Semestrální práce - PRPE

Nosník na pružném podloží

Vypracoval: Jiří Němeček

Analytické řešení základového pásu na Winklerovském pružném podkladu

Rovnice ohybové čáry:

$$\frac{d^4}{dx^4} w(x) = \frac{f_z}{E \cdot I_y}$$

Diferenciální rovnice prutu na winklerovském pružném podkladu:

$$\frac{d^4}{dx^4} w(x) + \frac{b \cdot C_1}{E \cdot I_y} w(x) = \frac{f_z}{E \cdot I_y}$$

Substituce:

$$s = \sqrt[4]{\frac{4 \cdot E \cdot I_y}{b \cdot C_1}} \quad \text{----->} \quad \frac{b \cdot C_1}{E \cdot I_y} = \frac{4}{s^4}$$

Řešení homogenní rovnice:

$$w^{IV}(x)+\frac{4}{s^4}\,w(x)=0$$

$$\lambda^4+\frac{4}{s^4}=0$$

$$\frac{\lambda^4\cdot s^4+4}{s^4}=0 \qquad s\neq 0$$

$$\frac{\left(\lambda^2\cdot s^2-2\cdot\lambda\cdot s+2\right)\cdot\left(\lambda^2\cdot s^2+2\cdot\lambda\cdot s+2\right)}{s^4}=0$$

1) $$\frac{\left(\lambda^2\cdot s^2-2\cdot\lambda\cdot s+2\right)}{s^2}=\lambda^2-\frac{2\cdot\lambda}{s}+\frac{2}{s^2} \qquad D=\frac{4}{s^2}-4\cdot\frac{2}{s^2}=-\frac{4}{s^2}$$

$$\lambda_1=\frac{\frac{2}{s}+\frac{2}{s}\,i}{2}=\frac{1+i}{s} \qquad \lambda_2=\frac{\frac{2}{s}-\frac{2}{s}\,i}{2}=\frac{1-i}{s}$$

2)

$$\frac{\left(\lambda^2 \cdot s^2 + 2 \cdot \lambda \cdot s + 2\right)}{s^2} = \lambda^2 + \frac{2 \cdot \lambda}{s} + \frac{2}{s^2}$$

$$D = \frac{4}{s^2} - 4 \cdot \frac{2}{s^2} = -\frac{4}{s^2}$$

$$\lambda_3 = \frac{-\frac{2}{s} + \frac{2}{s}\ i}{2} = -\frac{1-i}{s}$$

$$\lambda_4 = \frac{-\frac{2}{s} - \frac{2}{s}\ i}{2} = -\frac{1+i}{s}$$

$$w_H = A \cdot e^{-\frac{x}{s}} \cdot cos\left(\frac{x}{s}\right) + B \cdot e^{-\frac{x}{s}} \cdot sin\left(\frac{x}{s}\right) + C \cdot e^{\frac{x}{s}} \cdot cos\left(\frac{x}{s}\right) + D \cdot e^{\frac{x}{s}} \cdot sin\left(\frac{x}{s}\right)$$

$$\frac{x}{s} = \xi$$

$$w_H = A \cdot e^{-\xi} \cdot cos(\xi) + B \cdot e^{-\xi} \cdot sin(\xi) + C \cdot e^{\xi} \cdot cos(\xi) + D \cdot e^{\xi} \cdot sin(\xi)$$

Nehomogenní řešení:

$$f_z = p_0 + p_1 \cdot x$$

$$E \cdot I_y \cdot w^{IV}(x) + b \cdot C_1 \cdot w(x) = p_0 + p_1 \cdot x$$

$$E \cdot I_y \cdot w^{IV}(x) = 0$$

$$b \cdot C_1 \cdot w(x) = p_0 + p_1 \cdot x$$

$$w_P(x) = \frac{p_0 + p_1 \cdot x}{b \cdot C_1} = \frac{p_0 + p_1 \cdot s \cdot \xi}{b \cdot C_1}$$

Celkové řešení:

$$w = w_H + w_P$$

$$w = \frac{p_0 + p_1 \cdot s \cdot \xi}{b \cdot C_1} + A \cdot e^{-\xi} \cdot cos(\xi) + B \cdot e^{-\xi} \cdot sin(\xi) + C \cdot e^{\xi} \cdot cos(\xi) + D \cdot e^{\xi} \cdot sin(\xi)$$

