Stabilita tlačeného prutu

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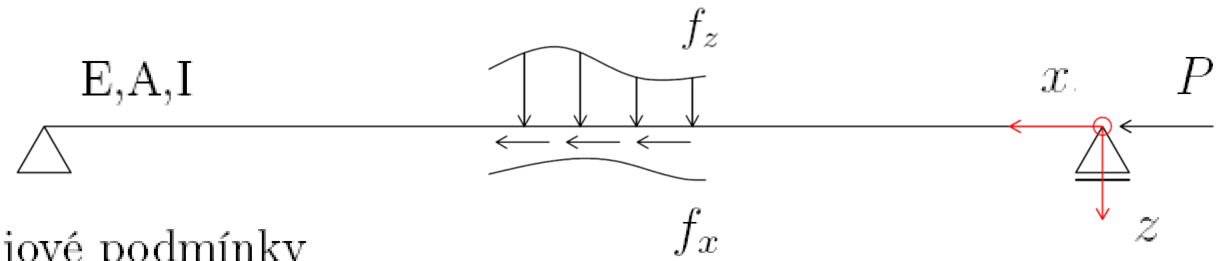
Řešená úloha



Okrajové podmínky

$$
\begin{aligned}
u(L) &= 0 \\
w(L) &= w(0) = 0 \\
M(0) &= M(L) = 0 \\
N(x) &= -P - \int_0^x f_x(s)\,ds
\end{aligned}
$$

Pole posunů

$$
\begin{aligned}
\tilde{u}(x,z) &= u(x) - w'(x)z \\
\tilde{w}(x,z) &= w(x)
\end{aligned}
$$

Celková potenciální energie soustavy

$$E_p(u,w) = E_{int} + E_{ext} = \int_V \frac{1}{2} E \varepsilon_x(u,w)\, dV - \int_L f_z w\, dx - \int_L f_x u\, dx - Pu(0)$$

Green-Lagrange tenzor deformace

$$E_{ij} = \frac{1}{2}\left(\frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i} + \frac{\partial \tilde{u}_k}{\partial x_i}\frac{\partial \tilde{u}_k}{\partial x_j}\right)$$

$$\begin{aligned}
\varepsilon_x(u,w) = E_{11} &= \frac{\partial \tilde{u}(x,z)}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial \tilde{u}(x,z)}{\partial x}\right)^2 + \left(\frac{\partial \tilde{w}(x,z)}{\partial x}\right)^2\right] \\
&= u'(x) - w''(x)z + \frac{1}{2}\left[(u'(x) - w''(x)z)^2 + w'^2(x)\right] \\
&\approx u'(x) - w''(x)z + \frac{1}{2} w'^2(x)
\end{aligned}$$

Podmínky stacionarity a minima

$$\Delta E_p = E_p(\mathbf{u} + \delta\mathbf{u}) - E_p(\mathbf{u}) = \delta E_p + \delta^2 E_p + \delta^3 E_p ...$$

Gâteauxův diferenciál

$$\delta^n E_p = \left. \frac{d^n E_p(\mathbf{u} + h\delta\mathbf{u})}{dh^n} \right|_{h=0}$$

Stacionarita

$$\delta E_p = 0$$

Minimum

$$\delta^2 E_p > 0$$

Podmínka stacionarity pro w(x,z)

$$\int_V E\left(\frac{1}{2}\delta w' w'^3 + \delta w' u' w' + \delta w'' w'' z^2 - \overbrace{\frac{1}{2}\delta w'' w'^2 z - \delta w'' u' z - \delta w' w' w'' z}^{0}\right) dV - \int_L \delta w f_z dx = 0$$

$$\int_L EA\left(\frac{1}{2}\delta w' w'^3 + \delta w' u' w'\right) dx + \int_L EI\left(\delta w'' w''\right) dx - \int_L \delta w f_z\, dx = 0$$

Per partes 1. člen

$$\begin{aligned}
\int_L EA\delta w'\left(\frac{1}{2}w'^3 + u'w'\right) dx &= \left[EA\delta w\left(\frac{1}{2}w'^3 + u'w'\right)\right]_0^L - \int_L \delta w\left(EA\left(\frac{1}{2}w'^3 + u'w'\right)\right)' dx \\
&= \left[EA\delta w\left(\frac{1}{2}w'^2 + u'\right) w'\right]_0^L - \int_L \delta w\left(EA\left(\frac{1}{2}w'^3 + u'w'\right)\right)' dx \\
&= \left[\delta w N w'\right]_0^L - \int_L \delta w\left(EA\left(\frac{1}{2}w'^3 + u'w'\right)\right)' dx
\end{aligned}$$

