Stochastic modelling of heterogeneous materials based on image analysis

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Seminar on Uncertainty Modelling in Engineering

Motivation

Modelling of heterogeneous materials with

- random spatial distribution of particular constituents
- characteristic length comparable to the macroscopic lengthscale

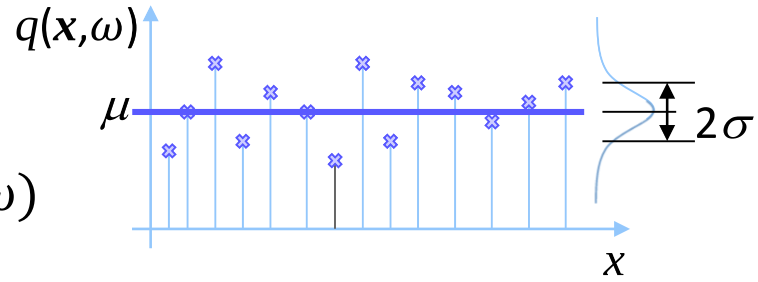
irregular masonry

→ assumptions required for application of homogenization-based modelling are inadequate

→ variation in material properties has significant influence on structural response and needs to be considered

Random fields

Definition of a random field $q(\boldsymbol{x}, \omega)$**:**



mean: $\mu_q(\boldsymbol{x}) = \mathbb{E}[q(\boldsymbol{x}, \omega)] = \int_\Omega q(\boldsymbol{x}, \omega)\, \mathbb{P}(\mathrm{d}\omega)$

covariance function: $C_q(\boldsymbol{x}, \boldsymbol{x}') = \mathbb{E}\left[\left(q(\boldsymbol{x}, \omega) - \mu_q(\boldsymbol{x})\right)\left(q(\boldsymbol{x}', \omega) - \mu_q(\boldsymbol{x}')\right)\right] =$

$$= \int_\Omega \left(q(\boldsymbol{x}, \omega) - \mu_q(\boldsymbol{x})\right)\left(q(\boldsymbol{x}', \omega) - \mu_q(\boldsymbol{x}')\right)\mathbb{P}(\mathrm{d}\omega)$$

Modelling of a random field:

- discretization: by FE mesh into a finite but high number of points

  covariance function $C_q(\boldsymbol{x}, \boldsymbol{x}')$ becomes covariance matrix $\mathbf{C}_q$

- dimensionality reduction: by solving eigenvalue problem: $\mathbf{C}_q \boldsymbol{\phi}_i = \zeta_i \boldsymbol{\phi}_i$

  leads to Karhunen-Loève (KL) expansion: $\hat{\boldsymbol{q}}(\omega) \approx \mu_q + \sum_{i=1}^{M} \sqrt{\zeta_i}\, \xi_i(\omega) \boldsymbol{\phi}_i$

Uncertainty propagation

[e.g. Ghanem & Spanos, 1991, SFE or Matthies & Keese, 2005, CMAME]

Demonstration for a linear system: $\boldsymbol{A}\left(q\left(\boldsymbol{\xi}(\omega)\right)\right)\boldsymbol{u}\left(\boldsymbol{\xi}(\omega)\right) = \boldsymbol{f}$

- system response can be approximated by polynomial chaos (PC) with respect to probability measure of $\boldsymbol{\xi}(\omega)$: $\tilde{\boldsymbol{u}}\left(\boldsymbol{\xi}(\omega)\right) = \sum_{\alpha\in\mathcal{J}} \boldsymbol{u}_\alpha H_\alpha\left(\boldsymbol{\xi}(\omega)\right)$

  e.g. Hermite polynomials $H_\alpha\left(\boldsymbol{\xi}(\omega)\right)$ are orthogonal in Gaussian measure

- using KL approximation of $q\left(\boldsymbol{\xi}(\omega)\right)$ and PC approximation of $\boldsymbol{u}\left(\boldsymbol{\xi}(\omega)\right)$:

$$\widehat{\boldsymbol{A}}\left(\hat{q}(\omega)\right)\tilde{\boldsymbol{u}}(\omega) = \boldsymbol{f}$$

- applying Bubnov-Galerkin projection we obtain

$$\forall\beta\in\mathcal{J}: \sum_{\alpha\in\mathcal{J}} \mathbb{E}\left[H_\beta\left(\boldsymbol{\xi}(\omega)\right)\widehat{\boldsymbol{A}}\left(\widehat{\boldsymbol{q}}\left(\boldsymbol{\xi}(\omega)\right)\right)H_\alpha\left(\boldsymbol{\xi}(\omega)\right)\right]\boldsymbol{u}_\alpha = \mathbb{E}\left(\boldsymbol{f}H_\beta\left(\boldsymbol{\xi}(\omega)\right)\right)$$

  which is a large system of equation for PC coefficients

Covariance function

How to determine covariance function of material properties?

