Overview of Micro-Elasticity theories with emphasis on strain gradient elasticity: Part I – Theoretical considerations

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Abstract

In terms of the atomic lattice model, elasticity incorporates only the nearest neighbor interactions, and that's it. The theory does not have any intrinsic length scale and as a result a 2cm slab behaves like a 10μm film, and there is no difference among a microcrack and a geological fault. Moreover, classical elasticity implies that wave speed of plane shear and dilatational waves in an unbounded medium is independent of frequency, the same is predicted for Rayleigh waves, and surface SH waves are not predicted by elasticity. So the question is if one could develop an elasticity theory with intrinsic length scale or scales that can predict such phenomena. One way to include a length scale in an elasticity theory is by considering higher gradients of displacements (i.e. 2^{nd} , 3^{rd} and so forth). The fundamental idea of considering not only the first, but also the higher gradients of the displacement field in the expression for the strain energy function of an elastic solid, can be traced back to J. Bernoulli (1654-1705) and L. Euler (1707-1783) in connection with their work on beam theory. In elementary beam theory there are associated two sets of kinematical quantities (a deformation vector and a rotation vector) and two sets of surface loads (tractions and bending couples) with a section of the bar. In plate theory the situation is similar. Mindlin's pioneering work in 1964 on gradient linear elasticity is strongly influenced from structural mechanics. On the other hand Casal in 1961 was the first to see the connection between surface tension effects and the anisotropic second gradient theory. Mindlin following another path in 1965 was enforced to embark in third gradient of displacement and triple stresses to capture the surface energy property of new surfaces in solids. In this respect, for pedagogical reasons, we first present a technical beam bending theory with surface energy that contains two length scales. One length scale arises from Timoshenko's correction to account for shear strains and the other to account for surface effects. It is shown that the surface energy length scale of the theory is responsible for (i) a "stiffening" effect of the beam similar to that produced by the presence of initial tension, and (ii) a size effect exhibited by the flexural strength of beams, namely the dependence of the flexural strength on the inverse length of the beam for the same aspect ratio, which is similar to that considered by Griffith in his celebrated paper in 1921. Then, an overview of the various formats of general gradient elasticity theories is displayed focusing on Mindlin's second gradient theory that is based on the idea of the "unit cell" inspired from the concept of deformable directors proposed by Ericksen and Truesdell; if these directors become rigid then the theory degenerates to the Cosserat model. The latter is useful in the direction of development of an elasticity beam theory by generalizing the results of the technical theory presented previously. The principal difficulties indeed are to discover the practical significance of these generalized theories, to design experimental methods to explore their physical validity and to identify the length scale parameters. However, it is shown here that a Mindlin-Casal type strain gradient elasticity constitutes the simplest, in energy consistent, non-local extension of Hooke's law that is able to capture the surface energy and used to predict interesting phenomena like size and dispersion effects and cusping of crack lips. So, the simplified theory with 4 material constants i.e. 2 Lame and 2 additional length scales,

is subsequently used in Part 2 to attack some basic elasticity problems, albeit of technological importance, that are presented in Part 2 of these series of lectures.

Contents

1. Introductory notes

First of all I want to thank CTU and the Department of Mechanics-Faculty of Civil Engineering for the invitation of the series of the two lectures and particularly my colleague Prof. Milan Jirasek for encouraging me to re-visit work I have been concerned with in the past, and witty comments he made during my January 2017 visit to him that are considered here. These notes are not in their final form, but they were necessary for the introduction to the interested person into a very interesting subject of theoretical and applied Mechanics.

In the Mechanics of Materials we distinguish among continuum and discrete models. Such a distinction is of course an ancient one, which can be traced back to the philosophical controversy between atomists and stoics (see e.g. (Lloyd, 1970)). In our times this controversy still persists among those who believe that quantities, which enter a continuum description should be seen as 'averages' of some other underlying 'microscopical' properties of the material, and those who do not accept this point of view. For example Truesdell and Noll in the introduction of *Non-Linear Field Theories of Mechanics* (Truesdell and Noll, 1965, sect.3), state explicitly that: "..Widespread is the misconception that those who formulate continuum theories believe matter 'really is' continuous, denying the existence of molecules. This is not so. Continuum physics presumes *nothing* regarding the structure of matter. It confines itself to relations among gross phenomena, neglecting the structure of the material on a smaller scale. Whether the continuum approach is justified, in any particular case, is a matter, not for the philosophy or methodology of science, but for the experimental test...".

Note: In terms of lattice theory, elasticity incorporates only the nearest neighbor interactions, and that's it. It is well known that classical continua consist of points having three translational dofs that is displacement in three directions u_x, u_y, u_z in a fixed Cartesian coordinate system Oxyz. The material response due to action of external loads is described by a symmetric stress tensor σ_{ij} and the transmission of loads is uniquely determined by a force vector, neglecting couples. The theory does not have any intrinsic length scale and as a result a 2cm slab behaves like a 10μm film, and there is no difference among a microcrack and a geological fault. Moreover, classical elasticity implies that wave speed of plane shear and dilatational waves in an unbounded medium is independent of frequency, and surface SH waves are not predicted by elasticity.

Real materials often have a number of important length scales, which should be included in a realistic model such as molecules in polymers, crystals in a metamorphic rock, grains and particles, fibers, cellular structures, building blocks in rock masses or in architectural monuments like Parthenon temple etc. A length scale is included only if the theory is extended to consider 2^{nd}, 3^{rd} and so forth neighbor interactions and the theory is then "non-local" in contrast to elasticity which is a "local" theory. The continuum version of the nth neighbor interaction lattice theory is the so-called grade-n theory. Surface tension phenomena are also not captured by elasticity. In conclusion, due to limited amount of physics that was built in elasticity the micromechanical phenomena we are so eager to understand are not derivable from this theory.

In the study of cracks Griffith's paper (Griffith, 1921) was the initiation of the independent discipline of Fracture Mechanics. Griffith started with the study of the size effect exhibited by the tensile strength of glass fibers as is depicted in Fig. 1.1. Griffith fitted on the test data the following empirical law

$$\sigma_t = 154.44 + \frac{17.27}{d} \tag{1.1}$$

This law is the same with that proposed by Karmarsch (1858) based on a best-fitting procedure of experimental data of tension tests on cylindrical metal wires with different diameters "*Mittheilungen des gew. Ver. Für Hannover*", 1858, pp. 138-155. From the engineering standpoint the chief interest of this size effect is to apply it for the production of high-strength composites i.e. glass and graphite fiber composites with very small diameter approaching ultra-high strengths bonded together with resin. Of equal importance in terms of safety of large structures is that the standard methods of strength estimation in the lab may lead in some cases to serious error.

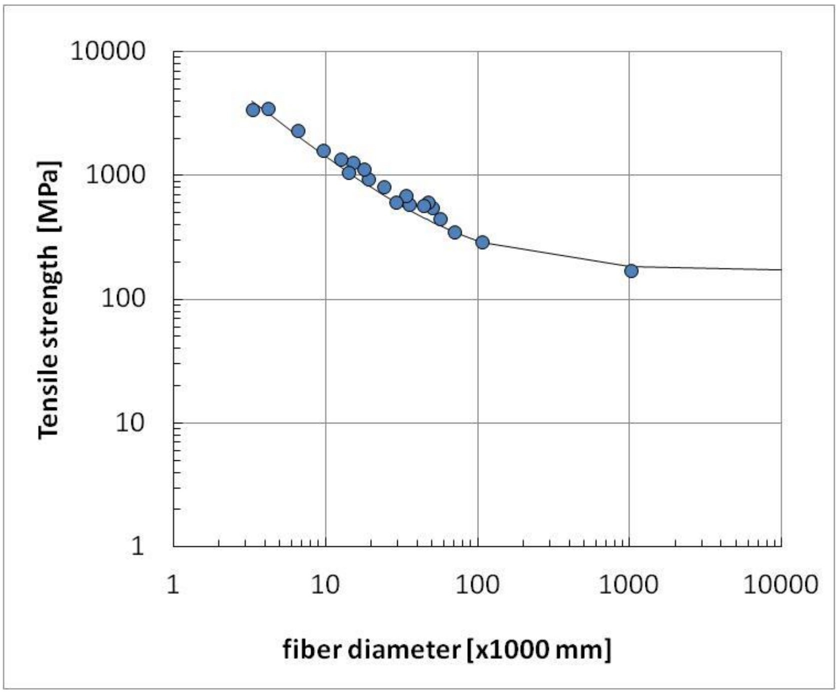


Fig. 1.1. Size effect exhibited by the tensile strength of glass fibers (circles) tested by Griffith (1921) and best-fitted inverse diameter relationship (continuous line).

Griffith, extended into theory by him by postulating an energy balance law considering the surface energy (or tension) 2γ of the crack in a plate under fixed-grip conditions, that is

$$-\frac{\partial U}{\partial a} = 2\gamma \tag{1.2}$$

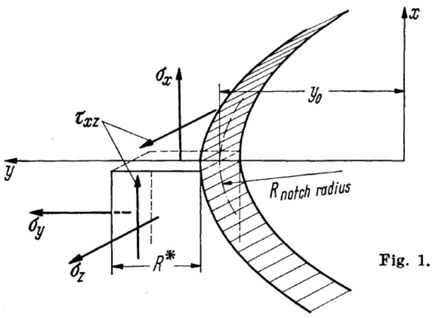
But as pointed out by Goodier (1968) in due course of calculating the strain energy in the crack plate Griffith neglected the stresses due to surface tension that are implied by the existence of surface energy on the crack surfaces. If surface energy is specified, the boundary value problem to be solved is not that of Inglis (1913), but rather one that would include a normal traction γ / r where r is the radius of curvature of the stress-free crack. Furthermore, because the specification of an energy balance then becomes redundant, that is it is guaranteed by the solution of a properly posed boundary value problem, some alternative criterion for fracture must be sought.

The presentation is organized as follows: In Section 2 a short historical overview of micro-mechanical theories is presented. Reference is made in Casal's 1D elastic bar model, 2D Cosserat theory and the extension of Timoshenko's beam theory. Section 3 is devoted to the presentation of Mindlin's gradient elasticity theory and his second gradient of strain elasticity theory. The latter includes a surface energy term in the potential energy density expression. Section 4 refers to the presentation of a Mindlin-Casal type first strain gradient elasticity theory which constitutes the simplest, in energy consistent, non-local extension of Hooke's law that is able to capture the surface energy phenomena.

2. Overview of simple micro-mechanical theories

2.1 Introductory notes

One of the researchers who empirically considered an intrinsic length scale based on the mean value of the nominal stress along the potential fracture path **and introduced a non-local integral-type theory** was Neuber (1936); this contribution is mentioned by Tanaka & Mura (1981) of fatigue crack initiation from notches. In fact Neuber used this modification of classical strength theory to better interpret metal fatigue experimental evidences.



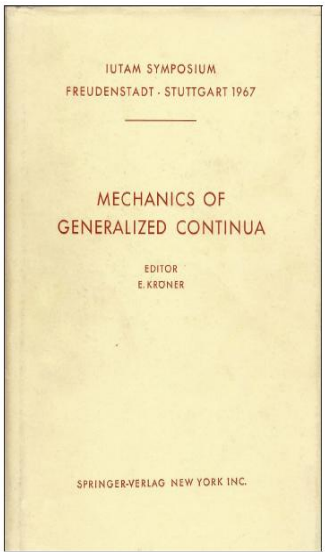


Fig. 2.1. Sketch taken from Neuber's paper presented in IUTAM 1967 conference (Kroener, 1968).

In his “stress-mean-value” theory he postulated that the strength of a material in the neighborhood of a notch (r = distance from point O of the notch tip, and Oz-axis is normal to the plane of the notch as is shown in Fig. 2.1) should be compared with the nominal stress σ_n whereas the kernel of the integral represents the local stress that is given as usual with Hooke’s law

$$\sigma_n = \frac{1}{R^*} \int_{r=0}^{R^*} \sigma_{zz}(r,0)dr$$

The length scale R^* appearing in this empirical model is an intrinsic length scale of the material. Similarly to Neuber, Eringen’s work (Eringen, 1983) is probably best known for advocating the use of **an integral-type spatial nonlocality**, where volume averages of state variables are computed. The nonlocal stress tensor, is defined as follows

$$\sigma_{ij}^g(\mathbf{x}) = \iiint_{\mathbf{V}} \alpha(\mathbf{x} - \mathbf{s})\sigma_{ij}^c(\mathbf{s})dV \qquad (2.1)$$

where α is a nonlocal weight function that is non-negative and attenuating for increasing values of the argument it encloses, and for isotropic elastic solid

$$\sigma_{ij}^c = \lambda\varepsilon_{kk}\delta_{ij} + 2G\varepsilon_{ij}$$

The meaning of Eq. (2.1) is that the nonlocal stress at point **x** is the weighted average of the local stress of all points in the neighborhood of **x**, the size of which is set via the definition of a(**s**). Both non-local and local stresses obey the stress equilibrium eqns, i.e. (in the absence of body forces)

$$\sigma_{ij,j}^g = 0,$$
$$\sigma_{ij,j}^c = 0$$

2.2 1D strain gradient models

We first may recall that the average value of an one-dimensional field *y=f(x)* (say an 1D stress field) in a representative length *L* is

$$\langle y \rangle = \frac{1}{L} \int_{-L/2}^{L/2} f(x+\xi)d\xi$$

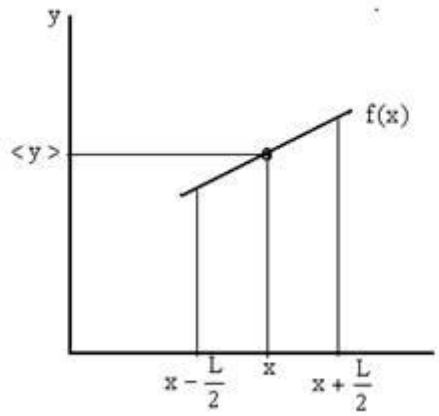


Fig. 2.2. Linearly varying stress field around material point x.

