Multi-fidelity Bayesian inference of Concrete Damage Plasticity Model parameters
or
Practical obstacles applying Bayesian inference

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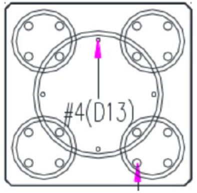
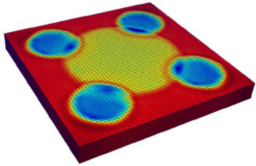
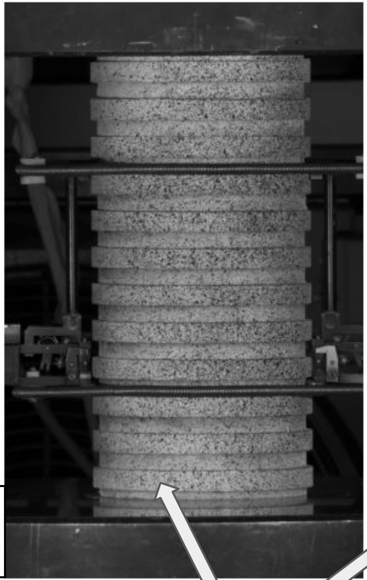
Goal: Design of triaxial loading conditions based on the uniaxial loading regime.

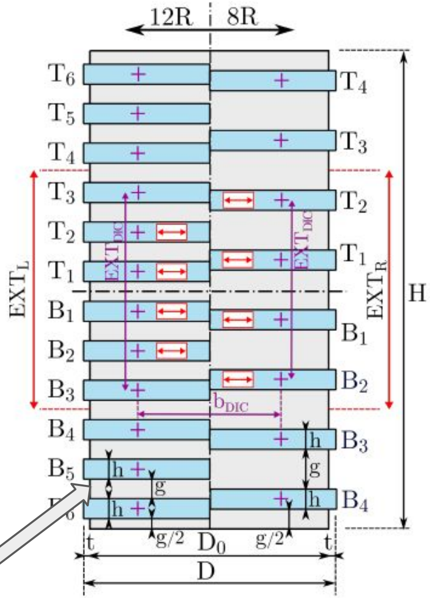
- Cheaper and less time consuming experimental setup.
- Development of multi-spiral reinforcement of concrete columns.

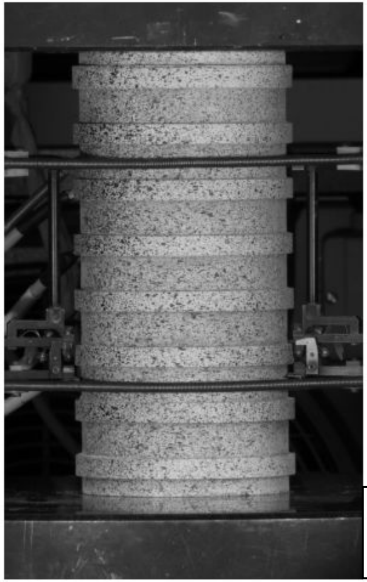






12R t3





8R t2

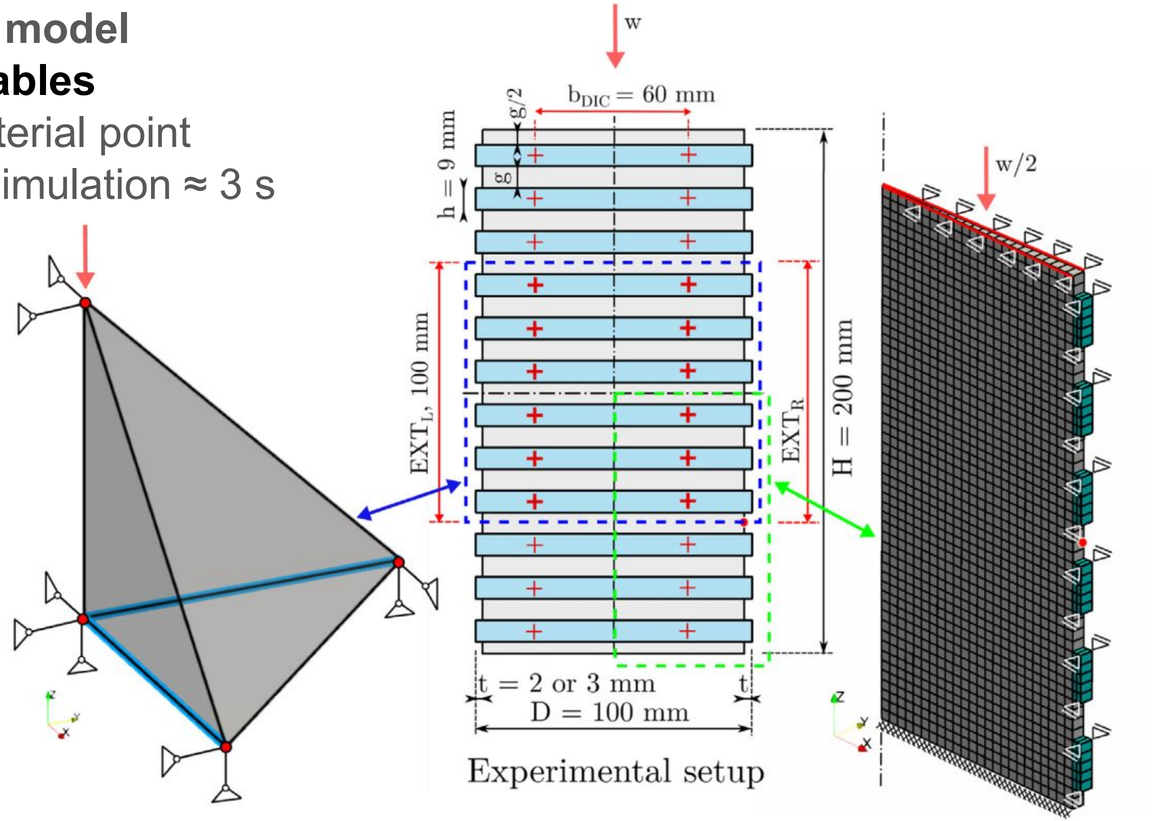
Aluminum rings differ in overall number (8 vs. 12) and thickness (2 vs. 3 mm)

Low-fidelity model

- **14 variables**
- one material point
- model simulation ≈ 3 s

High-fidelity model

- **15 material** parameters
- despite symmetry model simulation ≈ 1 h

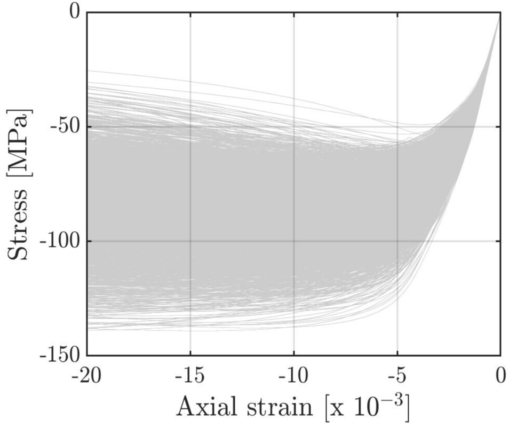


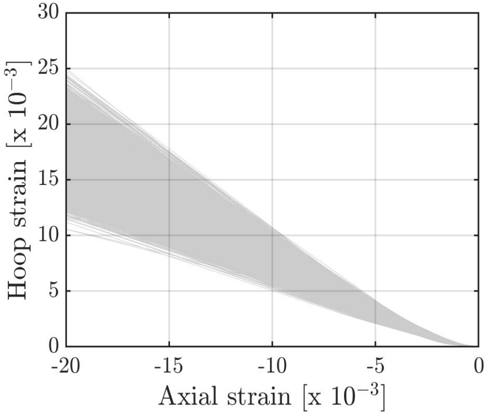
Low-fidelity model High-fidelity model

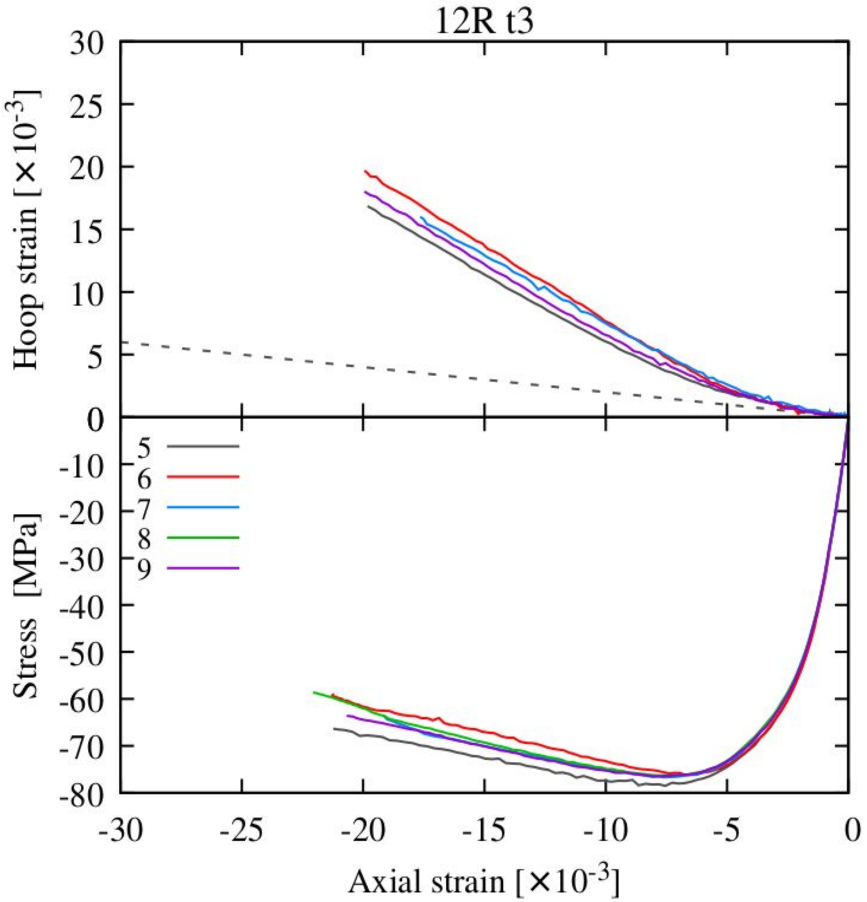
Numerical models of the experiments

- Experiment controlled by axial strain
- Observations:
  - Axial stress
  - Lateral/hoop strain

