Micro-structure in Linear Elasticity

R. D. MINDLIN

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Introduction

In this paper, there is formulated a linear theory of a three-dimensional, elastic continuum which has some of the properties of a crystal lattice as a result of the inclusion, in the theory, of the idea of the unit cell. The equations yield wave-dispersion relations with acoustic and optical branches of the same character as those found at long wave-lengths in crystal lattice theories and observed in neutron scattering experiments. Although specific solutions are not exhibited in detail, it is apparent from the form of the equations that there will be interesting surface effects under conditions of both motion and equilibrium.

The unit cell may also be interpreted as a molecule of a polymer, a crystallite of a polycrystal or a grain of a granular material. The mathematical model of the cell is a linear version of ERICKSEN & TRUESDELL's deformable directors [*1*]. If the cell is made rigid, the equations reduce to those of a linear COSSERAT continuum [*2*].

The method of derivation of the equations is analogous to one used in deducing two-dimensional equations of high frequency vibrations of plates from the three-dimensional equations of classical linear elasticity. By the same technique as that employed in passing from high order theories of plates to classical theories of plates, the equations are shown to reduce, at low frequencies and very long wave-lengths in isotropic materials, to those of an elastic continuum with potential energy-density dependent on strain and strain gradient and kinetic energy-density dependent on velocity and velocity gradient.

A linear form of TOUPIN's generalization [*8*, Section *7*] of couple-stress theory [*5*—*10*] is obtained by eliminating the difference between the deformations of the unit cell and the surrounding medium; and linear couple-stress theory itself is obtained by eliminating, further, the symmetric part of the strain gradient. Both of these special cases are also limited to low frequencies and very long wave-lengths.

1. Kinematics

Consider a material volume V, bounded by a surface S, with X_i, $i=1, 2, 3$, the rectangular components of the material position vector, measured from a fixed origin, and x_i the components, in the same rectangular frame, of the spatial position vector. The components of displacement of a material particle are defined as

$$u_i \equiv x_i - X_i. \tag{1.1}$$

Embedded in each material particle there is assumed to be a *micro-volume* V' in which X'_i and x'_i are the components of the material and spatial position vectors, respectively, referred to axes parallel to those of the x_i, with origin fixed in the particle: so that the origin of the coordinates x'_i moves with the displacement $\boldsymbol{u}$. A *micro-displacement* $\boldsymbol{u}'$ is defined; with components

$$u'_i \equiv x'_i - X'_i. \tag{1.2}$$

The absolute values of the displacement-gradients are assumed to be small in comparison with unity:

$$\left|\frac{\partial u_j}{\partial X_i}\right| \ll 1, \qquad \left|\frac{\partial u'_j}{\partial X'_i}\right| \ll 1, \tag{1.3}$$

so that we may write

$$\frac{\partial u_j}{\partial X_i} \approx \frac{\partial u_j}{\partial x_i} \equiv \partial_i u_j, \qquad u_j = u_j(x_i, t), \tag{1.4}$$

$$\frac{\partial u'_j}{\partial X'_i} \approx \frac{\partial u'_j}{\partial x'_i} \equiv \partial'_i u'_j, \qquad u'_j = u'_j(x_i, x'_i, t), \tag{1.5}$$

where t is the time.

Assume that the micro-displacement can be expressed as a sum of products of specified functions of the x'_i and arbitrary functions of the x_i and t. As an approximation, retain only a single, linear term of the series:

$$u'_j = x'_k \psi_{kj}, \tag{1.6}$$

where ψ_{kj} is a function of the x_i and t only. Then the displacement-gradient in the micro-medium is

$$\partial'_i u'_j = \psi_{ij}, \tag{1.7}$$

i.e., the *micro-deformation* ψ_{ij} is taken to be homogeneous in the *micro-medium* V' and non-homogeneous in the *macro-medium* V. In view of $(1.3)_2$, $|\psi_{ij}| \ll 1$. The symmetric part of ψ_{ij} is the *micro-strain:*

$$\psi_{(ij)} \equiv \tfrac{1}{2}(\psi_{ij} + \psi_{ji}) \tag{1.8}$$

and the antisymmetric part is the *micro-rotation:*

$$\psi_{[ij]} \equiv \tfrac{1}{2}(\psi_{ij} - \psi_{ji}). \tag{1.9}$$

An alternative interpretation of the micro-deformation is that the quantities ψ_{ij} are proportional to the components of the displacements of the tips of deformable *directors*, as described by ERICKSEN & TRUESDELL [*1*]. The $\psi_{[ij]}$ then are the components of rotation of the COSSERAT *trièdre* [*2*, p. 122].

We define the usual strain (now the *macro-strain*)

$$\varepsilon_{ij} \equiv \tfrac{1}{2}(\partial_i u_j + \partial_j u_i); \tag{1.10}$$

and also a *relative deformation* (the difference between the macro-displacement-gradient and the micro-deformation)

$$\gamma_{ij} \equiv \partial_i u_j - \psi_{ij}, \tag{1.11}$$

and a *micro-deformation gradient* (the macro-gradient of the micro-deformation):

$$\varkappa_{ijk} \equiv \partial_i \psi_{jk}. \tag{1.12}$$

All three of the tensors ε_{ij}, γ_{ij} and $\varkappa_{ijk}$ are independent of the micro-coordinates x'_i. Typical components of γ_{ij} and $\varkappa_{ijk}$ are illustrated in Figs. 1 and 2.

The u_i and ψ_{ij} are assumed to be single-valued functions of the x_i, leading to the compatibility equations

$$e_{mik} e_{nlj} \partial_i \partial_j \varepsilon_{kl} = 0, \tag{1.13}$$

$$e_{mij} \partial_i \varkappa_{jkl} = 0, \tag{1.14}$$

$$\partial_i(\varepsilon_{jk} + \omega_{jk} - \gamma_{jk}) = \varkappa_{ijk}, \tag{1.15}$$

where ω_{ij} is the *macro-rotation:*

$$\omega_{ij} \equiv \tfrac{1}{2}(\partial_i u_j - \partial_j u_i) \tag{1.16}$$

and e_{ijk} is the alternating tensor.

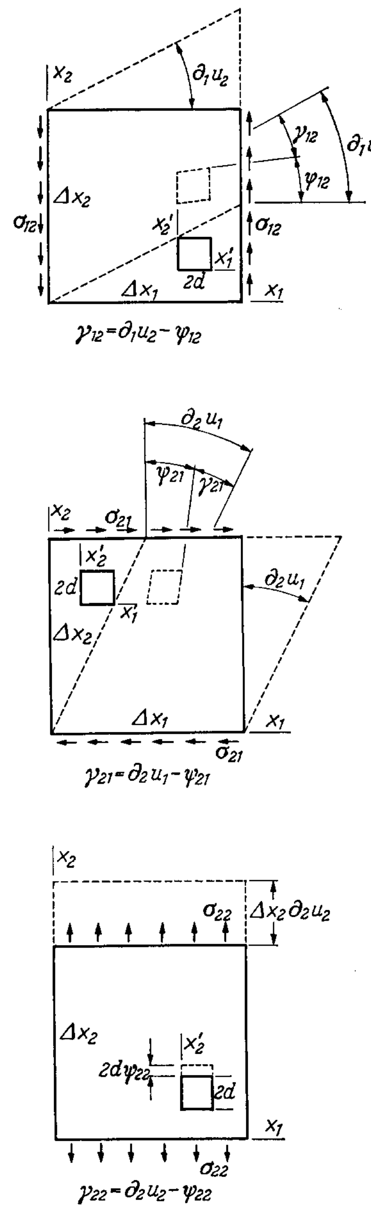

Fig. 1. Typical components of relative stress σ_{ij}, displacement-gradient $\partial_i u_j$, micro-deformation ψ_{ij} and relative deformation γ_{ij}

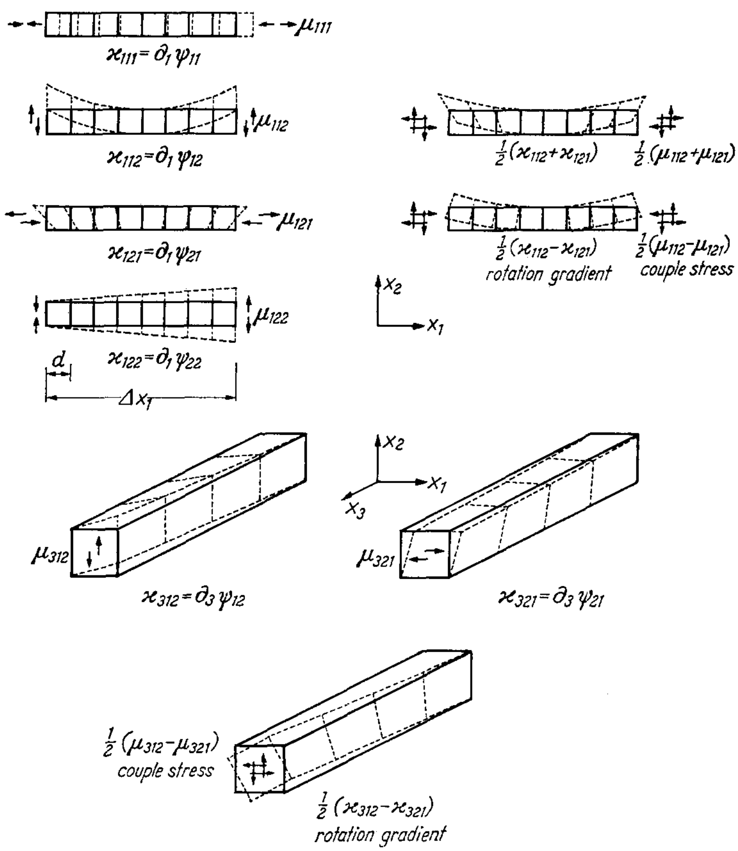

Fig. 2. Typical components of double stress μ_{ijk} and gradient of micro-deformation $\varkappa_{ijk}$

2. Kinetic and potential energies

Let the micro-medium be a parallelepiped with volume V' and edges of lengths $2d_i$ and direction cosines l_{ij} with respect to the axes x'_i. Let x''_i be oblique Cartesian coordinates parallel to the edges d_i, respectively. Then [3, p. 153]

$$x'_i = l_{ij} x''_j, \tag{2.1$_1$}$$

$$V' = 8 \, \|l_{ij} l_{ik}\|^{\frac{1}{2}} d_1 d_2 d_3, \tag{2.1$_2$}$$

$$dV' = \|l_{ij} l_{ik}\|^{\frac{1}{2}} dx''_1 \, dx''_2 \, dx''_3. \tag{2.1$_3$}$$

Let ϱ_M be the mass of macro-material per unit macro-volume and let ϱ' be the mass of micro-material per unit macro-volume. We define a kinetic energy-density (kinetic energy per unit macro-volume):

$$T \equiv \frac{1}{2} \varrho_M \dot{u}_j \dot{u}_j + \frac{1}{V'} \int\limits_{V'} \frac{1}{2} \varrho' (\dot{u}_j + \dot{u}'_j)(\dot{u}_j + \dot{u}'_j) \, dV', \tag{2.2}$$

where the dot designates differentiation with respect to time. Upon substituting (1.6) and (2.1) in (2.2) and performing the integration, we find

$$T = \tfrac{1}{2} \varrho \dot{u}_j \dot{u}_j + \tfrac{1}{6} \varrho' d_{kl}^2 \dot{\psi}_{kj} \dot{\psi}_{lj}, \tag{2.3$_1$}$$

where

$$\varrho \equiv \varrho_M + \varrho', \tag{2.3$_2$}$$

$$d^2_{kl} \equiv d_p d_q (\delta_{p1}\delta_{q1} l_{k1} l_{l1} + \delta_{p2}\delta_{q2} l_{k2} l_{l2} + \delta_{p3}\delta_{q3} l_{k3} l_{l3}) = d^2_{lk} \tag{2.3$_3$}$$

and δ_{ij} is the Kronecker symbol. In the case of a cube with edges of length $2d$ parallel to the axes of x'_i,

$$l_{ij} = \delta_{ij}, \qquad d_1 = d_2 = d_3 \equiv d. \tag{2.4}$$

Then the second term in $(2.3)_1$ reduces to $\frac{1}{6}\varrho' d^2 \dot\psi_{ij}\dot\psi_{ij}$. If the material is composed wholly of unit cells, $\varrho_M = 0$. Then $\varrho' = \varrho$.

For the potential energy-density (potential energy per unit macro-volume), we assume a function, W, of the forty-two variables $\varepsilon_{ij}, \gamma_{ij}, \varkappa_{ijk}$:

$$W = W(\varepsilon_{ij}, \gamma_{ij}, \varkappa_{ijk}). \tag{2.5}$$

A small, rigid rotation of the deformed body is described by a rotation, ω_{ij} = constant, of the macro-material and an equal rotation $\psi_{[ij]}$ of the micro-material. The associated displacements are

$$u_j = x_i \omega_{ij}, \qquad u'_j = x'_i \psi_{[ij]}. \tag{2.6}$$

The addition of such a displacement leaves W unchanged since the added $\varepsilon_{ij}, \gamma_{ij}$ and $\varkappa_{ijk}$ are zero.

The assumptions (1.6), (2.2) and (2.5) are the minimum that will lead to equations which yield the desired dispersion relations for plane waves: including longitudinal and transverse acoustic and optical branches. More or less than (1.6), (2.2) and (2.5) would be more or less than what is required.

The unit cell is taken to be a parallelepiped in order to represent the unit cell of a crystal lattice. However, another shape would only change the tensor d^2_{kl}. Also, the cell can be interpreted as a molecule of a polymer, a crystallite of a polycrystal or a grain of a granular material.

3. Variational equation of motion

We write Hamilton's principle for independent variations δu_i and $\delta\psi_{ij}$ between fixed limits of u_i and ψ_{ij} at times t_0 and t_1:

$$\delta \int_{t_0}^{t_1} (\mathscr{T} - \mathscr{W})\, dt + \int_{t_0}^{t_1} \delta\mathscr{W}_1\, dt = 0, \tag{3.1}$$

where $\mathscr{T}$ and $\mathscr{W}$ are the total kinetic and potential energies:

$$\mathscr{T} \equiv \int_V T\, dV, \qquad \mathscr{W} \equiv \int_V W\, dV \tag{3.2}$$

and $\delta\mathscr{W}_1$ is the variation of the work done by external forces.

