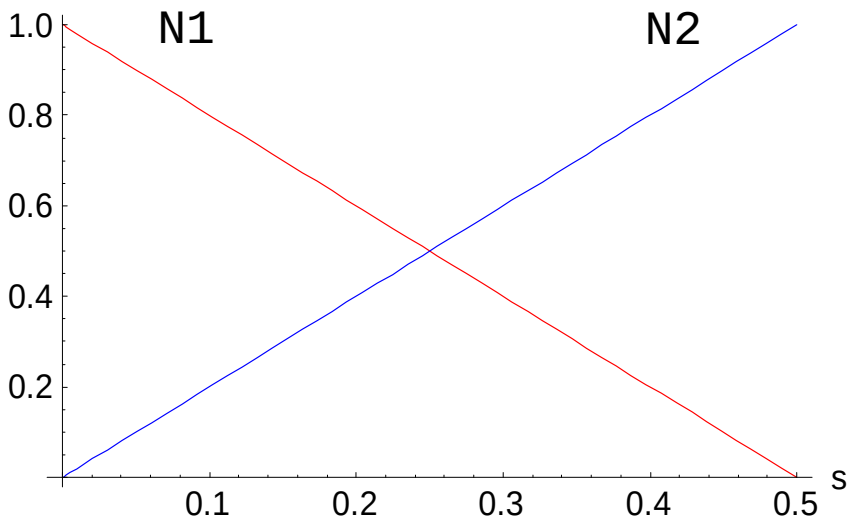
E.g.: $_{(2)}u = {}_2d = d_{1,2} = d_{2,1}$ $\mathbf{d}_2 = \left\{ {}_2d, {}_3d \right\}^T$

- Shape functions and element matrix of shape functions
  (s ... element local coordinate)

$$N_{e,1}(s) = 1 - \frac{s}{l_e}$$

$$N_{e,2}(s) = \frac{s}{l_e}$$

$$\mathbf{N}_e = \left[N_{e,1}(s), N_{e,2}(s) \right]$$
$$= \left[1 - \frac{s}{l_e}, \frac{s}{l_e} \right]$$



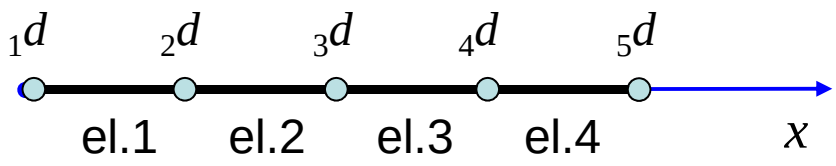
- matrix of shape functions derivatives

$$
\begin{aligned}
\mathbf{B}_e &= \frac{d}{dx}\left[N_{e,1}(s), N_{e,2}(s)\right] \\
&= \frac{d}{ds}\frac{ds}{dx}\left[N_{e,1}(s), N_{e,2}(s)\right] \\
&= \frac{d}{ds}\frac{ds}{dx}\left[1-\frac{s}{l_e}, \frac{s}{l_e}\right] \\
&= \left[-\frac{1}{l_e}, \frac{1}{l_e}\right]
\end{aligned}
$$

Virtual work principle

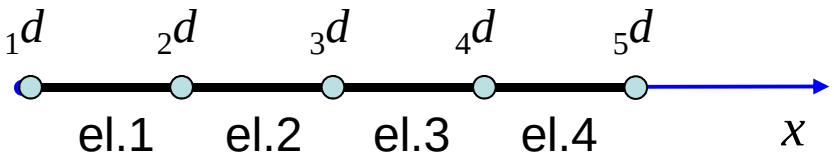
$$\int_{x=0}^{L}\left(EA\frac{du(x)}{dx}\frac{d\delta u(x)}{dx}-Ab(x)\delta u(x)\right)dx-\left.At\,\delta u(x)\right|_{x=0}=0$$

$$\int_{x=0}^{L}.....dx\rightarrow\sum_{e=1}^{4}\int_{s=0}^{l_e}.....ds$$ ... integration domain separated to elements



Note: since in this example we consider all elements to be the same, their $\mathbf{N}$ *and* $\mathbf{B}$ *matrices will be also the same and for the sake of brevity we will drop the indices* e *.*

$$\int_{x=0}^{L}\left(\overset{\boxed{1}}{EA\frac{du(x)}{dx}\frac{d\delta u(x)}{dx}}-\overset{\boxed{2}}{Ab(x)\delta u(x)}\right)dx-\overset{\boxed{3}}{At\,\delta u(x)\Big|_{x=0}}=0$$



	e.g. element 1	element 2
$\boxed{1}$	$\{\delta_1 d,\delta_2 d\}\int_{s=0}^{l_1}\mathbf{B}^T(s)EA\mathbf{B}(s)ds\begin{Bmatrix}{}_1d\\{}_2d\end{Bmatrix}$	$\{\delta_2 d,\delta_3 d\}\int_{s=0}^{l_e}\mathbf{B}^T(s)EA\mathbf{B}(s)ds\begin{Bmatrix}{}_2d\\{}_3d\end{Bmatrix}$
$\boxed{2}$	$\{\delta_1 d,\delta_2 d\}\int_{s=0}^{l_e}\mathbf{N}^T(s)A\bar{b}(s)ds$	$\{\delta_2 d,\delta_3 d\}\int_{s=0}^{l_e}\mathbf{N}^T(s)A\bar{b}(s)ds$
$\boxed{3}$	$\{\delta_1 d,\delta_2 d\}\begin{Bmatrix}A\bar{t}\\0\end{Bmatrix}$	$\{\delta_2 d,\delta_3 d\}\begin{Bmatrix}0\\0\end{Bmatrix}$