- Latin Hypercube Sampling derived from Halton sequence with 30,000 samples







Polynomial chaos approximation for low-fidelity model

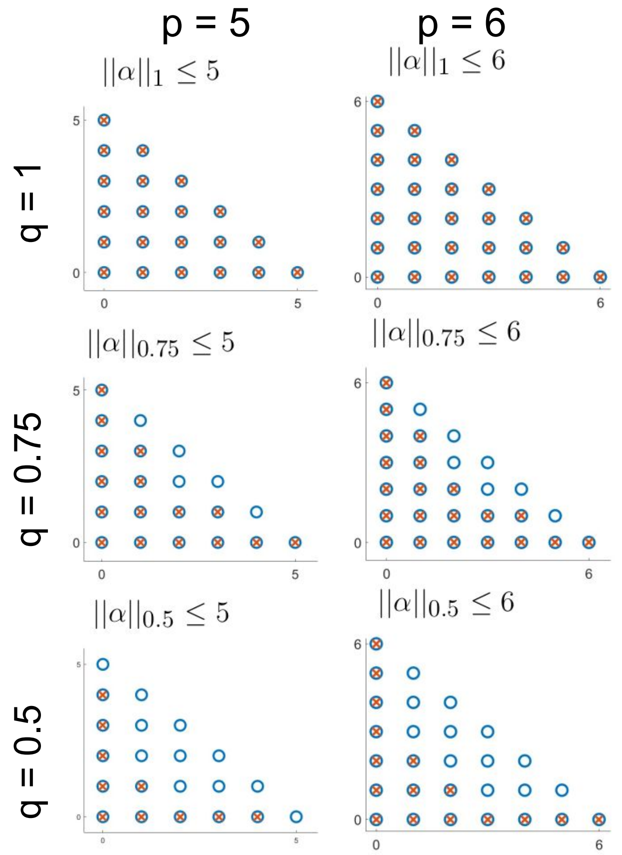
How to choose best bases of polynomial chaos
for 14 variables and 30,000 prior simulations?

$$\widehat{y}(\boldsymbol{X}) = \sum_{\boldsymbol{\alpha} \in \mathcal{A}} \widehat{\beta}_{\boldsymbol{\alpha}} \psi_{\boldsymbol{\alpha}}(\boldsymbol{X}) \qquad |\mathcal{A}^{n_X,p}| = \frac{(p+n_X)!}{p!\,n_X!}$$

- *hyperbolic truncation scheme* [Blatman & Sudret,2011]

$$\mathcal{A}^{n_X,p,q} = \left\{ \boldsymbol{\alpha} \in \mathbb{N}^{n_X} : \left(\sum_{k=1}^{n_X} \alpha_k^q \right)^{1/q} \leq p \right\}$$

- $p = 5, q = 1.00$, 11628 terms
- $p = 8, q = 0.77$, 11215 terms
- $p = 11, q = 0.63$, 11985 terms
- $p = 14, q = 0.57$, 11208 terms



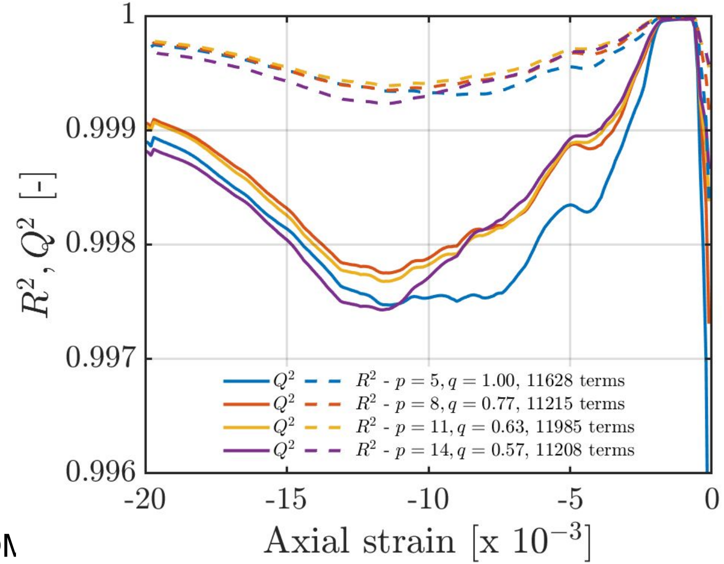
Polynomial chaos approximation for low-fidelity model

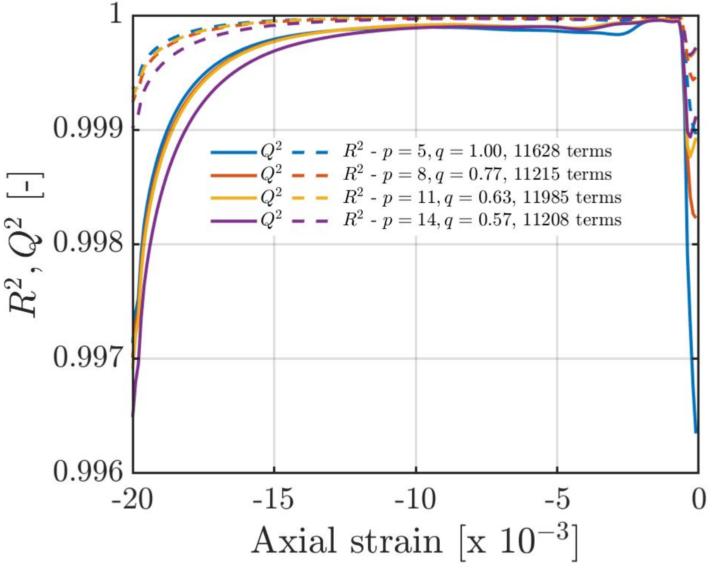
Error estimates: *coefficient of determination*:

$$R^2 = 1 - \frac{\frac{1}{n_s}\sum_{s=1}^{n_s}\left(\widehat{y}(\boldsymbol{x}_s) - y_s\right)^2}{\frac{1}{n_s}\sum_{s=1}^{n_s}\left(y_s - \bar{y}\right)^2} = \frac{s^2(\widehat{\boldsymbol{y}})}{s^2(\boldsymbol{y})}$$

Leave-one-out crossvalidation:

$$Q^2 = 1 - \frac{\frac{1}{n_s}\sum_{s=1}^{n_s}\left(\hat{\epsilon}(\boldsymbol{x}_s)\right)^2}{\frac{1}{n_s}\sum_{s=1}^{n_s}\left(y_s - \bar{y}\right)^2} = 1 - \frac{\frac{1}{n_s}\sum_{s=1}^{n_s}\left(\frac{\widehat{Y}(\boldsymbol{x}_s)-y_s}{1-h_{ss}}\right)^2}{s^2(\boldsymbol{y})}$$

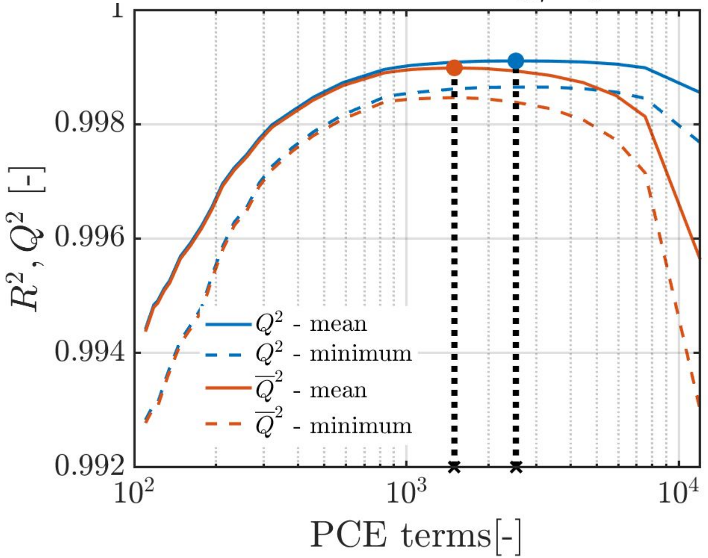




Cutting off terms with smallest relative variance: $$S_{\boldsymbol{\alpha}} = \frac{\mathrm{Var}\left[\widehat{y}_{\boldsymbol{\alpha}}(\boldsymbol{X})\right]}{\mathrm{Var}\left[\widehat{y}(\boldsymbol{X})\right]}$$

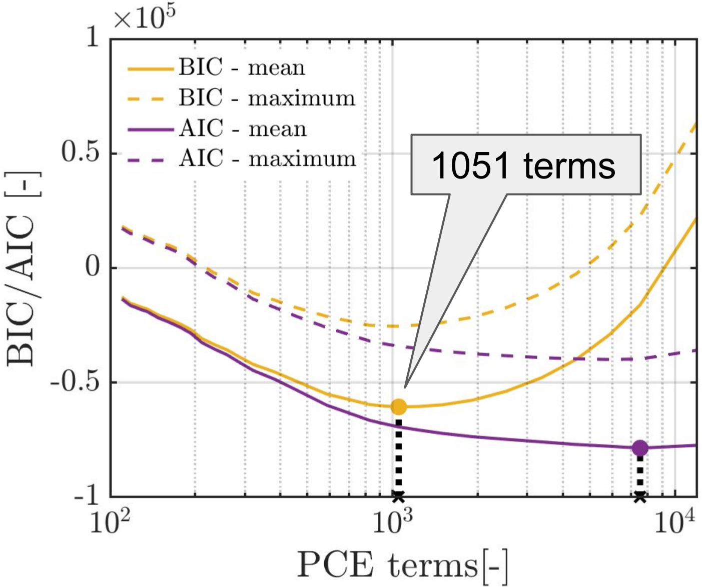
Adjusted cross-validation: [Chapelle et al.,2002]

$$\overline{Q}^2 = 1 - (1 - Q^2)\frac{1 + \mathrm{tr}\left((\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}\right)}{1 - n_\alpha / n_s}$$



Akaike and Bayesian information criteria:

$$\mathrm{AIC} = -2\ln(\hat{L}) + 2n_\alpha\,,$$
$$\mathrm{BIC} = -2\ln(\hat{L}) + n_\alpha \ln n_s$$
$$\hat{L} = \frac{1}{n_s}\sum_{s=1}^{n_s}(\widehat{y}(\boldsymbol{x}_s) - y_s)^2$$