Podmínka stacionarity pro w(x,z)

$$\int_V E\left(\frac{1}{2}\delta w' w'^3 + \delta w' u' w' + \delta w'' w'' z^2 - \overbrace{\frac{1}{2}\delta w'' w'^2 z - \delta w'' u' z - \delta w' w' w'' z}^{0}\right) dV - \int_L \delta w f_z dx = 0$$

$$\int_L EA\left(\frac{1}{2}\delta w' w'^3 + \delta w' u' w'\right) dx + \int_L EI\left(\delta w'' w''\right) dx - \int_L \delta w f_z\, dx = 0$$

Per partes 2. člen

$$\begin{aligned}
\int_L \left(EI\delta w'' w''\right) dx &= \left[EI\delta w' w''\right]_0^L - \int_L \delta w' \left(EIw''\right)' dx \\
&= \left[EI\delta w' w''\right]_0^L - \left[\delta w \left(EIw''\right)'\right] + \int_L \delta w \left(EIw''\right)'' dx \\
&= \left[M\delta w'\right]_0^L - \left[Q\delta w\right]_0^L + \int_L \delta w \left(EIw''\right)'' dx
\end{aligned}$$

Podmínka stacionarity pro w(x,z)

$$\int_V E\left(\frac{1}{2}\delta w' w'^3 + \delta w' u' w' + \delta w'' w'' z^2 - \overbrace{\frac{1}{2}\delta w'' w'^2 z - \delta w'' u' z - \delta w' w' w'' z}^{0}\right) dV - \int_L \delta w f_z dx = 0$$

$$\int_L EA\left(\frac{1}{2}\delta w' w'^3 + \delta w' u' w'\right) dx + \int_L EI\left(\delta w'' w''\right) dx - \int_L \delta w f_z \, dx = 0$$

Výsledná rovnice

$$[\delta w N w']_0^L - \int_L \delta w\left(EA\left(\frac{1}{2}w'^3 + u'w'\right)\right)' dx + [M\delta w']_0^L - [Q\delta w]_0^L + \int_L \delta w\left(EIw''\right)'' dx = \int_L \delta w f_z \, dx$$

$$\boxed{(EIw'')'' - \left(EA\left(\frac{1}{2}w'^3 + u'w'\right)\right)' = f_z}$$

Podmínka stacionarity pro u(x,z)

$$\int_V E\left(\delta u' u' + \frac{1}{2}\delta u' w'^2 - \delta u' w'' z\right) dV - \int_L f_x \delta u\, dx - P\delta u(0) = 0$$

$$\int_L EA\delta u'\left(u' + \frac{1}{2}w'^2\right) dx - \int_L f_x \delta u\, dx - P\delta u(0) = 0$$

Per partes 1. člen

$$\begin{aligned}\int_L EA\delta u'\left(u' + \frac{1}{2}w'^2\right) dx &= \left[EA\delta u\left(u' + \frac{1}{2}w'^2\right)\right]_0^L - \int_L \delta u\left(EA\left(u' + \frac{1}{2}w'^2\right)\right)' dx \\ &= [\delta u N]_0^L - \int_L \delta u\left(EA\left(u' + \frac{1}{2}w'^2\right)\right)' dx\end{aligned}$$

Podmínka stacionarity pro u(x,z)

$$\int_V E\left(\delta u' u' + \frac{1}{2}\delta u' w'^2 - \delta u' w'' z\right) dV - \int_L f_x \delta u \, dx - P\delta u(0) = 0$$

$$\int_L EA\delta u' \left(u' + \frac{1}{2}w'^2\right) dx - \int_L f_x \delta u \, dx - P\delta u(0) = 0$$

Výsledná rovnice

$$[\delta u N]_0^L - EA\int_L \delta u(u'' + w'w'')dx = P\delta u(0) + \int_L f_x \delta u \, dx$$

$$\boxed{-\left(EA\left(u' + \frac{1}{2}w'^2\right)\right)' = f_x}$$

$$\boxed{\begin{aligned}(EIw'')'' - \left(EA\left(\frac{1}{2}w'^3 + u'w'\right)\right)' &= f_z \\ -\left(EA\left(u' + \frac{1}{2}w'^2\right)\right)' &= f_x\end{aligned}}$$