- Gaussian covariance function:

$$C(\boldsymbol{x},\boldsymbol{x}') = \sigma_q^2 \exp\left(-\frac{(x-x')^2}{2L_x^2} - \frac{(y-y\prime)^2}{2L_y^2}\right)$$

- Exponential covariance function:

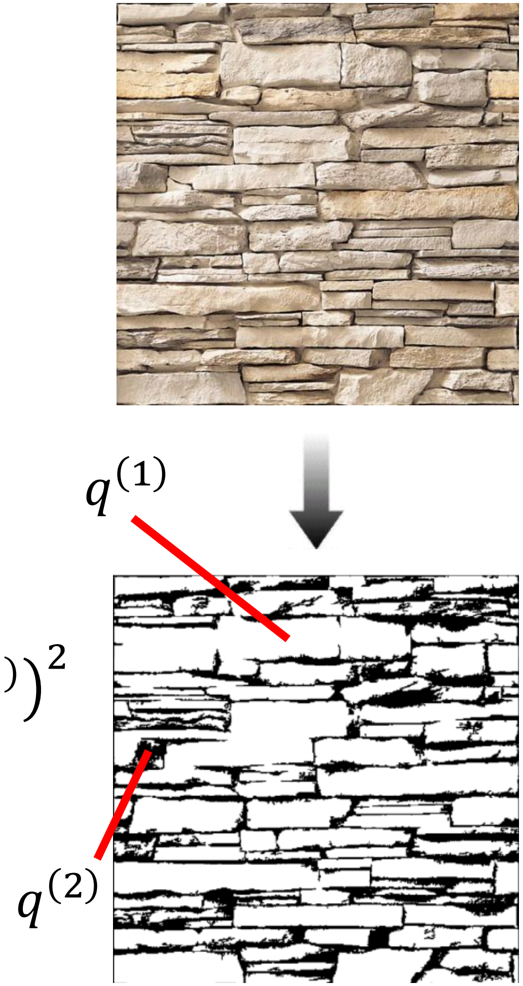
$$C(\boldsymbol{x},\boldsymbol{x}') = \sigma_q^2 \exp\left(-\left|\frac{x-x\prime}{L_x}\right| - \left|\frac{y-y\prime}{L_y}\right|\right)$$

- Image-based covariance function:

$$C(\boldsymbol{x},\boldsymbol{x}') = \left(S_2^{(1)}(\boldsymbol{x},\boldsymbol{x}') - \left(c^{(1)}\right)^2\right)\left(q^{(1)} - q^{(2)}\right)^2$$

$c^{(1)}$ is volume fraction of phase (1)

$S_2^{(1)}(\boldsymbol{x},\boldsymbol{x}')$ is two-point probability function



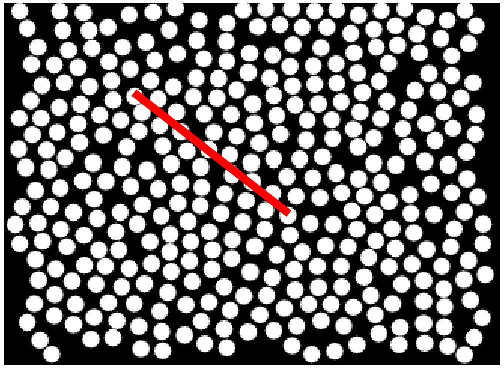
[Lombardo et al., 2009]

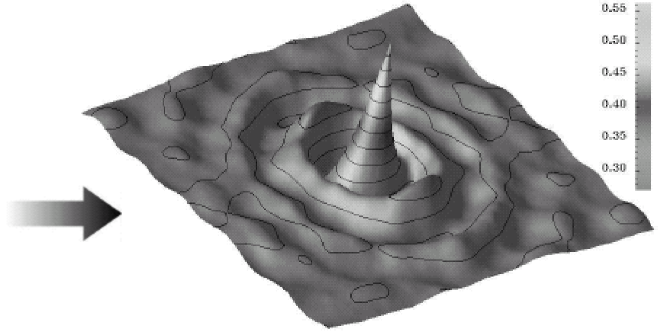
Covariance function

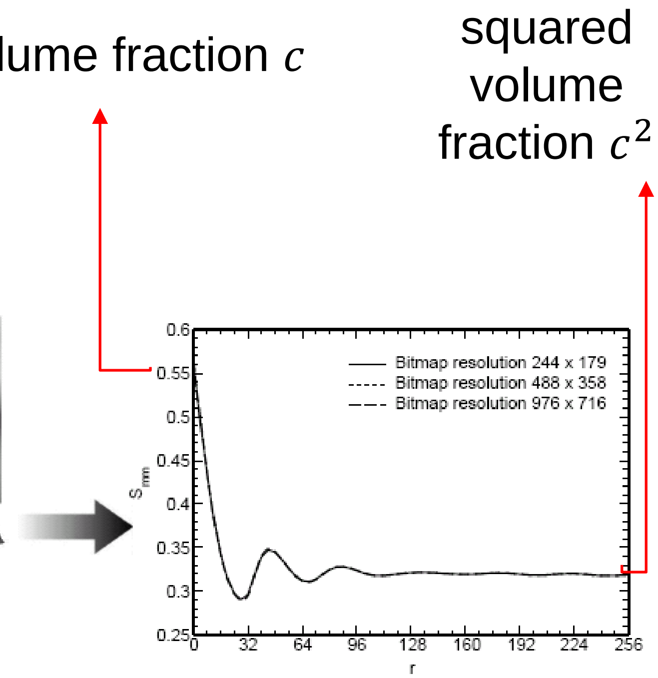
Two point probability function – probability that vector $\boldsymbol{x}\boldsymbol{x}'$ starts and ends in a given phase