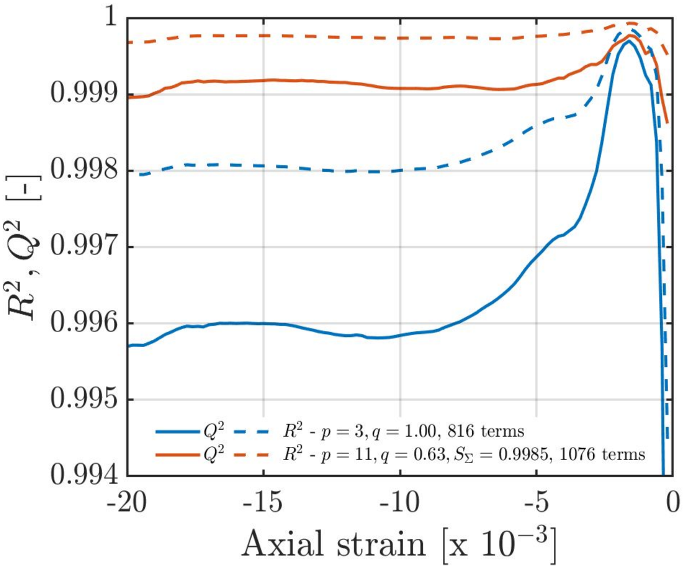


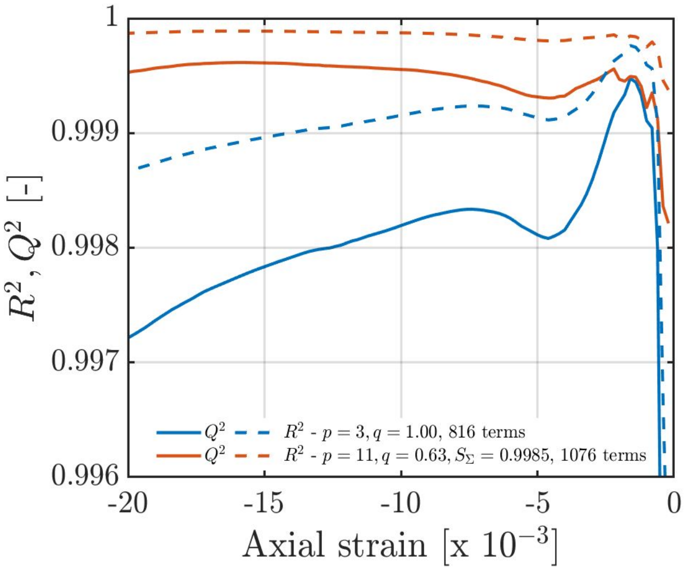
Polynomial chaos approximation for high-fidelity model

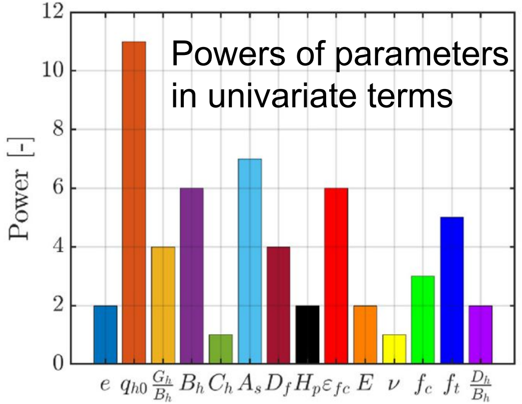
- Chosen topology for LF model extended by 25 terms related to additional 15th variable
  ⇨ **Sparse hyperbolic PCE up to 11th degree with 1076 terms in total**

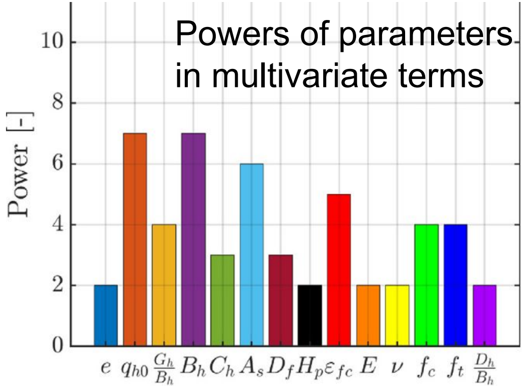
  ⇨ ≈ 3,000 HF model simulations of 4 experiments

- Comparison to full PCE up to 3rd degree with 816 terms



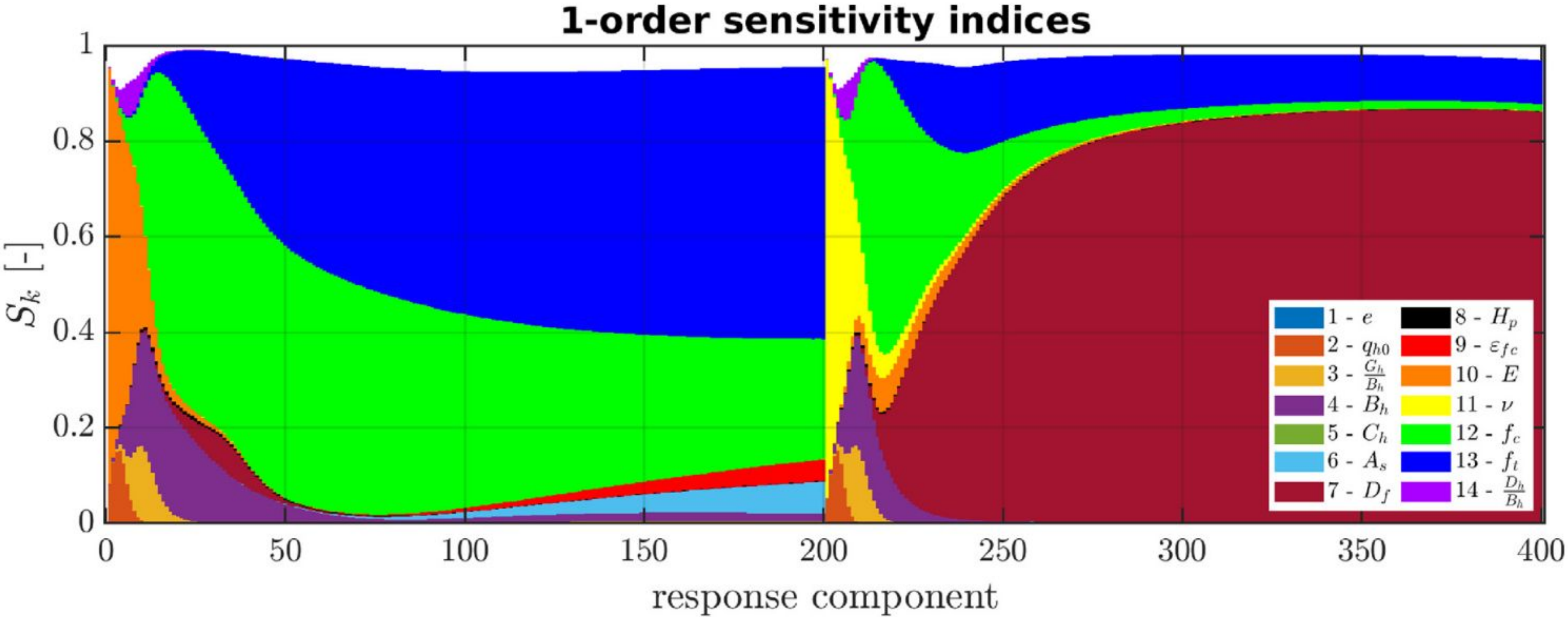






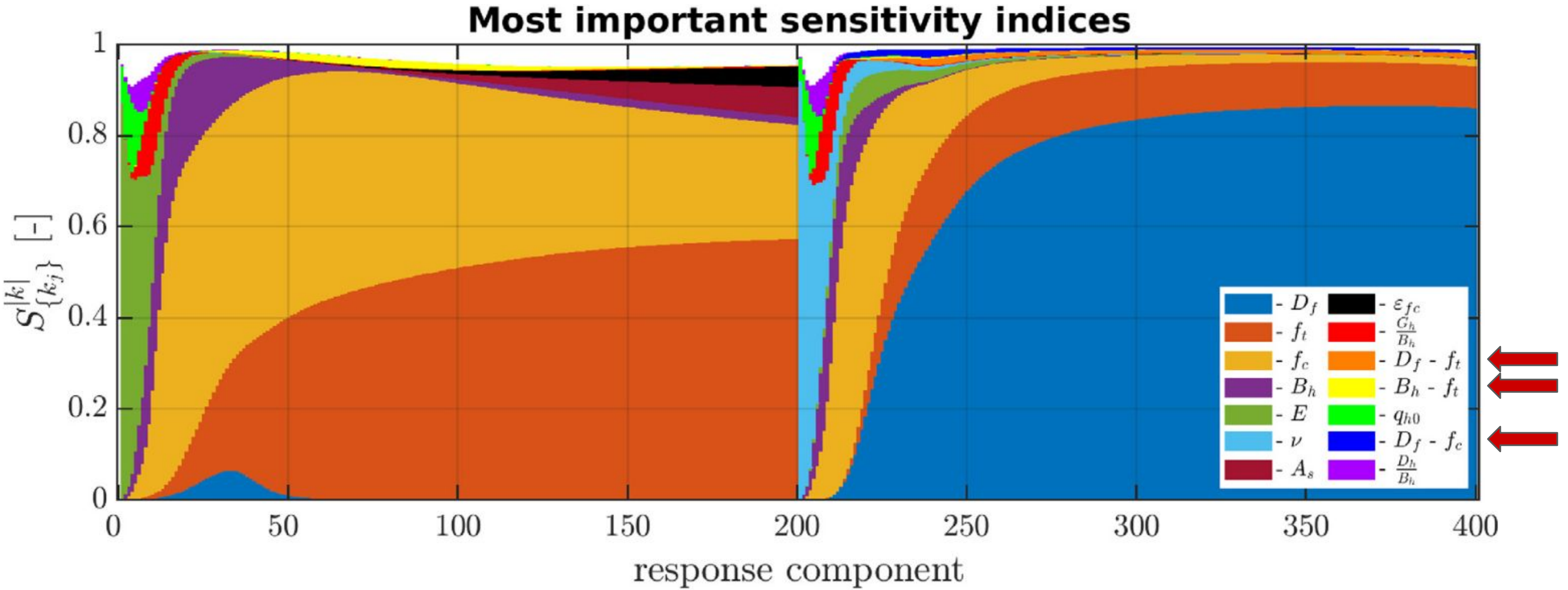
Sensitivity analysis - analysis of variance

- first-order Sobol' indices $$S_k = \sum_{\alpha \in \mathcal{A}_k} S_{\boldsymbol{\alpha}} \quad \mathcal{A}_k = \{\boldsymbol{\alpha} \in \mathcal{A} : \alpha_k > 0\,,\ \alpha_{j \neq k} = 0\}$$



- higher-order Sobol' indices

$$S_{\{k_j\}}^{|k|} = \sum_{\alpha \in \mathcal{A}_{\{k_j\}}^{|k|}} S_{\boldsymbol{\alpha}} \quad \mathcal{A}_{\{k_j\}}^{|k|} = \{\boldsymbol{\alpha} \in \mathcal{A} : \alpha_{k_j \in \boldsymbol{k}} > 0\,, \alpha_{k_j \notin \boldsymbol{k}} = 0\}$$



Bayesian inference - and now come troubles

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Bayes' rule:

$$p(\boldsymbol{X}|\boldsymbol{Z}) = \frac{p(\boldsymbol{Z}|\boldsymbol{X})p(\boldsymbol{X})}{p(\boldsymbol{Z})}$$

Prior distribution:

- **Generally so difficult to define!**

- Often several iterations/modifications
- Where the model is defined?
- Where the model is stable?
- Unrealistic parameter combinations - prior should often be correlated?