Element stiffness matrices

$$\{\delta_1 d, \delta_2 d\} \overset{1}{\boxed{\int_{s=0}^{l_e} \mathbf{B}^T(s) EA \mathbf{B}(s) ds}} \begin{Bmatrix} {}_1 d \\ {}_2 d \end{Bmatrix}$$

$$\mathbf{K}_e = \int_{s=0}^{l_e} \mathbf{B}^T(s) EA \mathbf{B}(s) ds$$

$$= \int_{s=0}^{l_e} \begin{bmatrix} -\frac{1}{l_e} \\ \frac{1}{l_e} \end{bmatrix} EA \left[-\frac{1}{l_e}, \frac{1}{l_e} \right] ds$$

$$= \begin{bmatrix} \frac{EA}{l_e} & -\frac{EA}{l_e} \\ -\frac{EA}{l_e} & \frac{EA}{l_e} \end{bmatrix}$$

```
(%i28) for e:1 thru nel do (
       Ke[e]:
       integrate(transpose(B(s)).matrix([E*A]).B(s),
       s,0,le)
       )$
```

```
(%i29) print(Ke[1],Ke[2],"...")$
```

$$\begin{bmatrix} \frac{A\,E}{le} & -\frac{A\,E}{le} \\ -\frac{A\,E}{le} & \frac{A\,E}{le} \end{bmatrix} \begin{bmatrix} \frac{A\,E}{le} & -\frac{A\,E}{le} \\ -\frac{A\,E}{le} & \frac{A\,E}{le} \end{bmatrix} \ldots$$

Element vectors of external nodal forces

2

$$\{\delta_1 d, \delta_2 d\} \int_{s=0}^{l_e} \mathbf{N}^T(s) A\bar{b}(s) ds$$

transform $s \rightarrow x$

```
(%i30) for e:1 thru nel do (
       fbe[e]:
       integrate(A*transpose(N(s)).matrix([b((e-1)*le+s)]),
       s,0,le)
       )$
```

el. node 1

el. node 2

```
(%i33) print(fbe[1],fbe[2],fbe[3],fbe[4])$
```

$$\begin{bmatrix} \frac{b0\,(le^2+3\,le)\,A}{6} \\ \frac{b0\,(2\,le^3+3\,le^2)\,A}{6\,le} \end{bmatrix} \begin{bmatrix} \frac{b0\,(4\,le^2+3\,le)\,A}{6} \\ \frac{b0\,(5\,le^3+3\,le^2)\,A}{6\,le} \end{bmatrix} \begin{bmatrix} \frac{b0\,(7\,le^2+3\,le)\,A}{6} \\ \frac{b0\,(8\,le^3+3\,le^2)\,A}{6\,le} \end{bmatrix} \begin{bmatrix} \frac{b0\,(10\,le^2+3\,le)\,A}{6} \\ \frac{b0\,(11\,le^3+3\,le^2)\,A}{6\,le} \end{bmatrix}$$

3

$$\{\delta_1 d, \delta_2 d\} \begin{Bmatrix} A\bar{t} \\ 0 \end{Bmatrix}$$

```
(%i31) for e:1 thru nel do (
       fte[e]:matrix([0],[0])
       )$
       fte[1]:matrix([A*t0],[0])$
```

el. node 1

el. node 2

```
(%i34) print(fte[1],fte[2],fte[3],fte[4])$
```

$$\begin{bmatrix} t0\,A \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\int_{x=0}^{L} \ldots\ldots dx \rightarrow \sum_{e=1}^{4} \int_{s=0}^{l_e} \ldots\ldots ds$$

$$\{\delta_1 d, \delta_2 d\}\mathbf{K}_1 \begin{Bmatrix} {}_1 d \\ {}_2 d \end{Bmatrix} + \{\delta_2 d, \delta_3 d\}\mathbf{K}_2 \begin{Bmatrix} {}_2 d \\ {}_3 d \end{Bmatrix} + \ldots$$

$$= \{\delta_1 d, \delta_2 d\} \begin{Bmatrix} {}_1 f_1^b + {}_1 f_1^t \\ {}_2 f_1^b + {}_2 f_1^t \end{Bmatrix} + \{\delta_2 d, \delta_3 d\} \begin{Bmatrix} {}_2 f_2^b + {}_2 f_2^t \\ {}_3 f_2^b + {}_3 f_2^t \end{Bmatrix} + \ldots$$

$$\{\delta_1 d, \delta_2 d, \ldots, \delta_5 d\}\mathbf{K} \begin{Bmatrix} {}_1 d \\ {}_2 d \\ \ldots \\ {}_5 d \end{Bmatrix} == \{\delta_1 d, \delta_2 d, \ldots, \delta_5 d\} \begin{Bmatrix} {}_1 f \\ {}_2 f \\ \ldots \\ {}_5 f \end{Bmatrix}$$

Assembly of global stiffness matrix and global force vector

```
(%i35) K:zeromatrix(nnode,nnode);
```

$$(\%o35)\quad \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

```
(%i36) f:zeromatrix(nnode,1);
```

$$(\%o36)\quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

initialize with zeros

```
(%i37) for e:1 thru nel do (
       for i:1 thru 2 do (
       for j:1 thru 2 do (
          K[(e-1)+i,(e-1)+j]: K[(e-1)+i,(e-1)+j] + Ke[e][i, j]
       )));
(%o37) done
```

allocate element matrices and vectors into global ones

```
(%i39) for e:1 thru nel do (
       for i:1 thru 2 do (
          f[(e-1)+i]: f[(e-1)+i] + fte[e][i, 1]+ fbe[e][i, 1]
       ));
(%o39) done
```

Global stiffness matrix $\mathbf{K}$

$$\begin{bmatrix} \frac{EA}{l_e} & -\frac{EA}{l_e} & 0 & 0 & 0 \\ -\frac{EA}{l_e} & \frac{2EA}{l_e} & -\frac{EA}{l_e} & 0 & 0 \\ 0 & -\frac{EA}{l_e} & \frac{2EA}{l_e} & -\frac{EA}{l_e} & 0 \\ 0 & 0 & -\frac{EA}{l_e} & \frac{2EA}{l_e} & -\frac{EA}{l_e} \\ 0 & 0 & 0 & -\frac{EA}{l_e} & \frac{EA}{l_e} \end{bmatrix}$$