By expanding *f(x+ξ)* into Taylor series and substituting the result in the integral and evaluating we find that the local value of y at the mid-point of the line of length *L* is

$$y = \langle y \rangle - \frac{L^2}{24} \frac{d^2 y}{dx^2}\Big|_x + O(L^4)$$

For uniform or linearly varying fields it is seen that the mid-point value coincides with the average over the whole length *L*. However for quadratically varying fields, i.e. stress distribution close to the tip of notch or crack, the second gradient of *y=f(x)* should be also taken into account. Based on the above observation and by induction we may generalize the result for the stress tensor in the 3D space. That is to say, we can easily show that for cubic REV with dimension **a**, the following approximation formula holds for the average stress,

$$\langle \sigma_{ij} \rangle_{REV} = \left(1 + \frac{a^2}{24}\nabla^2\right)\sigma_{ij}(x_k) \tag{2.2}$$

It is also legitimate to assume that Hooke's law does not hold for the local equilibrium stress but rather for an average stress that is defined over a Representative Elementary Volume (REV)

$$< \sigma_{ij} >_{REV} = \int_{(REV)} \sigma_{ij} dV = \lambda \varepsilon_{kk} \delta_{ij} + 2G\varepsilon_{ij} \tag{2.3}$$

By inverting the differential operator in Eq. (2.2) and substituting the average stress with the strains (Eq. (2.3)) we get the following expression of the mid-point stress tensor

$$\sigma_{ij} = \frac{1}{1 + \frac{a^2}{24}\nabla^2} \langle \sigma_{ij} \rangle_{REV} \approx \left(1 - \frac{a^2}{24}\nabla^2\right)\left(\lambda \varepsilon_{kk} \delta_{ij} + 2G\varepsilon_{ij}\right) \tag{2.4}$$

It may be shown that the above constitutive relation is a special form of gradient elasticity theory proposed by Mindlin (1964). Based on an appropriate and simple strain energy expression of the restricted form of this theory, the constitutive relations linking stresses with strains have the following form

$$\sigma_{ij} = \left(1-\ell^2\nabla^2\right)\left[\lambda\delta_{ij}\varepsilon_{kk} + 2G\varepsilon_{ij}\right] \tag{2.5}$$

This is a Laplacian-type strain gradient elasticity or Laplacian modification of Hooke's law with an intrinsic length scale proposed by Aifantis (Altan and Aifantis. 1992; Ru and Aifantis, 1993) to be used for the solution of static and dynamic problems in the frame of linear elasticity. The gradient length scale parameter ℓ may be associated with lattice characteristic dimension or some characteristic microstructural length scale of a solid, like mean grain or crystal size etc. In this case it may be shown that the theory leads to the prediction of size effects in static loading tests, or to dispersion in wave propagation problems.

The above averaging considerations find many applications in continuum mechanics. Typical example of a locally homogeneous one-dimensional continuum is the tension/compression bar. In this case we assume that the displacement of a normal section to the axis of the bar is adequately described by a displacement field $u = u(x)$, which, in a good approximation, is a linear function of position,

$$u \approx u_0 + \varepsilon x$$

In this case we call the deformation ***homogeneous***, and we describe it satisfactorily by the gradient of displacement.that we call the **strain**

$$\varepsilon = \nabla u = \left.\frac{du}{dx}\right|_x$$

In such a test it is sufficient to measure the relative displacement of two distant points of the bar along its axis and to interpolate linearly between them. From this measurement we can compute the axial elongation (or shortening) of the bar per unit length, , which coincides with the average and the local strain

$$\frac{u(x+L/2)-u(x-L/2)}{L} = <\varepsilon> = \varepsilon$$

The corresponding engineering model for tension/compression of long prismatic bars is classified as a **simple theory for an one-dimensional continuum**, which connects, through a constitutive relation, the axial force to the axial elongation,

$$N = N(\varepsilon)$$

Or in other words the axial force to the **first gradient of the deformation**,

$$N = N(\nabla u)$$

Accordingly the engineering theory of simple tension/compression of long prismatic bars is a **1st gradient continuum theory**.

On the contrary, in the frame of the **engineering beam-bending theory**, we get that a locally linear approximation of the deflection w(x) is not sufficient. In the case of pure bending we assume for example that

$$w \approx w_0 - \psi x - \frac{1}{2}\kappa x^2$$

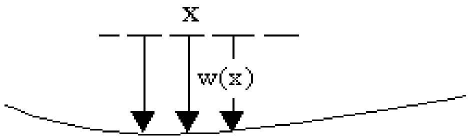


Fig. 2.3. Quadratically varying deformation around point x in a beam.

In this case the deformation cannot be assumed as being locally homogeneous. For the estimation of the curvature we need at least three measurements (at x and) and a parabolic interpolation of the data.

Accordingly the engineering bending theory of **long prismatic beams** is described by a constitutive relation, which connects the bending moment M to the curvature of the deflection of the beam,

$$M = M(\kappa) \Leftrightarrow M = M(\nabla^2 w)$$

where the bending curvature is

$$\kappa = -\nabla^2 w = -\left.\frac{d^2 w}{dx^2}\right|_x$$

This means that the engineering beam-bending theory is **a 2nd gradient continuum theory**.

Other well-known example of 2nd gradient continuum theory is the theory of capillary waves in ideal fluids with *surface tension*. In this case, through a simple membrane model of the free surface of the fluid, only the boundary condition on the free surface is modified, so as to include the effect of curvature.

1. the problem of surface tension T in fluids. For example in the study of *capillary surface waves in fluids* the information concerning the characteristic length of the problem, $\ell' = 2\pi\sqrt{T/(\rho g)}$, enters only through the dynamic boundary condition, that assigns on the surface of the fluid a fictitious pressure that is proportional to the mean curvature of that surface[1], $\Delta p = -T\,\nabla^2 \zeta(x,y)$; see Fig. 2.4.

[1] cf. L. A. Segel, *Mathematics Applied to Continuum Mechanics*, Dover, 1977.

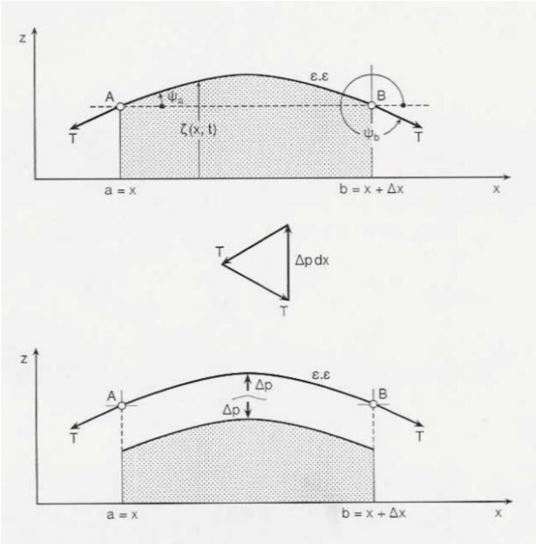


Fig. 2.4

2.3 Casal's elastic model for the rod with surface energy

Professor Germain has encouraged the communication o the French Academy of Sciences of the ideas of Casal (1961), which in turn seem to have inspired Germain's (1973a; 1973b) fundamental papers on the continuum mechanics structure of the grade-2 or higher grade theories. In our paper, we want to give full credit to Casal's original idea, who was first to see the connection between surface tension effects and the anisotropic grade-2 elasticity theory. Casal has extended the classical 1D Hookean definition of linear elastic solids by introducing additional terms of second order in displacements in the strain energy density expression (grade-2 or g2 theory) based on linear capillarity theory of liquids. Because this theory is anisotropic, it is possible to take into account surface free energy by multiplying a director (first-order tensor) with the strain-gradient (third-order tensor) in the expression for the strain energy density function. Accordingly, two material constants ℓ, ℓ' having the dimension of length, were introduced by Casal to characterize the internal and surface capillarity of the solid.

For demonstration purposes we consider first the 1D example of a tension bar of length $\boldsymbol{L}$ in a clamped end- free end configuration with the load acting on its free end along the x-axis. In the uniaxial case the strain energy of the tension bar was presented by Casal as follows

$$W = \frac{1}{2}\int_0^L E\left(\varepsilon^2 + \ell^2 \nabla\varepsilon\nabla\varepsilon\right)dx + \frac{1}{2}\left[E\ell'\varepsilon^2\right]_0^L \tag{2.6}$$

Casal's elastic strain energy ansatz (2.6) consists of two terms : (a) a "volume energy" term which includes the contribution of the strain gradient, and (b) a "surface energy" term. Accordingly, ℓ and ℓ'' are material lengths related to volume and surface elastic strain energy, respectively. Casal's expression for the global elastic strain energy of the tension bar was recovered by introducing an appropriate anisotropic, linear elastic, restricted Mindlin continuum. Since in a restricted continuum the relative deformation γ vanishes, the variation of the strain energy density becomes

$$\delta w = \tau\delta\varepsilon + \mu\delta\nabla\varepsilon \tag{2.7}$$

In this expression τ (Cauchy term) works on the strain, whereas μ (double stress) works on the strain gradient. In connection to the variation (2.7), Casal's model is equivalent with the following constitutive assumptions for the Cauchy- and double-stress

$$\begin{aligned} \tau &= \frac{\partial w}{\partial \varepsilon} = E\left(\varepsilon + \ell'\nabla\varepsilon\right) \\ \mu &= \frac{\partial w}{\partial \nabla\varepsilon} = E\left(\ell'\varepsilon + \ell^2\nabla\varepsilon\right) \end{aligned} \tag{2.8}$$

From eqns (2.6)-(2.8) the elastic strain density is derived, resulting in the following expression

$$\begin{aligned} \delta w &= \tau\delta\varepsilon + \mu\delta\varepsilon' = \frac{\partial w}{\partial \varepsilon}\delta\varepsilon + \frac{\partial w}{\partial \varepsilon'}\delta\varepsilon' = \\ &= E\left(\varepsilon + \ell'\varepsilon'\right)\delta\varepsilon + E\left(\ell'\varepsilon + \ell^2\varepsilon'\right)\delta\varepsilon' = \\ &= E(\varepsilon\delta\varepsilon + \ell^2\varepsilon'\delta\varepsilon') + E(\ell'\varepsilon\delta\varepsilon' + \ell'\varepsilon'\delta\varepsilon) = \\ &= \frac{1}{2}E\left\{\delta\left(\varepsilon^2\right) + \ell^2\delta(\varepsilon'^2)\right\} + \frac{1}{2}E\ell'\delta\left(\nabla\varepsilon^2\right) \end{aligned}$$

Hence the potential energy is recovered in the following manner

$$w = \frac{1}{2}E\left(\varepsilon^2 + \ell^2\nabla\varepsilon\nabla\varepsilon + \ell'\nabla\varepsilon^2\right) \tag{2.9}$$

It turns out that for positive strain energy density (w > 0) in the considered 1D case, the material lengths are restricted, such that

$$-1 < \frac{\ell'}{\ell} < 1$$

This means that if surface energy terms are included, then volume strain-gradient terms must also be included. It is worth noting that in Griffith's (1921) original theory of cracks only surface energy is considered, which is of course inadmissible in the sense of inequality (A.7); however, as was already mentioned above Griffith considered surface energy in an ad hoc manner.

2.4 The Cosserat continuum

First recall basics from tensors notation, for example Table 2.1.

Table 2.1. Indicial notation for tensors

- Tensors are indicated using indices $i, j, k, \ldots$

$$x_i,\ u_i,\ \tau_{ij},\ \kappa_{ijk},\ \ldots$$

- Indices from 1 to 3

$$x_i = \{x_1, x_2, x_3\}$$

- Repeated indices indicate summation

$$\tau_{ii} = \tau_{11} + \tau_{22} + \tau_{33}$$

- A comma indicates spatial differentiation

$$u_{i,j} = \partial u_i / \partial x_j$$

- Kronecker delta

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

- Permutation symbol

$$e_{ijk} = \begin{cases} 1 & 123,231,312 \\ -1 & 321,213,132 \\ 0 & \text{otherwise} \end{cases}$$

In natural or man-made constructions involving blocky rock structures like these illustrated in Fig. 2.5, the relative rotations between the building blocks could be equally important with the deformation of the structure or sliding between blocks.

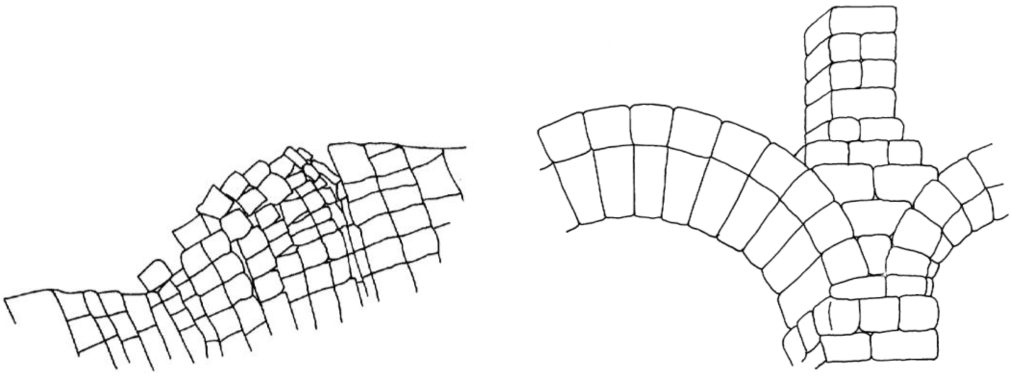


Fig. 2.5. Examples of natural and man-made blocky rock structures (sketches taken from the textbook of Goodman and Shi [2] on key block theory)

The study of such structures can be performed by a model of the so called oriented medium with rigid or deformable directors. The idea of couple stresses was originally introduced by Voigt and others, but the Cosserat brothers in 1909 (Cosserat E. and F., 1909) gave the first systematic treatment. They removed the connection between the rotation field and the displacement gradients and introduced an intrinsic length scale. This further generalization implies the introduction of couple stresses. A Cosserat continuum is equipped with the following ingredients:

Kinematics: Continuum of oriented ***rigid particles***, equipped with three perpendicular unit vectors called *trièdres rigides* (or rigid crosses) which Ericksen and Truesdell (1958) called "directors" of an "oriented merium". Each mtl point has with 6 d.o.f. i.e. 3 displacements u_i *and* 3 rotations (or micro-rotations) ω_i^c (different from the **rotational (anti-symmetric) part** ω_i **of displacement gradient**) (e.g. (Mindlin, 1964)).