Here:
uniform with several modifications of bounds

Parameter	Units	Minimum	Maximum
e	[-]	0.5	0.6
q_{h0}	[-]	0	0.4
G_h/B_h	[-]	10	30 (20)
B_h	$[10^{-3}]$	1	8 (5)
C_h	[-]	1	5
A_s	[-]	5	50
D_f	[-]	1.0	1.7
H_p	[-]	0	0.25
ε_{fc}	$[10^{-3}]$	0.05	0.35
E	[MPa]	35000	40000
ν	[-]	0.15 (0.17)	0.22
f_c	[MPa]	40	60 (50)
f_t	[MPa]	2	8 (6)
D_h/B_h	[-]	0	0.1
w_f	$[10^{-6}$ m]	30	400

$$\mathbf{Z}_e = \mathbf{F}\widehat{\boldsymbol{Y}}_e(\boldsymbol{X}) + \boldsymbol{\epsilon}_{\mathrm{M},e} + \boldsymbol{\epsilon}_{\mathrm{T},e}$$

Likelihood:

- all errors assumed to be Gaussian

$$p(\mathbf{Z}_e|\boldsymbol{X}) = \prod_c \frac{1}{\sqrt{(2\pi)^{n_z}|\mathbf{C}_e|}} \mathrm{e}^{-\frac{1}{2}(\mathbf{F}\widehat{\boldsymbol{Y}}_e(\boldsymbol{X}) - \mathbf{Z}_{e,c})^T \mathbf{C}_e^{-1} (\mathbf{F}\widehat{\boldsymbol{Y}}_e(\boldsymbol{X}) - \mathbf{Z}_{e,c})}$$

$$\mathbf{C}_e = \mathbf{C}_{\mathrm{M},e} + \mathbf{C}_{\mathrm{T},e}$$

Error of surrogate - based on leave-one-out cross-validation:

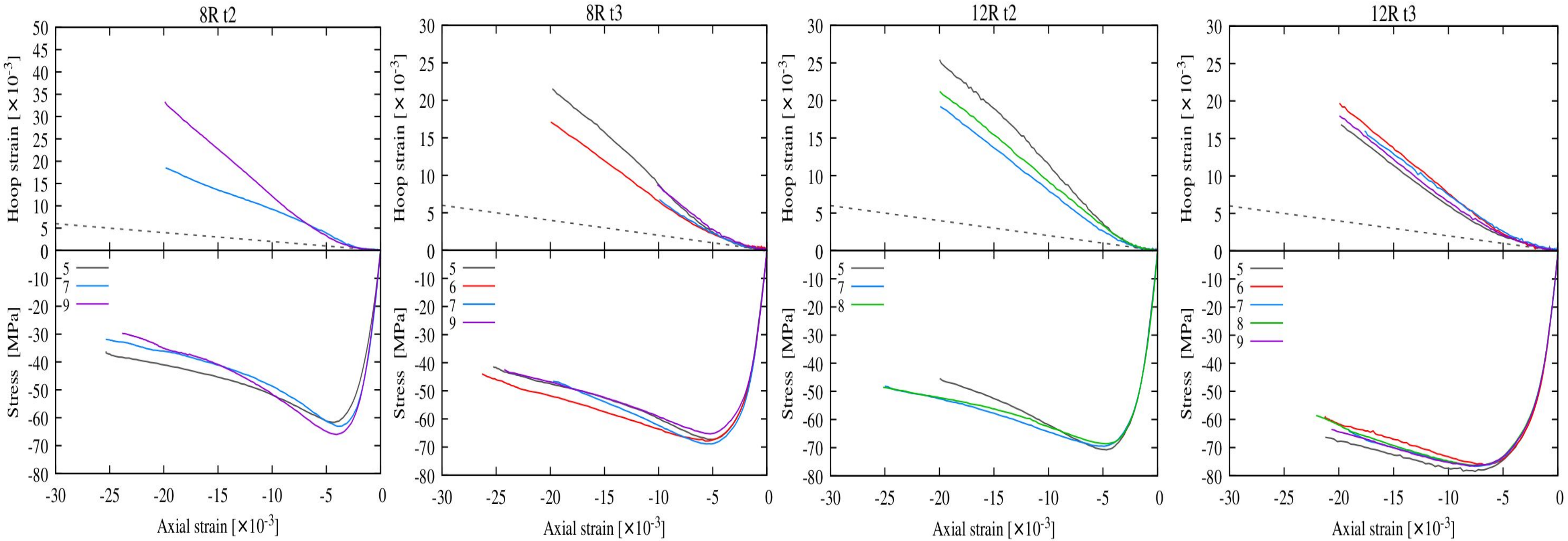
$$\mathbf{C}_{\mathrm{T},e} = \frac{\mathbf{F}\boldsymbol{\Delta}_e\boldsymbol{\Delta}_e^{\mathrm{T}}\mathbf{F}^{\mathrm{T}}}{n_s} \qquad \boldsymbol{\Delta}_e = \left(\widehat{\epsilon}_{\mathrm{T},e}(j,s)\right) \in \mathbb{R}^{n_y \times n_s} = \frac{y_{e;j,s} - \widehat{y}_{e;j,s}}{1 - h_{ss}}$$

Bayesian inference - experimental errors may be more tricky

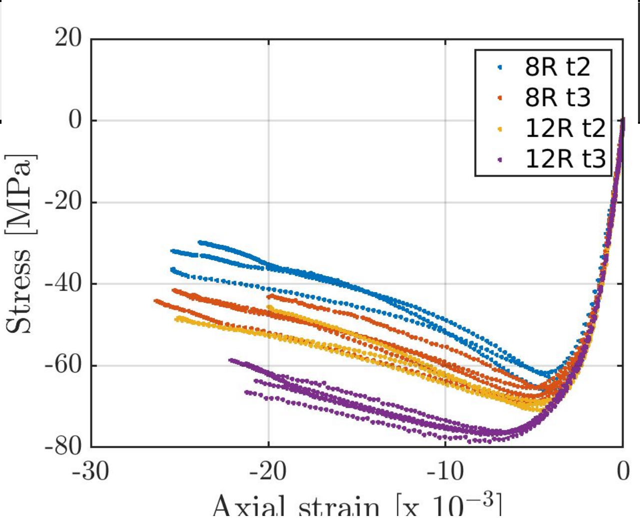
$$\mathbf{Z}_e = \mathbf{F}\widehat{\boldsymbol{Y}}_e(\boldsymbol{X}) + \boldsymbol{\epsilon}_{\mathrm{M},e} + \boldsymbol{\epsilon}_{\mathrm{T},e}$$

Experimental errors - how to choose their statistics?

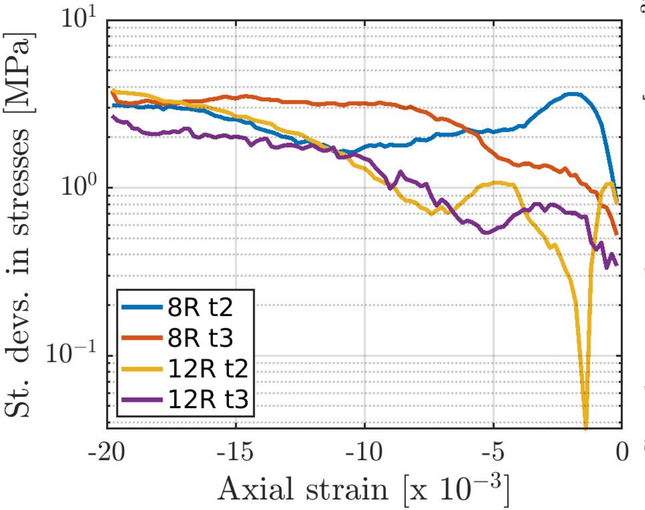
- 4 experiments with different boundary conditions - 8 or 12 rings being 2 or 3 mm thick
- 2 - 5 specimens for each experiment