Global vector of ext. nodal forces $\mathbf{f}$

$$\begin{Bmatrix} \frac{1}{6} Ab_0 l_e (3 + l_e) + At_0 \\ Ab_0 l_e + Ab_0 l_e^2 \\ Ab_0 l_e + 2Ab_0 l_e^2 \\ Ab_0 l_e + 3Ab_0 l_e^2 \\ \frac{Ab_0 l_e}{2} + \frac{11}{6} Ab_0 l_e^2 \end{Bmatrix}$$

Imposition of boundary conditions

$$\{\delta_1 d, \delta_2 d, ..., \delta_5 d\}\mathbf{K}\begin{Bmatrix} {}_1d \\ {}_2d \\ ... \\ {}_5d \end{Bmatrix} == \{\delta_1 d, \delta_2 d, ..., \delta_5 d\}\begin{Bmatrix} {}_1f \\ {}_2f \\ ... \\ {}_5f \end{Bmatrix}$$

... must be satisfied for any $\delta\mathbf{d}$ satisfying kinematic boundary conditions ($\delta_5 d = 0$)

$$\begin{bmatrix} \frac{EA}{l_e} & -\frac{EA}{l_e} & 0 & 0 & 0 \\ -\frac{EA}{l_e} & \frac{2EA}{l_e} & -\frac{EA}{l_e} & 0 & 0 \\ 0 & -\frac{EA}{l_e} & \frac{2EA}{l_e} & -\frac{EA}{l_e} & 0 \\ 0 & 0 & -\frac{EA}{l_e} & \frac{2EA}{l_e} & -\frac{EA}{l_e} \\ \cancel{0} & \cancel{0} & \cancel{0} & \frac{EA}{l_e} & \frac{EA}{l_e} \end{bmatrix}\begin{Bmatrix} {}_1d \\ {}_2d \\ {}_3d \\ {}_4d \\ \cancel{{}_5d} \end{Bmatrix} = \begin{Bmatrix} \frac{1}{6}Ab_0 l_e(3+l_e) + At_0 \\ Ab_0 l_e + Ab_0 l_e^2 \\ Ab_0 l_e + 2Ab_0 l_e^2 \\ Ab_0 l_e + 3Ab_0 l_e^2 \\ \cancel{\frac{Ab_0 l_e}{2} + \frac{11}{6}Ab_0 l_e^2} \end{Bmatrix}$$

... a system of linear algebraic equations

Solution of the system of equations

```
(%i44) sol:linsolve_by_lu(subst(par,Kr),subst(par,fr));
```

$$(\%o44)\quad \left[\begin{bmatrix} 0.005 \\ 0.007875 \\ 0.0085 \\ 0.006125 \end{bmatrix}, \text{false}\right]$$

```
(%i45) d:sol[1];
```

$$(\%o45)\quad \begin{bmatrix} 0.005 \\ 0.007875 \\ 0.0085 \\ 0.006125 \end{bmatrix}$$

... resulting nodal displacements

```
(%i46) d:addrow(d,[0]);
```

$$(\%o46)\quad \begin{bmatrix} 0.005 \\ 0.007875 \\ 0.0085 \\ 0.006125 \\ 0 \end{bmatrix}$$

... include the prescribed displacement

Calculation of strain and stress

- Strain and stress are analyzed locally on each element

$$\varepsilon_e(s) = \mathbf{B}(s)\mathbf{d}_e$$

```
(%i48) eps[1](s):=''(B(s).submatrix(3,4,5,d)) /* element 1, global nodes 1,2 */;
       eps[2](s):=''(B(s).submatrix(1,4,5,d)) /* element 2, global nodes 2,3 */;
       eps[3](s):=''(B(s).submatrix(1,2,5,d)) /* element 3, global nodes 3,4 */;
       eps[4](s):=''(B(s).submatrix(1,2,3,d)) /* element 4, global nodes 4,5 */;
```

(%o48) $eps_1(s):=0.00575$

(%o49) $eps_2(s):=0.00125$

(%o50) $eps_3(s):=-0.00475$

(%o51) $eps_4(s):=-0.01225$

... strain and stress are constant on each element
(result of the chosen order of approximation)

Calculation of strain and stress

- Strain and stress are analyzed locally on each element

$$\sigma_e(s) = E\varepsilon_e(s)$$

```
(%i52) sig[1](s):=''(subst(par,E*eps[1](s)));
       sig[2](s):=''(subst(par,E*eps[2](s)));
       sig[3](s):=''(subst(par,E*eps[3](s)));
       sig[4](s):=''(subst(par,E*eps[4](s)));
```