Odvozená rovnice pootočnení:

$$\varphi(x) = \frac{p_1}{b \cdot C_1} + \frac{1}{s} \left\langle e^{-\xi} \cdot \left((B-A) \cdot cos(\xi) - (B+A) \cdot sin(\xi) \right) + e^{\xi} \left((D+C) \cdot cos(\xi) + (D-C) \cdot sin(\xi) \right) \right\rangle$$

Odvozená rovnice momentu:

$$M(x) = \frac{b \cdot C_1 \cdot s^2}{2} \cdot \left\langle e^{-\xi} \cdot \left(B \cdot cos(\xi) - A \cdot sin(\xi) \right) - e^{\xi} \cdot \left(D \cdot cos(\xi) - C \cdot sin(\xi) \right) \right\rangle$$

Odvozená rovnice posouvající síly:

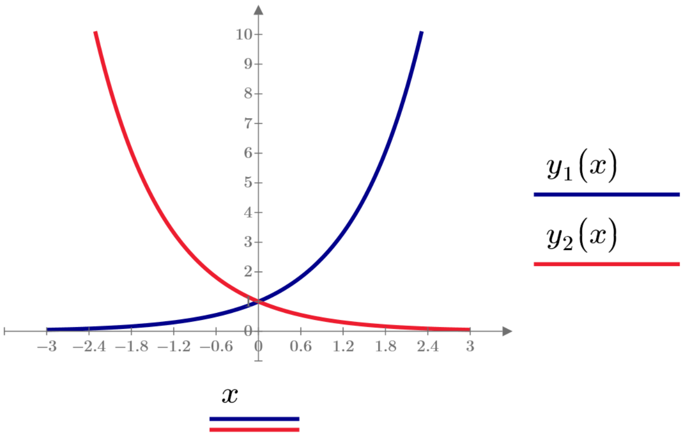
$$V(x) = -\frac{b \cdot C_1 \cdot s}{2} \cdot \left\langle e^{-\xi} \cdot \left((B+A) \cdot cos(\xi) + (B-A) \cdot sin(\xi) \right) + e^{\xi} \cdot \left((D-C) \cdot cos(\xi) - (D+C) \cdot sin(\xi) \right) \right\rangle$$

Nekonečně dlouhý nosník uprostřed zatížený silou F

Řešení pro x > 0 a x < 0:

$$y_1(x) := e^{x}$$

$$y_2(x) := e^{-x}$$



$$\lim_{x \to 0} e^{x} \to 1$$

$$\lim_{x \to \infty} e^{x} \to \infty$$

$$\lim_{x \to -\infty} e^{x} \to 0$$

$$\lim_{x \to 0} e^{-x} \to 1$$

$$\lim_{x \to \infty} e^{-x} \to 0$$

$$\lim_{x \to -\infty} e^{-x} \to \infty$$

Odvození pro x > 0

Okrajové podmínky:

$$w(\infty) = 0$$

$$\varphi(\infty) = 0 \qquad C = 0 \qquad D = 0$$

$$\varphi(0) = 0 \qquad 0 = \frac{1}{s}\,(B - A) \qquad \text{----->} \qquad B = A$$

$$V(0) = -\frac{F}{2} \qquad -\frac{F}{2} = -\frac{1}{2} \cdot b \cdot C_1 \cdot s \cdot (B + A)$$

$$A = B = \frac{F}{2 \cdot b \cdot C_1 \cdot s}$$

$$w(x) = e^{-\xi} \cdot \frac{F}{2 \cdot b \cdot C_1 \cdot s} \cdot (cos(\xi) + sin(\xi))$$

$$\varphi(x)=\frac{dw(x)}{dx}=\frac{F}{2\cdot b\cdot C_1\cdot s}\cdot\left(e^{-\xi}\cdot\left(-\frac{1}{s}\right)\cdot(cos(\xi)+sin(\xi))+e^{-\xi}\cdot\left(-sin\left(\frac{1}{s}\right)+cos\left(\frac{1}{s}\right)\right)\right)$$