[2] Goodman R.E. and Shi G. (1985), Block theory and its application to rock engineering, Prentice-Hall Inc.

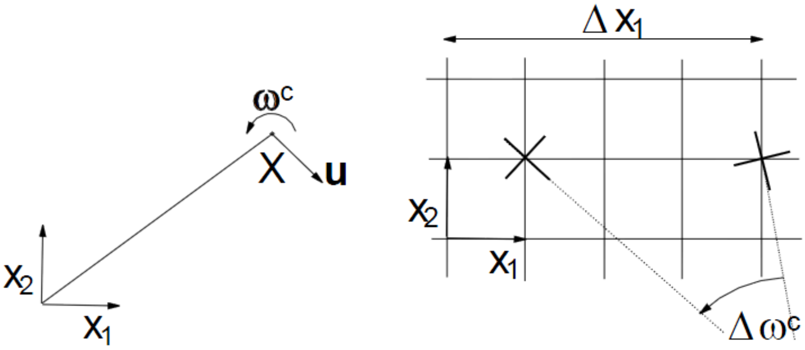


Fig. 2.6. Cosserat-continuum kinematics in 2-d; (a) Displacement u_i and rotation ω^c of a rigid cross (b) Relative rotation $\Delta\omega^c$ of two neighboring rigid crosses (curvature)

The deformation state is described by the *relative strain* γ_{ij} and the *Cosserat rotation gradient* κ_{ij} defined as

$$\begin{aligned} \gamma_{ij} &= u_{i,j} - e_{3ij}\omega^c, \\ \kappa_i &= \omega^c_{,i} \end{aligned} \tag{2.10}$$

The meaning of the relative strain may be realized if we split it into a symmetric and anti-symmetric part

$$\gamma_{ij} = \gamma_{(ij)} + \gamma_{[ij]}$$

$$\gamma_{(ij)} = \varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad , \quad \gamma_{(ij)} = \gamma_{(ji)} \tag{2.11}$$

$$\gamma_{[ij]} = (\omega_{ij} - \omega^c_{ij}), \quad \omega^c_{ij} = -e_{3ij}\omega^c$$

where

- ε_{qr} is defined as the usual symmetric infinitesimal macro-strain tensor defined in terms of the displacement vector u_q, $\partial_s \equiv \partial / \partial x_s$, the indices (q,r,s) span the range (1,2,3),

- $$\omega_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i}) \tag{2.12}$$

So, the *relative strain* γ_{ij} can be expressed through the usual macro-strain ε_{ij} and macro-rotation ω_{ij} *i*

$$\gamma_{ij} = \varepsilon_{ij} + (\omega_{ij} - \omega^c_{ij}) \tag{2.13}$$

The quantity $(\omega_{ij} - \omega^c_{ij})$ or "relative spin" in Eq. (2.13) represents the relative rotation of a material point with respect to the rotation of its neighborhood. If it is null it reduces to the usual strain ε_{ij}.

Statics: 9 force stresses σ_{ij} (*force per unit area*) associated with the displacements, and 9 **couple stresses** μ_{ij} (*torques per unit area*) associated with the rotations. In contrast to elasticity theory the stress tensor is not-symmetric.

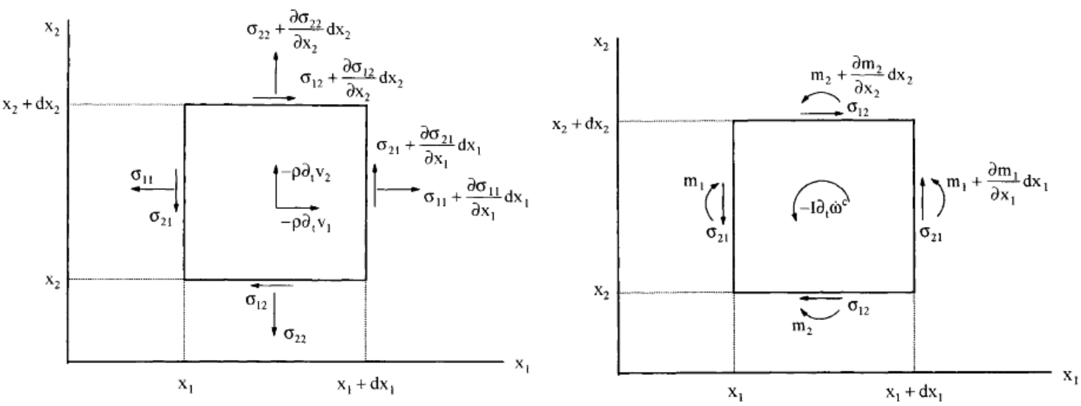
Based on the *virtual work principle* the following force and moment equilibrium conditions along Ox_1- and Ox_2-directions for a volume element V of the Cosserat medium in the static limit (Fig. 3**)** are derived first for the stresses

$$\begin{aligned} &\sigma_{ij,j} + f_i = 0, \\ &\mu_{ij,j} + e_{ijk}\sigma_{jk} = 0 \; in \, V \end{aligned} \qquad (2.14)$$

Equivalently,

$$\begin{aligned} &\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_2} = 0, \\ &\frac{\partial \tau_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} = \gamma, \\ &\frac{\partial \mu_{13}}{\partial x_1} + \frac{\partial \mu_{23}}{\partial x_2} + \tau_{12} - \tau_{21} = 0 \end{aligned} \qquad (2.15)$$

where , γ denotes the unit weight of the elastic medium and the symbol τ is sometimes here used to denote shear stress.



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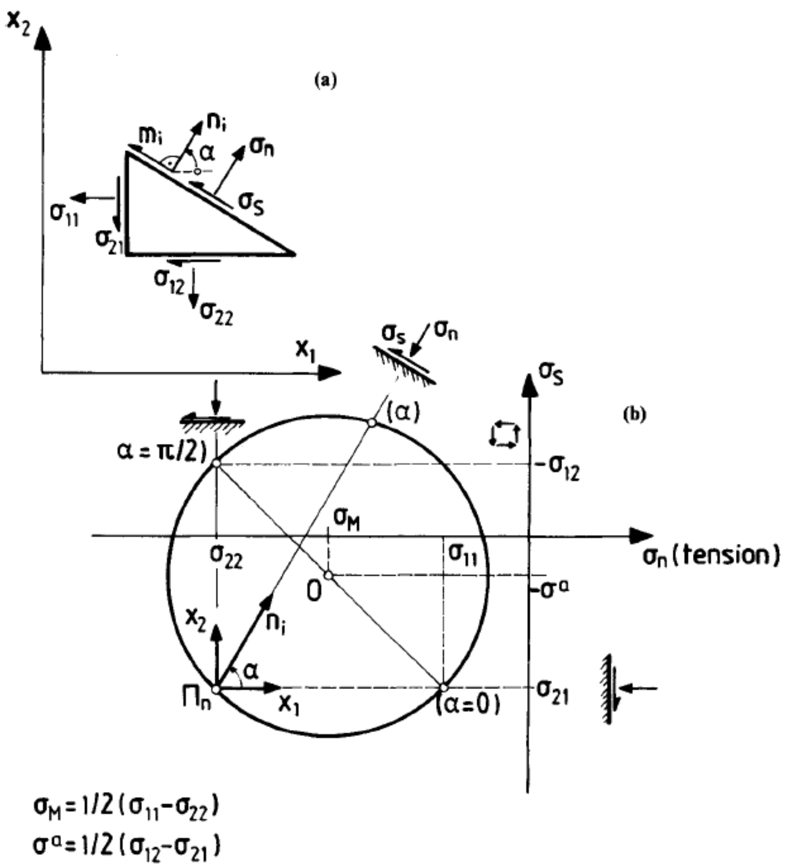


(b)

Fig. 2.7. (α) Stress and couple stress (μ_{i3}=m_i), *in the sense of Cauchy in 2D. (b) Statics of a 2D Cosserat asymmetric stress element represented by Mohr's circle (Vardoulakis and Sulem, 1997).*

The center M^{II} of the Mohr circle is shifted normal to the $O\sigma$ -axis by the amount of the asymmetry of the stress tensor, given by τ_a . From Fig. 4 we can also compute the angular displacement of the center M of the Mohr circle, expressed by the stress obliquity of the antisymmetric part of the stress tensor,

$$\tan\phi_\alpha = \frac{\tau_a}{p} = \frac{(\tau_{12} - \tau_{21})}{(\sigma_{11} + \sigma_{22})}$$

wherein $p = (\sigma_{xx} + \sigma_{zz})/2$ is the in-plane invariant mean normal stress and $\tau_a = (\tau_{xz} - \tau_{zx})/2$ is also an invariant and measure of the stress-tensor asymmetry.

Constitutive relations (in simplest isotropic Cosserat elasticity): The variation of potential energy of the Cosserat medium – following the terminology of Mindlin - is defined as

$$\delta w = \sigma_{(ij)}\delta\varepsilon_{ij} + \sigma_{[ij]}\delta\left(\omega_{ij} - \omega_{ij}^c\right) + m_i\delta\kappa_i$$

where

$\sigma_{(ij)}$ =Cauchy stress

$\sigma_{[ij]}$ =relative stress since it works on relative rotation.

The 2 elastic constants K, G of classical elasticity, the Cosserat shear modulus G^c linking the anti-symmetric shear stresses with the anti-symmetric part of the relative deformation and one additional elastic constant *with dimensions of* length (square root of the bending modulus with dimensions of force to the shear modulus) that is linked to the microstructure and may be identified experimentally by the method of size effects for a particular problem.

$$\begin{aligned}
&\sigma_{11} = (K+G)\varepsilon_{11} + (K-G)\varepsilon_{22} \\
&\sigma_{22} = (K-G)\varepsilon_{11} + (K+G)\varepsilon_{22} \\
&\sigma_{(12)} = \sigma_{(21)} = 2G\varepsilon_{12} \\
&\sigma_{[12]} = -\sigma_{[21]} = -2G^c\left(\omega - \omega^c\right) \\
&m_1 = M\kappa_1,\, m_2 = M\kappa_2
\end{aligned} \tag{2.16}$$

All theories in which the **stress tensor is not symmetric** are references as “polar-continua” or “micro-polar continua”. Tthe basic kinematic and static concepts of the “Cosserat” continuum that contains an intrinsic length scale were reworked in a milestone paper by late Professor Guenther (Guenther,1958) in Braunschweig. Günther's paper marks the rebirth of the so-called “continuum micro-mechanics” in the late 50’s and early 60's. Following this publication, several hundred papers were published all over the world on that subject. A variety of names have been invented and given to theories of various degrees of rigor and complexity: Cosserat continua or micro-polar media, oriented media, continuum theories with directors, multi-polar continua, micro-structured or micro-morphic continua, non-local media and others (Hermann, 1972). The state-of-the-art at this time was reflected in the collection of papers presented at the historical IUTAM Symposium on the "*Mechanics of Generalized Continua*", in Freudenstadt and Stuttgart in 1967 (Kroener, 1968). On the subject of Cosserat Elasticity recommendable for their clarity and didactical value are the papers by H. Schaeffer (1962, 1967) in German and of Koiter (1964) in English. Notable and of equal importance in relation to Gradient Elasticity is the milestone paper by Mindlin (1964) and two papers by Germain (1973a, b), the latter written partially in French and partially in English.

Table 2.2 presents the relations among the quantities involved in 3 theories, namely Cosserat, beam bending and form III of Mindlin’s theory that is a special case of the general theory of elasticity with deformable unit cells or deformable directors.

Table 2.2. Relations among the kinematic and static quantities of 3 theories.

2D Cosserat theory		Timoshenko's beam theory		Mindlin's theory	
Relative deformation (difference between macro- and micro-deformation)	$\gamma_{ij} = u_{i,j} - e_{3ij}\omega^c$	Shear strain γ	$\nabla w + \psi$	Relative deformation	$\gamma_{qr} \equiv \partial_q u_r - \psi_{qr}$
Rotation	ω^c	rotation	ψ	Anti-symmetric part of the micro-deformation of a particle	$\psi_{[qr]}$
rotation gradient or "curvature of deformation"	$\kappa_i = \omega^c_{,i}$	bending curvature	$\kappa = \psi,_x$	Micro-rotation gradient, curl of the strain	$\kappa_{i[jk]}$
"Balance" Stress tensor	σ_{ij}	*Normal stress*	σ	*"Balance" Stress tensor*	σ_{ij}
Couple stress tensor	$m_i = \mu_{3i}$	Bending moment	M	Couple stress tensor (anti-symmetric part of the double stress tensor)	$\mu_{i[jk]}$

2.5 Some remarks on technical beam theory

The fundamental idea of considering not only the first, but also the higher gradients of the displacement field in the expression for the strain energy function of an elastic solid, can be traced back to J. Bernoulli (1654-1705) and L. Euler (1707-1783) in connection with their work on beam theory.

Note:

- In elementary beam theory there are associated two sets of kinematical quantities (a deformation vector and a rotation vector) and two sets of surface loads (tractions and bending couples) with a section of the bar. In plate theory the situation is similar.
- As is shown in Fig. 2.8 in classical beam-bending theory *the microelement* is the cross section of the beam with 2 d.o.f., vertical displacement plus rotation.
- It is obvious that Mindlin's [20] pioneering work on gradient linear elasticity is strongly influenced from structural mechanics.
- It may be also shown that the Timoshenko beam is merely a 1-d Cosserat medium.

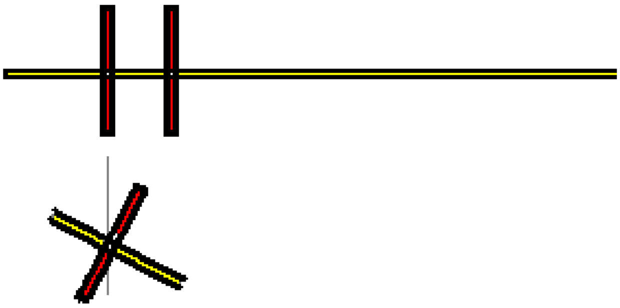


Fig. 2.8. Rotation of the cross-section of the beam. Cross-section and neutral axis form the rigid cross in Cosserat theory.