$$S_2^{(1)}(\boldsymbol{x},\boldsymbol{x}') = \mathrm{IDFT}\left\{\mathrm{DFT}\left\{\chi^{(1)}(\boldsymbol{x},\boldsymbol{x}')\right\}\overline{\mathrm{DFT}\{\chi^{(1)}(\boldsymbol{x},\boldsymbol{x}')\}}\right\}$$

$$\chi^{(1)}(\boldsymbol{x},\omega) = \begin{cases} 1 & \text{if } \boldsymbol{x} \in \mathcal{D}^{(1)}(\omega) \\ 0 & \text{otherwise} \end{cases}$$







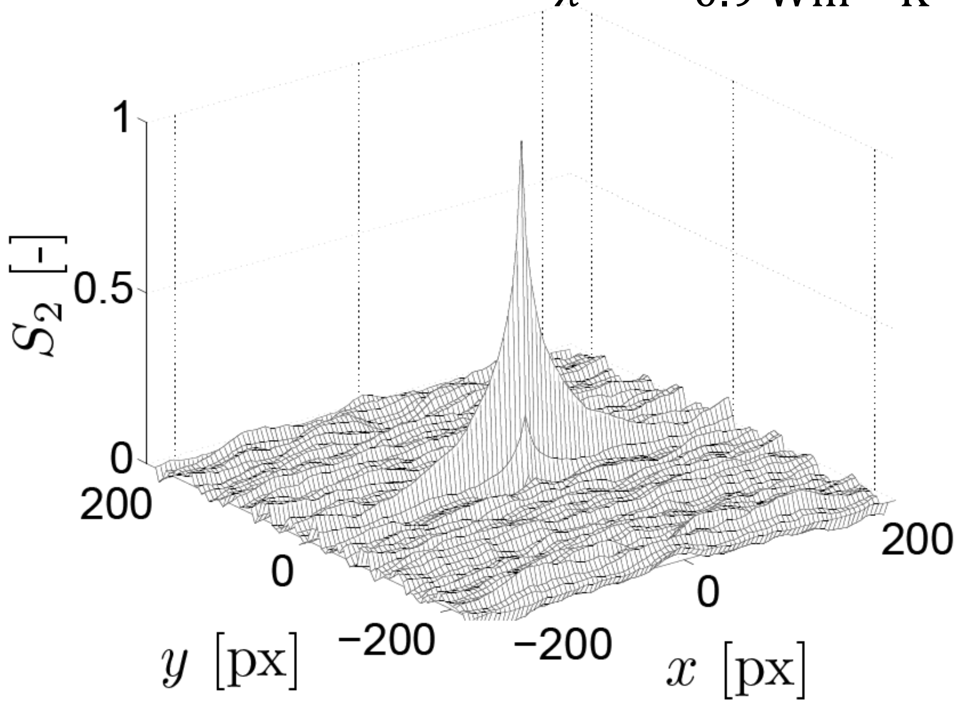
Numerical example – irregular masonry

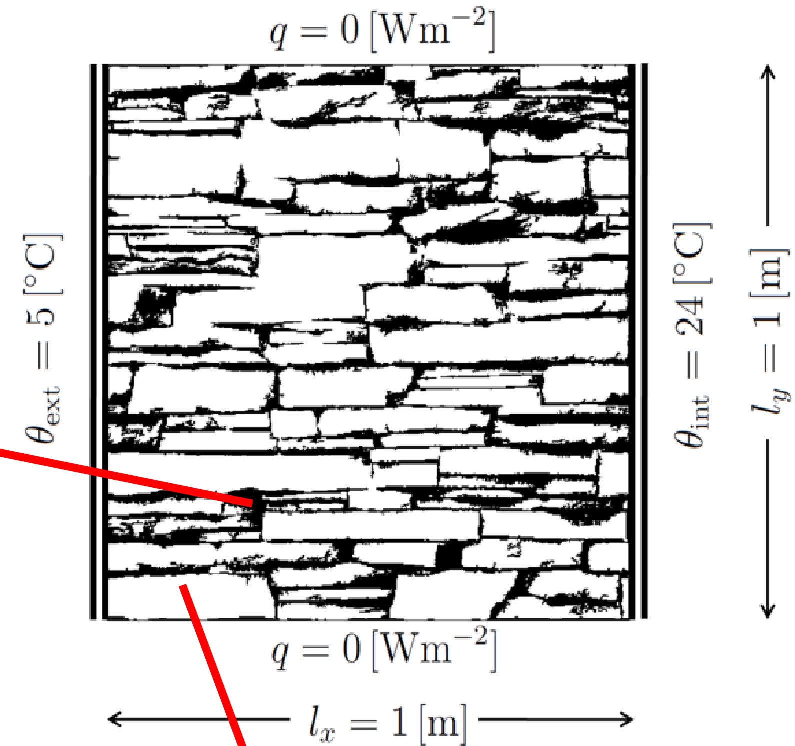
Linear heat conduction problem:

$$-\nabla\cdot(\lambda(x)\nabla\theta(x)) = f(x), \qquad x \in \mathcal{g} \subset \mathbb{R}^2$$

$$\theta(x) = g(x), \qquad x \in \partial\mathcal{g},$$

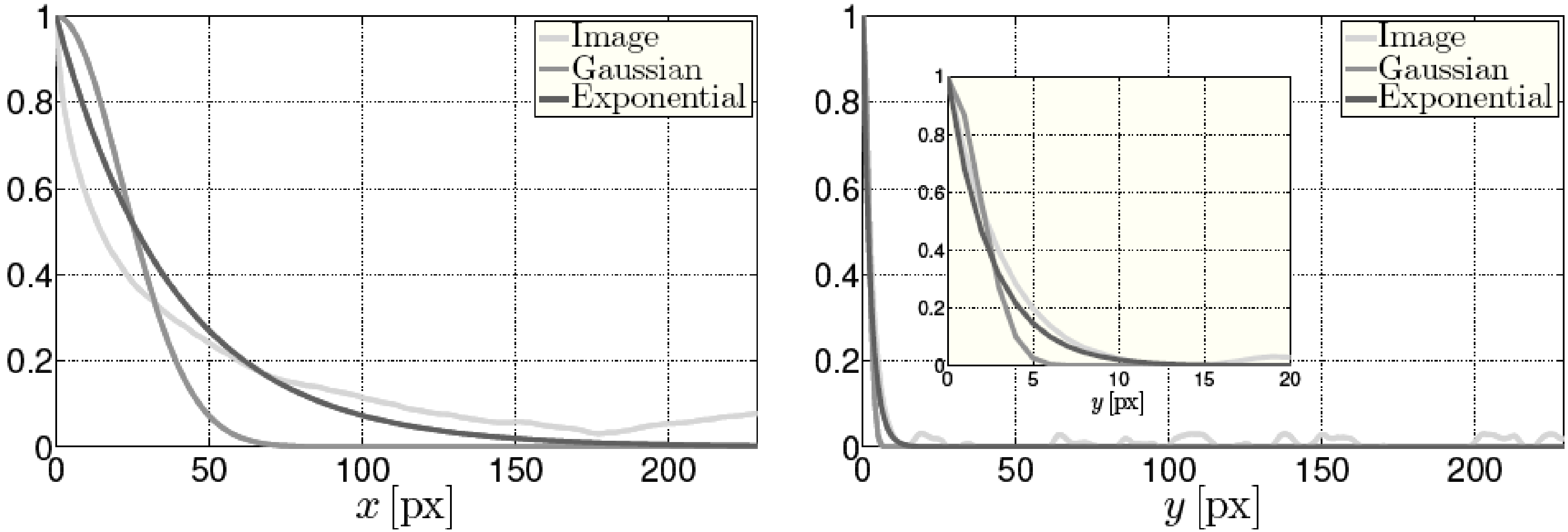
$\lambda^{(2)} = 0.9\ \mathrm{Wm^{-1}K^{-1}}$