In the usual way [4, p. 166], we find, from $(2.3)_1$ and $(3.2)_1$,

$$\delta \int_{t_0}^{t_1} \mathscr{T}\, dt = -\int_{t_0}^{t_1} dt \int_V (\varrho \ddot u_j\, \delta u_j + \tfrac{1}{3}\varrho' d^2_{jl} \ddot\psi_{lk}\, \delta\psi_{jk})\, dV. \tag{3.3}$$

As for the variation of potential energy, we first define

$$\tau_{ij} \equiv \frac{\partial W}{\partial \varepsilon_{ij}} = \tau_{ji}, \tag{3.4$_1$}$$

$$\sigma_{ij} \equiv \frac{\partial W}{\partial \gamma_{ij}}, \tag{3.4$_2$}$$

$$\mu_{ijk} \equiv \frac{\partial W}{\partial \varkappa_{ijk}}. \tag{3.4$_3$}$$

Then

$$\begin{aligned}\delta W &= \tau_{ij}\,\delta\varepsilon_{ij} + \sigma_{ij}\,\delta\gamma_{ij} + \mu_{ijk}\,\delta\varkappa_{ijk},\\ &= \tau_{ij}\,\partial_i\,\delta u_j + \sigma_{ij}(\partial_i\,\delta u_j - \delta\psi_{ij}) + \mu_{ijk}\,\partial_i\,\delta\psi_{jk},\\ &= \partial_i[(\tau_{ij}+\sigma_{ij})\,\delta u_j] - \partial_i(\tau_{ij}+\sigma_{ij})\,\delta u_j - \sigma_{ij}\,\delta\psi_{ij} + \partial_i(\mu_{ijk}\,\delta\psi_{jk}) -\\ &\quad - \partial_i\mu_{ijk}\,\delta\psi_{jk}.\end{aligned}$$

Applying the divergence theorem, we find

$$\begin{aligned}\delta\mathscr{W} &= \int_V \delta W\,dV\\ &= -\int_V \partial_i(\tau_{ij}+\sigma_{ij})\,\delta u_j\,dV - \int_V (\partial_i\mu_{ijk}+\sigma_{jk})\,\delta\psi_{jk}\,dV +\\ &\quad + \int_S n_i(\tau_{ij}+\sigma_{ij})\,\delta u_j\,dS + \int_S n_i\mu_{ijk}\,\delta\psi_{jk}\,dS.\end{aligned} \tag{3.5}$$

The form of (3.5) is the motivation for the adoption of the following form for the variation of work done by external forces:

$$\delta\mathscr{W}_1 = \int_V f_j\,\delta u_j\,dV + \int_V \Phi_{jk}\,\delta\psi_{jk}\,dV + \int_S t_j\,\delta u_j\,dS + \int_S T_{jk}\,\delta\psi_{jk}\,dS. \tag{3.6}$$

The definitions of u_j and ψ_{jk}, and the fact that the integrands of the volume and surface integrals represent variations of work per unit volume and area, yield the physical significances of the coefficients of δu_j and $\delta\psi_{jk}$. Thus, f_j is the body force per unit volume and t_j is the surface force per unit area (stress-vector or traction); Φ_{jk} is to be interpreted as a double force per unit volume [*4*, p. 187] and T_{jk} as a double force per unit area. The diagonal terms of Φ_{jk} and T_{jk} are double forces without moment and the off-diagonal terms are double forces with moment. The antisymmetric part $\Phi_{[jk]}$ of the *body double force* Φ_{jk} is the body couple. The antisymmetric part $T_{[jk]}$ of the *double traction* T_{jk} is the Cosserat couple-stress vector. In both Φ_{jk} and T_{jk}, the first subscript gives the orientation of the lever arm between the forces and the second subscript gives the orientation of the forces. Across a surface with its outward normal in the positive direction, the force at the positive end of the lever arm acts in the positive direction. ("Positive" refers to the positive sense of the coordinate axis parallel to the lever arm or force). Across a surface with its outward normal in the negative direction, the directions of the forces are reversed.

Substituting (3.3), (3.5) and (3.6) in (3.1), and dropping the integration with respect to time, we obtain the variational equation of motion:

$$\begin{aligned}&\int_V (\partial_i\tau_{ij} + \partial_i\sigma_{ij} + f_j - \varrho\ddot{u}_j)\,\delta u_j\,dV +\\ &\quad + \int_V (\partial_i\mu_{ijk} + \sigma_{jk} + \Phi_{jk} - \tfrac{1}{3}\varrho' d_{jl}^2\ddot{\psi}_{lk})\,\delta\psi_{jk}\,dV +\\ &\quad + \int_S [t_j - n_i(\tau_{ij}+\sigma_{ij})]\,\delta u_j\,dS + \int_S (T_{jk} - n_i\mu_{ijk})\,\delta\psi_{jk}\,dS = 0.\end{aligned} \tag{3.7}$$

4. Stress-equations of motion and boundary conditions

From the variational equation of motion, there follow immediately the twelve stress-equations of motion:

$$\partial_i(\tau_{ij}+\sigma_{ij})+f_j=\varrho\ddot{u}_j, \tag{4.1$_1$}$$

$$\partial_i\mu_{ijk}+\sigma_{jk}+\Phi_{jk}=\tfrac{1}{3}\varrho' d^2_{lj}\ddot{\psi}_{lk}, \tag{4.1$_2$}$$

and the twelve traction boundary conditions:

$$t_j=n_i(\tau_{ij}+\sigma_{ij}), \tag{4.2$_1$}$$

$$T_{jk}=n_i\mu_{ijk}. \tag{4.2$_2$}$$

In view of (3.4), (4.2) and the significance of $\varepsilon_{ij}, \gamma_{ij}, \varkappa_{ijk}$, appropriate terminology appears to be *Cauchy stress* for τ_{ij}, *relative stress* for σ_{ij} and *double stress* for μ_{ijk}. The twenty-seven components of μ_{ijk} are interpreted as double forces per unit area. The first subscript of a μ_{ijk} designates the normal to the surface across which the component acts; the second and third subscripts have the same significance as the two subscripts of T_{jk}. Typical components of μ_{ijk} are illustrated in Fig. 2.

The linear equations of a Cosserat continuum [*2*] are obtained by setting $\psi_{(ij)}=0$. Then $\sigma_{(ij)}=\tau_{ij}$ and $\mu_{i(jk)}=0$; and there remain $\mu_{i[jk]}$ (the Cosserat couple-stress) and $\sigma_{[ij]}$: which has been regarded as the antisymmetric part of an asymmetric stress τ_{ij}. However, in the present theory, the Cauchy stress, τ_{ij}, is symmetric and $\sigma_{[ij]}$ is the antisymmetric part of an asymmetric relative stress σ_{ij}.

Besides containing the linear equations of a Cosserat continuum as a special case, Eqs. (2.5), (3.4) and (4.1) also include, as low frequency, very long wave length approximations, linear versions of the equations of couple-stress theory [*5*—*10*] and TOUPIN's generalization of couple-stress theory [*8*, Section 7]. These are considered in Sections 9—12.

If additional terms were retained in the series expansion (1.6) of the micro-displacement u'_j, higher order stresses would appear. In addition to stresses corresponding to double forces per unit area, there would be stresses corresponding to n-tuple forces per unit area. All of the latter would be self equilibrating; whereas, of the μ_{ijk}, only the $\mu_{i(jk)}$ are self equilibrating.

5. Constitutive equations

For the potential energy-density we take a homogeneous, quadratic function of the forty-two variables $\varepsilon_{ij}, \gamma_{ij}, \varkappa_{ijk}$:

$$\begin{aligned} W=&\tfrac{1}{2}c_{ijkl}\varepsilon_{ij}\varepsilon_{kl}+\tfrac{1}{2}b_{ijkl}\gamma_{ij}\gamma_{kl}+\tfrac{1}{2}a_{ijklmn}\varkappa_{ijk}\varkappa_{lmn}+\\ &+d_{ijklm}\gamma_{ij}\varkappa_{klm}+f_{ijklm}\varkappa_{ijk}\varepsilon_{lm}+g_{ijkl}\gamma_{ij}\varepsilon_{kl}. \end{aligned} \tag{5.1}$$

Only $\frac{1}{2}\times 42\times 43=903$ of the $42\times 42=1764$ coefficients in (5.1) are independent. The number of coefficients, the relations among them and the number of independent ones are given in the following table:

$$
\begin{aligned}
c_{ijkl} = c_{klij} = c_{jikl}&: \quad 9\times 9 - 60 = 21\\
b_{ijkl} = b_{klij}&: \quad 9\times 9 - 36 = 45\\
a_{ijklmn} = a_{lmnijk}&: 27\times 27 - 351 = 378\\
d_{ijklm}&: \quad 9\times 27 = 243\\
f_{ijklm} = f_{ijkml}&: \quad 9\times 27 - 81 = 162\\
g_{ijkl} = g_{ijlk}&: \quad 9\times 9 - 27 = \underline{\ 54\ }\\
&\qquad\qquad\qquad\quad 903
\end{aligned}
\tag{5.2}
$$

From (5.1) and (3.4):

$$\tau_{pq} = c_{pqij}\varepsilon_{ij} + g_{ijpq}\gamma_{ij} + f_{ijkpq}\varkappa_{ijk}, \tag*{(5.3)$_1$}$$

$$\sigma_{pq} = g_{pqij}\varepsilon_{ij} + b_{ijpq}\gamma_{ij} + d_{pqijk}\varkappa_{ijk}, \tag*{(5.3)$_2$}$$

$$\mu_{pqr} = f_{pqrij}\varepsilon_{ij} + d_{ijpqr}\gamma_{ij} + a_{pqrijk}\varkappa_{ijk}. \tag*{(5.3)$_3$}$$

In the case of a centrosymmetric, isotropic material (referred to as isotropic in the sequel) the number of independent coefficients is greatly reduced. As there are no isotropic tensors of odd rank, d_{ijklm} and f_{ijklm} must vanish. The remaining coefficients must be homogeneous, linear functions of products of Kronecker deltas. There are three independent products of two Kronecker deltas and fifteen independent products of three Kronecker deltas. Hence

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu_1\delta_{ik}\delta_{jl} + \mu_2\delta_{il}\delta_{jk}, \tag*{(5.4)$_1$}$$

$$b_{ijkl} = b_1\delta_{ij}\delta_{kl} + b_2\delta_{ik}\delta_{jl} + b_3\delta_{il}\delta_{jk}, \tag*{(5.4)$_2$}$$

$$g_{ijkl} = g_1\delta_{ij}\delta_{kl} + g_2\delta_{ik}\delta_{jl} + g_3\delta_{il}\delta_{jk}, \tag*{(5.4)$_3$}$$

$$
\begin{aligned}
a_{ijklmn} = {} & a_1\delta_{ij}\delta_{kl}\delta_{mn} + a_2\delta_{ij}\delta_{km}\delta_{nl} + a_3\delta_{ij}\delta_{kn}\delta_{lm} +\\
& + a_4\delta_{jk}\delta_{il}\delta_{mn} + a_5\delta_{jk}\delta_{im}\delta_{nl} + a_6\delta_{jk}\delta_{in}\delta_{lm} +\\
& + a_7\delta_{ki}\delta_{jl}\delta_{mn} + a_8\delta_{ki}\delta_{jm}\delta_{nl} + a_9\delta_{ki}\delta_{jn}\delta_{lm} +\\
& + a_{10}\delta_{il}\delta_{jm}\delta_{kn} + a_{11}\delta_{jl}\delta_{km}\delta_{in} + a_{12}\delta_{kl}\delta_{im}\delta_{jn} +\\
& + a_{13}\delta_{il}\delta_{jn}\delta_{km} + a_{14}\delta_{jl}\delta_{kn}\delta_{im} + a_{15}\delta_{kl}\delta_{in}\delta_{jm}.
\end{aligned}
\tag*{(5.4)$_4$}
$$

The conditions (5.2) require the six relations

$$\mu_1 = \mu_2 \equiv \mu, \qquad g_2 = g_3,$$

$$a_1 = a_6, \qquad a_2 = a_9, \qquad a_5 = a_7, \qquad a_{11} = a_{12},$$

leaving eighteen independent coefficients. Thus, the potential energy density reduces to

$$
\begin{aligned}
W = {} & \tfrac{1}{2}\lambda\varepsilon_{ii}\varepsilon_{jj} + \mu\varepsilon_{ij}\varepsilon_{ij} + \tfrac{1}{2}b_1\gamma_{ii}\gamma_{jj} + \tfrac{1}{2}b_2\gamma_{ij}\gamma_{ij} +\\
& + \tfrac{1}{2}b_3\gamma_{ij}\gamma_{ji} + g_1\gamma_{ii}\varepsilon_{jj} + g_2(\gamma_{ij} + \gamma_{ji})\varepsilon_{ij} +\\
& + a_1\varkappa_{iik}\varkappa_{kjj} + a_2\varkappa_{iik}\varkappa_{jkj} + \tfrac{1}{2}a_3\varkappa_{iik}\varkappa_{jjk} + \tfrac{1}{2}a_4\varkappa_{ijj}\varkappa_{ikk} +\\
& + a_5\varkappa_{ijj}\varkappa_{kik} + \tfrac{1}{2}a_8\varkappa_{iji}\varkappa_{kjk} + \tfrac{1}{2}a_{10}\varkappa_{ijk}\varkappa_{ijk} + a_{11}\varkappa_{ijk}\varkappa_{jki} +\\
& + \tfrac{1}{2}a_{13}\varkappa_{ijk}\varkappa_{ikj} + \tfrac{1}{2}a_{14}\varkappa_{ijk}\varkappa_{jik} + \tfrac{1}{2}a_{15}\varkappa_{ijk}\varkappa_{kji},
\end{aligned}
\tag{5.5}
$$

and the constitutive equations become

$$\tau_{pq} = \lambda \delta_{pq} \varepsilon_{ii} + 2\mu \varepsilon_{pq} + g_1 \delta_{pq} \gamma_{ii} + g_2 (\gamma_{pq} + \gamma_{qp}), \tag{5.6$_1$}$$

$$\sigma_{pq} = g_1 \delta_{pq} \varepsilon_{ii} + 2 g_2 \varepsilon_{pq} + b_1 \delta_{pq} \gamma_{ii} + b_2 \gamma_{pq} + b_3 \gamma_{qp}, \tag{5.6$_2$}$$

$$\begin{aligned}\mu_{pqr} = {} & a_1 (\varkappa_{iip} \delta_{qr} + \varkappa_{rii} \delta_{pq}) + a_2 (\varkappa_{iiq} \delta_{pr} + \varkappa_{iri} \delta_{pq}) + a_3 \varkappa_{iir} \delta_{pq} + \\ & + a_4 \varkappa_{pii} \delta_{qr} + a_5 (\varkappa_{qii} \delta_{pr} + \varkappa_{ipi} \delta_{qr}) + a_8 \varkappa_{iqi} \delta_{pr} + a_{10} \varkappa_{pqr} + \\ & + a_{11} (\varkappa_{rpq} + \varkappa_{qrp}) + a_{13} \varkappa_{prq} + a_{14} \varkappa_{qpr} + a_{15} \varkappa_{rqp}.\end{aligned} \tag{5.6$_3$}$$

6. Displacement-equations of motion

We may obtain twelve equations of motion on the twelve variables u_i and ψ_{ij} by first inserting (1.10), (1.11) and (1.12) into the constitutive equations and then the latter into the stress-equations of motion (4.1). There is no necessity to assume spatially homogeneous material properties. In fact, the coefficients of elasticity and the densities may be taken as periodic functions, of oblique coordinates parallel to the edges of the unit cell, of periods $2d_i$. This would represent the periodic structure of a crystal lattice. However, the equations would then be highly intractable; whereas some of their main features are exhibited with *macro-homogeneous* material properties, at least for wave-lengths greater than the dimensions of the unit cell. The isotropic case is especially simple; but nevertheless it still contains many of the novel properties of the macro-homogeneous material.