To fix ideas we consider a simply supported beam of rectangular cross-section under the action of a vertical load P at its upper surface and at the mid-span. The longitudinal section of the beam is referred to a Cartesian coordinate system O(x,y,z) positioned on the neutral axis – which is the locus of centroids of cross-sections - with its origin at mid-span and with the Ox-axis directed along the neutral axis of the beam while Oz-axis extending vertically downwards, as it is illustrated in Fig. 2.9. Further, deformation quantities are assumed as infinitesimal, and the corresponding displacements of points in a cross-section along Ox and Oz directions are denoted by the symbols u, w respectively. Let the infinitesimal normal strains $\varepsilon_{xx}, \varepsilon_{zz}$ and the engineering shear strain γ_{xz} in the plane xOz to be defined as follows

$$\varepsilon_{xx} = \varepsilon = u,_x , \quad \varepsilon_{zz} = w,_z , \quad \gamma_{xz} = \gamma = u,_z + w,_x \tag{2.17}$$

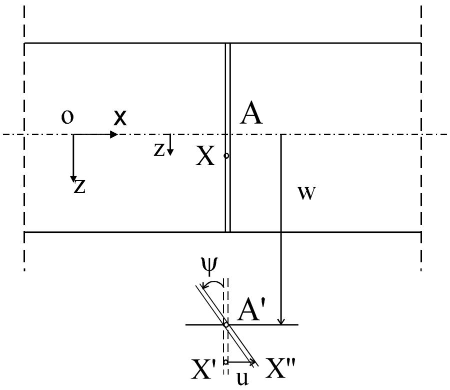


Fig. 2.9. Horizontal and vertical displacements u,w, respectively, and rotation ψ .

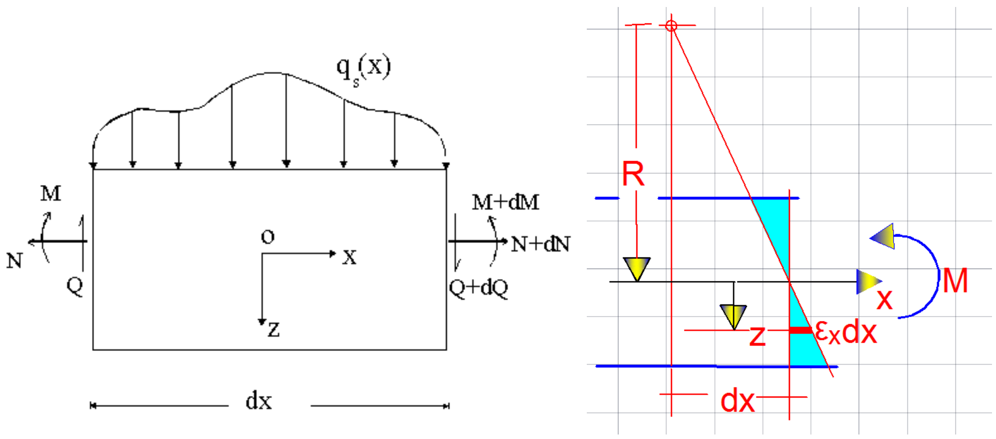


Fig. 2.10 Forces and moment, (b) Linear distribution of the horizontal strain along the height of the beam and definition of the radius of curvature.

For the achievement of a unique displacement solution, the Timoshenko extension of ***Bernoulli-Euler*** (B-E) theory rests on the following two kinematical assumptions:
Assumption #1: It is assumed that planar vertical cross-sections of the beam in the undeformed state, such as section A in Fig. 2.9, remain planar after loading (i.e. A' in Fig. 2.9). Consequently, as is displayed in Fig. 2.10, the horizontal displacement u along Ox-axis at a given vertical section of the beam should be a linear function of the vertical coordinate z, i.e.

$$u = \psi(x)z \Leftrightarrow \varepsilon = \psi,_x z \tag{2.18}$$

where ψ denotes the rotation (considered to be a small quantity) of the cross-section A of the beam at position x as is displayed in Fig. 2.9. Also, with the symbol κ we denote the gradient of the rotation angle (bending curvature) of the cross-section, that is

$$\kappa \equiv \psi,_x \,, \tag{2.19}$$

Substituting Eq. (2.19) into (2.18) we take the following expression for the horizontal strain

$$\varepsilon = \kappa z \tag{2.20}$$

The bending curvature $\kappa = 1/R$ in Fig. 2.10 is found as $\partial\varepsilon/\partial z = \psi_{,x}$.
Assumption #2: Every point lying at the vertical section x = *(ct)* is subjected to the same vertical displacement, that is to say

$$w = w(x) \tag{2.21}$$

This assumption means that the height of a given cross-section of the beam remains constant during deformation i.e. $\varepsilon_{zz} = 0$.

B-E assumption that vertical cross-sections perpendicular to the neutral beam axis remain orthogonal with the axis during the bending of the beam is relaxed in Timoshenko's theory (Fig. 2.11)

$$\gamma = \psi + \frac{dw}{dx} \neq 0 \tag{2.22}$$

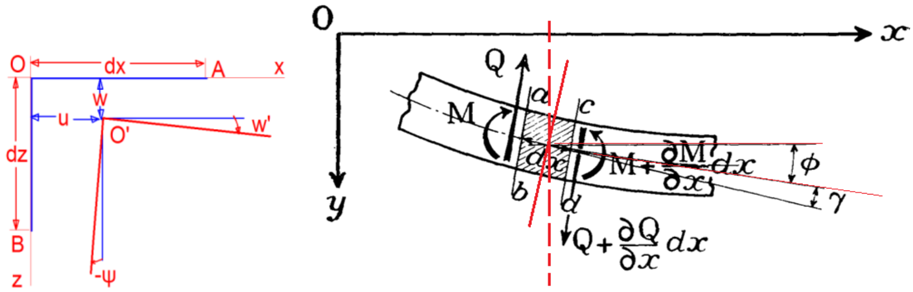


Fig. 2.11. Timoshenko beam with $\psi \neq -w,_x$.

Next the static equilibrium conditions of the beam are found by recourse to the *Principle of Virtual Work*. Define

$$Q = \frac{\partial \upsilon_b}{\partial \gamma}, \quad M = \frac{\partial \upsilon_b}{\partial \kappa} \tag{2.23}$$

wherein υ_b denotes the internal strain energy density of the beam.

The variation of the total strain energy δU_b along the length L of the beam with the absence of axial load ($N = 0$), takes the form

$$\delta U_b = 2\int_0^{L/2} \delta\upsilon_b dx = 2\int_0^{L/2} [Q\delta\gamma + M\delta\kappa]dx \Rightarrow \delta\upsilon_b = Q\delta\gamma + M\delta\kappa \tag{2.24}$$

The form of Eq. (2.24) is the motivation for the adoption of the following form for the variation of work δU_e done by external forces

$$\delta U_e = 2\int_0^{L/2} [q_s \delta w]dx + 2[Q\delta w + M\delta\psi]_{x=0}^{x=L/2} \tag{2.25}$$

where the factor 2 has been inserted because due to symmetry only the half-length of the beam could be considered. By requiring $\delta U_b = \delta U_e$ there are derived the static equilibrium equations

$$\frac{dQ}{dx} = -q_S(x), \quad \frac{dM}{dx} = Q, \tag{2.26}$$

$$Q = \int_{-z_c}^{z_t} \tau dz, \quad M = \int_{-z_c}^{z_t} \sigma z dz,$$

It may be easily shown that the representation of the strain energy density (potential) of the beam in the context of Timoshenko's beam bending theory is given by the following *ansatz*

$$\upsilon_T = \frac{1}{2} EI\left(\kappa^2 + \frac{\gamma^2}{\ell_v^2}\right) \tag{2.27}$$

where the term EI denotes the flexural rigidity or stiffness of the beam, I denotes the moment of inertia of the cross-section A of the beam, ℓ_v stands for a microstructural length scale of the beam material that considers the effect of the transverse shear stress contributing to the deflection $w = w(x)$ of the beam,

The following constitutive relationships for the bending moment and transverse shear force may be deduced

$$M = \frac{\partial \upsilon_T}{\partial \kappa} = EI\kappa, \quad Q = \frac{\partial \upsilon_T}{\partial \gamma} = EI\,\frac{\gamma}{\ell_v^2} \tag{2.28}$$

The first of Equations (2.28) forms the Bernoulli-Euler theorem depicting the analogy of the bending moment with the bending curvature of the beam, while the second is due to Timoshenko that considers the effect of the transverse shear forces on the beam deflection. **So, Timoshenko beam is merely a 1-d Cosserat medium.**

Herein an engineering gradient beam bending theory that has been previously presented by Vardoulakis *et al.* (1998) containing two material length scales and aiming at capturing the size effect exhibited by beams in bending, is re-visited. In fact we change the strain energy density (or potential) *ansatz* initially proposed in our previous work, with the following straightforward expression (Exadaktylos, 2017)

$$\upsilon_b = \frac{1}{2} EI\left(\kappa^2 + \frac{1}{\ell_v^2}\gamma^2 + \frac{2}{\ell_s}\kappa\gamma\right) \tag{2.29}$$

So, Bernoulli-Euler theory which leads to the proportionality of the bending moment with curvature, is expressed only by the first term, whereas **Timoshenko's beam bending theory that explains the effect of shear forces on beam deflection and bending curvature of the beam is expressed by the first two terms**.

The constitutive equations for the shear force Q and bending moment M, are also easily derived from the potential of Eq. (2.29) as follows

$$Q \equiv \frac{\partial \upsilon_b}{\partial \gamma} = EI\left(\frac{1}{\ell_v^2}\gamma + \frac{1}{\ell_s}\kappa\right) = EI\left(\frac{1}{\ell_v^2}\left[w' + \psi\right] + \frac{1}{\ell_s}\psi'\right) \tag{2.30}$$

$$M \equiv \frac{\partial \upsilon_b}{\partial \kappa} = EI\left(\kappa + \frac{1}{\ell_s}\gamma\right) = EI\left(\psi' + \frac{1}{\ell_s}\left[w' + \psi\right]\right) \tag{2.31}$$

The third term in the above strain energy density function has not been obtained arbitrarily, but rather on the simple and straightforward argument, namely that since the curvature and shear strain are already included by Bernoulli-Euler and Timoshenko, respectively, then their product should be also included for completeness of the representation. The above *ansatz* contains the last term that considers surface energy effects through the microstructural length scale ℓ_s, and also contains as a special case Timoshenko's beam bending theory through the length scale

ℓ_v as may be observed from Eq. (2.27). In fact, by applying Gauss' divergence theorem the total elastic strain energy of the beam takes the form

$$U_E = \frac{1}{2} EI \int_0^L \left(\kappa^2 + \frac{1}{\ell_v^2}\gamma^2 + \frac{2}{\ell_s}\kappa\gamma \right) dx =$$
$$= \frac{1}{2} EI \int_0^L \left(\kappa^2 + \frac{1}{\ell_v^2}\gamma^2 + \frac{2}{\ell_s}\kappa w' \right) dx + \frac{1}{2\ell_s} EI \int_0^L \nabla \psi^2 dx =$$
$$= \frac{1}{2} EI \int_0^L \left(\kappa^2 + \frac{1}{\ell_v^2}\gamma^2 + \frac{2}{\ell_s}\kappa w' \right) dx + \frac{1}{2\ell_s} EI \left[\psi^2 \right]_0^L$$

It may be easily shown that the positive-definiteness of the strain energy density is guaranteed if the following inequalities are valid

$$-1 < \frac{\ell_V}{\ell_S} < 1$$

This means, that surface energy length ℓ_s must not vanish if the length scale correcting B-E theory to account for shear strains is considered.

2.6 Closed-form solution of the simply supported beam in 3PB

It may be shown that the expression for the deflection of 3PB simply supported beam has the form (Exadaktylos, 2017)

$$w = \frac{w_c}{\eta^2 - 1} \left\{ 6\hat{\ell}_s \left(1 + 2\hat{\ell}_s\right) + \eta^2 - 12\hat{\ell}_s \left(1 + 2\hat{\ell}_s\right)\xi - 6\eta^2\xi^2 + 4\eta^2\xi^3 \right\},$$
$$0 \le \xi \le \frac{1}{2}$$

where we have used the following dimensionless quantities

$$\xi = \frac{x}{L}, \; w_c = \frac{PL^3}{48EI} \quad ,$$

$$\eta^2 = \left(\frac{\ell_s}{\ell_v}\right)^2 = \frac{5\hat{\ell}_s^2}{(1+\nu)\left(\frac{H}{L}\right)^2} \quad , \quad \eta^2 > 1$$

$$\hat{\ell}_s = \frac{\ell_s}{L}$$

w_c represents the maximum (i.e. mid-span) deflection derived from *Bernoulli-Euler* beam theory. For example for a rectangular cross-section with height H, we get that

ℓ_T essentially compares with the inverse of the aperture ratio of the beam, that is to say for a rectangular cross-section of the beam, Timoshenko (1921) found that ℓ_T compares with the inverse of the length to height ratio

$$\ell_T^2 = \frac{1}{5}(1+\nu)\left(\frac{H}{L}\right)^2 << 1 \quad for \quad H < L$$

Also, the rotation of the initially vertical cross-section of the beam could be found

$$\psi = \frac{\bar{\lambda}}{\eta^2 - 1}\xi\left\{2\hat{\ell}_s + \eta^2 - \eta^2\xi\right\} \qquad 0 \le \xi \le \frac{1}{2}$$

wherein

$$\bar{\lambda} = \frac{PL^2}{4EI}$$

The engineering shearing strain could be also found in the following manner

$$\gamma = w' + \psi = \frac{\bar{\lambda}}{\eta^2 - 1}\left\{-\hat{\ell}_s\left[1 + 2\hat{\ell}_s\right] + 2\hat{\ell}_s\xi\right\} \quad , \quad 0 \le \xi \le \frac{1}{2}$$

Note that ls should not vanish for positive definite potential energy of the beam. In addition, the bending curvature of the beam may be found by formal differentiation of Equation 94 as follows

$$\kappa = \frac{\bar{\lambda}}{L}\frac{1}{\eta^2 - 1}\left\{2\hat{\ell}_s + \eta^2 - 2\eta^2\xi\right\} \quad 0 \le \xi \le \frac{1}{2} \qquad (2.32)$$

As, it may be seen from Eq. (2.32), in contrast to classical theory, the present gradient theory with surface energy predicts always for any value of η^2 a finite and larger value of the beam curvature at its supporting ends (i.e. for $\xi = 1/2$). This is due to the presence of the surface energy term $2\hat{\ell}_s$ in the expression for the curvature that also is responsible for the inequality $\kappa \ne -\partial^2 w/\partial x^2$.