$\lambda^{(1)} = 1.9\ \mathrm{Wm^{-1}K^{-1}}$

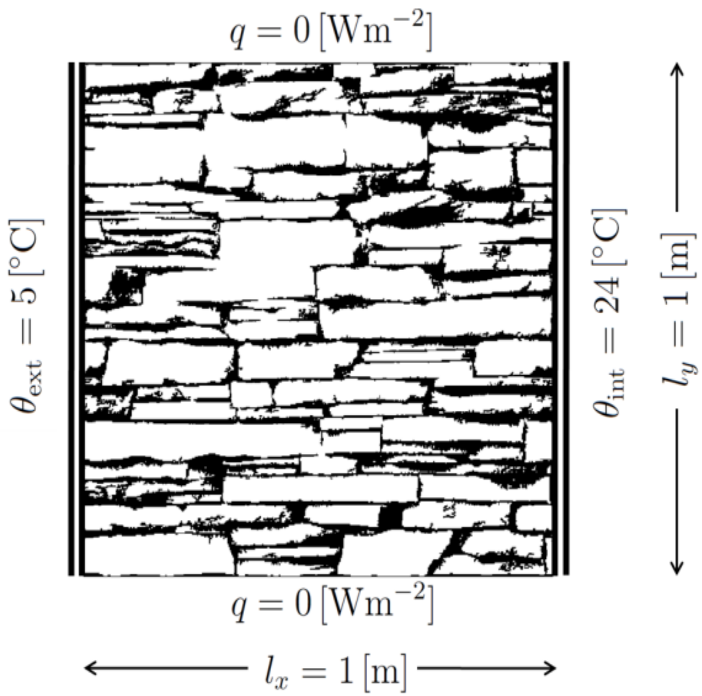
Numerical example – covariance kernel

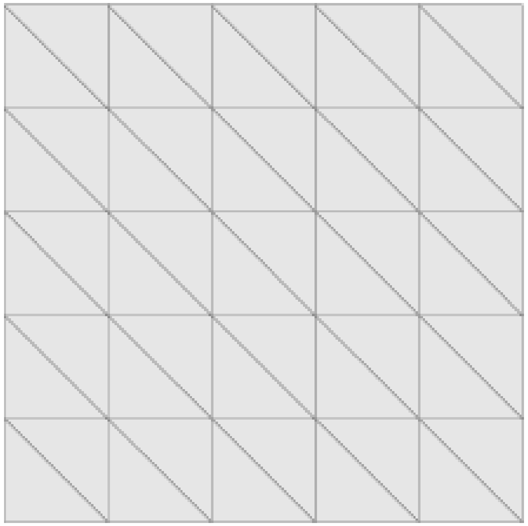


Correlation lengths of Gaussian and exponential covariance function were optimized so as to fit the corresponding covariance functions to image-based covariance function:

type	L_x	L_y
Gaussian	0.047552	0.004067
Exponential	0.083288	0.005689

Numerical example – FE discretization

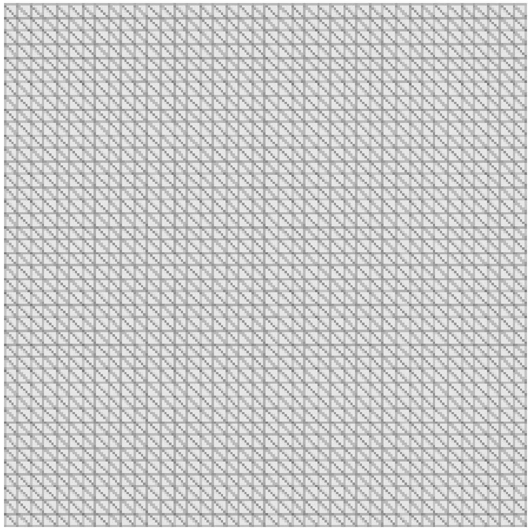




Element size

= 0.2 [m],

(36 FE nodes)