In the case of isotropy, the constitutive Eqs. (5.6) apply and also (2.4). Then, for the macro-homogeneous, isotropic material the equations on u_i and ψ_{ij} are

$$\begin{aligned}& (\mu + 2g_2 + b_2)\, \partial_j \partial_j u_i + (\lambda + \mu + 2g_1 + 2g_2 + b_1 + b_3)\, \partial_i \partial_j u_j - \\ & \quad - (g_1 + b_1)\, \partial_i \psi_{jj} - (g_2 + b_2)\, \partial_j \psi_{ji} - (g_2 + b_3)\, \partial_j \psi_{ij} + f_i = \varrho \ddot{u}_i,\end{aligned} \tag{6.1}$$

$$\begin{aligned}& (a_1 + a_5)\, (\partial_k \partial_l \psi_{kl} \delta_{ij} + \partial_i \partial_j \psi_{kk}) + (a_2 + a_{11})\, (\partial_j \partial_k \psi_{ki} + \partial_i \partial_k \psi_{jk}) + \\ & + (a_3 + a_{14})\, \partial_i \partial_k \psi_{kj} + a_4\, \partial_k \partial_k \psi_{ll} \delta_{ij} + (a_8 + a_{15})\, \partial_j \partial_k \psi_{ik} + \\ & + a_{10}\, \partial_k \partial_k \psi_{ij} + a_{13}\, \partial_k \partial_k \psi_{ji} + g_1\, \partial_k u_k \delta_{ij} + g_2 (\partial_i u_j + \partial_j u_i) + \\ & + b_1 (\partial_k u_k - \psi_{kk})\, \delta_{ij} + b_2 (\partial_i u_j - \psi_{ij}) + b_3 (\partial_j u_i - \psi_{ji}) + \Phi_{ij} = \tfrac{1}{3} \varrho' d^2 \ddot{\psi}_{ij}.\end{aligned} \tag{6.2}$$

7. Micro-vibrations

Consider solutions of (6.1) and (6.2) of the form

$$u_i = f_i = \Phi_{ij} = 0, \qquad \psi_{ij} = A_{ij}\, e^{i\omega t}, \tag{7.1}$$

where the A_{ij} are constants. Then (6.1) are satisfied identically and (6.2) become

$$b_1 \delta_{ij} A_{kk} + b_2 A_{ij} + b_3 A_{ji} = \tfrac{1}{3} \varrho' d^2 \omega^2 A_{ij}, \tag{7.2}$$

which admit the following solutions:

dilatational mode:

$$A_{11} = A_{22} = A_{33}; \qquad A_{ij} = 0, \qquad i \neq j; \tag{7.3$_1$}$$

$$\omega_d^2 = 3\,(3 b_1 + b_2 + b_3)/\varrho' d^2; \tag{7.3$_2$}$$

shear modes:

$$A_{ij}=A_{ji}, \qquad i\neq j; \qquad A_{ij}=0, \qquad i=j; \tag{7.4$_1$}$$

$$\omega_s^2=3\,(b_2+b_3)/\varrho' d^2; \tag{7.4$_2$}$$

equivoluminal extensional modes:

$$A_{ii}=0; \qquad A_{ij}=0, \qquad i\neq j; \tag{7.5$_1$}$$

$$\omega_s^2=3\,(b_2+b_3)/\varrho' d^2; \tag{7.5$_2$}$$

rotational modes:

$$A_{ij}=-A_{ji}, \tag{7.6$_1$}$$

$$\omega_r^2=3\,(b_2-b_3)/\varrho' d^2. \tag{7.6$_2$}$$

The restriction of the potential energy-density to be positive definite requires

$$3\,b_1+b_2+b_3>0, \qquad b_2+b_3>0, \qquad b_2-b_3>0.$$

Hence ω_d, ω_s and ω_r are real frequencies. The corresponding modes are analogous to the simple thickness-modes of vibration of a plate. Just as the latter are independent of the coordinates in the plane of the plate, so are the micro-modes independent of the coordinates x_i of the three-dimensional continuum with micro-structure. Extensional and flexural waves in a plate couple with thickness-modes, at high frequencies, to form the higher branches of the dispersion relations for a plate. Analogously, we may expect longitudinal and transverse acoustic waves, in the three-dimensional continuum with micro-structure, to couple with the micro-modes to form optical branches.

8. Plane waves, long wave-length

Consider solutions of (6.1) and (6.2) with f_i and Φ_{ij} zero and

$$u_i=u_i(x_1,t), \qquad \psi_{ij}=\psi_{ij}(x_1,t). \tag{8.1}$$

By means of linear combinations, the twelve equations may be composed into three independent equations and three independent systems of three equations each:

shear optical I:

$$(a_{10}+a_{13})\,\partial_1\partial_1\psi_{(23)}-(b_2+b_3)\,\psi_{(23)}=\tfrac{1}{3}\varrho' d^2\ddot{\psi}_{(23)}; \tag{8.2}$$

shear optical II: the same as (8.2) except that $\psi_{(23)}$ is replaced with $\psi_{22}-\psi_{33}$;

rotational optical:

$$(a_{10}-a_{13})\,\partial_1\partial_1\psi_{[23]}-(b_2-b_3)\,\psi_{[23]}=\tfrac{1}{3}\varrho' d^2\ddot{\psi}_{[23]}; \tag{8.3}$$

longitudinal system:

$$\begin{aligned}
k_{11}\,\partial_1\partial_1 u_1-k_{12}\,\partial_1\psi_{11}^D-k_{13}\,\partial_1\psi&=\varrho\ddot{u}_1,\\
k_{21}\,\partial_1 u_1+k_{22}\,\partial_1\partial_1\psi_{11}^D-k'_{22}\,\psi_{11}^D+k_{23}\,\partial_1\partial_1\psi&=\tfrac{1}{2}\varrho' d^2\ddot{\psi}_{11}^D,\\
k_{31}\,\partial_1 u_1+k_{32}\,\partial_1\partial_1\psi_{11}^D+k_{33}\,\partial_1\partial_1\psi-k'_{33}\,\psi&=\varrho' d^2\ddot{\psi},
\end{aligned} \tag{8.4}$$

where $\psi = \frac{1}{3}\psi_{ii}$, $\psi^D_{11} = \psi_{11} - \psi$ and

$$k_{11} = \lambda + 2\mu + 2g_1 + 4g_2 + b_1 + b_2 + b_3,$$
$$k_{22} = 2a_2 + a_3 + a_8 + \tfrac{3}{2}a_{10} + 2a_{11} + \tfrac{3}{2}a_{13} + a_{14} + a_{15},$$
$$k_{33} = 6a_1 + 2a_2 + a_3 + 9a_4 + 6a_5 + a_8 + 3a_{10} + 2a_{11} + 3a_{13} + a_{14} + a_{15},$$
$$k_{23} = k_{32} = 3a_1 + 2a_2 + a_3 + 3a_5 + a_8 + 2a_{11} + a_{14} + a_{15},$$
$$k_{31} = k_{13} = 3g_1 + 2g_2 + 3b_1 + b_2 + b_3,$$
$$k_{12} = k_{21} = 2g_2 + b_2 + b_3,$$
$$k'_{22} = \tfrac{3}{2}(b_2 + b_3),$$
$$k'_{33} = 3(3b_1 + b_2 + b_3);$$

transverse system I:

$$\begin{aligned} \bar{k}_{11}\partial_1\partial_1 u_2 - \bar{k}_{12}\partial_1\psi_{(12)} - \bar{k}_{13}\partial_1\psi_{[12]} &= \varrho\ddot{u}_2, \\ \bar{k}_{21}\partial_1 u_2 + \bar{k}_{22}\partial_1\partial_1\psi_{(12)} - \bar{k}'_{22}\psi_{(12)} + \bar{k}_{23}\partial_1\partial_1\psi_{[12]} &= \tfrac{2}{3}\varrho' d^2\ddot{\psi}_{(12)}, \\ \bar{k}_{31}\partial_1 u_2 + \bar{k}_{32}\partial_1\partial_1\psi_{(12)} + \bar{k}_{33}\partial_1\partial_1\psi_{[12]} - \bar{k}'_{33}\psi_{[12]} &= \tfrac{2}{3}\varrho' d^2\ddot{\psi}_{[12]}, \end{aligned} \tag{8.5}$$

where

$$\bar{k}_{11} = \mu + 2g_2 + b_2,$$
$$\bar{k}_{22} = 2a_2 + a_3 + a_8 + 2a_{10} + 2a_{11} + 2a_{13} + a_{14} + a_{15},$$
$$\bar{k}_{33} = -2a_2 + a_3 + a_8 + 2a_{10} - 2a_{11} - 2a_{13} + a_{14} + a_{15},$$
$$\bar{k}_{23} = \bar{k}_{32} = a_3 - a_8 + a_{14} - a_{15},$$
$$\bar{k}_{31} = \bar{k}_{13} = b_2 - b_3,$$
$$\bar{k}_{12} = \bar{k}_{21} = 2g_2 + b_2 + b_3,$$
$$\bar{k}'_{22} = 2(b_2 + b_3),$$
$$\bar{k}'_{33} = 2(b_2 - b_3),$$

transverse system II: the same as transverse system I except that u_2, $\psi_{(12)}$ and $\psi_{[12]}$ are replaced with u_3, $\psi_{(13)}$ and $\psi_{[13]}$, respectively.

If, now, (8.1) are specialized to the plane waves

$$u_i = A_i i \exp[i(\xi x_1 - \omega t)], \qquad \psi_{ij} = B_{ij}\exp[i(\xi x_1 - \omega t)], \tag{8.6}$$

four dispersion relations (ω *vs.* ξ) result:

shear optical waves (*SO*) (twice):

$$\tfrac{1}{3}\varrho' d^2\omega^2 = b_2 + b_3 + (a_{10} + a_{13})\xi^2; \tag{8.7}$$

rotational optical waves (*RO*):

$$\tfrac{1}{3}\varrho' d^2\omega^2 = b_2 - b_3 + (a_{10} - a_{13})\xi^2; \tag{8.8}$$

longitudinal waves (*LA*, *LO*, *LDO*):

$$\begin{Vmatrix} k_{11}\xi^2-\varrho\,\omega^2 & k_{12}\xi & k_{13}\xi \\ k_{21}\xi & k_{22}\xi^2+k'_{22}-\tfrac{1}{2}\varrho' d^2\omega^2 & k_{23}\xi^2 \\ k_{31}\xi & k_{32}\xi^2 & k_{33}\xi^2+k'_{33}-\varrho' d^2\omega^2 \end{Vmatrix}=0; \tag{8.9}$$

transverse waves (*TA*, *TO*, *TRO*) (twice):

$$\begin{Vmatrix} \bar{k}_{11}\xi^2-\varrho\,\omega^2 & \bar{k}_{12}\xi & \bar{k}_{13}\xi \\ \bar{k}_{21}\xi & \bar{k}_{22}\xi^2+\bar{k}'_{22}-\tfrac{2}{3}\varrho' d^2\omega^2 & \bar{k}_{23}\xi^2 \\ \bar{k}_{31}\xi & \bar{k}_{32}\xi^2 & \bar{k}_{33}\xi^2+\bar{k}'_{33}-\tfrac{2}{3}\varrho' d^2\omega^2 \end{Vmatrix}=0. \tag{8.10}$$

These dispersion relations are similar to those encountered in a second order theory of extensional waves in plates [*11*]. The relations (8.7) and (8.8) are like that for the second face-shear mode in the plate. The relations (8.9) and (8.10) are like that for the coupled extensional, thickness-stretch and symmetric thickness-shear modes in the plate; *i.e.*, one acoustic and two optical branches in each of (8.9) and (8.10).

In the dispersion relations (8.7) and (8.8) for the non-coupled modes, there are cut-off frequencies ω_s and ω_r, respectively, at which the group velocity ($d\omega/d\xi$) is zero. Positive definiteness of W requires

$$a_{10}+a_{13}>0, \qquad a_{10}-a_{13}>0. \tag{8.11}$$

Hence, the frequencies increase, from cut-off, with increasing real wave numbers. Below the cut-off frequencies, the wave numbers are pure imaginary with cut-off values

$$\xi=\pm i\left(\frac{b_2+b_3}{a_{10}+a_{13}}\right)^{\frac{1}{2}}, \qquad \xi=\pm i\left(\frac{b_2-b_3}{a_{10}-a_{13}}\right)^{\frac{1}{2}}, \tag{8.12}$$

respectively, at zero frequency.