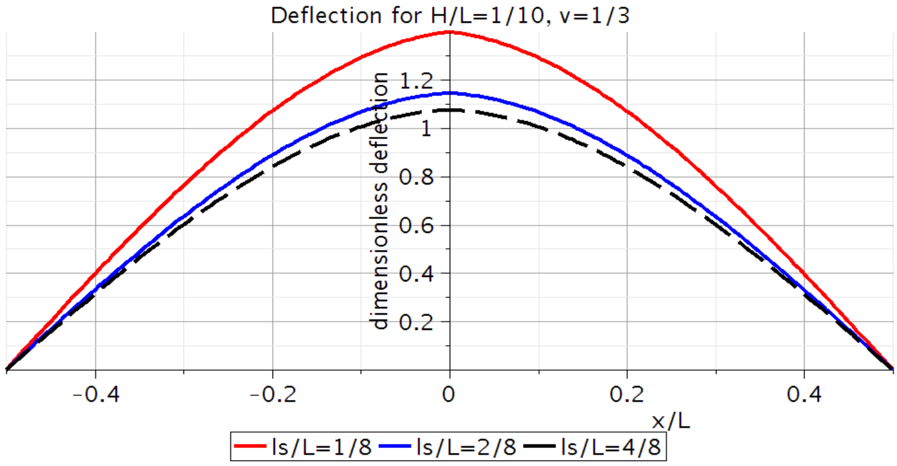


Fig. 2.12. "stiffening" effect of surface energy term Classical B-E theory elastic curve passes through 1 .

2.7 Size effect exhibited by the beam strength

Assuming that the *Poncelet - Saint Venant (PSV) failure hypothesis* is valid for granular brittle materials, then the fracture of the beam will occur when the horizontal extension strain at the mid-span of the bottom fiber of the beam denoted here as reaches the limit strain ε_f that is material property

$$\varepsilon_{xx} \geq \varepsilon_f \quad at \;\; \xi = 0, \;\; z = \frac{H}{2}$$

$$\sigma_{bu} = E\varepsilon_f = E\frac{H}{2}\kappa(0) = \sigma_{bu}^{B-E}\frac{\eta^2}{\eta^2 - 1}\left\{1 + \frac{2\ell_s}{\eta^2}\frac{1}{L}\right\}$$

$$\sigma_{bu}^{B-E} = \frac{P_f LH}{8I}$$

In the formula above, P_f denotes the value of the concentrated load at failure. For constant beam aperture ratio L/H, the extended beam bending theory accounting for surface effects, predicts a (-1)- power of the beam length dependence of the flexural strength of the beam, and (iii) this size effect law resembles Karmarsch's empirical law also used later by Griffith (i.e. Eq. (1.1)). Also we may say that *in contrast to Griffith's theory we do not need to consider ad hoc the existence of material defects.*

3. Mindlin's micro-elasticity theory

3.1 Kinematical considerations

In the sequel the basic formalism of the grade-2 theory of elasticity presented by Mindlin (1964) is outlined. With respect to a fixed Cartesian coordinate system $Ox_1x_2x_3$,

- Each material point has deformable micro-volume (e.g. Fig. 3.1) (multi-scale model).
- Two scales: micro-medium (unit cell) & macro-medium.
- Each scale has it's own deformation measure.
- Deformation inside unit cell (micro-volume) is homogeneous and inhomogeneous in the macro-volume V.

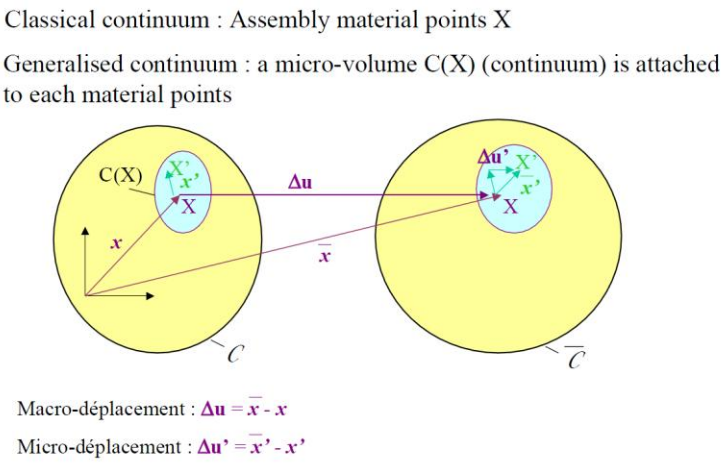


Fig. 3.1. Mindlin's 2 scales model.

The displacement of the macro-volume is defined as usually

$$u_i = x_i - X_i$$

On the other hand, in the micro-scale the micro-displacement is defined in the following manner

$$u_i' = x_i' - X_i'$$

Assuming that the absolute value of the displacement gradients is smaller than unity then the usual macro-deformation or macro-displacement gradient $\partial_r u_q$ (displacement gradient tensor) is recovered

$$\frac{\partial u_i}{\partial X_j} \approx \frac{\partial u_i}{\partial x_j} = \partial_j u_i, \quad u_i = u_i\left(x_j, t\right)$$

If the same holds true at the micro-scale then the micro-deformation tensor $\partial'_j u'_i$ is also recovered,

$$\frac{\partial u'_i}{\partial X'_j} \approx \frac{\partial u'_i}{\partial x'_j} = \partial'_j u'_i, \quad u'_i = u'_i(x_j, x_j', t)$$

That is assume that micro-displacements can be expressed as sums of products of specified functions of x'_i and arbitrary functions of x_i and time t, and as an approximation we keep only the linear term of the series

$$u'_k = x'_i \psi_{ik}$$

Where ψ_{ik} is a function of global coordinates x_i and time t only. The displacement-gradient of the micro-medium (element) is

$$\partial'_i u'_i = \psi_{ik}$$

with ψ_{qr} d**enoting the micro-deformation of a particle in the form of a grain or crystal for a granular or crystalline rock**, respectively, (Fig's 3.2 a, b),

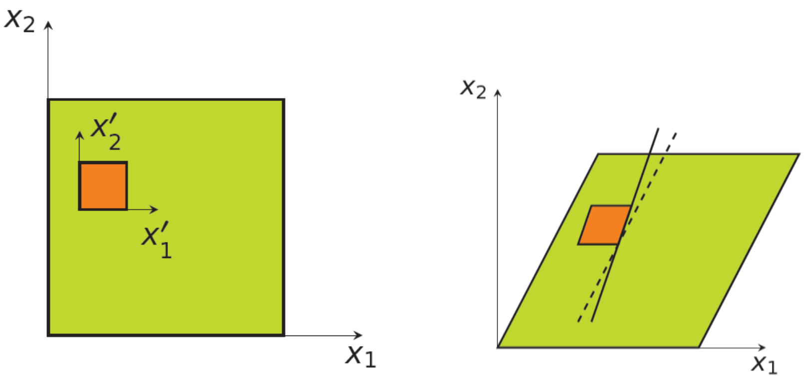


Fig. 3.2. (a) Multi-scale model, (b) relative deformation of the macro-volume w.r.t. the unit cell.

The micro-deformation gradient may be splitted it into a symmetric and an anti-symmetric part

$$\psi_{ij} = \psi_{(ij)} + \psi_{[ij]}$$

$$\psi_{(ij)} = \frac{1}{2}(\psi_{ij} + \psi_{ji}) \quad , \quad \psi_{(ij)} = \psi_{(ji)}$$

$$\psi_{[ij]} = \frac{1}{2}(\psi_{ij} - \psi_{ji})$$

The symmetric part is the **micro-strain** while the anti-symmetric part is the micro-rotation. Ericksen and Truesdell (1958) interpreted ψ as being proportional to the components of the displacements of the tips of deformable crosses, **so the component $\psi_{[ij]}$ describes the rotation of the rigid cross in Cosserat theory**.

Exact Theory of Stress and Strain in Rods and Shells

J. L. ERICKSEN & C. TRUESDELL

Dedicated to the memory of E. & F. COSSERAT

Contents

The usual strain is defined

$$\varepsilon_{qr} \equiv (1/2)(\partial_r u_q + \partial_q u_r)$$

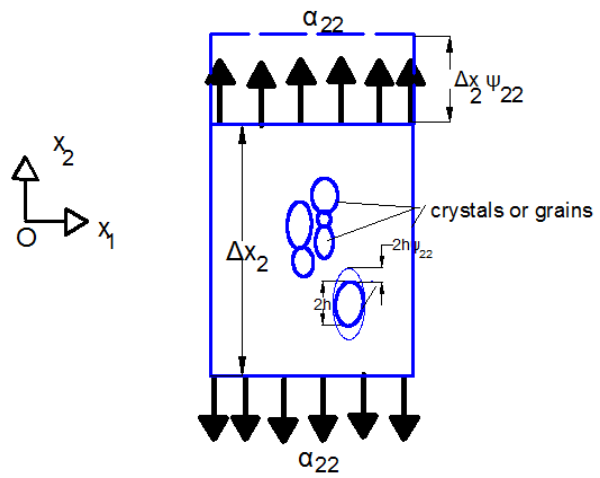
And also

$\gamma_{qr} \equiv \partial_q u_r - \psi_{qr}$ is the relative deformation

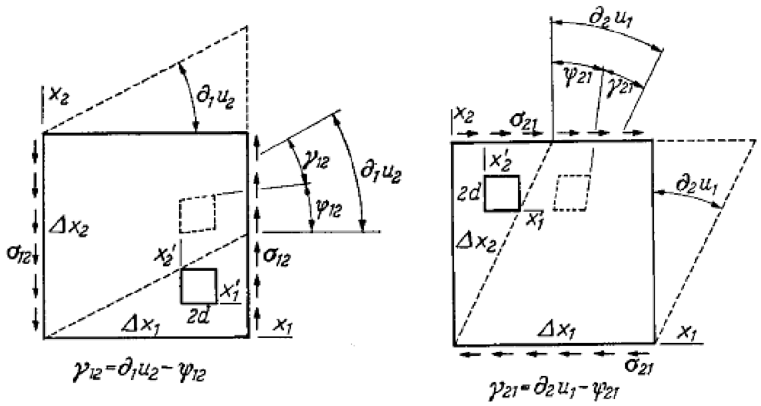
and

$\kappa_{qrs} \equiv \partial_q \psi_{rs}$ is the micro-deformation gradient or the macro-gradient of the micro-deformation

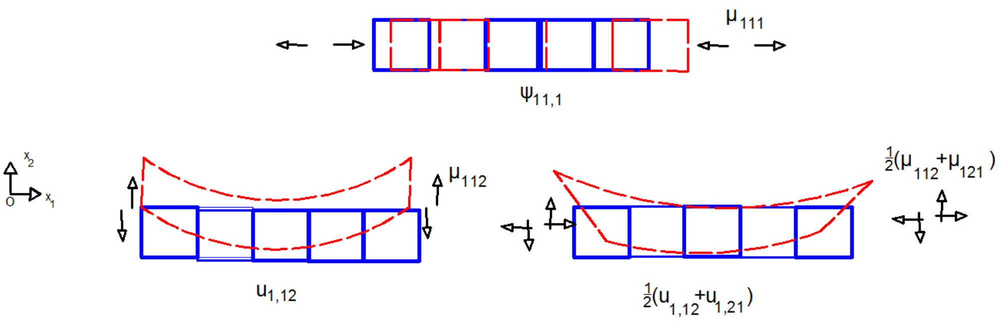
It is worth noting here that all three tensors $\varepsilon_{qr}, \gamma_{qr}, \kappa_{pqr}$ are independent of the micro-coordinates x_i'.