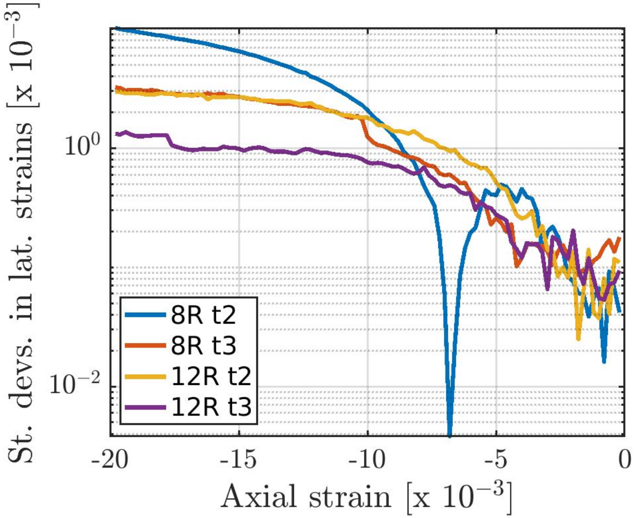


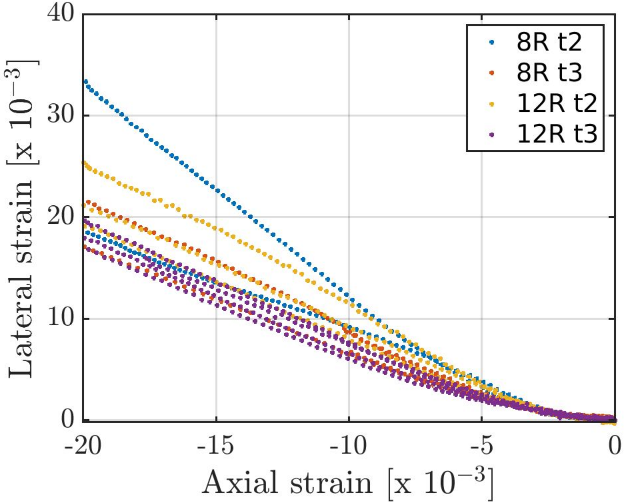
Experiment data and measurement error

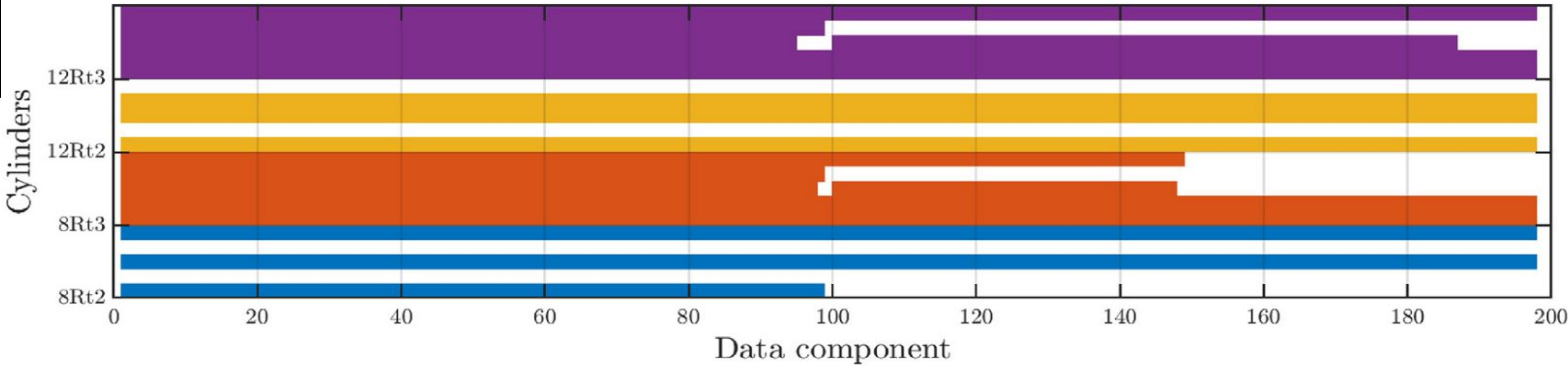


- All specimens are from same material
  - same concrete mix
  - we search for the material parameters of this mix
- Concrete is heterogeneous, every sample differs in morphology and exhibits different behavior
- Are observations correlated?





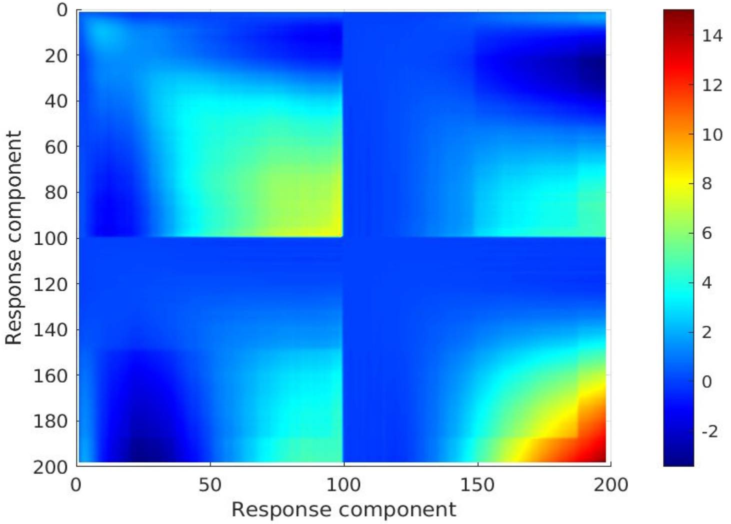




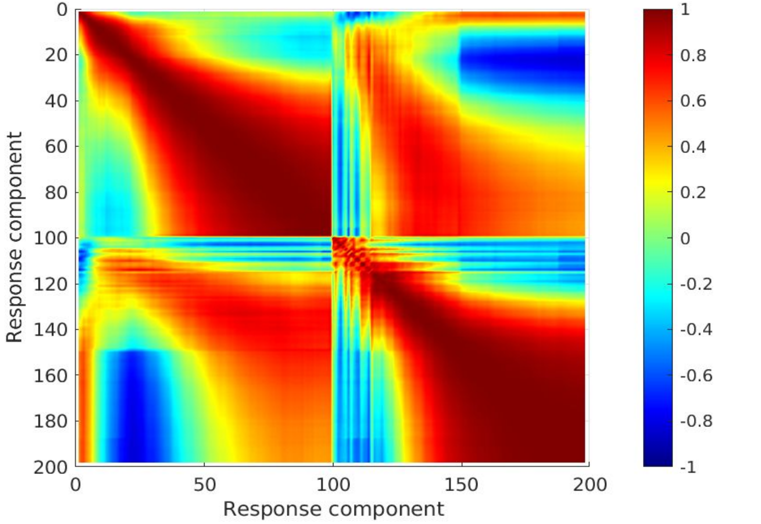
Data covariance

- indefinite matrix has 25 neg. eigenvalues

Covariance matrix



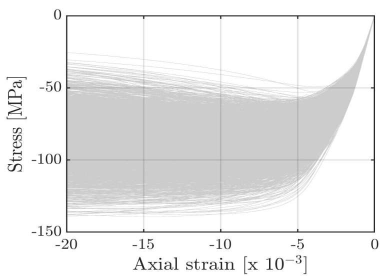
Correlation matrix

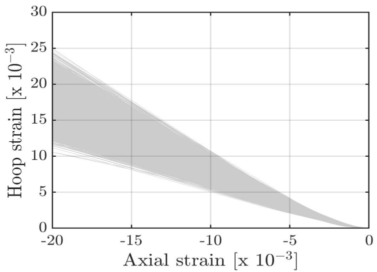


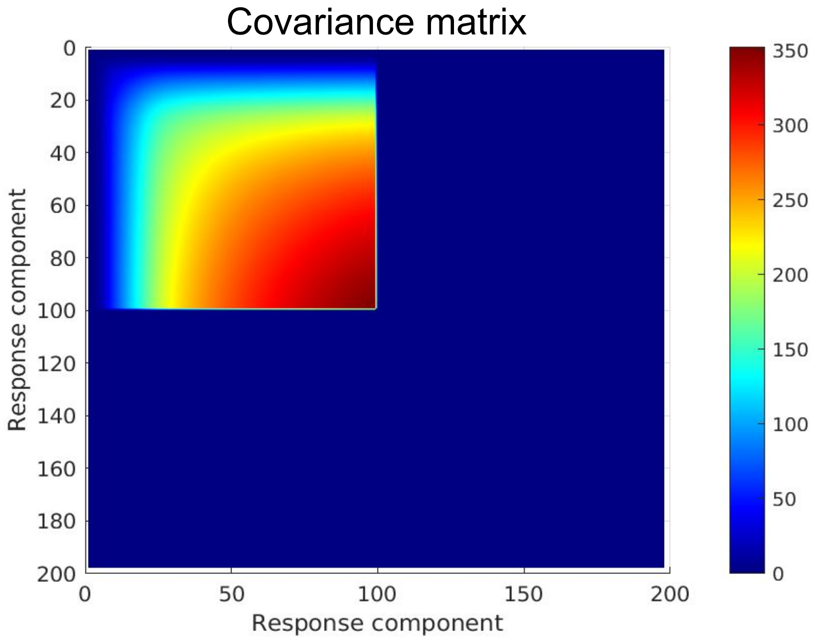
Covariance from prior simulations

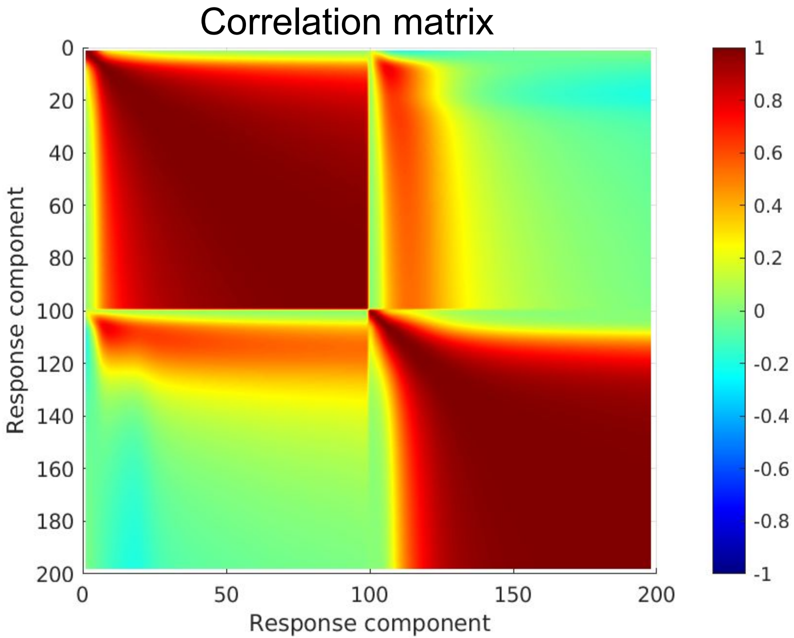
- enough simulations, positive definite matrix

Can we somehow use it?