The behavior of the acoustic branches in (8.9) and (8.10), at low frequencies, is described by ω'_i, ω''_i, ω'''_i ($i=1$ for longitudinal and $i=2$ for transverse): the values, at $\omega=0$ and $\xi=0$, of the first, second and third derivatives of ω with respect to ξ. We find

$$\omega'_i=\tilde{v}_i, \qquad \omega''_i=0, \qquad \omega'''_i=3\tilde{v}_i(\tilde{l}_i^2-h_i^2), \tag{8.13}$$

where

$$\tilde{v}_1^2=(\tilde{\lambda}+2\tilde{\mu})/\varrho, \qquad \tilde{v}_2^2=\tilde{\mu}/\varrho, \tag{8.14}$$

$$\tilde{\lambda}+2\tilde{\mu}=\lambda+2\mu-\frac{8g_2^2}{3(b_2+b_3)}-\frac{(3g_1+2g_2)^2}{3(3b_1+b_2+b_3)}, \tag{8.15$_1$}$$

$$\tilde{\mu}=\mu-\frac{2g_2^2}{b_2+b_3}, \tag{8.15$_2$}$$

$$\tilde{l}_1^2=2(\tilde{a}_1+\tilde{a}_2+\tilde{a}_3+\tilde{a}_4+\tilde{a}_5)/(\tilde{\lambda}+2\tilde{\mu}), \qquad \tilde{l}_2^2=2(\tilde{a}_3+\tilde{a}_4)/\tilde{\mu}, \tag{8.16}$$

$$h_1^2=\varrho' d^2[2\alpha^2+(\alpha+\beta)^2]/3\varrho, \qquad h_2^2=\varrho' d^2(1+\beta^2)/6\varrho, \tag{8.17}$$

$$\alpha=\frac{1}{b_2+b_3}\left(g_1-\frac{b_1(3g_1+2g_2)}{3b_1+b_2+b_3}\right), \qquad \beta=1+\frac{2g_2}{b_2+b_3}, \tag{8.18}$$

$$\begin{aligned}
\tilde{a}_1 &= \tfrac{1}{2}[(1+\beta)(3\alpha+\beta)a_1 + (1+2\alpha\beta+\beta^2)a_2 - \tfrac{1}{2}(1+\beta)(1-2\alpha-\beta)a_3 - \\
&\quad - (1-\beta)(3\alpha+\beta)a_5 - \tfrac{1}{2}(1-\beta)(1+2\alpha+\beta)a_8 + 2\alpha\beta a_{11} - \alpha(1-\beta)a_{14} + \\
&\quad + \alpha(1+\beta)a_{15}], \\
\tilde{a}_2 &= \tfrac{1}{2}\{-(1-2\alpha-\beta)(3\alpha+\beta)a_1 - \tfrac{1}{2}[1-(2\alpha+\beta)^2]a_2 + \tfrac{1}{4}(1-2\alpha-\beta)^2 a_3 + \\
&\quad + (3\alpha+\beta)^2 a_4 + (3\alpha+\beta)(1+2\alpha+\beta)a_5 + \tfrac{1}{4}(1+2\alpha+\beta)^2 a_8 + \\
&\quad + \alpha(3\alpha+2\beta)a_{10} + 2\alpha(\alpha+\beta)a_{11} + \alpha(3\alpha+2\beta)a_{13} + \alpha(1+\alpha+\beta)a_{14} - \\
&\quad - \alpha(1-\alpha-\beta)a_{15}\}, \\
\tilde{a}_3 &= \tfrac{1}{4}[-(1-\beta^2)a_2 + \tfrac{1}{2}(1+\beta)^2 a_3 + \tfrac{1}{2}(1-\beta)^2 a_8], \\
\tilde{a}_4 &= \tfrac{1}{4}[(1+\beta^2)a_{10} - (1-\beta^2)(a_{11}+a_{13}) + \tfrac{1}{2}(1+\beta)^2 a_{14} + \tfrac{1}{2}(1-\beta)^2 a_{15}], \\
\tilde{a}_5 &= \tfrac{1}{4}[-(1-\beta^2)a_{10} + (1+3\beta^2)a_{11} + (1+\beta^2)a_{13} - \tfrac{1}{2}(1+2\beta-3\beta^2)a_{14} - \\
&\quad - \tfrac{1}{2}(1-2\beta-3\beta^2)a_{15}].
\end{aligned} \tag{8.19}$$

Positive definiteness of W requires $\tilde{\mu}$, $\tilde{\lambda}+2\tilde{\mu}$, $\tilde{l}_i^2>0$, while $h_i^2>0$ by inspection.

It may be seen that the limiting group velocities $\tilde{v}_i$ are less than those that would be calculated from the strain-stiffnesses $\lambda+2\mu$ and μ. This phenomenon is due to the compliance of the unit cell and has been found in a theory of crystal lattices by Gazis & Wallis [*12*]. Inasmuch as $\omega_i''=0$, the group velocities at zero frequency are maxima or minima. Which one occurs depends on whether ω_i''' is greater or less than zero; and this, in turn, depends on whether $\tilde{l}^2$ is greater or less than h^2. Now $\tilde{l}_i^2$ and h_i^2 are positive quantities that are length-properties of the material — depending on stiffness ratios, density ratio and the size of the unit cell. Although d is probably smaller than the $\tilde{l}_i$, the density ratio ϱ'/ϱ and the stiffness ratios α and β can make either $\tilde{l}$ or h the greater. Hence, as the frequency increases from zero, both group velocities can increase or both can decrease or one can increase and the other decrease — depending on the properties of the material. There is no analogue in the theory of homogeneous plates because they do not have multiple stiffnesses and densities. The phenomenon does occur, however, in sandwich plates [*13*] and it has also been found in a theory of crystal lattices, with complex interatomic interactions, by Gazis & Wallis [*14*].

At the short wave-length limit ($\xi\to\infty$), the asymptotic values of the group velocities of the acoustic branches are

$$\frac{1}{2d}\left\{\frac{k_{22}+k_{33}-[(k_{22}-k_{33})^2+4k_{23}^2]^{\frac{1}{2}}}{\frac{1}{3}\varrho'}\right\}^{\frac{1}{2}} \tag{8.20}$$

from (8.9) and the same expression, with k replaced by $\bar{k}$, from (8.10). These can be much smaller than $(8.13)_1$ if

$$4(k_{22}k_{33}-k_{23}^2)\ll(k_{22}+k_{33})^2, \tag{8.21}$$

$$4(\bar{k}_{22}\bar{k}_{33}-\bar{k}_{23}^2)\ll(\bar{k}_{22}+\bar{k}_{33})^2, \tag{8.22}$$

which appear to be possible.

Regarding the optical branches in the longitudinal and transverse systems, the former have long wave-length cut-off frequencies ω_s and ω_d while the corresponding quantities for the latter are ω_s and ω_r. Thus, the two systems have one cut-off frequency in common. One of the two modes is shear and the other is equivoluminal extension. As in the case of plates, the group velocities of all four optical branches are zero at the long wave-length cut-off frequencies except in the unlikely circumstance of coincidence of cut-off frequencies within a system ($b_1=0$ for longitudinal modes or $b_3=0$ for transverse modes). Another parallel to the situation in plates is that, as ξ increases from zero, the behavior of the optical branches is very sensitive to small changes in the ratios of material properties. One possibility is that both lower optical branches have phase and group velocities of opposite sign; *i.e.*, diminishing ω with increasing ξ. With further increase of ξ, the absolute values of the group velocities would pass through maxima, then drop to zero and then increase; *i.e.*, the dispersion curves first would have a point of inflexion and then a minimum.

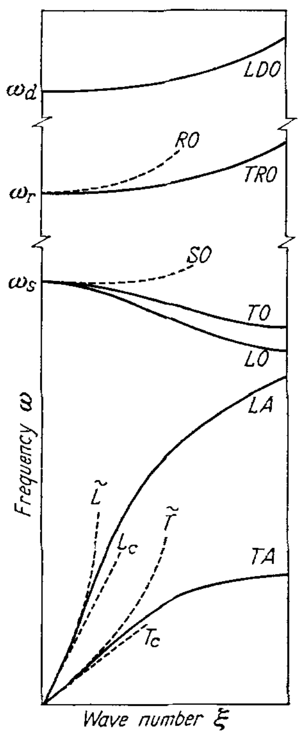

Fig. 3. Sketch of possible configuration of real branches of dispersion curves. $\boldsymbol{T}$ transverse; $\boldsymbol{L}$ longitudinal; $\boldsymbol{A}$ acoustic; $\boldsymbol{O}$ optical; $\boldsymbol{S}$ shear; $\boldsymbol{R}$ rotational; $\boldsymbol{D}$ dilatational; $\tilde{\boldsymbol{T}}, \tilde{\boldsymbol{L}}$ low frequency approximation; $\boldsymbol{T}_c\,\boldsymbol{L}_c$ classical elasticity

A sketch of a possible configuration of the real segments of the dispersion curves is shown in Fig. 3. The four lowest branches (*TA*, *LA*, *LO*, *TO*) are remarkably similar to those obtained by BROCKHOUSE & IYENGAR [*15*, Fig. 5] from measurements of neutron scattering in germanium.

At the corresponding stage in the development of equations of high frequency vibrations of plates, it is expedient to introduce correction factors to compensate, as well as possible within the framework of the theory, for errors introduced by the restrictive assumption regarding the variation of displacement through the thickness of the plate. The analogous restriction, here, is the assumed homogeneous deformation of the unit cell. The values of the correction factors are obtained, in the theory of plates, by matching appropriate points, slopes and curvatures of the dispersion curves with the corresponding quantities obtained from an exact solution of the three-dimensional equations. Since the analogue of the latter does not exist, in the present case, such an adjustment cannot be made. An alternative is to use experimental data.

There is another aspect of the theory of plates that should be mentioned. It is possible for a thickness-mode with $n+m$ nodal planes to have a frequency lower than one with n nodal planes. For example, in an isotropic plate, the thickness-shear mode with two nodal planes has a frequency lower than that of the thickness-stretch mode with one nodal plane if Poisson's ratio is greater than one-third. Thus, a better approximation is obtained if a sufficient number of terms is retained, in the series expansion of the displacement, to accomodate this contingency [*11*]. The analogue, in the present case, is the possibility of the appearance of a micro-mode with frequency lower than that of the dilatational micro-mode (7.3) if additional terms are retained in the series expansion (1.6). However, because of the complications that would ensue, such a step does not appear to be warranted at this time.

9. Low frequency, very long wave-length approximation: Form I

This section and the following three are devoted to discussions and derivations of equations of motion simpler than (6.1) and (6.2) but limited, in application, to much lower frequencies and much longer wave-lengths.

As noted previously, when thickness-shear and thickness-stretch deformations and the associated inertias are taken into account in the theory of plates [*16*, *17*, *18*], thickness-modes of vibration, analogous to the micro-modes, are obtained as well as flexural and extensional modes analogous to the transverse and longitudinal acoustic modes. At low frequencies, in comparison with the frequencies of the thickness-modes, and at long wave-lengths, in comparison with the thickness of the plate, the coupling of the flexural and extensional modes with the thickness-modes is negligible. As the frequencies of the flexural and extensional modes approach zero, the thickness-shear deformation approaches zero but the thickness-stretch deformation does not; rather, it is the stress associated with thickness-stretch that approaches zero. Thus, the antisymmetric and symmetric parts of deformation and stress have to be treated differently in passing from high frequency equations to the classical, low frequency equations. To obtain equations valid at low frequencies in the case of flexure, the thickness-shear deformation is made to approach zero by passing to a limit as the associated modulus of elasticity approaches infinity. The product of the two is indeterminate and this leaves the thickness-shear stress indeterminate in the constitutive equations. In the case of extension, the thickness-stress is set equal to zero and the resulting constitutive equation is used to eliminate the thickness-strain from the remaining equations.

In both flexure and extension of homogeneous plates, the thickness velocities are set equal to zero in the kinetic energy, for the low frequency approximation, because their contributions are negligibly small at the low frequencies to which the resulting equations are restricted owing to the suppression of the thickness-shear deformation and the omission of the thickness-stretch stress [*18*, *19*]. The same is not true of non-homogeneous plates. For example, in a sandwich plate the rotatory inertia of the facings, about the middle plane of the plate, can be of paramount importance, even at low frequencies, for certain combinations of stiffness ratios, density ratio and distance between facings [*13*].

Now, the thickness velocities are analogous to the micro-velocities $\dot{\psi}_{ij}$; the thickness of the plate is analogous to dimensions $2d_i$ of the unit cell; the thickness-shear deformation is analogous to the antisymmetric part of the relative deformation $(\gamma_{[ij]})$; the thickness-shear moduli are analogous to the $b_{ij[kl]}$; and the stress associated with thickness-stretch is analogous to the symmetric part of the relative stress $(\sigma_{(ij)})$. The known process of descending from high frequency equations of plates to low frequency, long wave-length approximations can serve as a guide to the treatment of analogous terms in the equations of the elastic continuum with micro-structure. However, regardless of the process or of the theory of plates, the test of the validity of the resulting equations of motion is that they yield the same dispersion relations that are found in the limit as $\omega\to 0$, $\xi\to 0$ for the acoustic branches of the general equations. Thus, in the isotropic case, the values of ω_i', ω_i'' and ω_i''', in (8.13), must be reproduced exactly. Attention will be confined, here, to this case because it is much simpler than the anisotropic form and because (8.13) are available for the final test.

Inspection of $(8.13)_3$ shows that, as in the case of sandwich plates, it is not permissible to discard the micro-velocities $\dot{\psi}_{ij}$. Their effect is contained in the h_i^2 and, as remarked in the discussion following (8.19), h^2 can be less than or greater than $\tilde{l}^2$ depending on stiffness ratios, the density ratio and the dimensions of the unit cell. Hence, omission of the $\dot{\psi}_{ij}$ would preclude the reproduction of the low frequency behavior. The remainder of the process, however, can follow the analogy with homogeneous plates.

Thus, we let

$$\sigma_{(ij)}=0, \tag{9.1}$$

$$b_2-b_3\to\infty, \qquad \gamma_{[ij]}\to 0, \tag{9.2}$$

and proceed to find the effect of these assumptions on the remaining terms.

The isotropic, constitutive equations for τ_{pq} and σ_{pq}, separated into symmetric and antisymmetric parts, are

$$\tau_{pq}=\lambda\,\delta_{pq}\varepsilon_{ii}+2\mu\,\varepsilon_{pq}+g_1\,\delta_{pq}\gamma_{ii}+2g_2\gamma_{(pq)}, \tag{9.3}$$

$$\sigma_{(pq)}=g_1\,\delta_{pq}\varepsilon_{ii}+2g_2\,\varepsilon_{pq}+b_1\,\delta_{pq}\gamma_{ii}+(b_2+b_3)\,\gamma_{(pq)}, \tag{9.4}$$

$$\sigma_{[pq]}=(b_2-b_3)\,\gamma_{[pq]}. \tag{9.5}$$

Then, with (9.2), $\sigma_{[pq]}$ is indeterminate in (9.5) and, with (9.1), (9.4) may be solved for $\gamma_{(pq)}$ in terms of ε_{pq}:

$$\gamma_{(pq)}=-\alpha\,\delta_{pq}\varepsilon_{ii}+(1-\beta)\,\varepsilon_{pq}, \tag{9.6}$$

where α and β are given in (8.18).