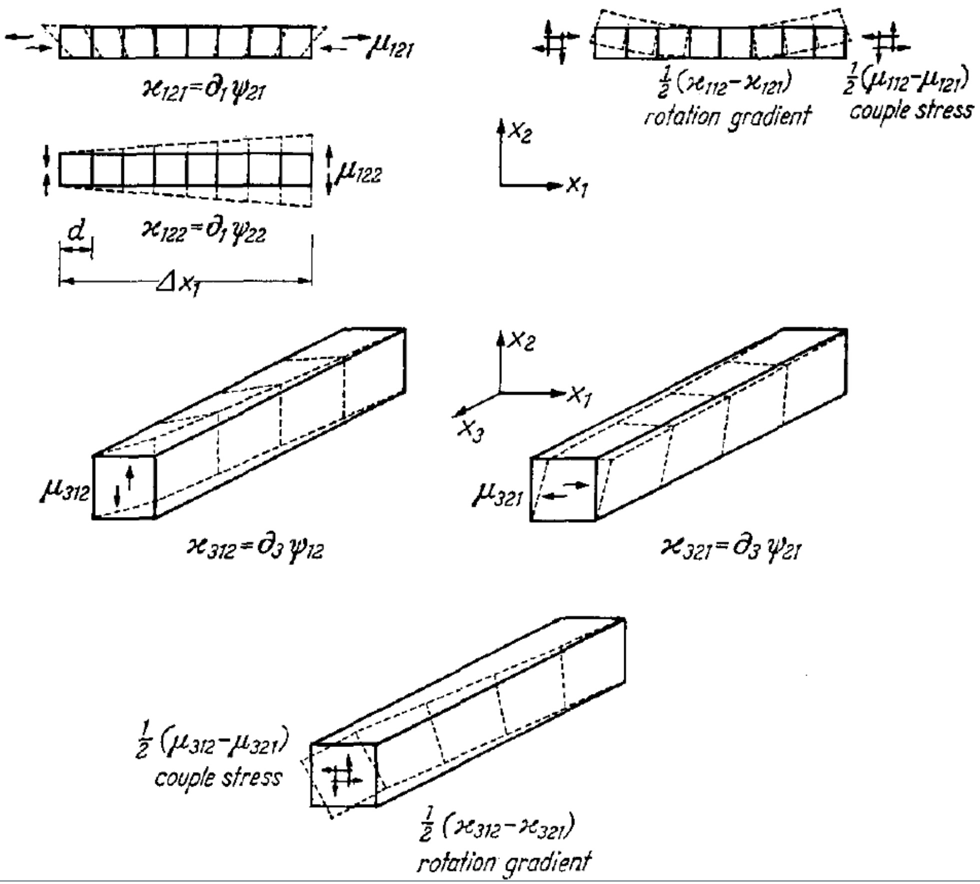

α22
Δx2 ψ22
x2
x1
O
crystals or grains
Δx2
2hψ22
2h
α22

(a)


x2
∂1u2
Δx2
σ12
x2'
x1'
2d
Δx1
σ12
x1
γ12
∂1u2
ψ12
γ12=∂1u2−ψ12
∂2u1
ψ21
γ21
x2
σ21
x2'
2d
x1'
Δx2
∂2u1
Δx1
x1
σ21
γ21=∂2u1−ψ21

(b)


μ111
ψ11,1
x2
x1
O
μ112
u1,12
½(μ112+μ121)
½(u1,12+u1,21)



(c)

Figure 3.3. (a,b) Typical components of relative stress α_{ij} ($\alpha_{ij} \equiv \sigma_{ij} - \tau_{ij}$,) displacement gradient $\partial_i u_j$, and micro-deformation ψ_{ij} for the simple case of uniaxial tension of a flat plate, and (c) various forms of micro-deformation gradients and associated double stresses and double stresses doublets (Mindlin, 1964).

3.2 Kinetic and potential energies

The kinetic energy density of the micro-medium may be derived as follows

$$T = \frac{1}{2}\rho' \dot{u}_i \dot{u}_i + \frac{1}{6}\rho d^2 \dot{\psi}_{ij} \dot{\psi}_{ij} = \rho' \partial_{tt} u_k + \frac{1}{6}\rho d^2 \partial_t \psi_{ij} \partial_t \psi_{ij}$$

where 2d is the size of the unit cell (element) of the microstructure, $\rho' = \rho_M + \rho$, ρ_M=density of the macro-volume and ρ the density of the micro-volume. Also the potential energy density (**potential energy per unit macro-volume**) is a homogeneous, quadratic function of the 42 variables $\varepsilon_{ij}, \gamma_{ij}, \kappa_{ijk}$

$$w = w(\varepsilon_{qr}, \gamma_{qr}, \kappa_{qrs})$$

Then, appropriate definitions for the stresses follow from the variation of w, i.e.

$$\tau_{qr} \equiv \frac{\partial w}{\partial \varepsilon_{qr}} = \tau_{rq}, \quad \alpha_{qr} \equiv \frac{\partial w}{\partial \gamma_{qr}}, \quad \mu_{qrs} \equiv \frac{\partial w}{\partial \kappa_{qrs}},$$

Then the following form of variation is assumed

$$\begin{aligned} \delta w &= \tau_{ij}\delta\varepsilon_{ij} + \alpha_{ij}\delta\gamma_{ij} + \mu_{ijk}\partial_i\delta\varepsilon_{jk} = \\ &= \tau_{ij}\partial_i\delta u_j + \alpha_{ij}\left(\partial_i\delta u_j - \delta\psi_{ij}\right) + \mu_{ijk}\partial_i\delta\psi_{jk} = \\ &= \partial_i\left[(\tau_{ij}+\alpha_{ij})\delta u_j\right] - \partial_i(\tau_{ij}+\alpha_{ij})\delta u_j - \alpha_{ij}\delta\psi_{ij} + \partial_i(\mu_{ijk}\delta\psi_{jk}) - \partial_i\mu_{ijk}\delta\psi_{jk} \end{aligned}$$

Integrating and applying the divergence theorem

$$\begin{aligned} \delta W = \delta\int_V w dV = -\int_V \partial_i\left[(\tau_{ij}+\alpha_{ij})\delta u_j\right]dV - \int_V (\alpha_{ij} + \partial_i\mu_{ijk}\delta\psi_{ij})dV + \\ + \int_{\partial V} n_i(\tau_{ij}+\alpha_{ij})\delta u_j dS + \int_{\partial V} n_i\mu_{ijk}\delta\psi_{jk} dS \end{aligned}$$

The form of the above equation motivates the adoption of the following form of the variation of external work

$$\delta W_{ext} = \iiint_V f_k\delta u_k dV + \iiint_V \Phi_{jk}\delta\psi_{jk} dV + \iint_{\partial V} t_k\delta u_k dS + \iint_{\partial V} T_{jk}\delta\psi_{jk} dS$$

Where we assumed the existence of the following set of external forces:

- body forces, $f_k dV$
- surface tractions, $t_k dS$
- double force per unit volume $\Phi_{ik} dV$
- double tractions, $T_{ik} dS$

In both Φ and T the first subscript indicates the direction of the lever arm between the forces and the second gives the orientation of the forces. Across a surface with its outward normal in the positive direction, the force at the positive end of the lever arm acts in the positive direction. “Positive” means the positive sense of the coordinate axis parallel to the lever arm of force. Across a surface with its outward normal in the negative direction, the directions of the forces are reversed.

Hamilton’s principle is written in terms of independent variations of the displacement δu_k and $\delta\psi_{kl}$ between fixed limits and of time also between fixed limits,

$$\delta\int_{t_0}^{t_1}\left(\hat{T} - W\right)dt + \int_{t_0}^{t_1}\delta W_{ext} dt = 0$$

$$\hat{\mathrm{T}} = \iiint_V T dV$$

From above relations **[Question: which?]** there follow the ($3+3^2=12$) stress-equations of motion in the volume V

$$\begin{aligned} &\partial_i(\tau_{ij}+\alpha_{ij})+f_j=\rho'\partial_{tt}u_j \\ &\partial_i\mu_{ijk}+\alpha_{jk}+\Phi_{jk}=\frac{1}{3}\rho d^2(\partial_{tt}\psi_{jk}) \end{aligned} \tag{4.1}$$

And the 12 (3 and $3^2=9$ respectively) traction boundary conditions read as follows

$$\begin{aligned} &t_j=n_i(\tau_{ij}+\alpha_{ij}) \\ &T_{jk}=n_i\mu_{ijk} \end{aligned} \tag{4.2}$$

In view of $\tau_{qr},\alpha_{qr},\mu_{qrs}$ denote the Cauchy stress (symmetric), relative stress (asymmetric), and double stress tensors, respectively. The second order stress tensor τ_{ij}, which is dual in energy to the macroscopic strain, is symmetric, i.e. $\tau_{ij}=\tau_{ji}$ whereas the third order tensor μ_{ijk}, which is dual in energy to the strain-gradient, is called the double stress. The τ_{ij} are like the components of the usual stress with the dimensions of force per unit area, however, they depend on the second gradient of strain in addition to the strain. The twenty-seven components μ_{kij} have the character of double forces per unit area. The first subscript of a double stress μ_{kij} designates the normal to the surface across which the component acts; the second and third subscripts have the same significance as the two subscripts of τ_{ij}. There are eight components of the deviator of the couple-stress or couples per unit area formed by the combinations $(1/2)(\mu_{pqr}-\mu_{prq})$, and ten independent combinations $(1/2)(\mu_{pqr}+\mu_{prq})$ called "double stress doublets", the latter being self-equilibrating (Mindlin, 1964;1965). Double force systems without moments are stress systems equivalent to two oppositely directed forces at the same point; such systems have direction but not net force and no resulting moment. Notice that singularities of this kind are discussed by Love (1927) and Eshelby (1951).

3.3 The anisotropic format

Constitutive equations may be found from the definition of the potential energy

$$\begin{aligned} w=&\frac{1}{2}c_{ijkl}\varepsilon_{ij}\varepsilon_{kl}+\frac{1}{2}b_{ijkl}\gamma_{ij}\gamma_{kl}+\frac{1}{2}a_{ijklmn}\kappa_{ijk}\kappa_{lmn}+ \\ &+d_{ijklm}\gamma_{ij}\kappa_{klm}+f_{ijklm}\kappa_{ijk}\varepsilon_{lm}+g_{ijkl}\gamma_{ij}\varepsilon_{kl} \end{aligned}$$

The constitutive tensors $c_{ijkl}, b_{ijkl}, a_{ijklmn}, d_{ijklm}, f_{ijklm}, g_{ijkl}$ contain 42x42=1764 coefficients with only 1/2x42x43=903 to be independent. From the definition of the stresses and the above ansatz the following constitutive eqns are found

$$\tau_{mn} = c_{mnkl}\varepsilon_{kl} + g_{klmn}\gamma_{kl} + f_{k\lim n}\kappa_{kli},$$
$$\alpha_{mn} = g_{mnkl}\varepsilon_{kl} + b_{klmn}\gamma_{kl} + d_{mnikl}\kappa_{ikl},$$
$$\mu_{mno} = f_{mnokl}\varepsilon_{kl} + d_{klmno}\gamma_{kl} + a_{mnoijk}\kappa_{ijk}$$

3.4 The isotropic format

For centro-symmetric isotropic materials the strain energy density could be considerably simplified since there are no isotropic tensors of odd rank and hence d_{ijklm}, f_{ijklm} must vanish. The remaining coefficients must be homogeneous, linear functions of products of Kronecker deltas. It is recalled that there are 3 independent products of 2 Kronecker deltas and 15 independent products of 3 Kronecker deltas. Finally the simplified form of the potential energy density **with 18 coefficients** is given as follows (Mindlin, 1964)

$$w = \frac{1}{2}\lambda\varepsilon_{ii}\varepsilon_{jj} + \mu\varepsilon_{ij}\varepsilon_{ij} + \frac{1}{2}b_1\gamma_{ii}\gamma_{jj} + \frac{1}{2}b_2\gamma_{ij}\gamma_{ij} +$$
$$+ \frac{1}{2}b_3\gamma_{ij}\gamma_{ji} + g_1\gamma_{ii}\varepsilon_{jj} + g_2\left(\gamma_{ij} + \gamma_{ji}\right)\varepsilon_{ij} +$$
$$+ a_1\kappa_{iik}\kappa_{kjj} + a_2\kappa_{iik}\kappa_{jkj} + \frac{1}{2}a_3\kappa_{iik}\kappa_{jjk} + \frac{1}{2}a_4\kappa_{ijj}\kappa_{ikk} +$$
$$+ \frac{1}{2}a_5\kappa_{ijj}\kappa_{kik} + \frac{1}{2}a_8\kappa_{iji}\kappa_{kjk} + \frac{1}{2}a_{10}\kappa_{ijk}\kappa_{ijk} + a_{11}\kappa_{ijk}\kappa_{jki} +$$
$$+ \frac{1}{2}a_{13}\kappa_{ijk}\kappa_{ikj} + \frac{1}{2}a_{14}\kappa_{ijk}\kappa_{jik} + \frac{1}{2}a_{15}\kappa_{ijk}\kappa_{kji}$$

The constitutive equations take the form

$$\tau_{pq} = \lambda\delta_{pq}\varepsilon_{kk} + 2\mu\varepsilon_{pq} + g_1\delta_{pq}\gamma_{kk} + 2g_2\gamma_{(pq)},$$
$$\alpha_{mn} = g_1\delta_{pq}\varepsilon_{kk} + 2g_2\varepsilon_{pq} + b_1\delta_{pq}\gamma_{kk} + b_2\gamma_{pq} + b_3\gamma_{qp},$$
$$\mu_{pqr} = a_1(\kappa_{iip}\delta_{qr} + \kappa_{rii}\delta_{pq}) + a_2(\kappa_{iip}\delta_{qr} + \kappa_{iri}\delta_{pq}) + a_3\kappa_{iir}\delta_{pq} +$$
$$+ a_4\kappa_{pii}\delta_{qr} + a_5(\kappa_{qii}\delta_{qr} + \kappa_{ipi}\delta_{qr}) + a_3\kappa_{iqi}\delta_{pr} + a_{10}\kappa_{pqr} +$$
$$+ a_{11}(\kappa_{rpq} + \kappa_{qrp}) + a_{13}\kappa_{prq} + a_{14}\kappa_{qpr} + a_{15}\kappa_{rqp}$$

[**Question: Show that the above expressions are valid**].

3.5 Limit case of Cosserat continuum

Note: The linear eqns of the Cosserat continumm are obtained by setting

$$\psi_{(ij)} = 0 \Rightarrow \alpha_{(ij)} = \tau_{ij} \wedge \mu_{i(jk)} = 0$$

[Question: why this is true?]

<u>Homework</u>: Derive under any simplifying assumptions the constitutive equations governing a Cosserat continuum.

3.6 Mindlin's restricted contimuum and the 3 forms of the theory

In the same paper Mindlin presented simpler models of the gradient elasticity employing simplifying assumptions by dropping the multi-scale character of the theory, that allow to express the strain energy density in terms of macroscopic displacement u_i only. This is achieved by setting zero the relative deformation tensor

$$\gamma_{qr} \equiv \partial_q u_r - \psi_{qr} = 0 \qquad (4.3)$$

This assumption makes the relative stress tensor a_{ij} to be workless. Also the eight components of the deviator of the couple-stress or couples per unit area formed by the combinations $(1/2)(\mu_{pqr} - \mu_{prq})$ are all equal to zero in the present gradient dependent elasticity theory.

The Forms I, II, and III presented by Mindlin and Eshel differ in the assumed relation between the microscopic deformation gradient κ_{ijk} and the macroscopic displacements u_i.

Form I: The microscopic deformation gradient is equal to the 2nd gradient of the macroscopic displacements u_i, i.e. $\tilde{\kappa}_{ijk} = \partial_i \partial_j u_k$ and the strain energy density is $\tilde{w} = \tilde{w}(\varepsilon_{ij}, \tilde{\kappa}_{ijk})$.