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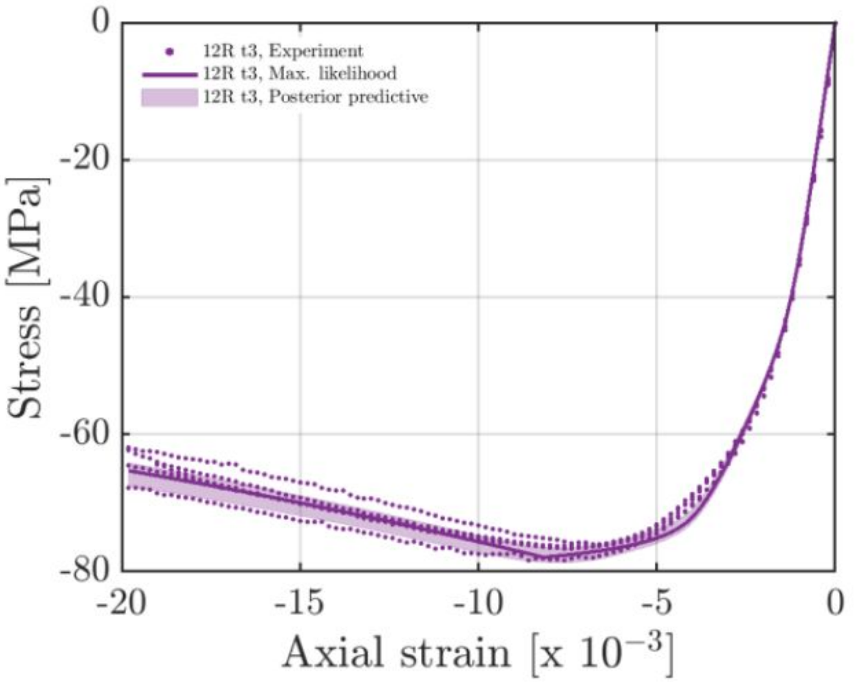
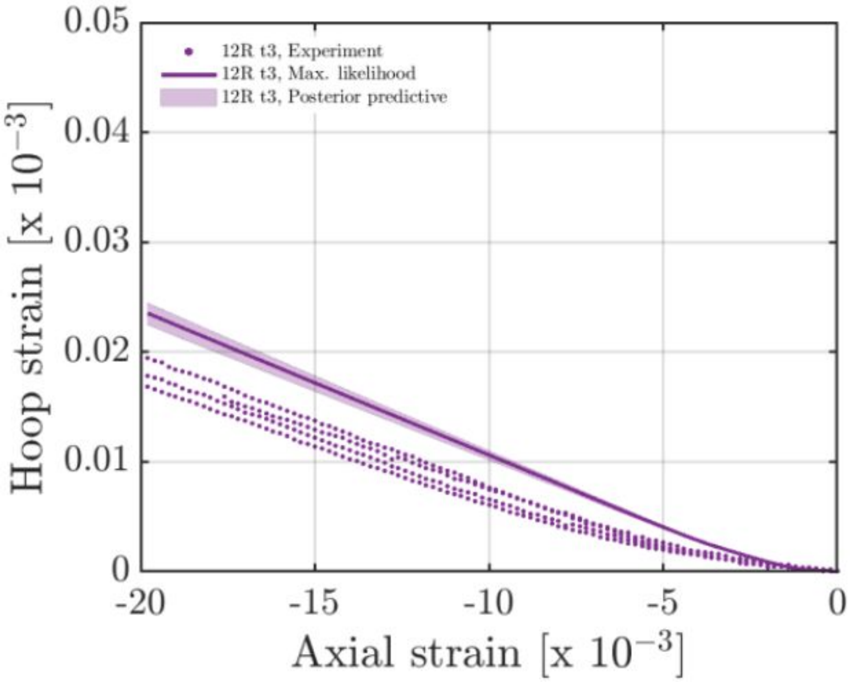
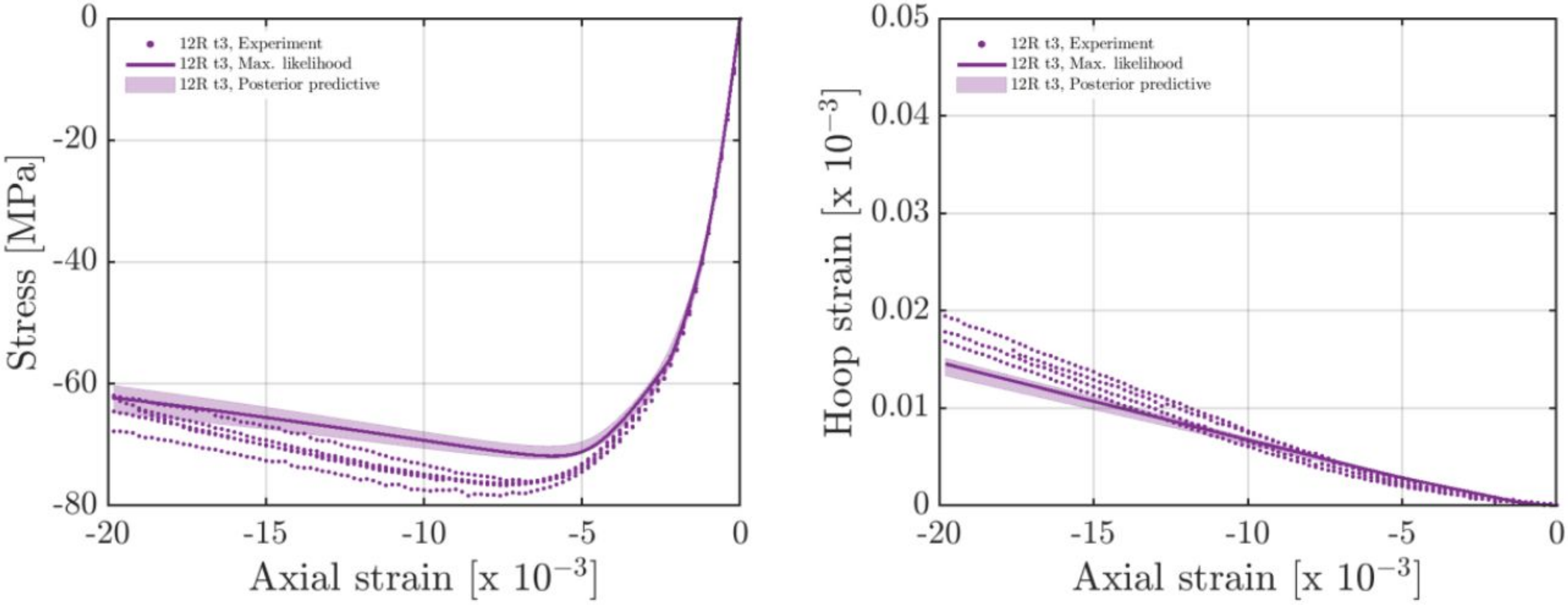
Data Covariance corrected using Covariance obtained for prior simulations