With regard to $\varkappa_{ijk}$, we note first that, since

$$\gamma_{pq}=\partial_p u_q-\psi_{pq}$$

and $\gamma_{[pq]}$ is now zero, we are left with

$$\psi_{[pq]}=\omega_{pq}, \qquad \psi_{(pq)}=\varepsilon_{pq}-\gamma_{(pq)}; \tag{9.7}$$

or, using the expression (9.6) for $\gamma_{(pq)}$, we have

$$\psi_{(pq)}=\alpha\,\delta_{pq}\varepsilon_{ii}+\beta\,\varepsilon_{pq}. \tag{9.8}$$

Accordingly, $\varkappa_{ijk}\equiv\partial_i\psi_{jk}=\partial_i\psi_{(jk)}+\partial_i\psi_{[jk]}$ reduces to

$$\varkappa_{ijk}\to\alpha\,\tilde{\varkappa}_{ill}\delta_{jk}+\tfrac{1}{2}(1+\beta)\,\tilde{\varkappa}_{ijk}-\tfrac{1}{2}(1-\beta)\,\tilde{\varkappa}_{ikj}, \tag{9.9}$$

where

$$\tilde{\varkappa}_{ijk}\equiv\partial_i\partial_j u_k=\tilde{\varkappa}_{jik}. \tag{9.10}$$

Thus, the part of the potential energy-density that is a function of $\varkappa_{ijk}$ becomes a function of $\tilde{\varkappa}_{ijk}$: the second gradient of the displacement. The eighteen components of $\tilde{\varkappa}_{ijk}$ may be resolved, in more than one way, into tensors whose components are independent linear combinations of the $\partial_i\partial_j u_k$; so that other forms of the energy-density, for the low frequency approximation, are possible. These are treated in Sections 11 and 12.

Upon inserting (9.9), (9.6) and (9.2) in (5.5), we find Form I of the low frequency approximation for the potential energy-density:

$$W \to \tilde{W} = \tfrac{1}{2}\tilde{\lambda}\,\varepsilon_{ii}\varepsilon_{jj} + \tilde{\mu}\,\varepsilon_{ij}\varepsilon_{ij} + \tilde{a}_1\tilde{\varkappa}_{iik}\tilde{\varkappa}_{kjj} + \tilde{a}_2\tilde{\varkappa}_{ijj}\tilde{\varkappa}_{ikk} + \\ + \tilde{a}_3\tilde{\varkappa}_{iik}\tilde{\varkappa}_{jjk} + \tilde{a}_4\tilde{\varkappa}_{ijk}\tilde{\varkappa}_{ijk} + \tilde{a}_5\tilde{\varkappa}_{ijk}\tilde{\varkappa}_{kji}, \tag{9.11}$$

where $\tilde{\lambda}$ and $\tilde{\mu}$ are given in (8.15) and $\tilde{a}_1 \ldots \tilde{a}_5$ in (8.19). The appearance of these coefficients is preliminary evidence of the validity of the process.

We define new stresses:

$$\tilde{\tau}_{ij} \equiv \frac{\partial \tilde{W}}{\partial \varepsilon_{ij}} = \tilde{\tau}_{ji}, \tag*{$(9.12)_1$}$$

$$\tilde{\mu}_{ijk} \equiv \frac{\partial \tilde{W}}{\partial \tilde{\varkappa}_{ijk}} = \tilde{\mu}_{jik}. \tag*{$(9.12)_2$}$$

Then

$$\tilde{\tau}_{pq} = \tilde{\lambda}\,\delta_{pq}\varepsilon_{ii} + 2\tilde{\mu}\,\varepsilon_{pq}, \tag*{$(9.13)_1$}$$

$$\tilde{\mu}_{pqr} = \tfrac{1}{2}\tilde{a}_1(\tilde{\varkappa}_{iip}\delta_{qr} + 2\tilde{\varkappa}_{rii}\delta_{pq} + \tilde{\varkappa}_{iiq}\delta_{pr}) + \tilde{a}_2(\tilde{\varkappa}_{pii}\delta_{qr} + \tilde{\varkappa}_{qii}\delta_{pr}) + \\ + 2\tilde{a}_3\tilde{\varkappa}_{iir}\delta_{pq} + 2\tilde{a}_4\tilde{\varkappa}_{pqr} + \tilde{a}_5(\tilde{\varkappa}_{rqp} + \tilde{\varkappa}_{rpq}). \tag*{$(9.13)_2$}$$

The variational equation of motion is now obtained from Hamilton's principle with independent variations δu_i alone since, by $(9.7)_1$ and (9.8), the ψ_{ij} are no longer independent of the u_i.

The variation of the potential energy-density is

$$\begin{aligned} \delta W &= \tilde{\tau}_{ij}\delta\varepsilon_{ij} + \tilde{\mu}_{ijk}\delta\tilde{\varkappa}_{ijk}, \\ &= \tilde{\tau}_{ij}\partial_i\delta u_j + \tilde{\mu}_{ijk}\partial_i\partial_j\delta u_k, \\ &= \partial_j[(\tilde{\tau}_{jk} - \partial_i\tilde{\mu}_{ijk})\,\delta u_k] - \partial_j(\tilde{\tau}_{jk} - \partial_i\tilde{\mu}_{ijk})\,\delta u_k + \partial_i(\tilde{\mu}_{ijk}\partial_j\delta u_k). \end{aligned} \tag{9.14}$$

Hence

$$\int_V \delta\tilde{W}\,dV = \int_S n_j(\tilde{\tau}_{jk} - \partial_i\tilde{\mu}_{ijk})\,\delta u_k\,dS - \\ - \int_V \partial_j(\tilde{\tau}_{jk} - \partial_i\tilde{\mu}_{ijk})\,\partial u_k\,dV + \int_S n_i\tilde{\mu}_{ijk}\partial_j\delta u_k\,dS. \tag{9.15}$$

Now, in the last integral of (9.15), the variation $\partial_j\delta u_k$ is not independent of δu_j on S: only its normal component $n_j\partial_j\delta u_k$ is independent. We separate the latter:

$$n_i\tilde{\mu}_{ijk}\partial_j\delta u_k = n_i\tilde{\mu}_{ijk}D_j\delta u_k + n_i\tilde{\mu}_{ijk}n_j D\,\delta u_k, \tag{9.16}$$

where

$$D_j \equiv (\delta_{jl} - n_j n_l)\,\partial_l, \qquad D \equiv n_l\partial_l. \tag{9.17}$$

The terms in (9.16) may be resolved, further, in more than one way. In this section we follow Toupin [8] and reserve an alternative resolution [9] for Section 12. Thus, for the first term on the right hand side of (9.16), which contains the non-independent variation $D_j\delta u_k$, we write

$$n_i\tilde{\mu}_{ijk}D_j\delta u_k = D_j(n_i\tilde{\mu}_{ijk}\delta u_k) - n_i D_j\tilde{\mu}_{ijk}\delta u_k - (D_j n_i)\,\tilde{\mu}_{ijk}\delta u_k. \tag{9.18}$$

The last two terms in (9.18) now contain the independent variation δu_k. For the preceding term, we note that, on the surface S,

$$D_j(n_i\tilde{\mu}_{ijk}\delta u_k) = (D_l n_l)\,n_j n_i\tilde{\mu}_{ijk}\delta u_k + n_q e_{qpm}\partial_p(e_{mlj}n_l n_i\tilde{\mu}_{ijk}\delta u_k). \tag{9.19}$$

By Stokes's theorem, the integral, over a smooth surface, of the last term in (9.19) vanishes. If the surface has an edge C, formed by the intersection of two portions, S_1 and S_2, of S, Stokes's theorem gives

$$\int_S n_q e_{qpm} \partial_p (e_{mlj} n_l n_i \tilde{\mu}_{ijk} \delta u_k) \, dS = \oint_C [n_i m_j \tilde{\mu}_{ijk}] \, \delta u_k \, ds, \tag{9.20}$$

where $m_j = e_{mlj} s_m n_l$ and the s_m are the components of the unit vector tangent to C. The bold face brackets [ ] in (9.20) indicate that the enclosed quantity is the difference between the values on S_1 and S_2.

Finally, we note that, in the first surface integral in (9.15), we may write

$$n_j \partial_i \tilde{\mu}_{ijk} = n_j D_i \tilde{\mu}_{ijk} + n_i n_j D \tilde{\mu}_{ijk}. \tag{9.21}$$

Then, assembling the results in (9.15)–(9.21), we find

$$\begin{aligned} \int_V \tilde{W} \, dV = &-\int_V \partial_j (\tilde{\tau}_{jk} - \partial_i \tilde{\mu}_{ijk}) \, \delta u_k \, dV + \\ &+ \int_S [n_j \tilde{\tau}_{jk} - n_i n_j D \tilde{\mu}_{ijk} - 2 n_j D_i \tilde{\mu}_{ijk} + (n_i n_j D_l n_l - D_j n_i) \tilde{\mu}_{ijk}] \, \delta u_k \, dS + \\ &+ \int_S n_i n_j \tilde{\mu}_{ijk} D \, \delta u_k \, dS + \oint_C [n_i m_j \tilde{\mu}_{ijk}] \, \delta u_k \, ds. \end{aligned} \tag{9.22}$$

This form suggests, for the variation of work done by external forces,

$$\delta \mathscr{W}_1 = \int_V F_k \, \delta u_k \, dV + \int_S \tilde{P}_k \, \delta u_k \, dS + \int_S \tilde{R}_k D \, \delta u_k \, dS + \oint_C \tilde{E}_k \, \delta u_k \, ds. \tag{9.23}$$

As for the kinetic energy, the micro-velocity $\dot{\psi}_{ij}$, in $(2.3)_1$, must be replaced by a linear function of macro-velocity gradients:

$$\dot{\psi}_{ij} \to h_{ijkl} \, \partial_k \dot{u}_l, \tag{9.24}$$

where

$$h_{ijkl} \equiv \tfrac{1}{2} (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) + \alpha \, \delta_{ij} \delta_{kl} + \tfrac{1}{2} \beta (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}), \tag{9.25}$$

so as to satisfy $(9.7)_1$ and (9.8). Then the kinetic energy-density $(2.3)_1$ becomes

$$\begin{aligned} T &= \tfrac{1}{2} \varrho \dot{u}_j \dot{u}_j + \tfrac{1}{6} \varrho' \tilde{d}^2_{pkmn} \partial_m \dot{u}_n \partial_p \dot{u}_k, \\ &= \tfrac{1}{2} \varrho \dot{u}_j \dot{u}_j + \tfrac{1}{6} \partial_p [\varrho' \tilde{d}^2_{pkmn} (\partial_m \dot{u}_n) \dot{u}_k] - \tfrac{1}{6} \partial_p (\varrho' \tilde{d}^2_{pkmn} \partial_m \dot{u}_n) \dot{u}_k, \end{aligned} \tag{9.26}$$

where

$$\begin{aligned} \tilde{d}^2_{pkmn} &\equiv d^2_{jl} h_{lqpk} h_{jqmn} = \tilde{d}^2_{mnpk} \\ &= \tfrac{1}{2} d^2 [\delta_{pm} \delta_{kn} - \delta_{pn} \delta_{km} + 2\alpha (3\alpha + 2\beta) \delta_{pk} \delta_{mn} + \beta^2 (\delta_{pm} \delta_{kn} + \delta_{pn} \delta_{km})]. \end{aligned} \tag{9.27}$$

The total kinetic energy is

$$\begin{aligned} \mathscr{T} = \int_V T \, dV = \int_V [\tfrac{1}{2} \varrho \dot{u}_j \dot{u}_j - \tfrac{1}{6} \partial_p (\varrho' \tilde{d}^2_{pkmn} \partial_m \dot{u}_n) \dot{u}_k] \, dV + \\ + \int_S \tfrac{1}{6} \varrho' n_p \tilde{d}^2_{pkmn} (\partial_m \dot{u}_n) \dot{u}_k \, dS; \end{aligned} \tag{9.28}$$

from which

$$\begin{aligned} \delta \int_{t_0}^{t_1} \mathscr{T} \, dt = &-\int_{t_0}^{t_1} dt \int_V [\varrho \ddot{u}_k - \tfrac{1}{3} \partial_p (\varrho' d^2_{pkmn} \partial_m \ddot{u}_n)] \, \delta u_k \, dV - \\ &- \int_{t_0}^{t_1} dt \int_S \tfrac{1}{3} \varrho' n_p d^2_{pkmn} (D_m \ddot{u}_n + n_m D \ddot{u}_n) \, \delta u_k \, dS. \end{aligned} \tag{9.29}$$

The variational equation of motion is formed from (9.22), (9.23) and (9.29), from which follow the stress-equations of motion and boundary conditions:

$$\partial_j(\tilde{\tau}_{jk} - \partial_i \tilde{\mu}_{ijk}) + F_k = \varrho \ddot{u}_k - \tfrac{1}{3}\partial_p(\varrho' \tilde{d}^2_{pkmn}\partial_m \ddot{u}_n), \tag{9.30$_1$}$$

$$\begin{aligned} n_j \tilde{\tau}_{jk} - n_i n_j D \tilde{\mu}_{ijk} - 2 n_j D_i \tilde{\mu}_{ijk} + (n_i n_j D_l n_l - D_j n_i)\tilde{\mu}_{ijk} + \\ + \tfrac{1}{3}\varrho' n_p \tilde{d}^2_{pkmn}(D_m \ddot{u}_n + n_m D \ddot{u}_n) = \tilde{P}_k, \end{aligned} \tag{9.30$_2$}$$

$$n_i n_j \tilde{\mu}_{ijk} = \tilde{R}_k, \tag{9.30$_3$}$$

$$[n_i m_j \tilde{\mu}_{ijk}] = \tilde{E}_k. \tag{9.30$_4$}$$

The displacement-equations of motion are obtained by first replacing $2\varepsilon_{ij}$ with $\partial_i u_j + \partial_j u_i$ and $\tilde{\varkappa}_{ijk}$ with $\partial_i \partial_j u_k$, in (9.13), and then substituting the latter in $(9.30)_1$. The result is

$$\begin{aligned} (\tilde{\lambda} + 2\tilde{\mu})(1 - \tilde{l}_1^2 \nabla^2)\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\boldsymbol{u} - \tilde{\mu}(1 - \tilde{l}_2^2 \nabla^2)\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{u} + \boldsymbol{F} \\ = \varrho(\ddot{\boldsymbol{u}} - h_1^2 \boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\ddot{\boldsymbol{u}} + h_2^2 \boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\ddot{\boldsymbol{u}}), \end{aligned} \tag{9.31}$$

where the $\tilde{l}_i^2$ and h_i^2 are defined in (8.16) and (8.17).