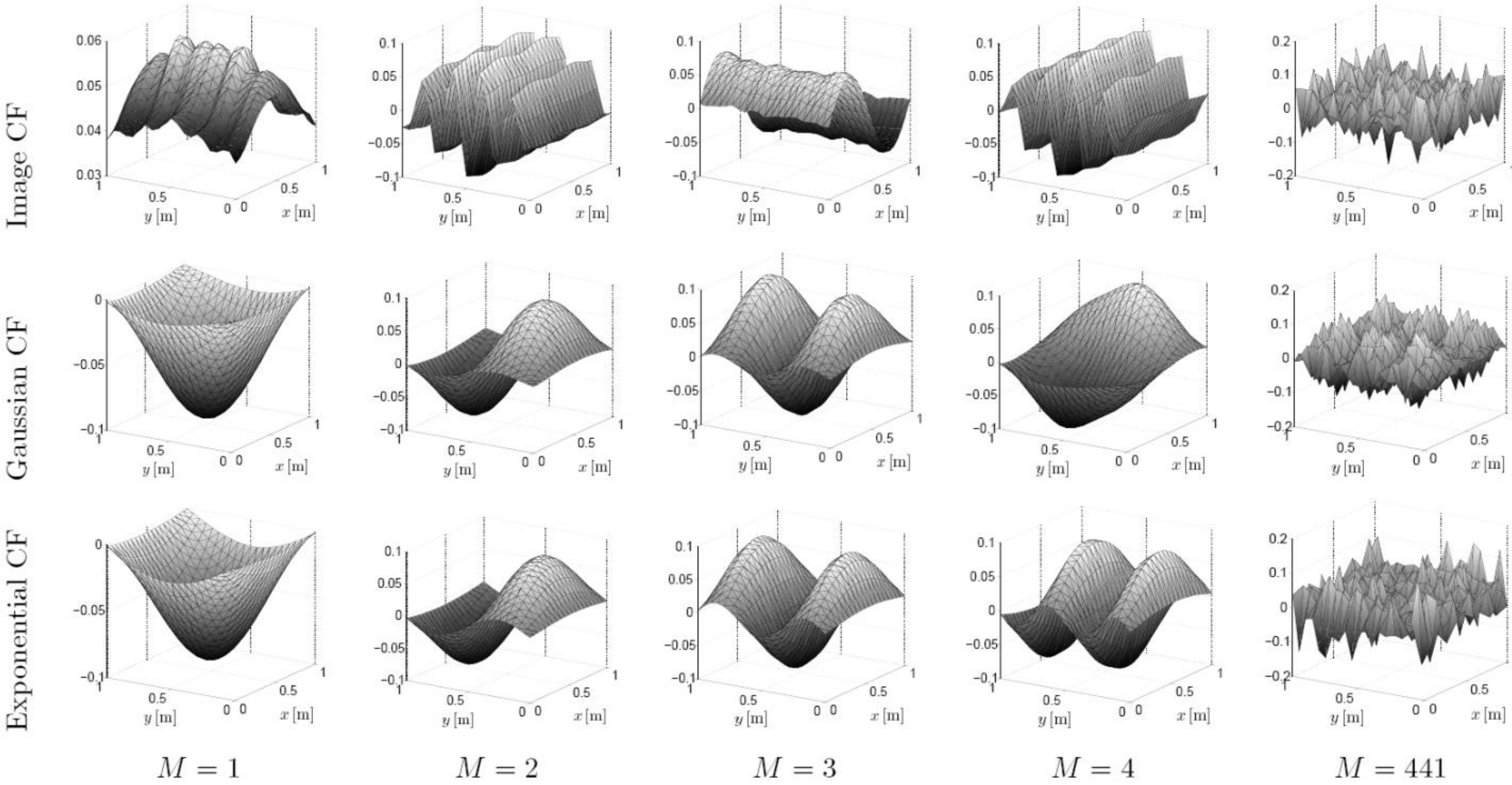


Element size

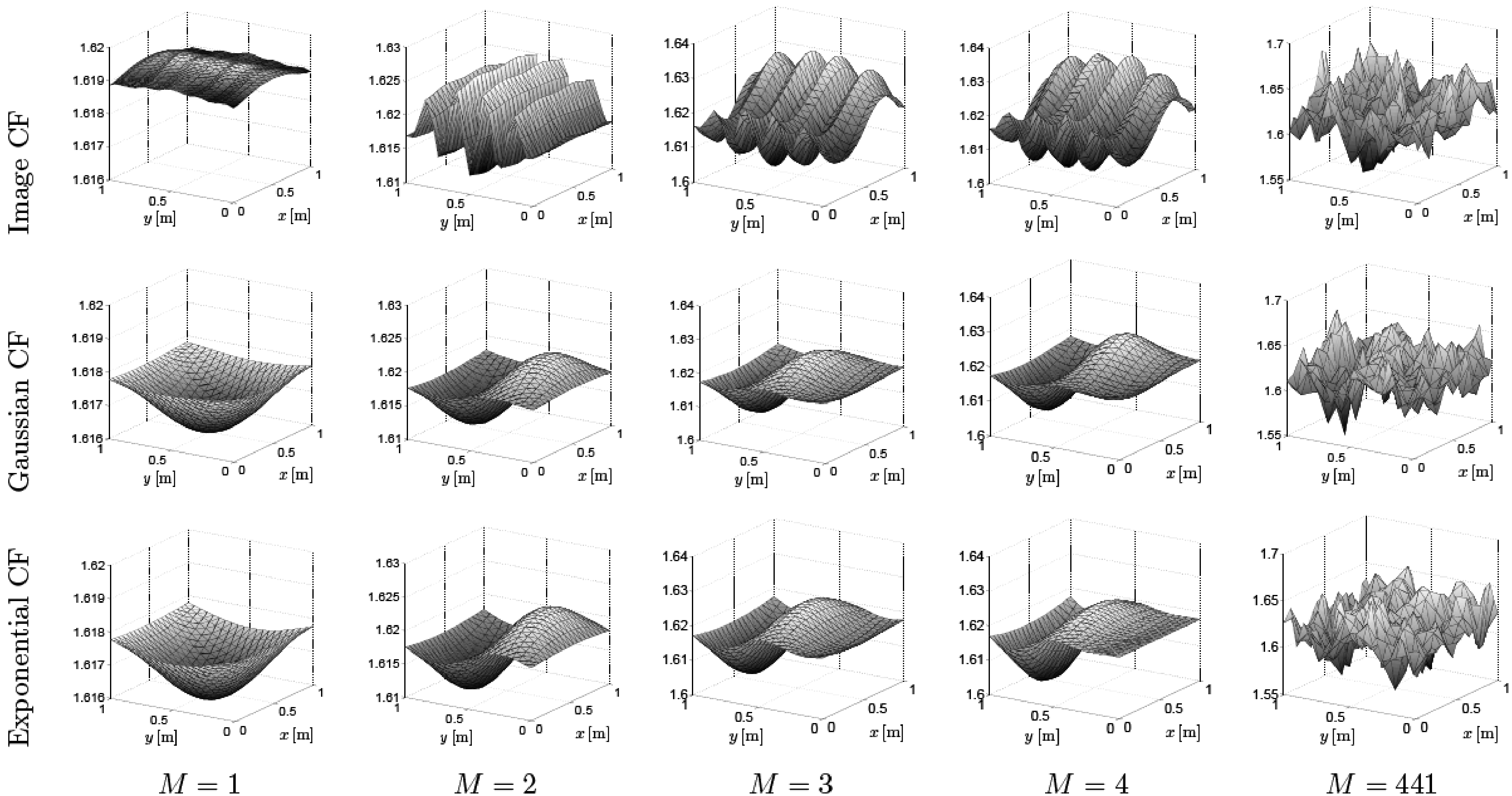
= 0.025 [m],

(1681 FE nodes)