(%o52) $sig_1(s):=11.5$

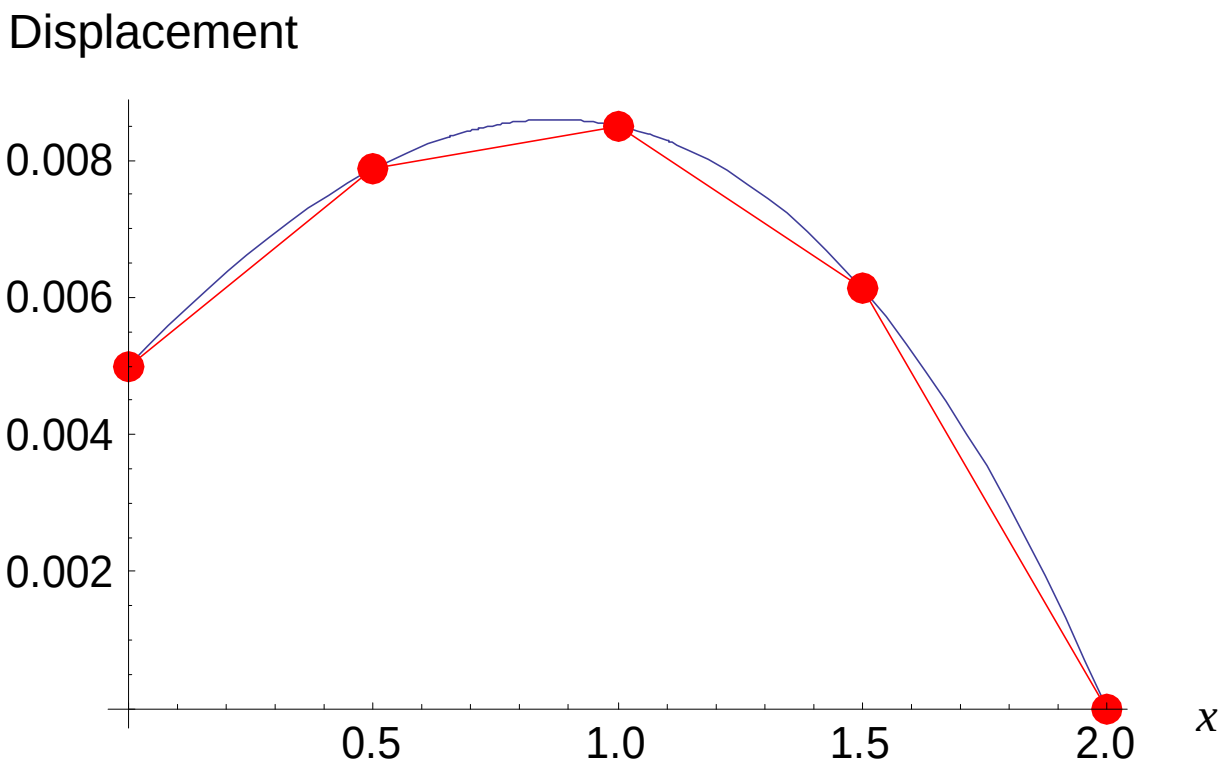
(%o53) $sig_2(s):=2.500000000000002$

(%o54) $sig_3(s):=-9.500000000000002$

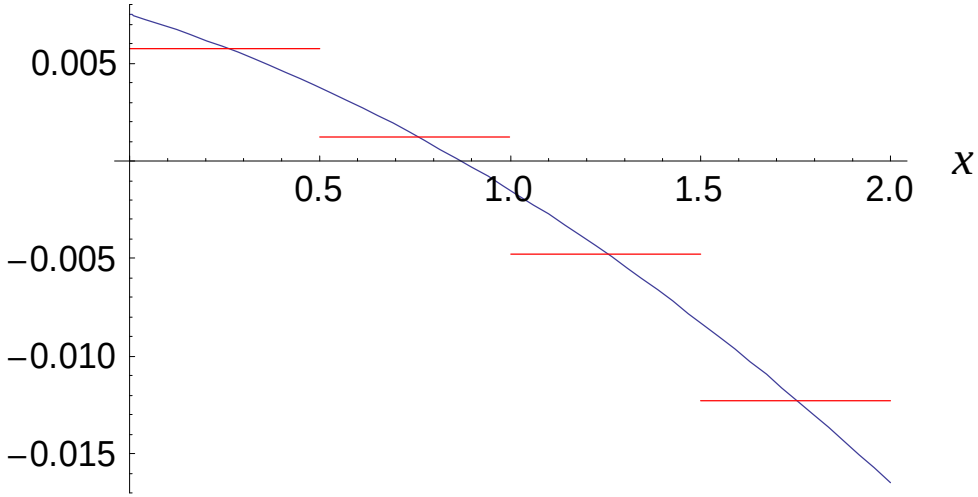
(%o55) $sig_4(s):=-24.5$

... strain and stress are constant on each element
(result of the chosen order of approximation)