Omitting the body force and taking the divergence and curl of (9.31), we find the equations governing the propagation of dilatation and rotation:

$$\tilde{v}_1^2(1 - \tilde{l}_1^2 \nabla^2)\nabla^2 \boldsymbol{\nabla}\cdot\boldsymbol{u} = (1 - h_1^2 \nabla^2)\boldsymbol{\nabla}\cdot\ddot{\boldsymbol{u}}, \tag{9.32$_1$}$$

$$\tilde{v}_2^2(1 - \tilde{l}_2^2 \nabla^2)\nabla^2 \boldsymbol{\nabla}\times\boldsymbol{u} = (1 - h_2^2 \nabla^2)\boldsymbol{\nabla}\times\ddot{\boldsymbol{u}}, \tag{9.32$_2$}$$

where the $\tilde{v}_i^2$ are defined in (8.14). For the plane waves

$$(\boldsymbol{\nabla}\cdot\boldsymbol{u}, \nabla\times\boldsymbol{u}) = (A, \boldsymbol{A})\exp[i(\xi\,\boldsymbol{n}\cdot\boldsymbol{r} - \omega t)], \tag{9.33}$$

the dispersion relations are

$$\omega_i^2 = \tilde{v}_i^2 \xi^2 (1 + \tilde{l}_i^2 \xi^2)/(1 + h_i^2 \xi^2), \tag{9.34}$$

from which follow exactly the properties (8.13). Thus the validity of the approximate equations for low frequencies and very long wave lengths is established. The dispersion relations (9.34) are illustrated in Fig. 3 by the dashed curves labelled $\tilde{L}$ and $\tilde{T}$.

10. Relation to Toupin's generalization of couple-stress theory

The theory of elasticity with couple-stresses, which is considered in [*5*—*10*], is based on the same kinematics as is classical elasticity; but the potential energy-density is assumed to be a function of the strain and the curl of the strain instead of the strain alone. In the linear theory, the components of the curl of the strain are the same as the components of the gradient of the rotation: eight independent linear combinations of the eighteen components of the second gradient of the displacement. For the equilibrium case, Toupin [*8*, Section 7] has generalized the theory to include all eighteen components. If the inertia terms are omitted, (9.30) are identical, in form, with Toupin's Eqs. (7.8)—(7.11)*.

* Note that, by definition, Toupin's β^{pqr} is symmetric in the second and third indices, whereas $\tilde{\mu}_{ijk}$ is symmetric in the first two indices. Note, also, that b should be replaced by $-b$ in Toupin's Eqs. A, B, (7.9) and (7.19).

However, although the form is the same, there are some significant differences. Equations (9.30) pertain to a low frequency, very long wave-length approximation to the equations of a material with micro-structure and the effect of the micro-structure survives, in both the potential and kinetic energy-densities, through the contribution of the symmetric part of the relative deformation. This part, $\gamma_{(ij)}$, can, in fact be traced to $\tilde{\tau}_{ij}$ and $\tilde{\mu}_{ijk}$, in (9.30), through the coefficients α and β in $\tilde{W}$. Similarly the contribution of $\gamma_{(ij)}$ to the acceleration terms in (9.30) can be traced through the coefficients α and β in h_{ijkl} and then in $\tilde{d}^2_{ijkl}$. On the other hand, TOUPIN's equations do not stem from considerations of microstructure.

The equations of the material with micro-structure can be reduced to those of a material without micro-structure (*i.e.*, to a *micro-homogeneous* material) by causing the micro-medium to merge with the macro-medium. This may be accomplished (just as readily, in this case, for the anisotropic as for the isotropic medium) by passing to the limit as

$$b_{ijkl} \to \infty, \qquad \gamma_{ij} \to 0, \tag{10.1}$$

instead of (9.1) and (9.2); and, at the same time, $\varrho \to \varrho'$, so as to remove the distinction between micro- and macro-densities $\left(i.e.,\ \varrho_M \to 0 \text{ in } (2.3)_2\right)$. Then

$$\psi_{ij} \to \partial_i u_j, \qquad \varkappa_{ijk} \to \partial_i \partial_j u_k \equiv \tilde{\varkappa}_{ijk}, \tag{10.2}$$

instead of $(9.7)_1$, (9.8) and (9.9). Accordingly, from (5.1),

$$W \to \tilde{W}^0 = \tfrac{1}{2} \mathring{c}_{ijkl}\, \varepsilon_{ij}\, \varepsilon_{kl} + \tfrac{1}{2} \mathring{a}_{ijklmn} \tilde{\varkappa}_{ijk} \tilde{\varkappa}_{lmn} + \mathring{f}_{ijklm} \tilde{\varkappa}_{ijk}\, \varepsilon_{lm} \tag{10.3}$$

and we define new stresses

$$\tilde{\tau}^0_{ij} \equiv \frac{\partial \tilde{W}^0}{\partial \varepsilon_{ij}} = \tilde{\tau}^0_{ji}, \tag{10.4$_1$}$$

$$\tilde{\mu}^0_{ijk} \equiv \frac{\partial \tilde{W}^0}{\partial \tilde{\varkappa}_{ijk}} = \tilde{\mu}^0_{jik}. \tag{10.4$_2$}$$

Also, in the kinetic energy-density (9.26),

$$\tilde{d}^2_{pkmn} \to \tilde{d}^2_{pm}\, \delta_{kn}, \tag{10.5}$$

where the d^2_{pm} are again given by $(2.3)_3$.

In the isotropic case $d^2_{pm} = d^2 \delta_{pm}$ and $\tilde{W}^0, \tilde{\tau}^0_{ij}, \tilde{\mu}^0_{ijk}$ have the same form as $\tilde{W}, \tilde{\tau}_{ij}, \tilde{\mu}_{ijk}$ but the coefficients are $\lambda, \mu, \mathring{a}_1 \ldots \mathring{a}_5$ instead of $\tilde{\lambda}, \tilde{\mu}, \tilde{a}_1 \ldots \tilde{a}_5$.

The formulation of the variational equation of motion proceeds as before and we arrive at the stress-equations of motion and boundary conditions:

$$\partial_j (\tilde{\tau}^0_{jk} - \partial_i \tilde{\mu}^0_{ijk}) + F_k = \varrho' \ddot{u}_k - \tfrac{1}{3} \partial_p (\varrho' d^2_{pm} \partial_m \ddot{u}_k), \tag{10.6$_1$}$$

$$\begin{aligned} n_j \tilde{\tau}^0_{jk} - n_i n_j D \tilde{\mu}^0_{ijk} - 2 n_j D_i \tilde{\mu}^0_{ijk} + (n_i n_j D_l n_l - D_j n_i) \tilde{\mu}^0_{ijk} + \\ + \tfrac{1}{3} \varrho' n_p d^2_{pm} (D_m \ddot{u}_k + n_m D \ddot{u}_k) = \tilde{P}^0_k, \end{aligned} \tag{10.6$_2$}$$

$$n_i n_j \tilde{\mu}^0_{ijk} = \tilde{R}^0_k, \tag{10.6$_3$}$$

$$[n_i m_j \tilde{\mu}^0_{ijk}] = \tilde{E}^0_k. \tag{10.6$_4$}$$

Without the acceleration terms, (10.6) are now precisely the linear form of TOUPIN's equations; and (9.30), without the acceleration terms, differ only in that the coefficients in $\tilde{W}$ and $\tilde{W}^0$ (and hence in the stresses) have different meanings. However, with the acceleration terms included, the *forms* are actually different: the fourth rank tensor $\tilde{d}^2_{pkmn}$ in (9.30) is replaced by the second rank tensor d^2_{pm} in (10.6) so that there are fewer coefficients in the latter.

In the isotropic case, it is only necessary to let

$$\tilde{\lambda} \to \lambda, \qquad \tilde{\mu} \to \mu, \qquad \alpha \to 0, \qquad \beta \to 1 \quad (\text{or } g_1 \to 0,\ g_2 \to 0) \tag{10.7}$$

in the equations of Section 9, to reach the equations of the micro-homogeneous medium. Thus, (9.31) reduces to

$$(\lambda + 2\mu)(1 - \mathring{l}_1^2 \nabla^2) \nabla\nabla\cdot\boldsymbol{u} - \mu(1 - \mathring{l}_2^2 \nabla^2) \nabla\times\nabla\times\boldsymbol{u} + \boldsymbol{F} = \varrho'(1 - \tfrac{1}{3} d^2 \nabla^2)\,\ddot{\boldsymbol{u}}, \tag{10.8}$$

where $\mathring{l}_1^2$ and $\mathring{l}_2^2$ are obtained from $\tilde{l}_1^2$ and $\tilde{l}_2^2$, in (8.16), by employing (10.7). It will be observed that the left hand sides of (9.31) and (10.8) have the same form, but the right hand sides have different forms because, with (10.7), $h_1^2 = h_2^2 = \frac{1}{3} d^2$. The dispersion relations for plane waves, from (10.8) are

$$\omega_i^2 = v_i^2 \xi^2 (1 + \mathring{l}_i^2 \xi^2)/(1 + \tfrac{1}{3} d^2 \xi^2), \tag{10.9}$$

where $v_1^2 = (\lambda + 2\mu)/\varrho'$, $v_2^2 = \mu/\varrho'$; and the properties at $\omega = 0$, $\xi = 0$ are

$$\omega_i' = v_i, \qquad \omega_i'' = 0, \qquad \omega_i''' = 3 v_i (\mathring{l}_i^2 - \tfrac{1}{3} d^2) \tag{10.10}$$

instead of (8.13).

The difference between the equations of the micro-homogeneous medium and the equations of the low frequency, very long wave-length approximation is similar to the difference between plane strain and plane stress; or, more appropriately in the present context, to the difference between equations of low frequency extensional vibrations of plates with the thickness of the plate constrained and not constrained to remain constant. In the case of equilibrium, the difference is solely in the physical interpretation of the elastic stiffnesses. With stiffnesses determined by experiments falling within the restrictions of the equations, the two equilibrium theories would be indistinguishable. For example, the numerical quantity that would be assigned to the stiffness $\tilde{\mu}$, in one theory, would be assigned to μ in the other. In the case of motion, however, there is an essential difference as the number of coefficients is not the same in the two theories unless $g_1 = g_2 = 0$, which is analogous to zero Poisson's ratio.

11. Low frequency, very long wave-length approximation: Form II

As noted before, the eighteen components $\partial_i \partial_j u_k$ may be arranged in independent linear combinations which form tensors. One such, indicated by Toupin [*8*, p. 404], is the gradient of the strain:

$$\hat{\varkappa}_{ijk} \equiv \partial_i \varepsilon_{jk} = \tfrac{1}{2}(\partial_i \partial_j u_k + \partial_i \partial_k u_j) = \hat{\varkappa}_{ikj}. \tag{11.1}$$

The potential energy-density, for the low frequency, very long wave-length approximation, may be expressed as a function of ε_{ij} and $\hat{\varkappa}_{ijk}$ by setting

$$\tilde{\varkappa}_{ijk} = \hat{\varkappa}_{ijk} + \hat{\varkappa}_{jki} - \hat{\varkappa}_{kij} \tag{11.2}$$

in (9.11), with the result:

$$\begin{aligned} W \to \hat{W} = \tfrac{1}{2} \tilde{\lambda}\, \varepsilon_{ii} \varepsilon_{jj} + \tilde{\mu}\, \varepsilon_{ij} \varepsilon_{ij} + \hat{a}_1 \hat{\varkappa}_{iik} \hat{\varkappa}_{kjj} + \hat{a}_2 \hat{\varkappa}_{ijj} \hat{\varkappa}_{ikk} + \hat{a}_3 \hat{\varkappa}_{iik} \hat{\varkappa}_{jjk} + \\ + \hat{a}_4 \hat{\varkappa}_{ijk} \hat{\varkappa}_{ijk} + \hat{a}_5 \hat{\varkappa}_{ijk} \hat{\varkappa}_{kji}, \end{aligned} \tag{11.3}$$

where

$$\hat{a}_1 = 2\tilde{a}_1 - 4\tilde{a}_3, \qquad \hat{a}_2 = -\tilde{a}_1 + \tilde{a}_2 + \tilde{a}_3,$$
$$\hat{a}_3 = 4\tilde{a}_3, \qquad \hat{a}_4 = 3\tilde{a}_4 - \tilde{a}_5, \qquad \hat{a}_5 = -2\tilde{a}_4 + 2\tilde{a}_5. \tag{11.4}$$

New stresses are defined by

$$\hat{\tau}_{ij} = \frac{\partial \hat{W}}{\partial \varepsilon_{ij}} = \hat{\tau}_{ji}, \tag{11.5$_1$}$$

$$\hat{\mu}_{ijk} = \frac{\partial \hat{W}}{\partial \hat{\varkappa}_{ijk}} = \hat{\mu}_{ikj}, \tag{11.5$_2$}$$

whence

$$\hat{\tau}_{pq} = \tilde{\lambda}\,\delta_{pq}\,\varepsilon_{ii} + 2\tilde{\mu}\,\varepsilon_{pq}, \tag{11.6$_1$}$$

$$\begin{aligned}\hat{\mu}_{pqr} = {} & \tfrac{1}{2}\hat{a}_1(\delta_{pq}\hat{\varkappa}_{rii} + 2\delta_{qr}\hat{\varkappa}_{iip} + \delta_{rp}\hat{\varkappa}_{qii}) + 2\hat{a}_2\delta_{qr}\hat{\varkappa}_{pii} + \\ & + \hat{a}_3(\delta_{pq}\hat{\varkappa}_{iir} + \delta_{pr}\hat{\varkappa}_{iiq}) + 2\hat{a}_4\hat{\varkappa}_{pqr} + \hat{a}_5(\hat{\varkappa}_{rpq} + \hat{\varkappa}_{qrp}).\end{aligned} \tag{11.6$_2$}$$

The variation of the potential energy-density is

$$\begin{aligned}\delta\hat{W} & = \hat{\tau}_{ij}\,\delta\varepsilon_{ij} + \hat{\mu}_{ijk}\,\delta\hat{\varkappa}_{ijk} = \hat{\tau}_{ij}\,\partial_i\delta u_j + \hat{\mu}_{ijk}\,\partial_i\partial_j\delta u_k \\ & = \partial_j[(\hat{\tau}_{jk} - \partial_i\hat{\mu}_{ijk})\,\delta u_k] - \partial_j(\hat{\tau}_{jk} - \partial_i\hat{\mu}_{ijk})\,\delta u_k + \partial_i(\hat{\mu}_{ijk}\,\partial_j\delta u_k).\end{aligned} \tag{11.7}$$

Now, $(11.7)_3$ has the same form as $(9.14)_3$ and the kinetic energy density (9.26) is unchanged. Hence, the variational equation of motion has the same form as in Section 9 and leads to boundary conditions like those in (9.30), but with $2n_jD_i$ replaced by $(n_jD_i + n_iD_j)$, and stress-equations of motion:

$$\partial_j(\hat{\tau}_{jk} - \partial_i\hat{\mu}_{ijk}) + F_k = \varrho\ddot{u}_k - \tfrac{1}{3}\partial_p(\varrho'\tilde{d}^2_{pkmn}\,\partial_m\ddot{u}_n). \tag{11.8}$$

Recalling that $\tilde{\mu}_{ijk} = \tilde{\mu}_{jik}$ whereas $\hat{\mu}_{ijk} = \hat{\mu}_{ikj}$, the quantity in parentheses on the left hand side of $(9.30)_1$ is not symmetric but the corresponding quantity in (11.8) is symmetric. The latter is a more convenient form for the introduction of a stress function of the Airy type.