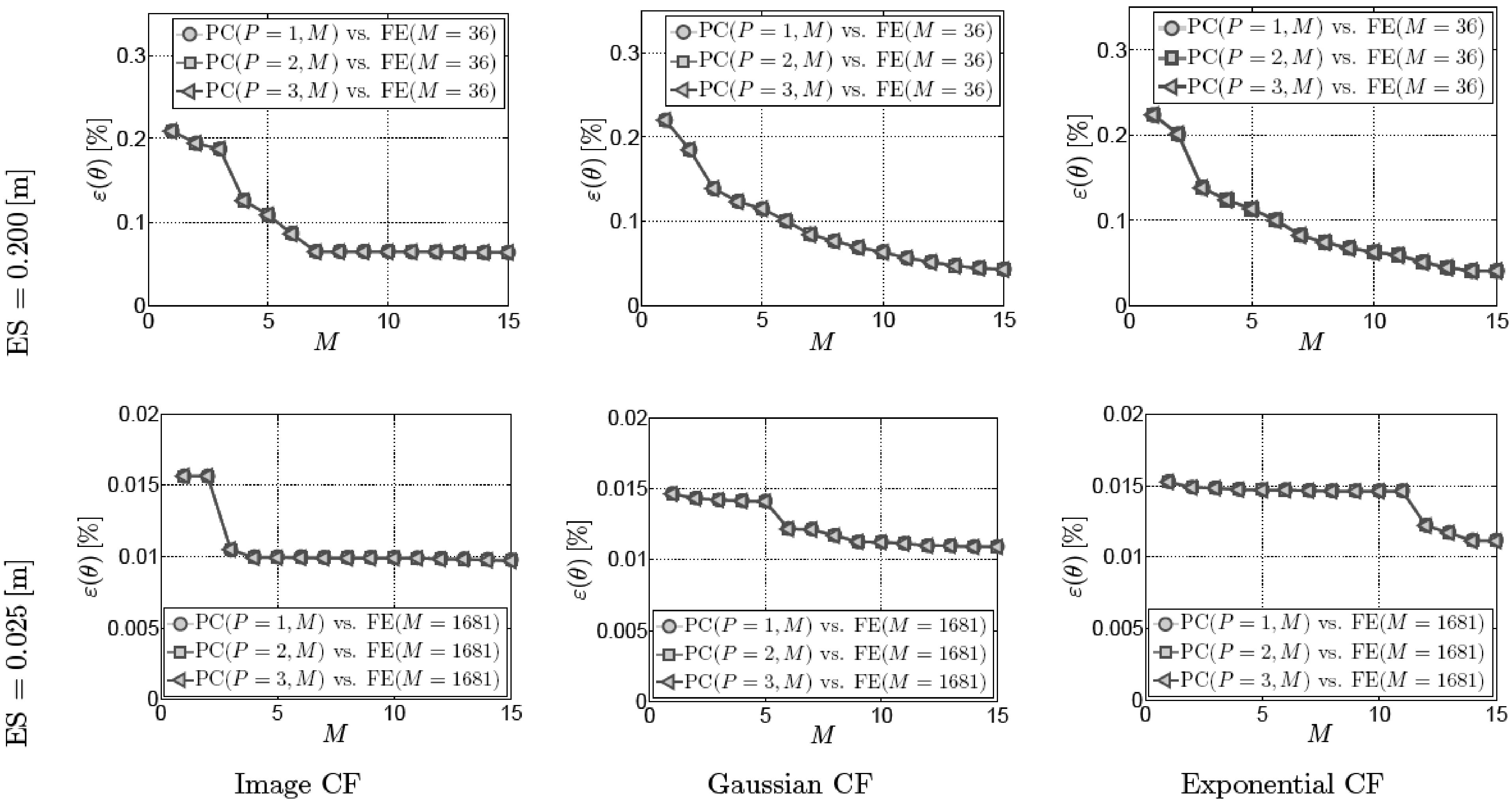
Numerical example – Karhunen-Loève modes