Vyjádření u' z druhé rovnice

$$EA\left(u' + \frac{1}{2}w'^2\right) = -P - \int_0^x f_x(s)ds$$

$$u' = \frac{-P - \int_0^x f_x(s)ds}{EA} - \frac{1}{2}w'^2$$

$$u' = \frac{P(x)}{EA} - \frac{1}{2}w'^2$$

$$\boxed{(EIw'')'' + f_x w' + \left(P + \int_0^x f_x(s)ds\right) w'' = f_z}$$

Podmínka minima pro w(x,z)

$$
\begin{aligned}
\delta^2 E_p(u,\delta^2 w,w) &= \int_L EA\left(\frac{3}{2}\delta w'^2 w'^2 + \delta w'^2 u'\right)dx + \int_L EI\delta w''^2 dx \\
&= \int_L EA\left(\frac{3}{2}\delta w'^2 w'^2 + \delta w'^2\left(\frac{-P}{EA} - \frac{1}{2}w'^2\right)\right)dx + \int_L EI\delta w''^2 dx \\
&= \int_L EA\delta w'^2 w'^2 dx - \int_L P\delta w'^2 dx + \int_L EI\delta w''^2 dx
\end{aligned}
$$

$$
P = \min_{\substack{\delta w \\ w}}\left(\frac{\int_L EI\delta w''^2 dx + \int_L EA\delta w'^2 w'^2 dx}{\int_L \delta w'^2 dx}\right)
$$

Podmínka minima pro u(x,z)

$$
\begin{aligned}
\delta^2 E_p(\delta^2 u, u, w) &= \int_V E\delta u'^2 dV \\
&= \int_L EA\delta u'^2 dx \\
&= EA \parallel \delta u' \parallel^2 \geq \frac{2EA}{L^2} \parallel \delta u \parallel^2 > 0 \quad pro\, EA > 0
\end{aligned}
$$

Uvážení smykového zkosení

$$E_p(u,w) = E_{int} + E_{ext} = \int_V \frac{1}{2} E \varepsilon_x^2 \, dV + \int_V G\gamma^2 \, dV + E_{ext}$$

Pole posunů

$$\begin{aligned}\tilde{u}(x,z) &= u(x) + \phi(x)z \\ \tilde{w}(x,z) &= w(x) \\ \tilde{\phi}(x,z) &= \phi(x)\end{aligned}$$

Normálová deformace

$$\varepsilon_x = E_{11} = u' + \phi' z + \frac{1}{2}\left((u' + \phi' z)^2 + w'^2\right)$$

Smyková deformace

$$\gamma = E_{13} = \frac{1}{2}\left(\phi + w' + (u' + \phi z)\phi\right)$$

Uvážení smykového zkosení

$$E_p(u,w) = E_{int} + E_{ext} = \int_V \frac{1}{2} E \varepsilon_x^2 \, dV + \int_V G\gamma^2 \, dV + E_{ext}$$

Pole posunů

$$\begin{aligned} \tilde{u}(x,z) &= u(x) + \phi(x)z \\ \tilde{w}(x,z) &= w(x) \\ \tilde{\phi}(x,z) &= \phi(x) \end{aligned}$$

Normálová deformace

$$\varepsilon_x = E_{11} = u' + \phi' z + \frac{1}{2}\left((u' + \phi' z)^2 + w'^2\right)$$

Smyková deformace

$$\gamma = E_{13} = \frac{1}{2}\left(\phi + w' + (u' + \phi z)\phi\right)$$

Výsledné rovnice

$$\begin{aligned}
(GA(\phi + w' + \phi u'))' + \left(EA\left(u'w' + \frac{1}{2}w'^3\right)\right)' &= -f_z \\
\big(GA(\phi^2 + \phi^2 u' + \phi w')\big)' + \left(EA\left(u' + \frac{1}{2}w'^2\right)\right)' &= -f_x \\
(EI\phi')' + \big(GI\phi^2\phi'\big)' &= 0
\end{aligned}$$

Děkuji za pozornost