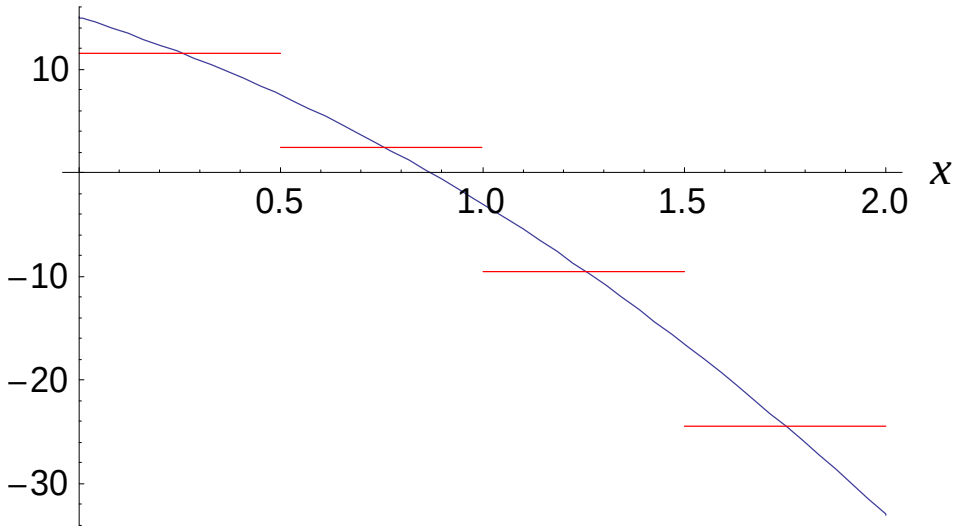
Compare FEM with analytical solution



Strain



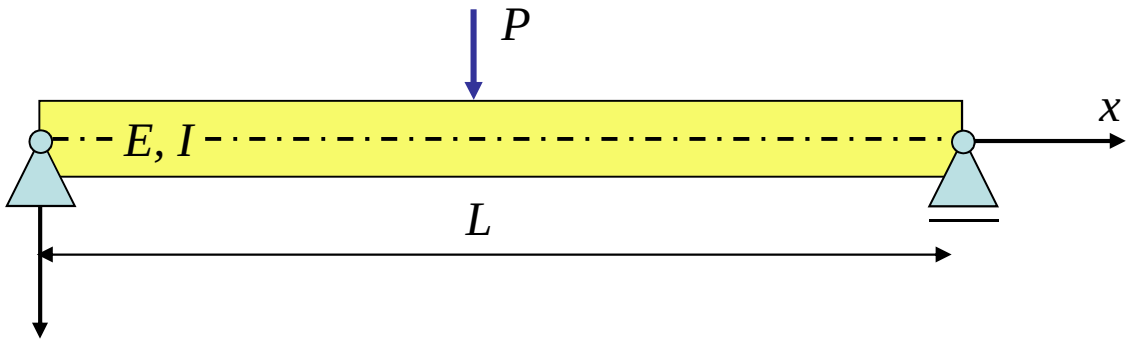
Stress



Example 2 – beam bending

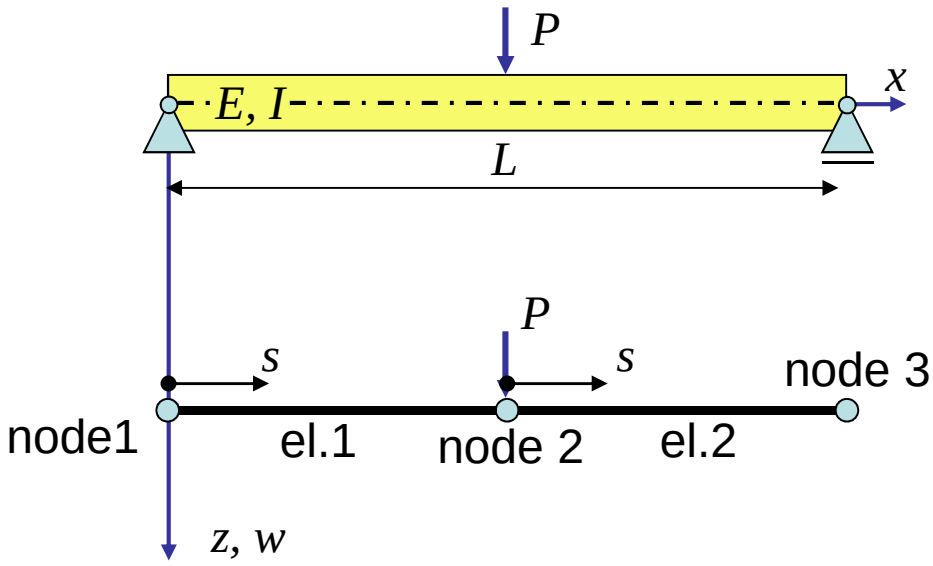
Problem description

Use the FEM to determine the displacement and bending moment distribution of a beam loaded by a point load in mid-span.



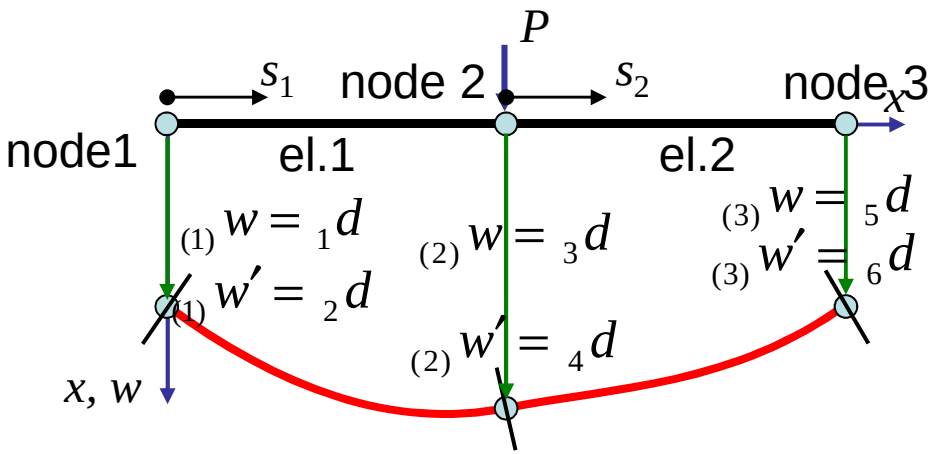
Approximation of displacement

- Discretization of the domain into 2 elements, 3 nodes
- In each element, introduce local coordinate s :



$$s = \frac{x - x_{e,(1)}}{l_e} \qquad \left(\text{i.e. } s \in \langle 0,1 \rangle\right)$$

- In each element, assume approximation of displacement by the same type of function – 3rd order polynomial. Displacements and their derivatives (negative rotations) at nodes are its coefficients (degrees of freedom):



$$w(s) = \left(1 - 3s^2 + 2s^3\right) w_{e,(1)} + \left(s - 2s^2 + s^3\right) l_e w'_{e,(1)} + \left(3s^2 - 2s^3\right) w_{e,(2)}$$
$$+ \left(-s^2 + s^3\right) l_e w'_{e,(2)}$$

$$w(s) = \left[N_{e,1}(s), N_{e,2}(s), N_{e,3}(s), N_{e,4}(s)\right] \begin{Bmatrix} d_{e,1} \\ d_{e,2} \\ d_{e,3} \\ d_{e,4} \end{Bmatrix} = \mathbf{N}_e \mathbf{d}_e$$

$$d_{e,1} = w_{e,(1)}, d_{e,2} = w'_{e,(1)}, d_{e,3} = w_{e,(2)}, d_{e,4} = w'_{e,(2)}$$

Method of total potential energy

- To determine the unknown nodal degrees of freedom, we use the method of total potential energy
- The total potential energy is $E_{pot} = E_{int} + E_{ext}$, where

$$E_{int} = \int_{x=0}^{L} \left(\int M(x) d\kappa \right) dx = \int_{x=0}^{L} \left(\int EI\kappa(x) d\kappa \right) dx = \int_{x=0}^{L} \frac{EI}{2} \kappa^2(x)\, dx$$

$$= \underbrace{\int_{s=0}^{1} \frac{E_1 I_1}{2} \kappa^2(s)\, l_1 ds}_{\text{element 1}} + \underbrace{\int_{s=0}^{1} \frac{E_2 I_2}{2} \kappa^2(s)\, l_2 ds}_{\text{element 2}}$$

$$E_{ext} = -Pw(L/2) = -P_3 d$$

Note:

$$s = \frac{x - x_{e,(1)}}{l_e} \rightarrow x = s\, l_e + x_{e,(1)}$$

$$dx = \frac{dx}{ds} ds = l_e ds$$

- The generalized strain can be expressed from approximated DOF's as

$$\kappa(x) = -\frac{d^2 w(x)}{dx^2} = -\frac{d^2 w(s)}{l_e^2 ds^2} = -\frac{d^2 \mathbf{N}_e(s)}{l_e^2 ds^2}\mathbf{d}_e =$$

$$= \left[\frac{6}{l_e^2} - \frac{12s}{l_e^2}, \frac{4}{l_e} - \frac{6s}{l_e}, -\frac{6}{l_e^2} + \frac{12s}{l_e^2}, \frac{2}{l_e} - \frac{6s}{l_e}\right]\mathbf{d}_e$$

$$= \mathbf{B}_e\,\mathbf{d}_e$$

Note:

$$\frac{d^2 w}{dx^2} = \frac{d}{dx}\left(\frac{dw}{dx}\right) = \frac{d}{ds}\frac{ds}{dx}\left(\frac{dw}{ds}\frac{ds}{dx}\right)$$

$$= \left(\frac{ds}{dx}\right)^2 \frac{d^2 w}{ds^2} = \frac{1}{l_e^2}\frac{d^2 w}{ds^2}$$