$$\varphi(x)=\frac{F}{2\cdot b\cdot C_1\cdot s^2}\cdot e^{-\xi}\cdot(-2\cdot sin(\xi))$$

$$\varphi(x)=-\frac{F}{b\cdot C_1\cdot s^2}\cdot e^{-\xi}\cdot sin(\xi)$$

$$M(x)=-E\cdot I_y\cdot\frac{d\varphi(x)}{dx}=-\frac{F\cdot\left(-E\cdot I_y\right)}{b\cdot C_1\cdot s^2}\cdot\left(e^{-\xi}\cdot\left(-\frac{1}{s}\right)\cdot sin(\xi)+e^{-\xi}\cdot cos(\xi)\cdot\frac{1}{s}\right)$$

$$M(x)=\frac{F\cdot E\cdot I_y}{b\cdot C_1\cdot s^3}\cdot e^{-\xi}\cdot(cos(\xi)-sin(\xi))=\frac{F\cdot s^4}{4\cdot s^3}\cdot e^{-\xi}\cdot(cos(\xi)-sin(\xi))$$

$$M(x)=\frac{F\cdot s}{4}\cdot e^{-\xi}\cdot(cos(\xi)-sin(\xi))$$

$$V(x) = \frac{dM(x)}{dx} = \frac{F \cdot s}{4} \cdot \left(e^{-\xi} \cdot \left(-\frac{1}{s}\right) \cdot (cos(\xi) - sin(\xi)) + e^{-\xi} \cdot \left(-sin(\xi) \cdot \frac{1}{s} - cos(\xi) \cdot \frac{1}{s}\right)\right)$$

$$V(x) = -\frac{F}{2} \cdot e^{-\xi} \cdot cos(\xi)$$

Pro x < 0:

$$w(-x) = w(x) \qquad M(-x) = M(x)$$

$$\varphi(-x) = -\varphi(x) \qquad V(-x) = -V(x)$$

Numerické řešení:

$E := 50\ MPa$

$I_y := 0.00014\ m^4$

$F := 20\ kN$

$C := 35\ MPa$

$C_1 \cdot b = C$

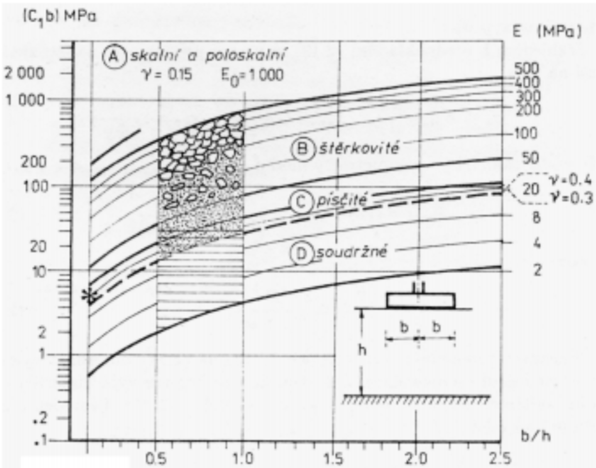
$b := 50\ cm$

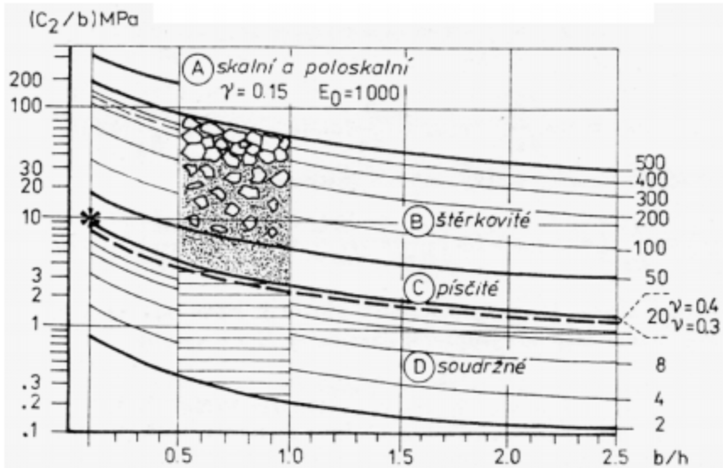
$h := 15\ cm$

$\frac{C_2}{b} = 8\ MPa$

$$I_y = \frac{b \cdot h^3}{12}$$

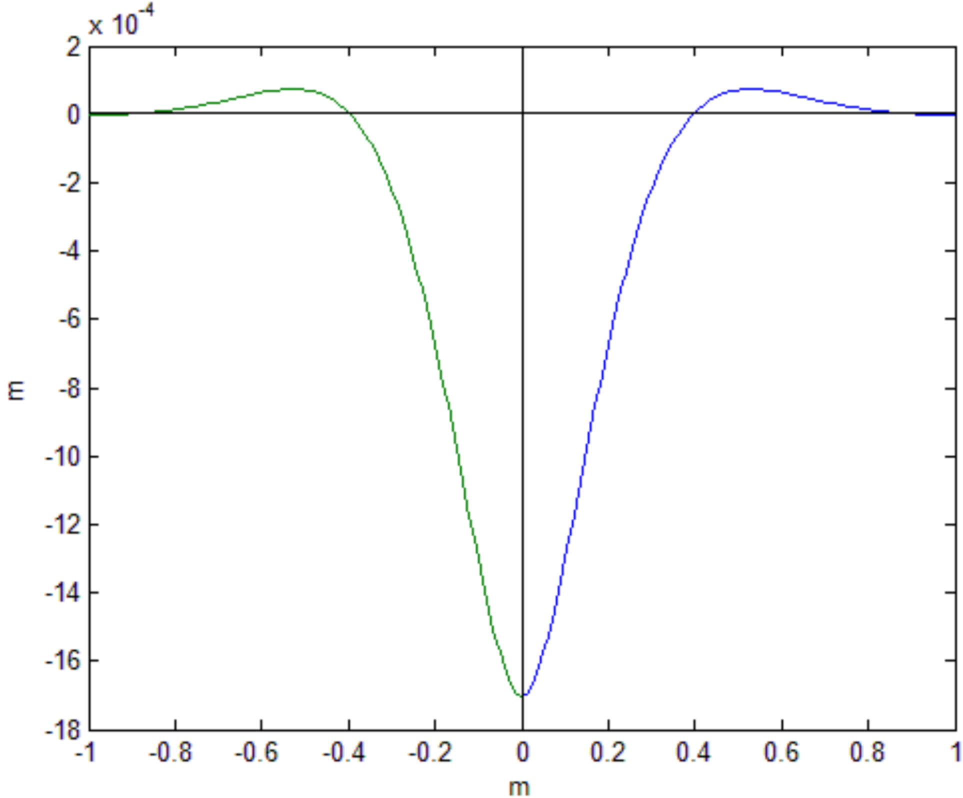
Hodnoty konstant z grafu:



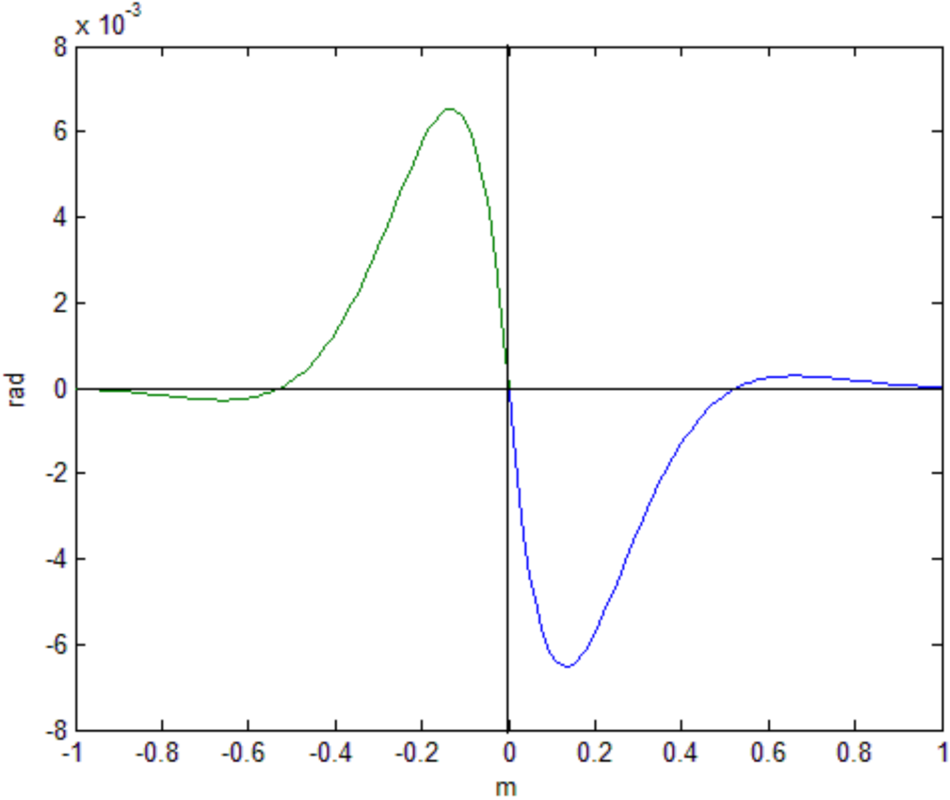


Průhyb:

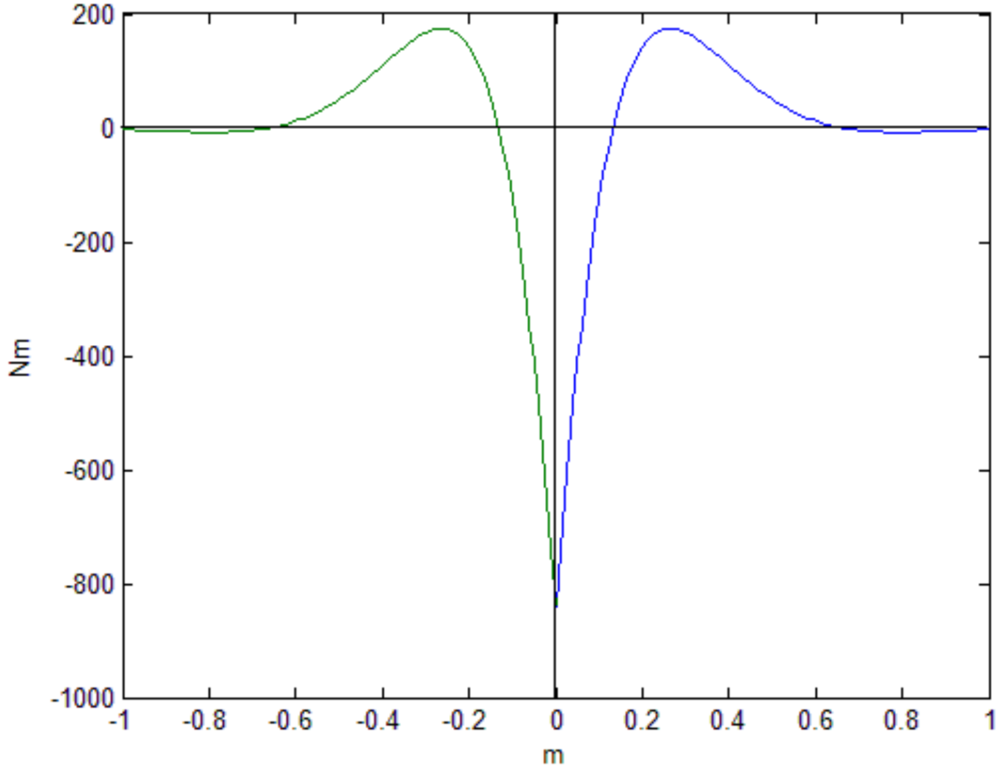
$$w(0) = 0.0017\ m$$



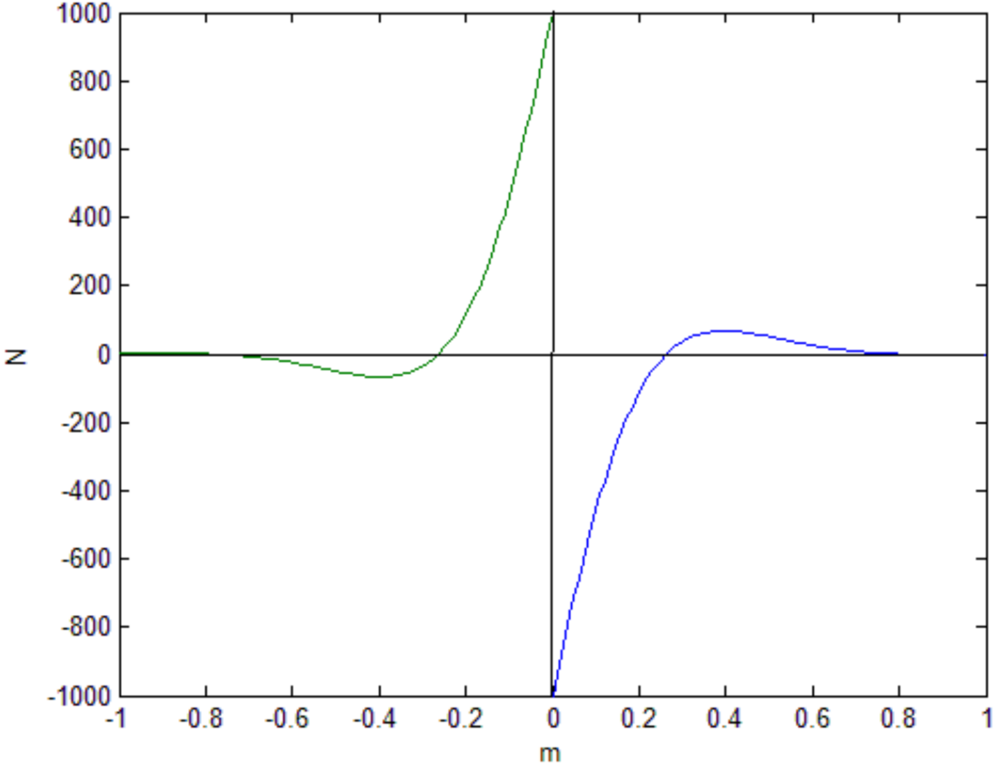
Pootočení:



Ohybový moment:



Posouvající síla:



Nekonečně dlouhý nosník uprostřed zatížený jedním momentem

Okrajové podmínky a jejich aplikace:

$M := 20\ kN \cdot m$

Pro x > 0:

$w(\infty) = 0$

$\varphi(\infty) = 0 \qquad C = 0 \qquad D = 0$

$w(0) = 0 \qquad 0 = e^{-0} \cdot (A \cdot cos(0) + B \cdot sin(0))$ -----> $A = 0$

$M(0) = \frac{M}{2} \qquad \frac{M}{2} = \frac{b \cdot C_1 \cdot s^2}{2}$ -----> $B = \frac{M}{b \cdot C_1 \cdot s^2}$

$$w(x) = e^{-\xi} \cdot \frac{M}{b \cdot C_1 \cdot s^2} \cdot sin(\xi)$$

$$\varphi(x) = \frac{dw(x)}{dx} = e^{-\xi} \cdot \frac{M}{b \cdot C_1 \cdot s^2} \cdot sin(\xi) \cdot \left(-\frac{1}{s}\right) + e^{-\xi} \cdot \frac{M}{b \cdot C_1 \cdot s^2} \cdot cos(\xi) \cdot \frac{1}{s}$$

$$\varphi(x) = e^{-\xi} \cdot \frac{M}{b \cdot C_1 \cdot s^3} \cdot (cos(\xi) - sin(\xi))$$

$$M(x) = -E \cdot I_y \cdot \frac{d\varphi(x)}{dx} = -\frac{M \cdot E \cdot I_y}{b \cdot C_1 \cdot s^3} \cdot \left(e^{-\xi} \cdot \left(-\frac{1}{s}\right) \cdot (cos(\xi) - sin(\xi)) + e^{-\xi} \cdot \left(-sin(\xi) \cdot \frac{1}{s} - cos(\xi) \cdot \frac{1}{s}\right)\right)$$

$$M(x) = -\frac{M \cdot E \cdot I_y}{b \cdot C_1 \cdot s^4} \cdot e^{-\xi} \cdot (-2 \cdot cos(\xi)) = \frac{M \cdot E \cdot I_y}{b \cdot C_1 \cdot \frac{4 \cdot E \cdot I_y}{b \cdot C_1}} \cdot e^{-\xi} \cdot 2 \cdot cos(\xi)$$

$$M(x) = \frac{M}{2} \cdot e^{-\xi} \cdot cos(\xi)$$

$$V(x) = \frac{dM(x)}{dx} = \frac{M}{2} \cdot \left(e^{-\xi} \cdot \left(-\frac{1}{s}\right) \cdot cos(\xi) + e^{-\xi} \cdot \frac{1}{s} \cdot (-sin(\xi))\right)$$