To get the displacement-equation of motion, substitute $(11.1)_2$ and (1.10) in (11.6) and the latter in (11.8). The result is

$$\begin{aligned}(\tilde{\lambda} + 2\tilde{\mu})(1 - \hat{l}_1^2\nabla^2)\,\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\boldsymbol{u} - \tilde{\mu}(1 - \hat{l}_2^2\nabla^2)\,\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{u} + \boldsymbol{F} \\ = \varrho(\ddot{\boldsymbol{u}} - h_1^2\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\ddot{\boldsymbol{u}} + h_2^2\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\ddot{\boldsymbol{u}}),\end{aligned} \tag{11.9}$$

where

$$\hat{l}_1^2 = 2(\hat{a}_1 + \hat{a}_2 + \hat{a}_3 + \hat{a}_4 + \hat{a}_5)/(\tilde{\lambda} + 2\tilde{\mu}), \qquad \hat{l}_2^2 = (\hat{a}_3 + 2\hat{a}_4 + \hat{a}_5)/2\tilde{\mu}. \tag{11.10}$$

In view of (11.4), $\hat{l}_i^2 = \tilde{l}_i^2$, so that the displacement-equations of motion (11.9) and (9.31) are identical.

12. Low frequency, very long wave-length approximation: Form III

For some purposes it is advantageous to separate the curl of the strain (or the gradient of the rotation):

$$\bar{\varkappa}_{ij} \equiv e_{jlm}\,\partial_l\,\varepsilon_{mi} = \tfrac{1}{2}e_{jlm}\,\partial_i\,\partial_l\,u_m, \tag{12.1}$$

from $\partial_i\partial_j u_k$ as $\bar{\varkappa}_{ij}$ is the part of $\partial_i\partial_j u_k$ that gives rise to couple-stresses. The double stress is separated, thereby, into a non-self-equilibrating part and a self-equilibrating part. Now, $\bar{\varkappa}_{ii} = 0$. Hence $\bar{\varkappa}_{ij}$ has only eight independent components. (They are the components of the dyadic $\boldsymbol{\varkappa}$ in Reference [9].) The remaining ten linear combinations of the $\partial_i \partial_j u_k$ were considered separately by JARAMILLO [20]. They can be expressed as

$$\bar{\bar{\varkappa}}_{ijk} \equiv \hat{\varkappa}_{ijk} + \tfrac{1}{3} e_{ilj} \bar{\varkappa}_{kl} + \tfrac{1}{3} e_{ilk} \bar{\varkappa}_{jl} = \tfrac{1}{3} (\partial_i \partial_j u_k + \partial_k \partial_i u_j + \partial_j \partial_k u_i). \quad (12.2)$$

Thus, $\bar{\bar{\varkappa}}_{ijk} = \bar{\bar{\varkappa}}_{kij} = \bar{\bar{\varkappa}}_{jki} = \bar{\bar{\varkappa}}_{jik}$; i.e., $\bar{\bar{\varkappa}}_{ijk}$ is fully symmetric.

The potential energy-density, for the low frequency, very long wave-length approximation, may be expressed as a function of ε_{ij}, $\bar{\varkappa}_{ij}$ and $\bar{\bar{\varkappa}}_{ijk}$ by setting

$$\hat{\varkappa}_{ijk} = \bar{\bar{\varkappa}}_{ijk} - \tfrac{1}{3} e_{ilj} \bar{\varkappa}_{kl} - \tfrac{1}{3} e_{ilk} \bar{\varkappa}_{jl} \quad (12.3)$$

in (11.3). The result is

$$\begin{aligned} W \to \bar{W} = {} & \tfrac{1}{2} \tilde{\lambda} \varepsilon_{ii} \varepsilon_{jj} + \tilde{\mu}\, \varepsilon_{ij} \varepsilon_{ij} + 2\bar{d}_1 \bar{\varkappa}_{ij} \bar{\varkappa}_{ij} + 2\bar{d}_2 \bar{\varkappa}_{ij} \bar{\varkappa}_{ji} + \\ & + \tfrac{3}{2} \bar{a}_1 \bar{\bar{\varkappa}}_{iij} \bar{\bar{\varkappa}}_{kkj} + \bar{a}_2 \bar{\bar{\varkappa}}_{ijk} \bar{\bar{\varkappa}}_{ijk} + \bar{f} e_{ijk} \bar{\varkappa}_{ij} \bar{\bar{\varkappa}}_{kll}, \end{aligned} \quad (12.4)$$

where

$$\begin{aligned} & 18\bar{d}_1 = -2\hat{a}_1 + 4\hat{a}_2 + \hat{a}_3 + 6\hat{a}_4 - 3\hat{a}_5, \\ & 18\bar{d}_2 = 2\hat{a}_1 - 4\hat{a}_2 - \hat{a}_3, \qquad 3\bar{a}_1 = 2(\hat{a}_1 + \hat{a}_2 + \hat{a}_3), \\ & \bar{a}_2 = \hat{a}_4 + \hat{a}_5, \qquad 3\bar{f} = \hat{a}_1 + 4\hat{a}_2 - 2\hat{a}_3. \end{aligned} \quad (12.5)$$

The definitions

$$\bar{\tau}_{ij} \equiv \frac{\partial \bar{W}}{\partial \varepsilon_{ij}} = \bar{\tau}_{ji}, \quad (12.6)_1$$

$$\bar{\mu}_{ij} \equiv \frac{\partial \bar{W}}{\partial \bar{\varkappa}_{ij}}, \qquad \bar{\mu}_{ii} = 0, \quad (12.6)_2$$

$$\bar{\bar{\mu}}_{ijk} \equiv \frac{\partial \bar{W}}{\partial \bar{\bar{\varkappa}}_{ijk}} = \bar{\bar{\mu}}_{kij} = \bar{\bar{\mu}}_{jki} = \bar{\bar{\mu}}_{jik}, \quad (12.6)_3$$

where $\bar{\mu}_{ij}$ is the couple-stress deviator, lead to

$$\bar{\tau}_{pq} = \tilde{\lambda}\, \delta_{pq} \varepsilon_{ii} + 2\tilde{\mu} \varepsilon_{pq}, \quad (12.7)_1$$

$$\bar{\mu}_{pq} = 4\bar{d}_1 \bar{\varkappa}_{pq} + 4\bar{d}_2 \bar{\varkappa}_{qp} + \bar{f} e_{pqi} \bar{\bar{\varkappa}}_{ijj}, \quad (12.7)_2$$

$$\begin{aligned} \bar{\bar{\mu}}_{pqr} = {} & \bar{a}_1 (\bar{\bar{\varkappa}}_{iir} \delta_{pq} + \bar{\bar{\varkappa}}_{iip} \delta_{qr} + \bar{\bar{\varkappa}}_{iiq} \delta_{rp}) + 2\bar{a}_2 \bar{\bar{\varkappa}}_{pqr} + \\ & + \tfrac{1}{3} \bar{f} \bar{\varkappa}_{ij} (\delta_{pq} e_{ijr} + \delta_{qr} e_{ijp} + \delta_{rp} e_{ijq}). \end{aligned} \quad (12.7)_3$$

The variation of the potential energy-density now takes the form

$$\begin{aligned} \delta \bar{W} & = \bar{\tau}_{ij} \delta \varepsilon_{ij} + \bar{\mu}_{ij} \delta \bar{\varkappa}_{ij} + \bar{\bar{\mu}}_{ijk} \delta \bar{\bar{\varkappa}}_{ijk}, \\ & = \partial_j [(\bar{\tau}_{jk} - \partial_i \bar{\mu}^*_{ijk}) \delta u_k] - \partial_j (\bar{\tau}_{ij} - \partial_i \bar{\mu}^*_{ijk}) \delta u_k + \partial_i (\bar{\mu}^*_{ijk} \partial u_k), \end{aligned} \quad (12.8)$$

where

$$\bar{\mu}^*_{ijk} \equiv \tfrac{1}{2} e_{jkl} \bar{\mu}_{il} + \bar{\bar{\mu}}_{ijk}. \quad (12.9)$$

Again, (12.8) has the same form as (9.14) and so we can find stress-equations of motion and boundary conditions of the same form as (9.30), but with $\tilde{\mu}_{ijk}$ replaced by $\bar{\mu}^*_{ijk}$. Such a form of the boundary conditions cannot be compared directly with the results of Reference [9] because, there, one of the independent variations was taken to be the tangential component of rotation — which here is embedded in the normal derivative $D\delta u_k$. To get the alternative form of the boundary conditions, we can return to (9.16) and further resolve:

$$D\,\delta u_k = 2\,\delta w_i n_j e_{ijk} + D_k(n_i\,\delta u_i) - (D_k n_i)\,\delta u_i + n_k\,\delta\varepsilon_{nn}, \qquad (12.10)$$

where $w_i (\equiv \frac{1}{2} e_{ilm} \partial_l u_m)$ is the rotation and ε_{nn} (not summed) is the normal component of strain $n_i n_j \varepsilon_{ij}$. Then integration by parts and application of the divergence theorem and Stokes's theorem leads, in the notation of Reference [9], to

$$\begin{aligned}\delta\overline{W} = & -\int_V (\nabla\cdot\overline{\boldsymbol{\tau}} + \tfrac{1}{2}\nabla\times\nabla\cdot\overline{\boldsymbol{\mu}} - \nabla\cdot\overline{\overline{\boldsymbol{\mu}}}\cdot\nabla)\cdot\delta\boldsymbol{u}\,dV + \\ & + \int_S \{\boldsymbol{n}\cdot\overline{\boldsymbol{\tau}} + \tfrac{1}{2}\boldsymbol{n}\times(\nabla\cdot\overline{\boldsymbol{\mu}} - \nabla\bar{\mu}_{nn}) - (\nabla\cdot\overline{\overline{\boldsymbol{\mu}}})\cdot\boldsymbol{n} - \\ & - \boldsymbol{n}\cdot\nabla\times[\boldsymbol{n}\times(\boldsymbol{n}\cdot\overline{\overline{\boldsymbol{\mu}}} + \boldsymbol{n}\cdot\overline{\overline{\boldsymbol{\mu}}}\cdot\boldsymbol{n}\boldsymbol{n})]\}\cdot\delta\boldsymbol{u}\,dS + \\ & + \int_S [\boldsymbol{n}\cdot\overline{\boldsymbol{\mu}}\times\boldsymbol{n} + 2\boldsymbol{n}\times(\boldsymbol{n}\cdot\overline{\overline{\boldsymbol{\mu}}}\cdot\boldsymbol{n})\times\boldsymbol{n}]\cdot(\delta\boldsymbol{w}\times\boldsymbol{n})\,dS + \int_S \boldsymbol{n}\boldsymbol{n}:\overline{\overline{\boldsymbol{\mu}}}\cdot\boldsymbol{n}\,\delta\varepsilon_{nn}\,dS + \\ & + \oint_C [\tfrac{1}{2}\overline{\boldsymbol{\mu}}_{nn}\boldsymbol{s} + (\boldsymbol{s}\times\boldsymbol{n})\cdot(\boldsymbol{n}\cdot\overline{\overline{\boldsymbol{\mu}}} + \boldsymbol{n}\cdot\overline{\overline{\boldsymbol{\mu}}}\cdot\boldsymbol{n}\boldsymbol{n})]\cdot\delta\boldsymbol{u}\,ds,\end{aligned} \qquad (12.11)$$

where $\boldsymbol{s}$ is a unit vector tangent to the edge C.

The variation of work done by external forces is now taken to be

$$\delta\mathscr{W}_1 = \int_V \boldsymbol{F}\cdot\delta\boldsymbol{u}\,dV + \int_S (\boldsymbol{P}\cdot\delta\boldsymbol{u} + \boldsymbol{Q}\cdot\delta\boldsymbol{w}\times\boldsymbol{n} + R\,\delta\varepsilon_{nn})\,dS + \oint_C \boldsymbol{E}\cdot\delta\boldsymbol{u}\,ds. \qquad (12.12)$$

Here $\boldsymbol{Q}$ is the tangential component of the couple-stress vector and R is a double force per unit area, without moment, normal to S.

The variational equation of motion then yields the stress-equation of motion

$$\nabla\cdot\overline{\boldsymbol{\tau}} + \tfrac{1}{2}\nabla\times\nabla\cdot\overline{\boldsymbol{\mu}} - \nabla\cdot\overline{\overline{\boldsymbol{\mu}}}\cdot\nabla + \boldsymbol{F} = \varrho\,\ddot{\boldsymbol{u}} - \tfrac{1}{3}\nabla\cdot(\varrho'\,\tilde{\boldsymbol{d}}^2:\nabla\ddot{\boldsymbol{u}}) \qquad (12.13)$$

and the boundary conditions

$$\begin{aligned}&\boldsymbol{n}\cdot\overline{\boldsymbol{\tau}} + \tfrac{1}{2}\boldsymbol{n}\times(\nabla\cdot\overline{\boldsymbol{\mu}} - \nabla\bar{\mu}_{nn}) - \\ &\quad - (\nabla\cdot\overline{\overline{\boldsymbol{\mu}}})\cdot\boldsymbol{n} - \boldsymbol{n}\cdot\nabla\times[\boldsymbol{n}\times(\boldsymbol{n}\cdot\overline{\overline{\boldsymbol{\mu}}} + \boldsymbol{n}\cdot\overline{\overline{\boldsymbol{\mu}}}\cdot\boldsymbol{n}\boldsymbol{n})] + \tfrac{1}{3}\varrho'\,\boldsymbol{n}\cdot\tilde{\boldsymbol{d}}^2:\nabla\ddot{\boldsymbol{u}} = \boldsymbol{P},\end{aligned} \qquad (12.14)_1$$

$$\boldsymbol{n}\cdot\overline{\boldsymbol{\mu}}\times\boldsymbol{n} + 2\boldsymbol{n}\times(\boldsymbol{n}\cdot\overline{\overline{\boldsymbol{\mu}}}\cdot\boldsymbol{n})\times\boldsymbol{n} = \boldsymbol{Q}, \qquad (12.14)_2$$

$$\boldsymbol{n}\boldsymbol{n}:\overline{\overline{\boldsymbol{\mu}}}\cdot\boldsymbol{n} = R, \qquad (12.14)_3$$

$$[\tfrac{1}{2}\bar{\mu}_{nn}\boldsymbol{s} + (\boldsymbol{s}\times\boldsymbol{n})\cdot(\boldsymbol{n}\cdot\overline{\overline{\boldsymbol{\mu}}} + \boldsymbol{n}\cdot\overline{\overline{\boldsymbol{\mu}}}\cdot\boldsymbol{n}\boldsymbol{n})] = \boldsymbol{E}. \qquad (12.14)_4$$