- We denote $\left[E_e\, I_e\right] = \mathbf{D}_e$

- Then

$$E_{pot} = \int_{s=0}^{1} \frac{1}{2}\mathbf{D}_1\left(\mathbf{B}_1\mathbf{d}_1\right)^2 l_1\, ds + \int_{s=0}^{1} \frac{1}{2}\mathbf{D}_2\left(\mathbf{B}_2\mathbf{d}_2\right)^2 l_2\, ds - P_3 d$$

$$= \int_{s=0}^{1} \frac{1}{2}\mathbf{d}_1^T\mathbf{B}_1^T\mathbf{D}_1\mathbf{B}_1\mathbf{d}_1\, l_1\, ds + \int_{s=0}^{1} \frac{1}{2}\mathbf{d}_2^T\mathbf{B}_2^T\mathbf{D}_2\mathbf{B}_2\mathbf{d}_2\, l_1\, ds - P_3 d$$

- The unknown DOF's (${}_1d$, ${}_2d$, ... ${}_6d$) are determined from:
  a) kinematic boundary conditions ${}_1d = {}_{(1)}w = 0$ and ${}_5d = {}_{(3)}w = 0$
  b) minimization of the total potential energy with respect to the remaining unknowns (${}_2d$, ${}_3d$, ${}_4d$, ${}_6d$)

$$\frac{\partial E_{pot}}{\partial\, {}_2d} = 0 \quad \frac{\partial E_{pot}}{\partial\, {}_3d} = 0 \quad \frac{\partial E_{pot}}{\partial\, {}_4d} = 0 \quad \frac{\partial E_{pot}}{\partial\, {}_6d} = 0$$

$$\frac{\partial E_{pot}}{\partial_2 d} = \int_{s=0}^{1} \frac{d}{d_2 d}\left(\frac{1}{2}\mathbf{d}_1^T\mathbf{B}_1^T\mathbf{D}_1\mathbf{B}_1\mathbf{d}_1\right) l_1\, ds + \int_{s=0}^{1} \frac{d}{d_2 d}\left(\frac{1}{2}\mathbf{d}_2^T\mathbf{B}_2^T\mathbf{D}_2\mathbf{B}_2\mathbf{d}_2\right) l_2\, ds - \frac{d}{d_2 d}\left(P_3 d\right)$$

$$= \int_{s=0}^{1} \frac{1}{2}\left(\{0,1,0,0\}\mathbf{B}_1^T\mathbf{D}_1\mathbf{B}_1\mathbf{d}_1 + \mathbf{d}_1^T\mathbf{B}_1^T\mathbf{D}_1\mathbf{B}_1\begin{Bmatrix}0\\1\\0\\0\end{Bmatrix}\right) l_1\, ds + 0 - 0$$

$$= \int_{s=0}^{1} \frac{1}{2} 2B_{1,2}\mathbf{D}_1\mathbf{B}_1\mathbf{d}_1\, l_1\, ds = \int_{s=0}^{1} B_{1,2}\mathbf{D}_1\mathbf{B}_1\mathbf{d}_1\, l_1\, ds = 0 \qquad (1)$$

where

$$\mathbf{B}_1 = \left[\frac{6}{l_1^2} - \frac{12s}{l_1^2}, \overset{B_{1,2}}{\boxed{\frac{4}{l_1} - \frac{6s}{l_1}}}, -\frac{6}{l_1^2} + \frac{12s}{l_1^2}, \frac{2}{l_1} - \frac{6s}{l_1}\right]$$

Similarly

$$\frac{\partial E_{pot}}{\partial_3 d} = \int_{s=0}^{1} B_{1,3}\mathbf{D}_1\mathbf{B}_1\mathbf{d}_1\, l_1\, ds + \int_{s=0}^{1} B_{2,1}\mathbf{D}_2\mathbf{B}_2\mathbf{d}_2\, l_2\, ds - P = 0 \qquad (2)$$

$$\frac{\partial E_{pot}}{\partial_4 d} = \int_{s=0}^{1} B_{1,4}\mathbf{D}_1\mathbf{B}_1\mathbf{d}_1\, l_1\, ds + \int_{s=0}^{1} B_{2,2}\mathbf{D}_2\mathbf{B}_2\mathbf{d}_2\, l_2\, ds = 0 \qquad (3)$$

$$\frac{\partial E_{pot}}{\partial_6 d} = \int_{s=0}^{1} B_{2,4}\mathbf{D}_2\mathbf{B}_2\mathbf{d}_2\, l_2\, ds = 0 \qquad (4)$$

element 1

element 2

After evaluating the integrals, equations (1) to (4) can be rearranged:

$$\begin{bmatrix} \frac{8EI}{L} & -\frac{24EI}{L^2} & \frac{4EI}{L} & 0 \\ -\frac{24EI}{L^2} & \frac{192EI}{L^3} & 0 & \frac{24EI}{L^2} \\ \frac{4EI}{L} & 0 & \frac{16EI}{L} & \frac{4EI}{L} \\ 0 & \frac{24EI}{L^2} & \frac{4EI}{L} & \frac{8EI}{L} \end{bmatrix} \begin{Bmatrix} {}_2d \\ {}_3d \\ {}_4d \\ {}_6d \end{Bmatrix} = \begin{Bmatrix} 0 \\ P \\ 0 \\ 0 \end{Bmatrix}$$

from which we solve

$${}_2d = {}_{(1)}w' = \frac{L^2P}{16EI}, \quad {}_3d = {}_{(2)}w = \frac{L^3P}{48EI}, \quad {}_4d = {}_{(2)}w' = 0, \quad {}_6d = {}_{(3)}w' = -\frac{L^2P}{16EI}$$