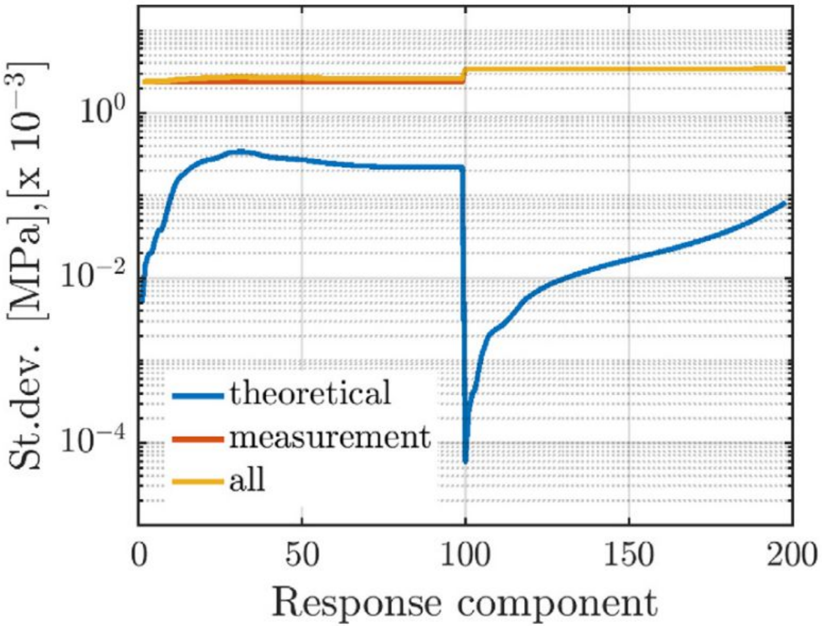


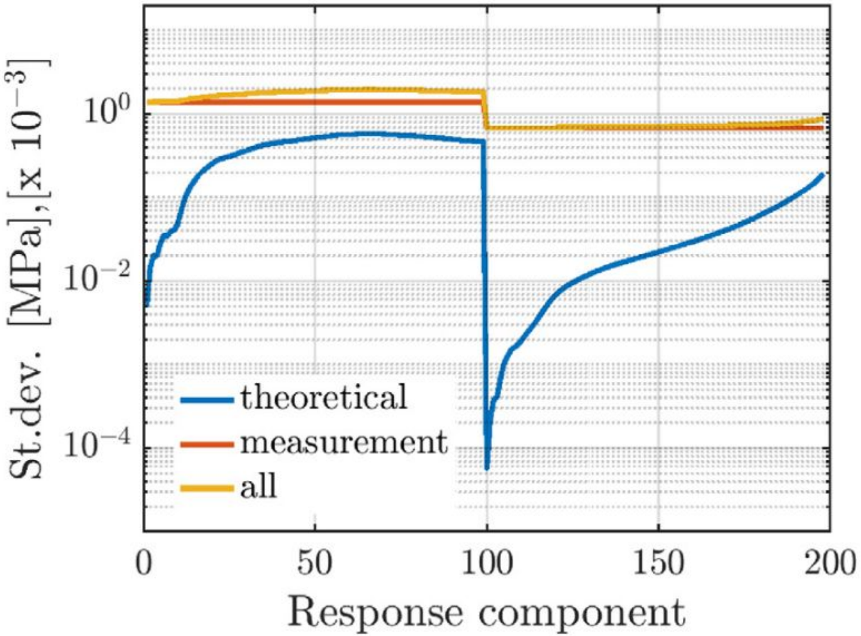


Data Covariance computed from data with gaps filled by mean



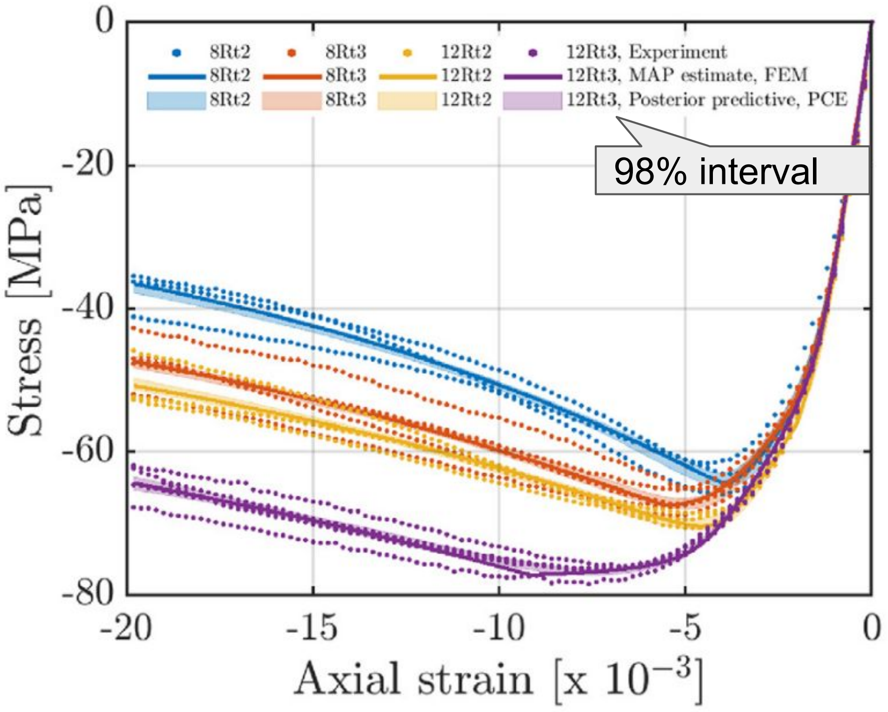
Experiment data and measurement error

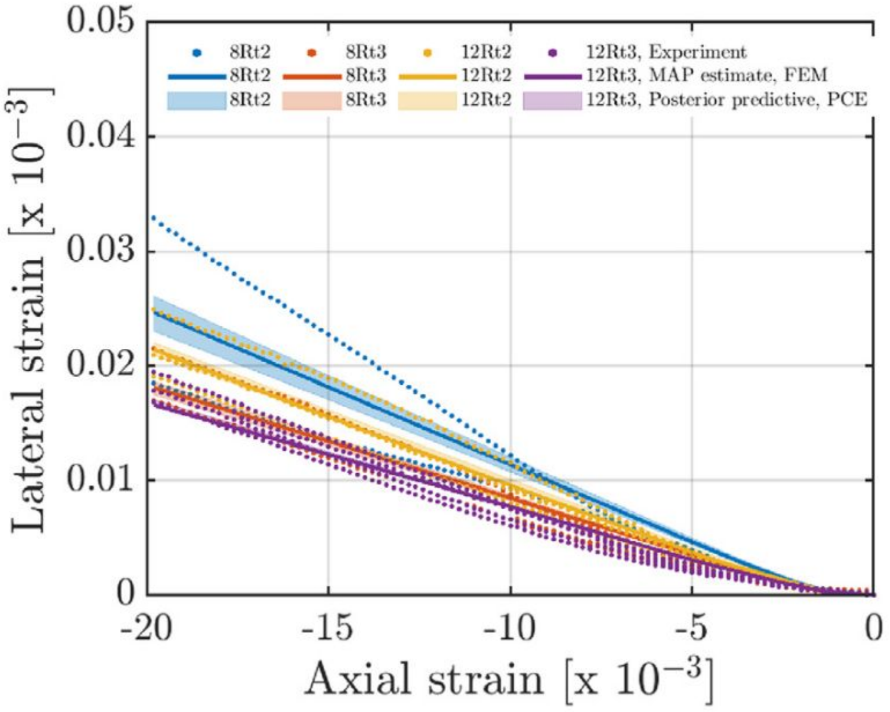




Final decision: neglect all correlations and use only diagonal covariance matrix

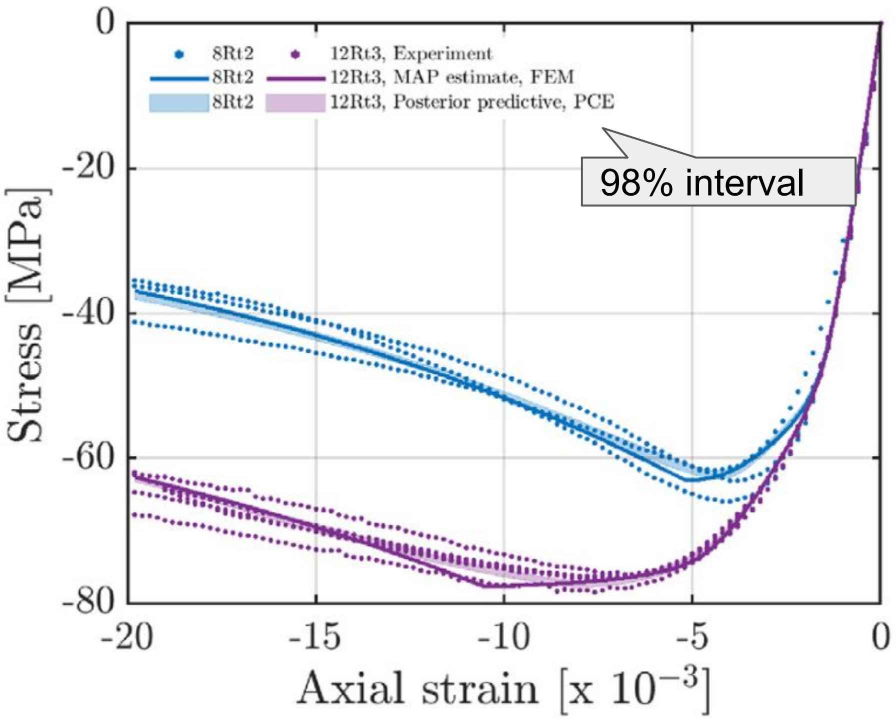
Best fits for all experiments

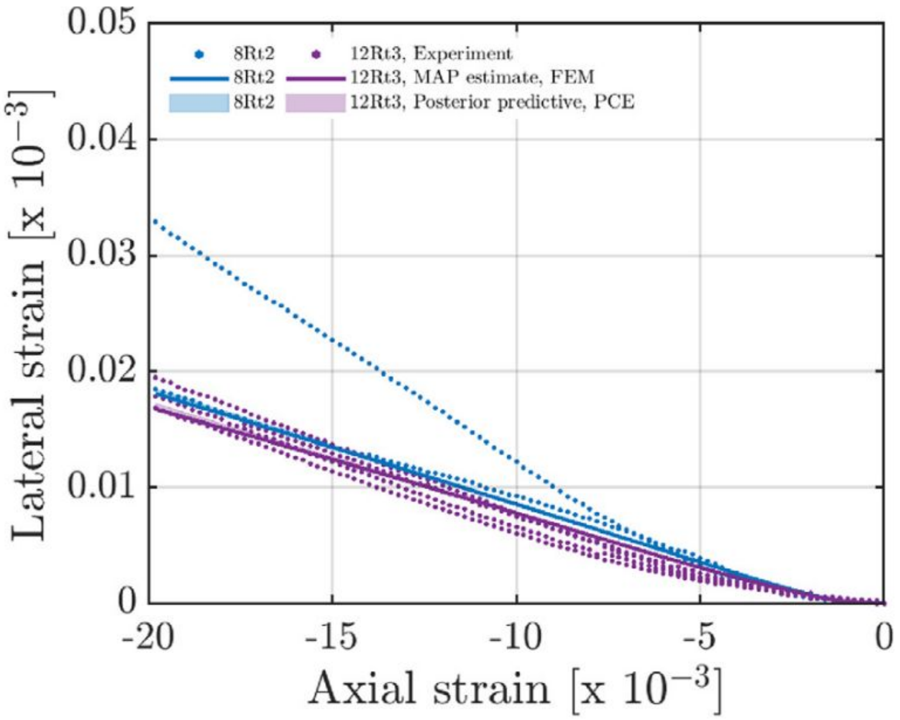




Posterior predictive considering uncorrelated data

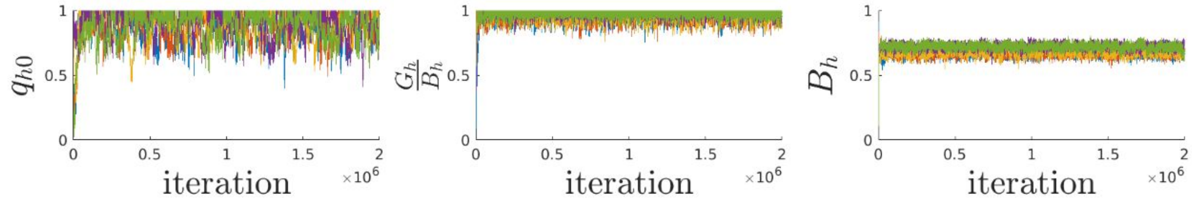
Best fits of both experiments 8Rt2 and 12Rt3

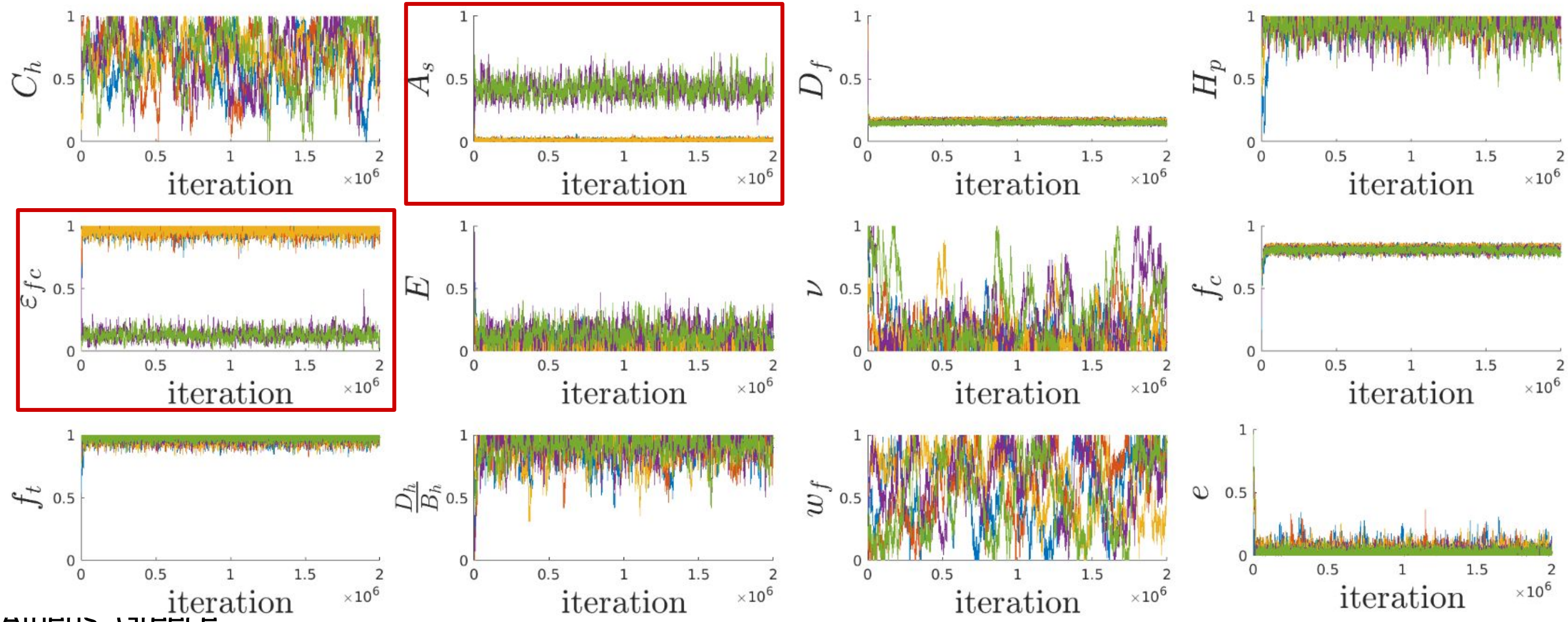




Markov chain Monte Carlo sampling of posterior

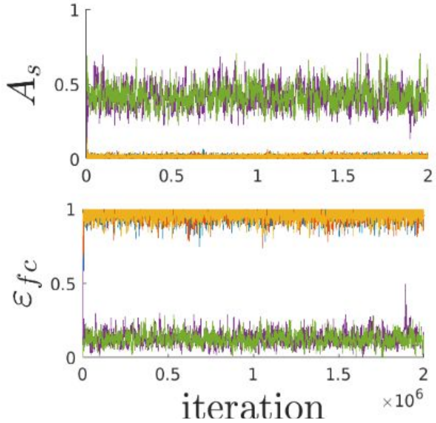
- normally distributed proposal
- acceptance rate 20% - 50%
- 5 independent chains starting at random