Form II: The microscopic deformation gradient is assumed to be equal to the first gradient of strain $\hat{\kappa}_{ikj} = \partial_i \varepsilon_{jk}$ so the strain energy density function depends only on the strain and its gradient $\hat{w} = \hat{w}(\varepsilon_{ij}, \hat{\kappa}_{ijk})$.

Form III: In this case the microscopic deformation gradient was splitted into 2 parts namely the gradient of microdeformation $\bar{\kappa}_{ij} = \frac{1}{2}\varepsilon_{jlk}\partial_l \partial_i u_k$ where ε_{jlk} denotes the Levi-Civita permutation tensor and the **symmetric part of the second gradient of macroscopic displacement** $\bar{\bar{\kappa}}_{ijk} = \frac{1}{3}\left(\partial_i \partial_j u_k + \partial_j \partial_k u_i + \partial_k \partial_i u_j\right)$. The potential energy density in this form becomes $\bar{w} = \bar{w}(\varepsilon_{ij}, \bar{\kappa}_{ij}, \bar{\bar{\kappa}}_{ijk})$.

Mindlin (1964) has shown that all 3 forms reduce to the same displacement equations of motion for isotropic materials, whereas Mindlin and Eshel (1968) demonstrated the same result for the stress-equations of motion.

3.7 Surface effects

Surface effects in Mindlin's theory are possible if one adds the initial stress terms in the potential energy density expression for the isotropic material

$$c_0 \varepsilon_{kk} + b_0 \gamma_{kk} \qquad (4.4)$$

If we take $c_0 = -b_0$ both constants, then there is added a constant in both expression for the Cauchy stress τ_{ij} and the relative stress α_{ij}, or in other words a homogeneous and isotropic initial stress, but the total stress remains unchanged and so the traction vector t_i at the surface vanishes under no action of external stress. However, from the equilibrium Eq. $(4.1)_2$ with zero Φ (body double forces) there will be a non-zero double force and so at the boundaries of the body there will emerge according to Eq. (4.2) double tractions T_{jk} acting on them. The removal of these double tractions will cause the appearance of displacements localized at the surface of the solid, so there will be an energy trapped close to the surface that is a surface energy. In crack equilibrium and propagation problems this is the energy accociated with new surface created by crack propagation.

[**Exercise: find the displacement equilibrium equation and it solution**]

3.8 Mindlin's second gradient elastic model

In another paper Mindlin (1965) embarked in a third gradient of displacement theory with **triple stresses** to capture the surface energy property of new surfaces in solids,

$$\tau_{qr} \equiv \frac{\partial w}{\partial \varepsilon_{qr}}, \quad \tau_{qrs} \equiv \frac{\partial w}{\partial \varepsilon_{qrs}}, \quad \tau_{qrst} \equiv \frac{\partial w}{\partial \varepsilon_{qrst}}, \qquad (4.5)$$

where we have set

$$\varepsilon_{qrs} = u_{s,qr}, \varepsilon_{pqrs} = u_{s,pqr} \qquad (4.6)$$

$$\delta \int_V w dV = \int_V (\tau_{ij}\partial_j \delta u_i + \tau_{ijk}\partial_k \partial_j \delta u_i + \tau_{ijkl}\partial_l \partial_k \partial_j \delta u_i) dV \qquad (4..7)$$

The right-hand-side of the above integral may be reduced to include a surface integral by application of the chain rule of differentiation and Gauss's divergence theorem.

The potential energy density **with 18 coefficients** i.e. 2 Lame constants and 16 additional constants a, b, c, b_0 is given by the 2nd degree polynomial as follows (Mindlin, 1965)

$$\begin{aligned} w = & \frac{1}{2}\lambda \varepsilon_{ii}\varepsilon_{jj} + \mu \varepsilon_{ij}\varepsilon_{ij} + a_1 \varepsilon_{iik}\varepsilon_{kjj} + a_2 \varepsilon_{iik}\varepsilon_{kjj} + \\ & + a_3 \varepsilon_{iik}\varepsilon_{jjk} + a_4 \varepsilon_{ijk}\varepsilon_{ijk} + a_5 \varepsilon_{ijk}\varepsilon_{kji} + \\ & + b_1 \varepsilon_{iijj}\varepsilon_{kkll} + b_2 \varepsilon_{ijkk}\varepsilon_{ijll} + b_3 \varepsilon_{iijk}\varepsilon_{jkll} + \\ & + b_4 \varepsilon_{iijk}\varepsilon_{llkj} + b_5 \varepsilon_{iijk}\varepsilon_{lljk} + b_6 \varepsilon_{ijkl}\varepsilon_{ijkl} + \\ & + b_7 \varepsilon_{ijkl}\varepsilon_{jkli} + c_1 \varepsilon_{ii}\varepsilon_{jjkk} + c_2 \varepsilon_{ij}\varepsilon_{ijkk} + \\ & + c_3 \varepsilon_{ij}\varepsilon_{kkij} + b_0 \varepsilon_{iikk} \end{aligned} \qquad (4.8)$$

The last term $b_0 \varepsilon_{iikk}$ was called by Mindlin "surface energy term" since gives rise to surface phenomena. Subsequently, if one writes the total potential deformation energy of the system according to the generalized Clapeyron's theorem (Love, 1927), and considers vanishing external forces it will find the result

$$U = W - W_{ext} = W = \iiint_V w dV = b_0 \iint_{\partial V} \partial_n u_{i,i} dS$$

where U is the total potential energy of the system. So the interesting result is found namely that if one finds a proper displacement field that will be self-equilibrated then the above Gibbs or Helmholtz free energy is not null for a traction-free surface of the solid.

4. A strain gradient elasticity with surface energy

4.1 Introduction

Here we present a special version of Mindlin's linear Elasticity Theory with Microstructure. The basic feature of this theory is that it predicts surface energy in the frame of a strain gradient theory, i.e. without recourse to second gradient of strain. Vardoulakis and Sulem, Exadaktylos, Georgiadis have applied this theory in a number papers, where static and dynamic boundary-value problems have been addressed and solved analytically and numerically by Aravas, Zervos, Papanicolopoulos. It is demonstrated that a Mindlin-type linear gradient elasticity with surface energy, constitutes the simplest, in energy consistent, **non-local extension of Hooke's law**.

In case of a *restricted* Mindlin continuum the relative deformation γ_{ij} vanishes since the macroscopic strain coincides with the micro-deformation, i.e. Eq. (4.3). In this particular type of micro-homogeneous material and considering Mindlin's Form II, the micro-deformation gradient (i.e. the macro-gradient of the micro-deformation) κ_{ijk} is identical with the gradient of strain, that is to say

$$\kappa_{ijk} = \kappa_{ikj} \equiv \partial_i \varepsilon_{jk}$$

4.2 Variational equations of motion

Next, the following *ansatz* for the potential energy density *w* (potential energy per unit macro-volume) is taken

$$w = w(\varepsilon_{ij}, \partial_k \varepsilon_{ij}) \tag{4.9}$$

that corresponds to form II (Mindlin and Eshel, 1968). The variation of the total potential energy in volume V of the body is defined as follows

$$\delta \int_V w dV = \int_V (\tau_{ij} \delta\varepsilon_{ij} + \mu_{ijk} \partial_i \delta\varepsilon_{jk}) dV \tag{4.10}$$

where we define

$$\tau_{ij} \equiv \frac{\partial w}{\partial \varepsilon_{ij}}, \qquad \mu_{ijk} \equiv \frac{\partial w}{\partial(\partial_i \varepsilon_{jk})} \tag{4.11}$$

To prepare for the formulation of a variational principle, we apply the chain rule of differentiation and the divergence theorem; furthermore, we resolve $\partial_i u_j$ on the boundary ∂V of V into a *smooth surface - gradient* and a *normal-gradient. Let* n_i be the unit normal to ∂V and pointing away from V, then the following are applicable on ∂V

$$\partial_i \delta u_j \equiv D_i \delta u_j + n_i D\delta u_j, D_i \equiv \left[\frac{1}{R_1}+\frac{1}{R_2}\right] n_i - (\delta_{ik} - n_i n_k)\partial_k, \\ D \equiv n_k \partial_k, \tag{4.12}$$

where n_k is the outward unit normal on the boundary ∂V. After applying the divergence theorem, the final expression for the variation in potential energy of a smooth boundary ∂V reads

$$\delta W = \int_V \delta w dV = -\int_V \partial_j(\tau_{jk} - \partial_i \mu_{ijk})\delta u_k dV + \int_{\partial V} n_j(\tau_{jk} - \partial_i \mu_{ijk})\delta u_k dS \\ + \int_{\partial V} \left(\left[\frac{1}{R_1}+\frac{1}{R_2}\right] n_j - D_j \right) n_i \mu_{ijk} \delta u_k dS + \int_{\partial V} n_i n_j \mu_{ijk} D\delta u_k dS \tag{4.13}$$

where $(1/R_1 + 1/R_2)$ is the mean curvature of the bounding surface. Looking at the structure of (4.13) we now postulate the following form for the variation of work W_{ext} done by external forces

$$\delta W_{ext} = \iiint_V f_k \delta u_k dV + \iint_{\partial V} t_k \delta u_k dS + \iint_{\partial V} n_l m_k \partial_l \delta u_k dS \tag{4.14}$$

where f_k is the body force per unit volume, t_k, m_k are the specified tractions and double tractions, respectively, on the smooth surface ∂V.

Next, we write Hamilton's principle for independent variations $\delta u_i, \partial_i \delta u_j$ between fixed limits of u_i and $\partial_i \delta u_j$ at times t_0 and t_1 (Love, 1927)

$$\delta \int_{t_0}^{t_1} (T - W) dt + \int_{t_0}^{t_1} \delta W_{ext} dt = 0 \tag{4.15}$$

where T is the total kinetic energy of the system. It can be shown that for the case of the restricted Mindlin continuum the following relationship is valid (Mindlin, 1964)

$$\delta\int_{t_0}^{t_1} T dt = -\int_{t_0}^{t_1} dt \int_V \left[\rho\partial_{tt}u_k - \frac{1}{3}\rho d^2 \partial_i\left(\partial_{tt}\partial_i u_k\right)\right]\delta u_k dV$$
$$-\int_{t_0}^{t_1} dt \int_{\partial V} \frac{1}{3}\rho d^2 n_i \left(D_i\partial_{tt}u_k + n_i D\partial_{tt}u_k\right)\delta u_k dS \tag{4.16}$$

From relations (4.13), (4.14), (4.15) and (4.16) there follow the only surviving stress-equations of motion in the volume V [**please derive yourself!**]

$$\partial_i\sigma_{ij} + f_j = \rho\partial_{tt}u_j - \frac{1}{3}\rho d^2 \partial_i(\partial_{tt}\partial_i u_j) \tag{4.17}$$

where we have set

$$\sigma_{ij} \equiv \tau_{ij} - \partial_k\mu_{kij} \tag{4.18}$$

Since the workless 2nd order relative stress tensor aij in a restricted Mindlin continuum without double body forces is in equilibrium with double stress

$$\alpha_{ij} + \partial_k\mu_{kij} = 0$$

We notice that according to (4.17) and in the static limit, the new stress tensor σ_{ij} is identified with the common macroscopic equilibrium stress tensor.

The surface ∂V of the considered volume V is divided into two complementary parts ∂V_u and ∂V_σ such that on ∂V_u kinematic data whereas on ∂V_σ static data are prescribed. In classical continua these are constraints on displacements and tractions, respectively. For the stresses the following set of six traction and double traction boundary conditions on a smooth surface ∂V_σ are also derived from the virtual work principle - i.e. equations (4.13), (4.14) and (4.16)

$$n_j\tau_{jk} - n_j\partial_i\mu_{ijk} + \left(\left[\frac{1}{R_1} + \frac{1}{R_2}\right]n_j - D_j\right)n_i\mu_{ijk} +$$
$$+\frac{1}{3}\rho d^2 n_j\left(D_j\partial_{tt}u_k + n_j D\partial_{tt}u_k\right) = t_k \tag{4.19}$$

$$n_i n_j \mu_{ijk} = m_k \tag{4.20}$$

Since strain gradients are considered into the constitutive description, additional kinematic data must be prescribed on ∂V_u. With the displacement already given in ∂V_u, then **according to Gauss' theorem** only its normal derivative with respect to that boundary is unrestricted. This means that on ∂V_u the normal derivative of the displacement should also be given, i.e.

$$u_i = w_i \quad \text{on} \quad \partial V_{u1} \tag{4.21}$$
$$Du_i = r_i \quad \text{on} \quad \partial V_{u2} \tag{4.22}$$

The most general form of the strain energy density function for a linear, macroscopically homogeneous and isotropic, grade-2 elastic material is the following 2nd degree polynomial

$$\begin{aligned} w = & \frac{1}{2}\lambda\varepsilon_{ii}\varepsilon_{jj} + G\varepsilon_{ij}\varepsilon_{ji} + \ell_{1k}\partial_k(\varepsilon_{ii}\varepsilon_{jj}) + \ell_{2k}\partial_k(\varepsilon_{ij}\varepsilon_{ji}) \\ & + \ell_{3j}\partial_k(\varepsilon_{ij}\varepsilon_{ik}) + \ell_{4k}\partial_i(\varepsilon_{ij}\varepsilon_{kj}) + \ell_{5k}\partial_j(\varepsilon_{ii}\varepsilon_{kj}) + a_1\partial_j\varepsilon_{ij}\partial_k\varepsilon_{ik} \\ & + a_2\partial_k\varepsilon_{ii}\partial_j\varepsilon_{kj} + a_3\partial_k\varepsilon_{ii}\partial_k\varepsilon_{jj} + a_4\partial_k\varepsilon_{ij}\partial_k\varepsilon_{ij} + a_5\partial_k\varepsilon_{ij}\partial_i\varepsilon_{kj} \end{aligned} \tag{4.23}$$

where the five a_n are the additional constants which appear in Toupin's strain-gradient theory (Toupin, 1962; Mindlin, 1964), and ℓ_{nk} (n=1,...,5; k=1,2,3) are the five additional directors in order to include the effect of terms linear in the strain gradient, $\partial_i\varepsilon_{jk}$, in the strain energy density expression. The choice of the above polynomial expression for the strain energy density of the material with microstructure implies that terms of higher degree are small in comparison with those retained.