- By substituting to $w(s) = \mathbf{N}_e \mathbf{d}_e$ we obtain, e.g. for element 1:

$$w(x) = \frac{PL^3}{48EI}\left(3\frac{x}{L} - 4\frac{x^3}{L^3}\right)$$

... which is in this special case equal to the exact solution of the strong form

Generalization

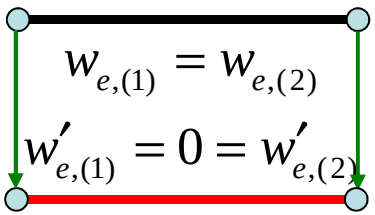
Approximation of displacement – choice of interpolation functions

- General requirements

  In order to assure convergence of the FE solution to the exact solution as the number of elements increases, the shape functions must satisfy the following requirements:

  1. The shape functions must not allow a strain to appear if the nodal variables correspond to rigid body motion (displacement, rotation)

For example:



$$\kappa = \left[\boxed{\frac{6}{l_e^2} - \frac{12s}{l_e^2}}, \frac{4}{l_e} - \frac{6s}{l_e}, \boxed{-\frac{6}{l_e^2} + \frac{12s}{l_e^2}}, \frac{2}{l_e} - \frac{6s}{l_e} \right] \begin{Bmatrix} w_{e,(1)} \\ 0 \\ w_{e,(2)} \\ 0 \end{Bmatrix} = 0$$

2. Completeness condition: If the nodal variables correspond to uniform strain, then the shape functions must yield this uniform strain.

For example:

$$w_{e,(1)} = 0 = w_{e,(2)}$$

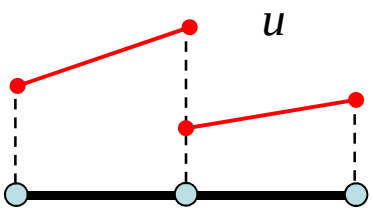
$$-w'_{e,(1)} = w'_{e,(2)}$$

$$\kappa = \left[\frac{6}{l_e^2} - \frac{12s}{l_e^2}, \boxed{\frac{4}{l_e} - \frac{6s}{l_e}}, -\frac{6}{l_e^2} + \frac{12s}{l_e^2}, \boxed{\frac{2}{l_e} - \frac{6s}{l_e}} \right] \begin{Bmatrix} 0 \\ w'_{e,(1)} \\ 0 \\ -w'_{e,(2)} \end{Bmatrix} = \frac{2}{l_e} w'_{e,(1)}$$

3. Compatibility condition: The shape functions should be chosen so that strain remains finite at the elements boundaries. Therefore:

- If generalized strain involves 1st derivative of displacement, displacement must be continuous across the element boundary.

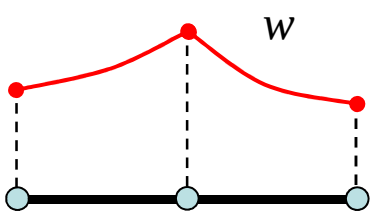
  For example:



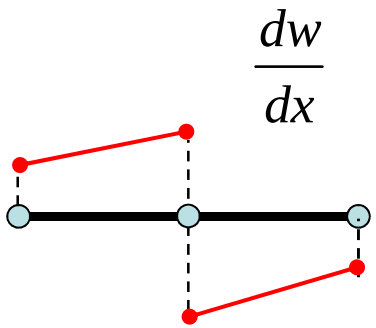
not acceptable

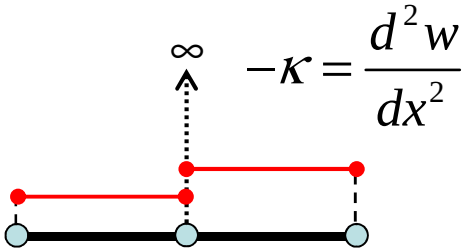
- If generalized strain involves derivatives of displacement up to order n, then continuity of displacement and its derivatives up to order n-1 is required.

  For example:



not acceptable





- Elements satisfying the condition 3. are called conformable elements. The degree of compatibility achieved by the shape functions at element boundaries is commonly called as:
    - C^0 continuity ... if only displacement and none of its derivatives is continuous
    - C^1 continuity ... if displacement and its 1st derivatives are continuous
    - C^n continuity ... if displacement and its derivatives up to n^{th} order are continuous

Assembly of the global stiffness matrix and external load vector

- The total potential energy of a an elastic structure discretized into n elements can be written as:

$$\pi = \sum_{e=1}^{n}\left[\frac{1}{2}\mathbf{d}_e^T\left(\boxed{\int_v \mathbf{B}_e^T\mathbf{D}_e\mathbf{B}_e J_e dv}\right)\mathbf{d}_e - \mathbf{d}_e^T\boxed{\int_v \mathbf{N}_e^T\mathbf{b}\, J_e dv} - \mathbf{d}_e^T\boxed{\int_c \mathbf{N}_e^T\mathbf{t}\, J_{ce} dc\Big|_{S_t}}\right]$$

$$= \sum_{e=1}^{n}\left(\frac{1}{2}\mathbf{d}_e^T\mathbf{K}_e\mathbf{d}_e - \mathbf{d}_e^T\mathbf{f}_e\right) \qquad (*)$$

on element boundary
with static b.c.