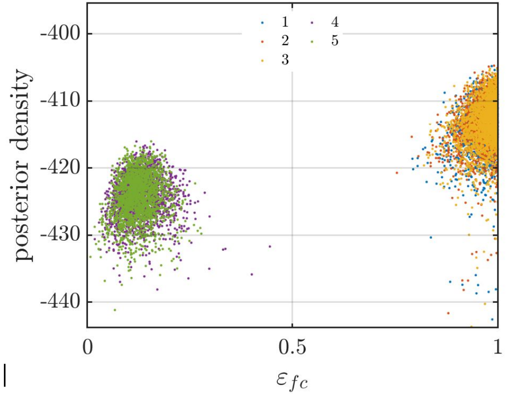


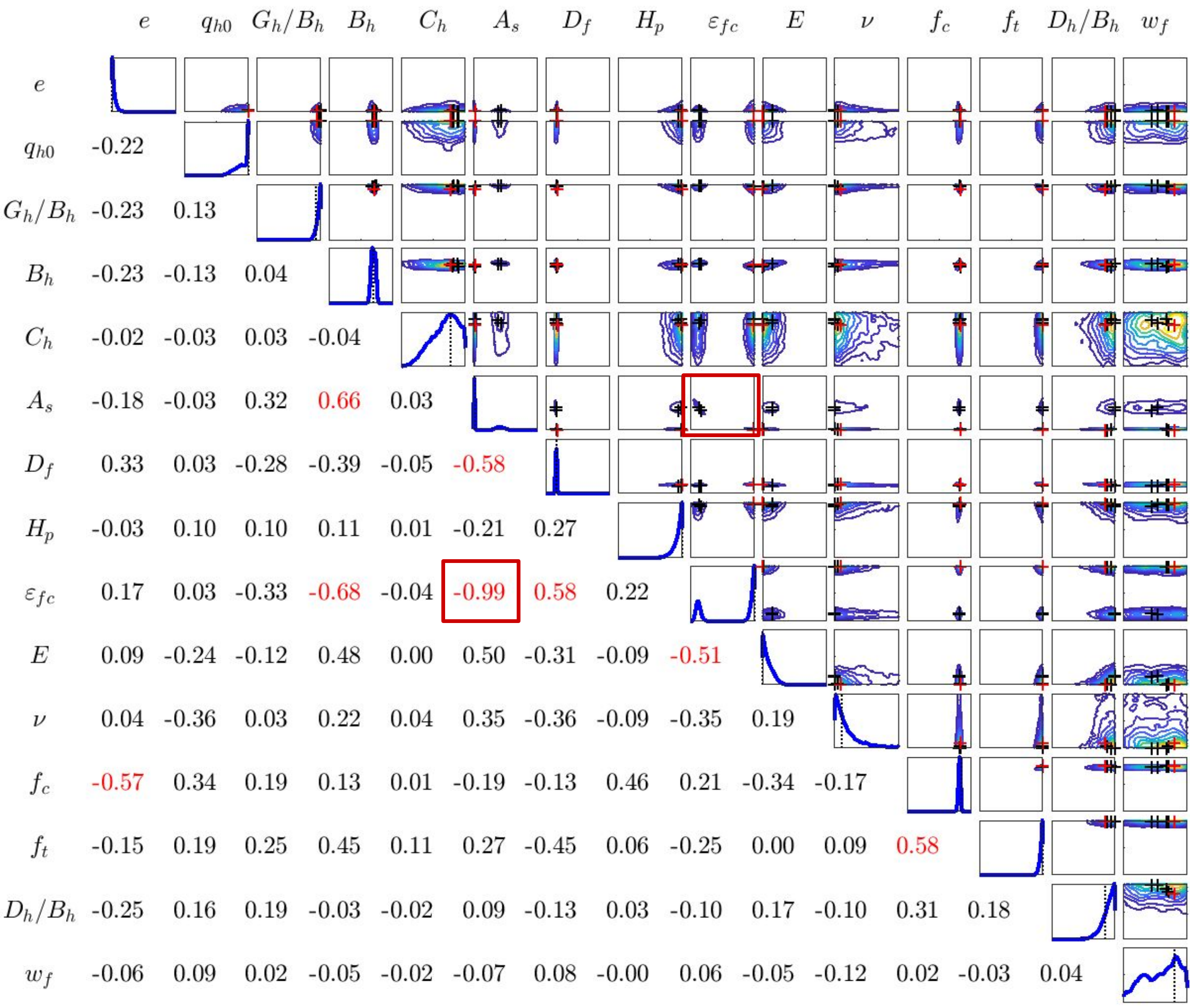


Markov chain Monte Carlo sampling of posterior

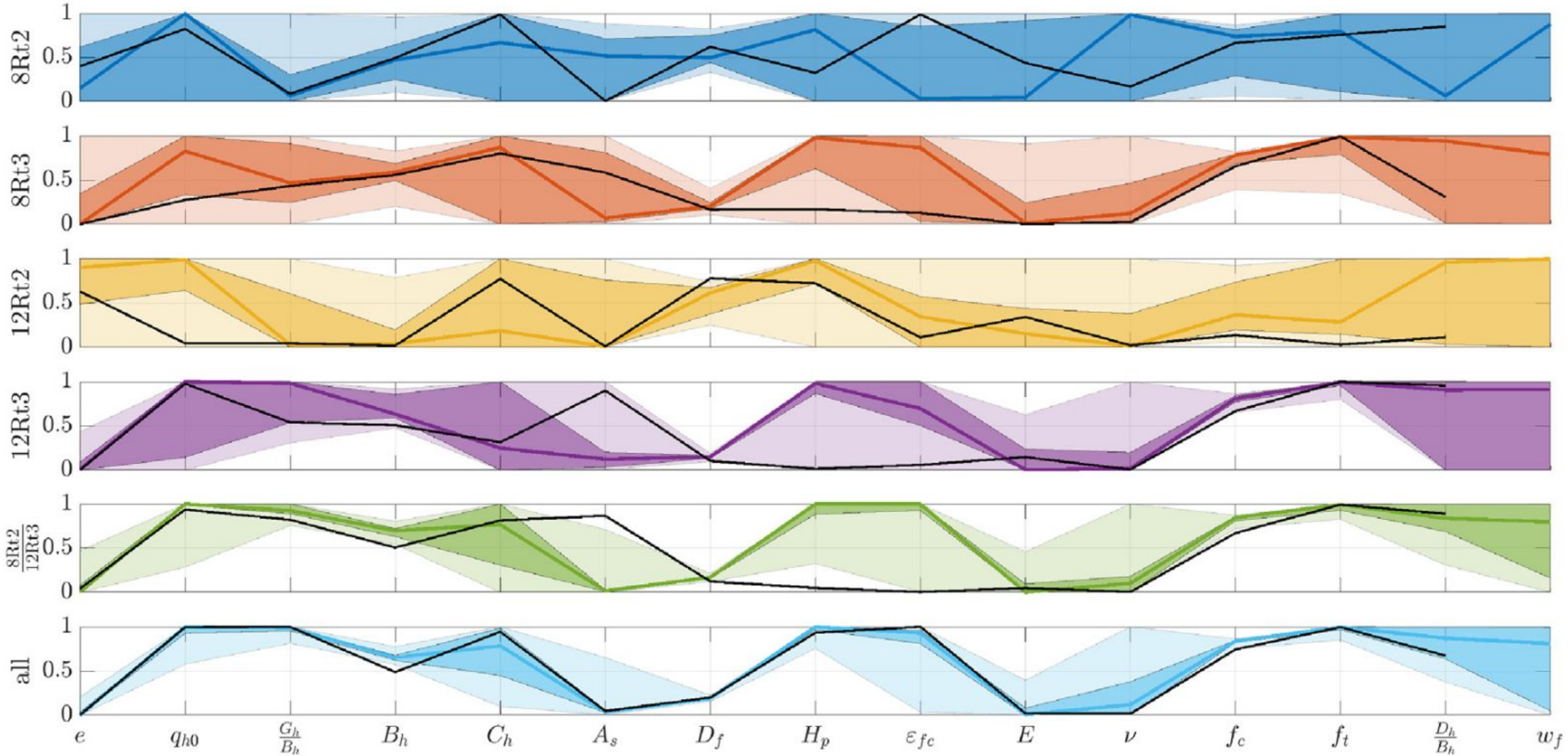
Experiments 8Rt2 and 12Rt3







Parameters in 98% and 0.1% posterior intervals identified using HF model (black line = max posterior identified using LF model)



Conclusions

- Yes,
  - It is possible to identify 11 out of 15 parameters of complex damage- plastic model for concrete using two similar and cheap compression tests with a satisfactory accuracy.

    A. Kučerová, J. Sýkora, P. Havlásek, D. Jarušková and M. Jirásek: Efficient probabilistic multi-fidelity calibration of a damage-plastic model for confined concrete. *Computer Methods in Applied Mechanics and Engineering*, Volume 412, 1 July 2023, 116099.

  - Accuracy of a surrogate and computational costs are becoming manageable in many tasks of material parameter inference.

- But,
  - Interpretation of posterior uncertainties needs to be done very carefully as it is very dependent of often very weak formulation of prior distribution and also uncertainties in likelihood.

  - Estimation of correlations in data can be more tricky than it seems.