The displacement equation of motion is obtained by substituting (1.10), $(12.1)_2$ and $(12.2)_2$ in (12.7) and the latter in (12.13). The result is

$$\begin{aligned}(\tilde{\lambda} + 2\tilde{\mu})\,(1 - \bar{l}_1^2\,\nabla^2)\,\nabla\nabla\cdot\boldsymbol{u} - \tilde{\mu}\,(1 - \bar{l}_2^2\,\nabla^2)\,\nabla\times\nabla\times\boldsymbol{u} + \boldsymbol{F} \\ = \varrho\,(\ddot{\boldsymbol{u}} - h_1^2\nabla\nabla\cdot\ddot{\boldsymbol{u}} + h_2^2\nabla\times\nabla\times\ddot{\boldsymbol{u}}),\end{aligned} \qquad (12.15)$$

where

$$\bar{l}_1^2 = (3\bar{a}_1 + 2\bar{a}_2)/(\tilde{\lambda} + 2\tilde{\mu}), \qquad \bar{l}_2^2 = (3\,\bar{d}_1 + \bar{a}_1 + 2\bar{a}_2 - \bar{f})/3\tilde{\mu}. \qquad (12.16)$$

In view of (12.5) and (11.4),

$$\bar{l}_i^2 = \hat{l}_i^2 = \tilde{l}_i^2 = l_i^2, \quad \text{say}; \qquad (12.17)$$

so that all three forms of the low frequency, very long wave-length approximation yield the same displacement-equation of motion. Necessary and sufficient conditions for positive definiteness of $\overline{W}(=\hat{W}=\widetilde{W})$ are

$$\begin{aligned} &\tilde{\mu}>0, \quad 3\tilde{\lambda}+2\tilde{\mu}>0,\\ &\bar{d}_1>0, \quad -\bar{d}_1<\bar{d}_2<\bar{d}_1,\\ &\bar{a}_2>0, \quad 3\bar{a}_1+2\bar{a}_2>0, \quad \bar{f}<0. \end{aligned} \tag{12.18}$$

Hence $l_i^2>0$ and we have already seen that $h_i^2>0$.

The equations that were considered in Reference [9] are obtained from those of this section by setting $\bar{\bar{\varkappa}}_{ijk}=0$ in the potential energy-density and $\varrho'=0$ in the kinetic energy density. Accordingly, the limitation to low frequencies, long wave-lengths and large dimensions is more severe than was apparent previously.

13. Solution of the approximate equations of equilibrium

In this section it is proved that any solution, $\boldsymbol{u}$, of the equation

$$(\tilde{\lambda}+2\tilde{\mu})(1-l_1^2\nabla^2)\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\boldsymbol{u}-\tilde{\mu}(1-l_2^2\nabla^2)\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{u}+\boldsymbol{F}=0, \tag{13.1}$$

in a region V bounded by a surface S, can be expressed as

$$\boldsymbol{u}=\boldsymbol{B}-l_2^2\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\boldsymbol{B}-\tfrac{1}{2}(k_1-l_1^2\nabla^2)\boldsymbol{\nabla}[\boldsymbol{r}\cdot(1-l_2^2\nabla^2)\boldsymbol{B}+B_0], \tag*{(13.2)$_1$}$$

where

$$\tilde{\mu}(1-l_2^2\nabla^2)\nabla^2\boldsymbol{B}=-\boldsymbol{F}, \tag*{(13.2)$_2$}$$

$$\tilde{\mu}(1-l_1^2\nabla^2)\nabla^2 B_0=\boldsymbol{r}\cdot(1-l_1^2\nabla^2)\boldsymbol{F}-4l_1^2\boldsymbol{\nabla}\cdot\boldsymbol{F}, \tag*{(13.2)$_3$}$$

$$k_1=(\tilde{\lambda}+\tilde{\mu})/(\tilde{\lambda}+2\tilde{\mu})$$

and $\boldsymbol{r}$ is the position vector.

Consider a field point $P(x, y, z)$ and a source point $Q(\xi, \eta, \zeta)$ and define

$$4\pi\boldsymbol{U}_P\equiv-\int_V r_1^{-1}\boldsymbol{u}_Q\,dV_Q, \tag{13.3}$$

where

$$r_1^2=(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2, \quad dV_Q=d\xi\,d\eta\,d\zeta.$$

Then $\nabla^2\boldsymbol{U}=\boldsymbol{u}$ [3, p. 210], or

$$\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\boldsymbol{U}-\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{U}=\boldsymbol{u}. \tag{13.4}$$

Define

$$\psi\equiv\boldsymbol{\nabla}\cdot\boldsymbol{U}, \quad \boldsymbol{H}=-\boldsymbol{\nabla}\times\boldsymbol{U}, \quad (\boldsymbol{\nabla}\cdot\boldsymbol{H}=0). \tag{13.5}$$

Then, from (13.4),

$$\boldsymbol{u}=\nabla\psi+\boldsymbol{\nabla}\times\boldsymbol{H}, \quad \boldsymbol{\nabla}\cdot\boldsymbol{H}=0, \tag{13.6}$$

which is Helmholtz's resolution. Substituting (13.6) in (13.2)$_1$, we have

$$\tilde{\mu}\,\nabla^2[k(1-l_1^2\nabla^2)\boldsymbol{\nabla}\psi+(1-l_2^2\nabla^2)\boldsymbol{\nabla}\times\boldsymbol{H}]+\boldsymbol{F}=0, \tag{13.7}$$

where $k=(\tilde{\lambda}+2\tilde{\mu})/\tilde{\mu}$.

Define

$$4\pi l_2^2\boldsymbol{B}'_P\equiv\int_V r_1^{-1}e^{-r_1/l_2}[k(1-l_1^2\nabla^2)\boldsymbol{\nabla}\psi+(1-l_2^2\nabla^2)\boldsymbol{\nabla}\times\boldsymbol{H}]_Q\,dV_Q. \tag{13.8}$$

Then [3, p. 210]

$$(1 - l_2^2 \nabla^2)\boldsymbol{B}' = k(1 - l_1^2 \nabla^2)\boldsymbol{\nabla}\psi + (1 - l_2^2 \nabla^2)\boldsymbol{\nabla}\times\boldsymbol{H} \tag{13.9}$$

and, from (13.9) and (13.7),

$$\tilde{\mu}(1 - l_2^2 \nabla^2)\nabla^2\boldsymbol{B}' = -\boldsymbol{F}. \tag{13.10}$$

Also, the divergence of (13.9) yields

$$(1 - l_2^2 \nabla^2)\boldsymbol{\nabla}\cdot\boldsymbol{B}' = k(1 - l_1^2 \nabla^2)\nabla^2\psi. \tag{13.11}$$

Define

$$2k\psi^* \equiv \boldsymbol{r}\cdot(1 - l_2^2 \nabla^2)\boldsymbol{B}'. \tag{13.12}$$

Then, using (13.10), we have

$$2k\tilde{\mu}(1 - l_1^2 \nabla^2)\nabla^2\psi^* = 4l_1^2\boldsymbol{\nabla}\cdot\boldsymbol{F} - \boldsymbol{r}\cdot(1 - l_1^2 \nabla^2)\boldsymbol{F} + 2\tilde{\mu}(1 - l_2^2 \nabla^2)\boldsymbol{\nabla}\cdot\boldsymbol{B}'. \tag{13.13}$$

Define

$$B_0 = 2k(\psi - \psi^*) \tag{13.14}$$

and find

$$\tilde{\mu}(1 - l_1^2 \nabla^2)\nabla^2 B_0 = \boldsymbol{r}\cdot(1 - l_1^2 \nabla^2)\boldsymbol{F} - 4l_1^2\boldsymbol{\nabla}\cdot\boldsymbol{F} \tag{13.15}$$

by using (13.11) and (13.13). Also, from (13.14) and (13.12),

$$2k\psi = \boldsymbol{r}\cdot(1 - l_2^2 \nabla^2)\boldsymbol{B}' + B_0. \tag{13.16}$$

Now, define

$$\boldsymbol{D} \equiv \boldsymbol{B}' - l_2^2\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot B' - k(1 - l_1^2 \nabla^2)\boldsymbol{\nabla}\psi. \tag{13.17}$$

By (13.11), $\boldsymbol{\nabla}\cdot\boldsymbol{D} = 0$; which is a necessary and sufficient condition for the existence of a function $\boldsymbol{H}^*$ such that $\boldsymbol{\nabla}\times\boldsymbol{H}^* = \boldsymbol{D}$, *i.e.*,

$$\boldsymbol{\nabla}\times\boldsymbol{H}^* = \boldsymbol{B}' - l_2^2\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\boldsymbol{B}' - k(1 - l_1^2 \nabla^2)\boldsymbol{\nabla}\psi. \tag{13.18}$$

Making use of (13.9) and (13.11) we find

$$(1 - l_2^2 \nabla^2)\boldsymbol{\nabla}\times\boldsymbol{H}^* = (1 - l_2^2 \nabla^2)\boldsymbol{\nabla}\times\boldsymbol{H}. \tag{13.19}$$

Next, define

$$\boldsymbol{B}'' \equiv \boldsymbol{\nabla}\times\boldsymbol{H} - \boldsymbol{\nabla}\times\boldsymbol{H}^* \tag{13.20}$$

and note that, by (13.19) and (13.20),

$$(1 - l_2^2 \nabla^2)\boldsymbol{B}'' = 0, \qquad \boldsymbol{\nabla}\cdot\boldsymbol{B}'' = 0. \tag{13.21}$$

From (13.20), using (13.18) and then (13.16),

$$\nabla\times\boldsymbol{H} = \boldsymbol{B}'' + \boldsymbol{B}' - l_2^2\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\boldsymbol{B}' - \tfrac{1}{2}(1 - l_1^2 \nabla^2)\boldsymbol{\nabla}[\boldsymbol{r}\cdot(1 - l_2^2 \nabla^2\boldsymbol{B}') + B_0]. \tag{13.22}$$

Then substitute (13.22) and (13.16) in $(13.6)_1$ to get

$$\boldsymbol{u} = \boldsymbol{B}'' + \boldsymbol{B}' - l_2^2\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot(\boldsymbol{B}'' + \boldsymbol{B}') - \tfrac{1}{2}(k_1 - l_1^2 \nabla^2)\boldsymbol{\nabla}[\boldsymbol{r}\cdot(1 - l_2^2 \nabla^2)\boldsymbol{B}' + B_0]. \tag{13.23}$$

Finally, define

$$\boldsymbol{B} \equiv \boldsymbol{B}' + \boldsymbol{B}''. \tag{13.24}$$

In view of (13.21), we may write (13.23) and (13.10) in the form of $(13.2)_1$ and $(13.2)_2$; and (13.15) is already in the form $(13.2)_3$. Thus Eqs. (13.2) are a complete solution of (13.1). If $l_1^2 = 0$, (13.1) reduces to the equilibrium equation of couple-stress theory and (13.2) is the solution found in [9]. If both l_1^2 and l_2^2 are zero, (13.1) is the classical equation of equilibrium with body force and (13.2) is the solution found in [*21*]. If, in addition, the body force is zero, (13.2) is PAPKOVITCH's solution [*22*]. The proof follows, generally, that in [*23*] but with an improvement as a result of an illuminating criticism by E. STERNBERG.

14. Concentrated force according to the approximate equations

In an infinite region V, let the body force be zero outside a finite region V_0 which contains the origin and a non-vanishing field of parallel forces $\boldsymbol{F}$. A concentrated force is defined by

$$\boldsymbol{P} = \lim_{V_0 \to 0} \int_V \boldsymbol{F}_Q \, dV_Q. \tag{14.1}$$

In [*9*] it was shown that, in an infinite region, solutions of equations of the type $(13.2)_2$ and $(13.2)_3$ are

$$4\pi \tilde{\mu} \boldsymbol{B} = \int_V r_1^{-1} (1 - e^{-r_1/l_2}) \boldsymbol{F}_Q \, dV_Q, \tag{14.2}$$

$$4\pi \tilde{\mu} B_0 = -\int_V r_1^{-1} (1 - e^{-r_1/l_1}) [\boldsymbol{r}' \cdot (1 - l_1^2 V_Q^2) \boldsymbol{F}_Q - 4 l_1^2 \boldsymbol{\nabla}_Q \cdot \boldsymbol{F}_Q] \, dV_Q, \tag{14.3}$$

where $r' = \sqrt{\xi^2 + \eta^2 + \zeta^2}$. Now

$$\lim_{V_0 \to 0} r_1 = r, \qquad \lim_{V_0 \to 0} r' = 0. \tag{14.4}$$

Hence, for the concentrated force, (14.2) reduces to

$$4\pi \tilde{\mu} \boldsymbol{B} = r^{-1} (1 - e^{-r/l_2}) \boldsymbol{P}. \tag{14.5}$$

In (14.3) the term in the integrand of the form $\psi(r_1) \boldsymbol{\nabla}_Q \cdot \boldsymbol{F}_Q$ is transformed according to

$$\int_V \psi \boldsymbol{\nabla}_Q \cdot \boldsymbol{F}_Q \, dV_Q = \int_V [\boldsymbol{\nabla}_Q \cdot (\psi \boldsymbol{F}_Q) - \boldsymbol{F}_Q \cdot \boldsymbol{\nabla}_Q \psi] \, dV_Q = \int_S \psi \boldsymbol{n} \cdot \boldsymbol{F}_Q \, dS - \int_V \boldsymbol{F}_Q \cdot \boldsymbol{\nabla}_Q \psi \, dV_Q. \tag{14.6}$$

The surface integral in (14.6) vanishes because $\boldsymbol{F} = 0$ outside V_0. Also

$$\lim_{V_0 \to 0} \int_V \boldsymbol{F}_Q \cdot \boldsymbol{\nabla}_Q \psi(r_1) \, dV_Q = -\boldsymbol{P} \cdot \boldsymbol{\nabla} \psi(r). \tag{14.7}$$

Hence, for the concentrated force,

$$\pi \tilde{\mu} B_0 = l_1^2 \boldsymbol{P} \cdot \boldsymbol{\nabla} [r^{-1} (1 - e^{-r/l_1})]. \tag{14.8}$$

Equations (14.5) and (14.8) constitute the solution of (13.1) for the concentrated force. If $l_1^2 = 0$ then $B_0 = 0$ and the solution reduces to that found in [*9*]. If, in addition, $l_2^2 = 0$, the solution reduces to KELVIN's [*4*, p. 183].

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Department of Civil Engineering
Columbia University
New York 27, New York

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