$\mathbf{d}_e$ ... vector of element nodal DOF's
$\mathbf{D}_e$ ... matrix that relates generalized stress and generalized strain
$\mathbf{B}_e$ ... matrix that relates generalized strain to nodal DOF's
$\mathbf{b}$ ... body force
$\mathbf{t}$ ... surface traction
$\mathbf{K}_e$ ... element stiffness matrix
$\mathbf{f}_e$ ... element vector of external forces
J_e ... determinant of Jacobian matrix

- Note on Jacobian matrix:

  Recall the transformation from global to local coordinates and vice versa

$$s = \frac{x - x_{e,(1)}}{l_e} \rightarrow x = s\, l_e + x_{e,(1)}$$

$$dx = \frac{dx}{ds} ds = l_e ds = J\ ds$$

  In multidimensional case $\mathbf{J}$ becomes a matrix. It is called Jacobian matrix.

  Further details will be discussed later.

- The unknown nodal degrees of freedom that form vectors $\mathbf{d}_e$ are determined by using the prescribed kinematic constraints and minimizing π with respect to all remaining m DOF's.
- We arrange all the unknown DOF's into a single vector

$$\mathbf{d} = \begin{Bmatrix} {}_1d \\ {}_2d \\ {}_3d \\ {}_4d \\ {}_5d \\ \vdots \\ {}_md \end{Bmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \text{element 1} \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \text{element 2} \end{matrix}$$

- Then minimization of π implies

$$\frac{\partial \pi}{\partial\, {}_id} = 0$$

... number of equations corresponds to the number of unknown DOF's

- By applying the derivatives to equation (*) we can write, for example for elements with 4 DOF's

$$\begin{bmatrix} \begin{bmatrix}\mathbf{K}_1\end{bmatrix} & \begin{matrix}0 & 0 & \cdots\\ 0 & 0 & \cdots\\ 0 & 0 & \cdots\\ 0 & 0 & \cdots\end{matrix} \\ \begin{matrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots\end{matrix} & \begin{matrix}0 & 0 & \cdots\\ 0 & 0 & \cdots\\ 0 & 0 & \cdots\\ \vdots & \vdots & \ddots\end{matrix}\end{bmatrix} \begin{Bmatrix} {}_1d\\ {}_2d\\ {}_3d\\ {}_4d\\ {}_5d\\ \vdots\\ {}_md \end{Bmatrix} + \begin{bmatrix} \begin{matrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots\end{matrix} & \begin{matrix}0 & 0 & \cdots\\ 0 & 0 & \cdots\\ 0 & 0 & \cdots\\ 0 & 0 & \cdots\\ 0 & 0 & \cdots\\ 0 & 0 & \cdots\\ \vdots & \vdots & \ddots\end{matrix} \\ \begin{matrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots\end{matrix} & \begin{matrix}0 & 0 & \cdots\\ 0 & 0 & \cdots\\ 0 & 0 & \cdots\\ \vdots & \vdots & \ddots\end{matrix}\end{bmatrix} \begin{Bmatrix} {}_1d\\ {}_2d\\ {}_3d\\ {}_4d\\ {}_5d\\ \vdots\\ {}_md \end{Bmatrix} + \ldots$$

$$= \begin{Bmatrix} \begin{Bmatrix}\mathbf{f}_1\end{Bmatrix}\\ 0\\ \vdots\\ 0 \end{Bmatrix} + \begin{Bmatrix} 0\\ 0\\ \begin{Bmatrix}\mathbf{f}_2\end{Bmatrix}\\ 0 \end{Bmatrix} + \cdots$$

- In a compact form

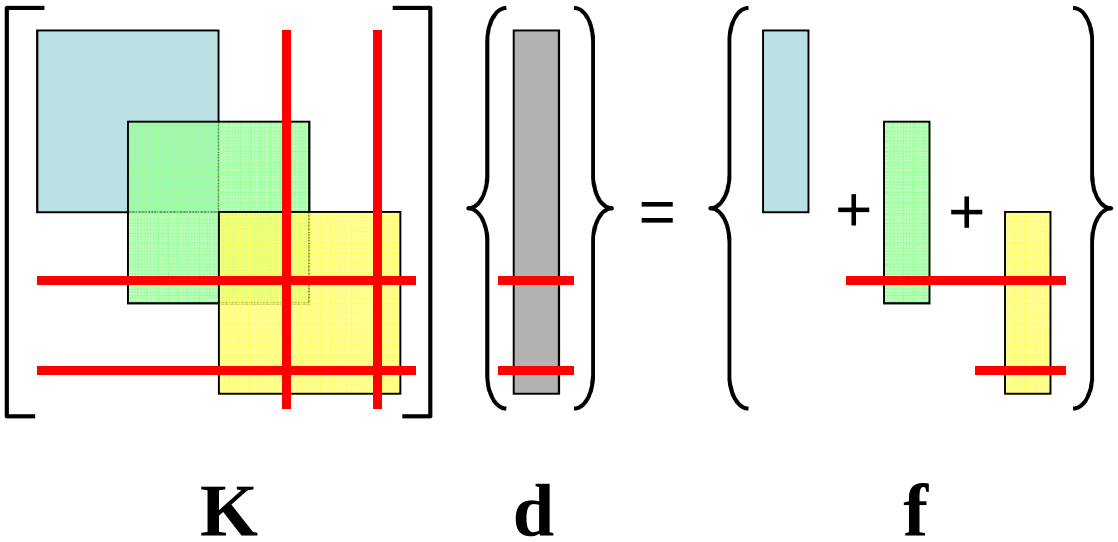
$$\left[\mathbf{K}\right]_{m\times m}\left\{\mathbf{d}\right\}_{m}=\left\{\mathbf{f}\right\}_{m}$$

... global system of equations, from which d is solved

$\mathbf{d}$ ... global vector of element nodal DOF's
$\mathbf{K}$ ... global stiffness matrix
$\mathbf{f}$ ... global vector of external forces

- Notes:
  - The same result is obtained if the principle of virtual work is applied. The advantage of using the principle of virtual work as opposed to the method of total potential energy is that it is valid generally for any constitutive relationship, not only elastic.

- The global matrices can be first assembled regardless of kinematic boundary conditions. Subsequently the homogenous kinematic boundary conditions can be applied by striking out the corresponding rows and columns in the global vectors:



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Remark

This document is designated solely as a teaching aid for students of CTU in Prague, Faculty of Civil Engineering, course Numerické metody v inženýrských úlohách.

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