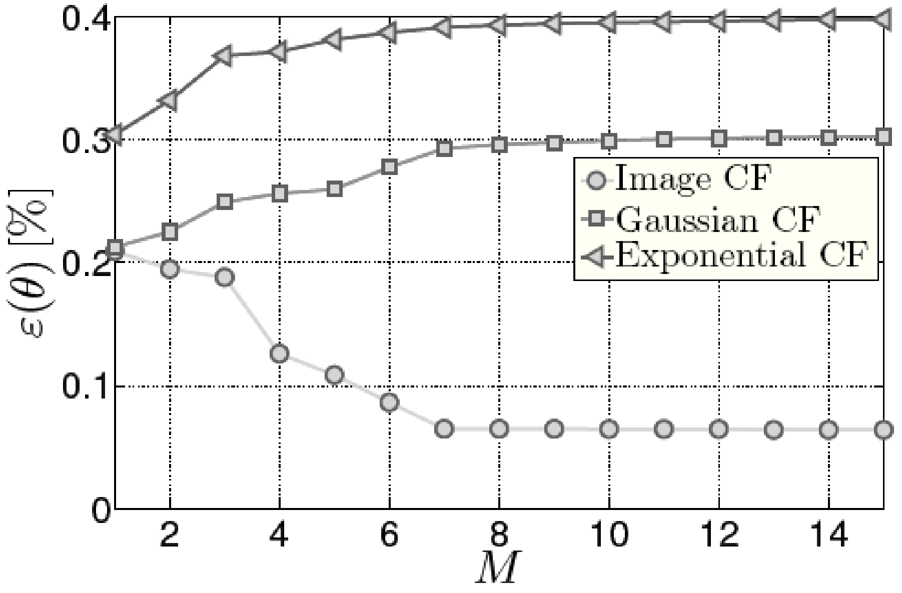


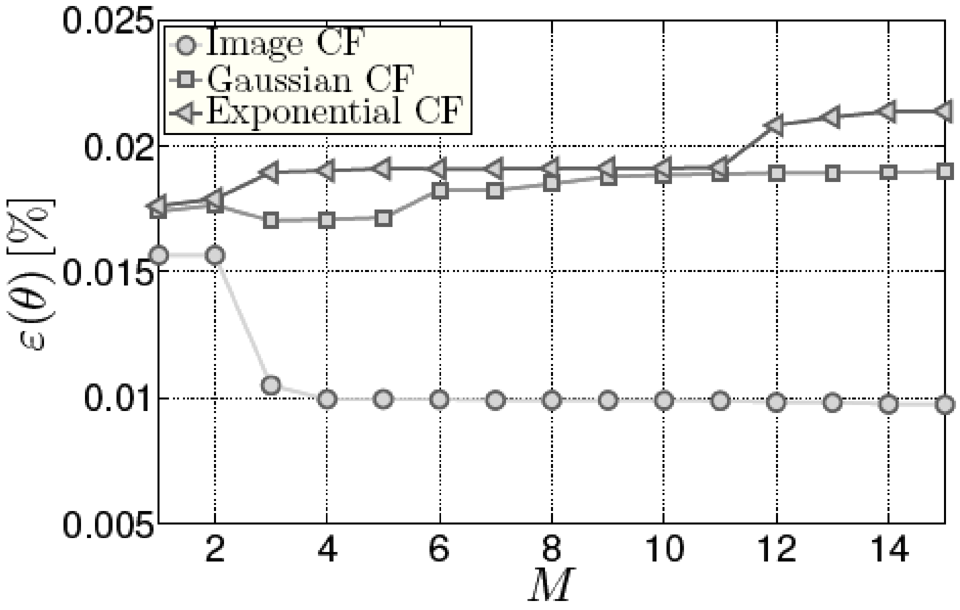
Numerical example – RF realisations



Numerical example – PC + KL error





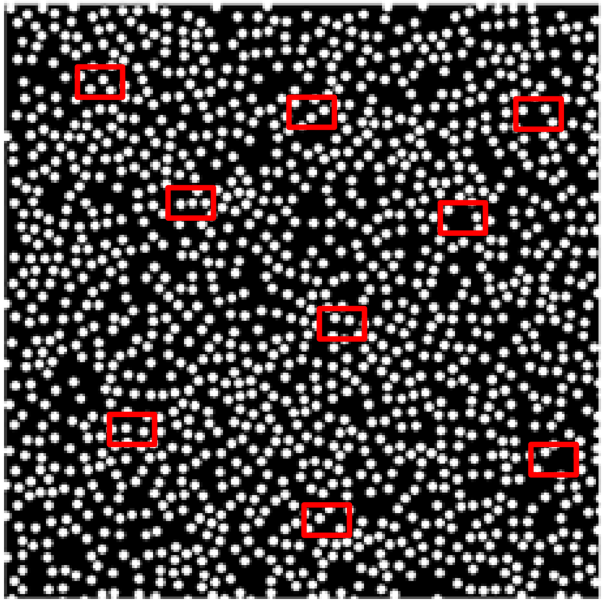


$$\epsilon(\theta) = \frac{\left\| \theta_M - \theta_{441}^{\text{image}} \right\|_{l^2(\mathcal{g}\times\Omega)}}{\left\| \theta_{441}^{\text{image}} \right\|_{l^2(\mathcal{g}\times\Omega)}}$$

Discretized by 1000 samples

Conclusions and future work

- Demonstrated significant difference in temperature field modelled using ad-hoc chosen covariance function and image-based covariance function
- Hard to determine, which covariance function truly better represents the reality comparison needs to be done on samples randomly cut from a microstructure



Conclusions

THANK YOU FOR YOUR ATTENTION.

Aknowledgements

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