In (4.23) ℓ_{nk} are characteristic directors such that

$$\ell_{nk} \equiv \bar{\ell}_n \nu_k, \qquad \nu_k\nu_k = 1; \qquad n = 1,...,5, \quad k = 1,2,3.$$

Accordingly Eq. (4.23) defines a gradient anisotropic elasticity with constant characteristic directors ℓ_{nk}. Also, the terms in (4.23) that are associated with these directors have the meaning of surface energy, since by using the divergence theorem one may find the following relations

$$\begin{aligned} \int_V \partial_k(\ell_{1k}\varepsilon_{ii}\varepsilon_{jj})dV &= \bar{\ell}_1 \int_{\partial V} (\varepsilon_{ii}\varepsilon_{jj})(\nu_k n_k)dS, \\ \int_V \partial_k(\ell_{2k}\varepsilon_{ij}\varepsilon_{ji})dV &= \bar{\ell}_2 \int_{\partial V} (\varepsilon_{ij}\varepsilon_{ji})(\nu_k n_k)dS, \\ \int_V \partial_k(\ell_{3j}\varepsilon_{ij}\varepsilon_{jk})dV &= \bar{\ell}_3 \int_{\partial V} (\varepsilon_{ij}\varepsilon_{ik})(\nu_j n_k)dS, \\ \int_V \partial_i(\ell_{4k}\varepsilon_{ij}\varepsilon_{kj})dV &= \bar{\ell}_4 \int_{\partial V} (\varepsilon_{ij}\varepsilon_{kj})(\nu_k n_i)dS, \\ \int_V \partial_j(\ell_{5k}\varepsilon_{ii}\varepsilon_{kj})dV &= \bar{\ell}_5 \int_{\partial V} (\varepsilon_{ii}\varepsilon_{kj})(\nu_k n_j)dS. \end{aligned}$$

The ratios ℓ_{nk}/G have the dimension of length whereas the ratios a_n/G have the dimension of square of length. The constitutive equations for the Cauchy and total stresses, as well as the double stresses are then derived by recourse to (4.10), (4.11), (4.18) and (4.23) as it will be demonstrated in the next paragraphs.

4.2 Constitutive equations of strain gradient theory with surface energy

Significant simplification of the theory results if we set:

$$\ell_{1k} = \frac{\lambda}{2}\ell_k,\ \ell_{2k} = \mu\ell_k,\ a_3 = \frac{\lambda}{2}\ell^2,\ a_4 = \mu\ell^2$$

Then the elastic-strain energy density function becomes,

$$w = \frac{1}{2}\lambda\varepsilon_{mm}\varepsilon_{nn} + G\varepsilon_{mn}\varepsilon_{nm} + \ell^2\left(\frac{1}{2}\lambda\partial_k\varepsilon_{mm}\partial_k\varepsilon_{nn} + G\partial_k\varepsilon_{mn}\partial_k\varepsilon_{nm}\right) + \partial_k\left(\frac{1}{2}\lambda(\varepsilon_{mm}\varepsilon_{nn}) + G(\varepsilon_{mn}\varepsilon_{nm})\right)\ell'_k$$

In the above energy expression ℓ and ℓ'_k are material constants with dimension of length. The director ℓ'_k takes on the boundary the direction of the unit outward normal, i.e. if

$$\ell'_k = \ell' n_k \quad \text{on } \partial V$$

Hence, the above theory is anisotropic (due to directors).

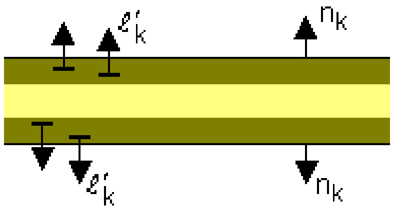


Fig. 4.1.Piece-wise constant director field in a strip confined into the two boundary layers.

As can be seen from the following expression, in this case the elastic-stain energy density function describes a 2nd-gradient elastic material equipped with surface energy,

$$W = \iiint_V \left(\frac{1}{2}\lambda\varepsilon_{mm}\varepsilon_{nn} + G\varepsilon_{mn}\varepsilon_{nm} + \ell^2\left(\frac{1}{2}\lambda\partial_k\varepsilon_{mm}\partial_k\varepsilon_{nn} + G\partial_k\varepsilon_{mn}\partial_k\varepsilon_{nm}\right)\right)dV + \iint_{\partial V} \ell'\left(\frac{1}{2}\lambda(\varepsilon_{mm}\varepsilon_{nn}) + G(\varepsilon_{mn}\varepsilon_{nm})\right)dS$$

Notice that from the requirement of positive elastic-strain energy density we get a restriction of the relative surface length scale ℓ'/ℓ. It seems that the limits of this ratio depend on the type of boundary value problem.

4.3 Skin effect and surface free energy

Our purpose here is to show that a basic feature of the present strain gradient elasticity theory with surface energy is the appearance of a skin effect associated with the volume energy parameter ℓ and a pre-tension as is done by Mindlin's (1964,1965) theory. Furthermore, it will be shown that the relative surface energy parameter ℓ' / ℓ gives rise to surface energy associated with a new cut in the material .

The deformation of an isotropic semi-infinite body $x_1 \geq 0$ due to a large uniform tensile stress $\sigma_{22} = \sigma, (\sigma > 0)$, parallel with the surface with outward unit normal vector (n_1 n_2 n_3)=(-1 0 0) with the Cartesian coordinates be x_1, x_2, and x_3, is considered as was done in (Exadaktylos & Vardoulakis, 1998). Starting from a stress-free configuration, C_0, the body is stressed uniaxially under plane strain conditions, and C is the resultant configuration. Then, the pre-stressed body is incrementally deformed and let its current configuration state to be that of C'. The problem under consideration is formulated in terms of the 1st Piola-Kirchhoff stress π_{ij} with respect to current configuration C', with $\Delta\pi_{ij}$ being its increment referred to the deformed initially stressed state C. Assuming infinitesimal strain elasticity, the Jaumann stress increments $\overset{o}{\Delta}\sigma_{ij}$ of the total stress are related directly to the strain increments through the following constitutive for the total stress, Cauchy stress and double stress tensors, respectively in the frame of Casal-Mindlin theory [**please derive**]

$$\left.\begin{aligned} \sigma_{ij} &= \lambda\delta_{ij}\varepsilon_{kk} + 2G(\varepsilon_{ij} - \ell^2\nabla^2\varepsilon_{ij}) \\ \tau_{ij} &= \lambda\delta_{ij}\varepsilon_{kk} + 2G\varepsilon_{ij} + 2G\ell_k\partial_k\varepsilon_{ij} \\ \mu_{kij} &= 2G\ell_k\varepsilon_{ij} + 2G\ell^2\partial_k\varepsilon_{ij} \end{aligned}\right\} \tag{4.24}$$

For the traction-free surface of the half-space the following incremental boundary conditions are valid

$$\Delta\pi_{11} = \Delta\pi_{21} = \mu_{111} = \mu_{112} = 0 \qquad \text{on} \;\; x_1 = 0 \tag{4.26}$$

It is possible to assume, without loss of generality (it can be shown that, in this problem, the quantities u_1, u_3 do not couple with u_2; these quantities satisfy homogeneous equations with homogeneous boundary conditions and therefore vanish identically) the following displacement field

$$u_2 = u_2(x_1), \; u_1 = u_3 = 0 \tag{4.27}$$

and the only non-zero initial stress σ_{22} to act along x_2-axis. Upon substituting the strain-displacement relation into the the stress-strain relations and the resulting expressions for the stresses into the stress-equation of equilibrium $\partial_j\Delta\pi_{ij} = 0$, we find only the following surviving displacement equation of equilibrium

$$\left(1-\frac{\ell^2}{(1+\xi)}\frac{d^2}{dx_1^2}\right)\frac{d^2}{dx_1^2}u_2=0 \tag{4.28}$$

where we have set $\xi=-\sigma_{22}/2G$. The solution of Eq. (4.28), vanishing at infinity, is merely,

$$u_2(x_1)=c\exp\left(-\frac{\sqrt{1+\xi}}{\ell}x_1\right) \tag{4.29}$$

where *c* denotes an integration constant. The first three boundary conditions described by Eq. (4.26) are satisfied identically, whereas the only remaining boundary condition along $x_1=0$ takes the form

$$\mu_{112}=2G\left\{\ell'\frac{d}{dx_1}+\ell^2\frac{d^2}{dx_1^2}\right\}u_2=0 \quad \text{on } x_1=0 \tag{4.30}$$

which holds true for $\nu_r\equiv -n_r$ and gives the following equation

$$c\left[-\frac{\ell'}{\ell}+\sqrt{1+\xi}\right]=0$$

(4.31)

From Eq. (4.31) one may deduce that the only case which gives non-zero and exponentially decaying displacement with distance from the surface of the solid, that is $c\neq 0$, is the following

$$\frac{\ell'}{\ell}=\sqrt{1+\xi}\Leftrightarrow \quad \xi=-[1-(\frac{\ell'}{\ell})^2] \tag{4.32}$$

The above relation elucidates the importance of the surface strain gradient term ℓ' in determining surface effects. Eq. (4.32) depicts that the effect of the surface energy parameter is equivalent to the effect of an *initial stress*. The dependence of initial stress ξ on the relative surface energy parameter ℓ'/ℓ is shown in Fig. 4.2. From this figure it may be seen that if $\ell'/\ell=0$ the half-space is under surface tension, with this surface tension to be maximum. As ℓ'/ℓ increases from the value of zero the initial tension or in other words the *surface tension* of the medium decreases reaching the value of zero for $\ell'/\ell=1$. At $\ell'/\ell=1$ the initial stress changes sign and for $\ell'/\ell>1$ becomes compressive in nature. That is, for values of the relative surface energy parameter higher than the value of one, the medium is under surface compression and it is no longer in a state of elastic equilibrium, or in other words as it is also shown by the inequality $-1<\ell'/\ell<1$ its strain energy density function is negative definite.

The elastic strain energy density of the considered 1D configuration reduces to Casal's original expression recovered above and is given by

$$w=G\{\varepsilon^2+\ell^2\nabla\varepsilon\nabla\varepsilon+2\ell'\varepsilon\nabla\varepsilon\},\quad \nabla\equiv d/dx_1 \tag{4.33}$$

Substituting in Eq. (4.33) the values for the strain and the strain-gradient, we find

$$\hat{w} = \{1-(\frac{\ell'}{\ell})^2\}(\frac{\ell'}{\ell})^2(\frac{c}{\ell})^2\exp(-2\frac{\sqrt{1+\xi}}{\ell}x_1), \quad \hat{w} = w/G \tag{4.34}$$

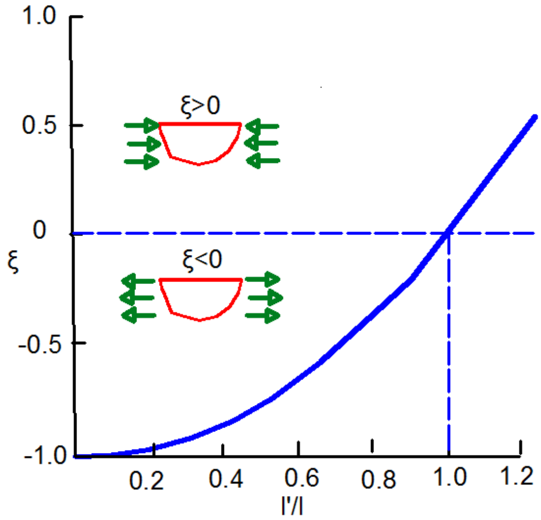


Fig. 4.2. Graphical representation of the relation of the dimensionless pre-stress ξ *with the relative surface energy parameter* ℓ'/ℓ *(Exadaktylos & Vardoulakis, 1998).*

By adopting the following definition for the average surface stress (or surface free energy)

$$\gamma_{se} = \int_V w dV / A \tag{4.35}$$

where A is the area of the free smooth surface, we may find after some manipulations

$$\gamma_{se} = \frac{G}{2}\{1-\left(\frac{\ell'}{\ell}\right)^2\}\frac{\ell' c^2}{\ell^2} \tag{4.36}$$

This is also, for each surface, the energy per unit area required to separate the body along a smooth plane and $\gamma_{se} > 0$ if inequality

$$-1 < \frac{\ell'}{\ell} < 1$$

holds true. A plot of the dependence of the dimensionless surface energy

$$\hat{\gamma} = 2\ell\frac{\gamma_{se}}{Gc^2}$$

on the surface energy length scale is shown in Fig. 4.3. If $\ell' = 0$ then no surface energy is assigned to the new surface, but on the other hand the presence of initial stress that makes the material to be stiffer, is effectively modeled.

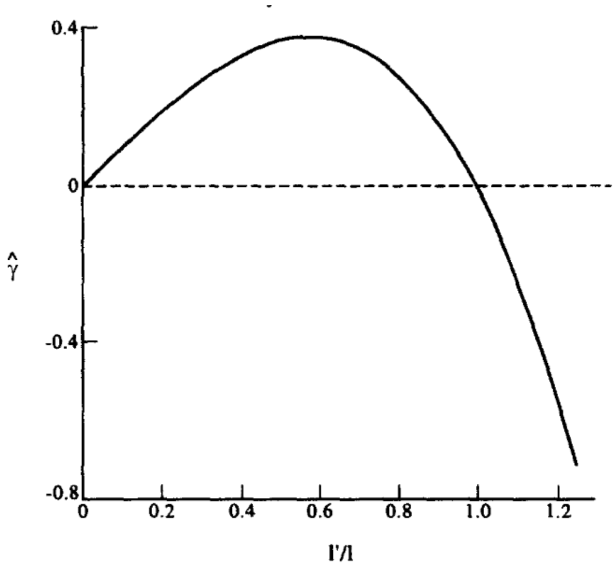


Fig. 4.3 Plot of normalized specific surface energy vs the relative surface energy length scale (Exadaktylos and Vardoulakis, 1998).

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