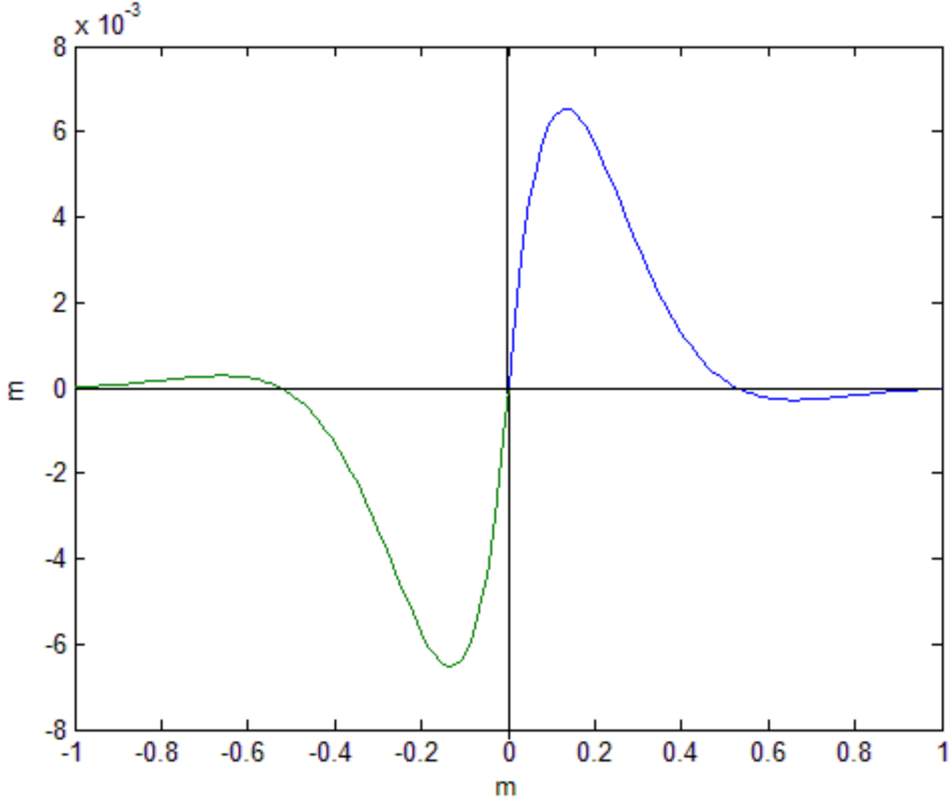
$$V(x) = -\frac{M}{2 \cdot s} \cdot e^{-\xi} \cdot (cos(\xi) + sin(\xi))$$

Pro x < 0:

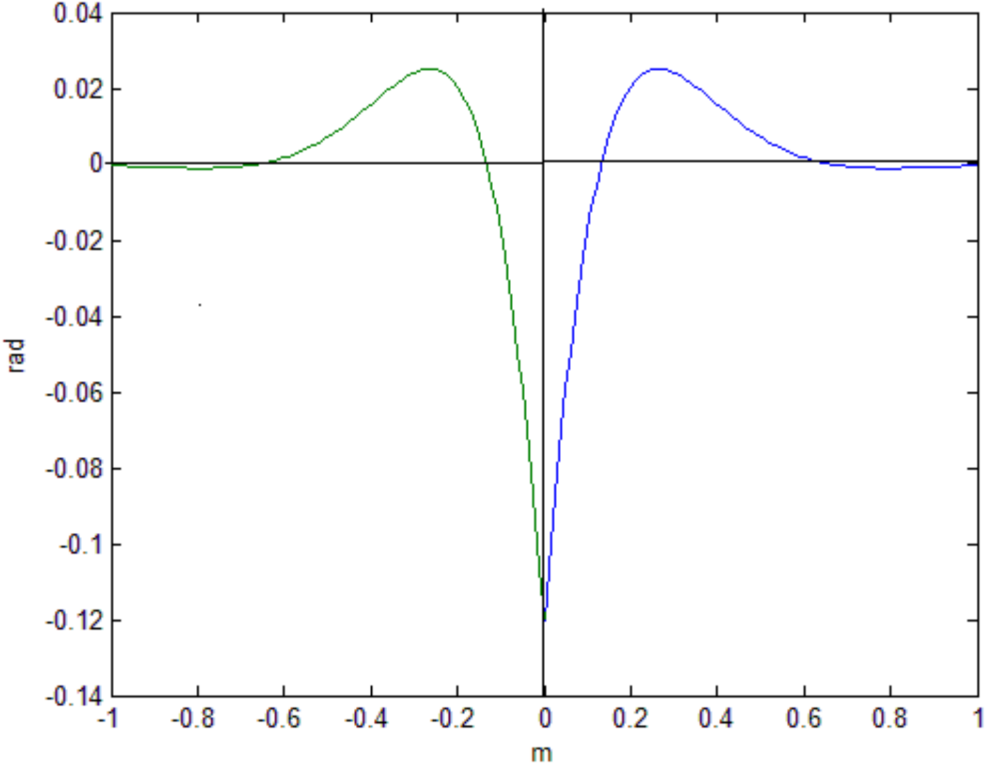
$$w(-x) = -w(x) \qquad M(-x) = -M(x)$$

$$\varphi(-x) = \varphi(x) \qquad V(-x) = V(x)$$

Průhyb:

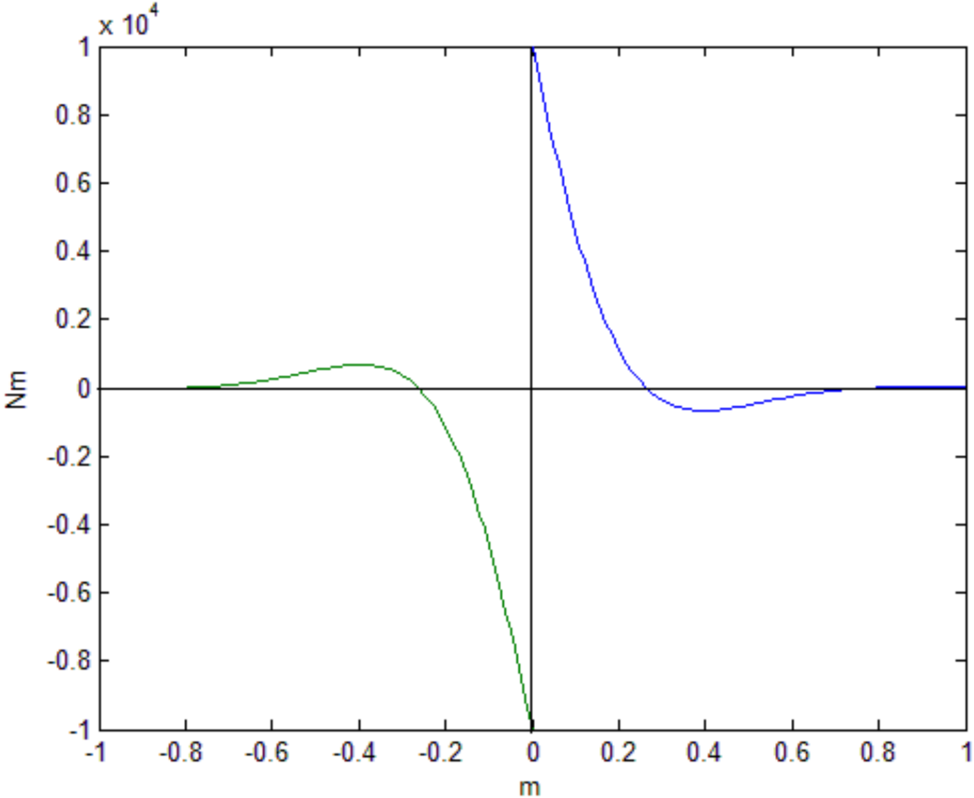


Pootočení:

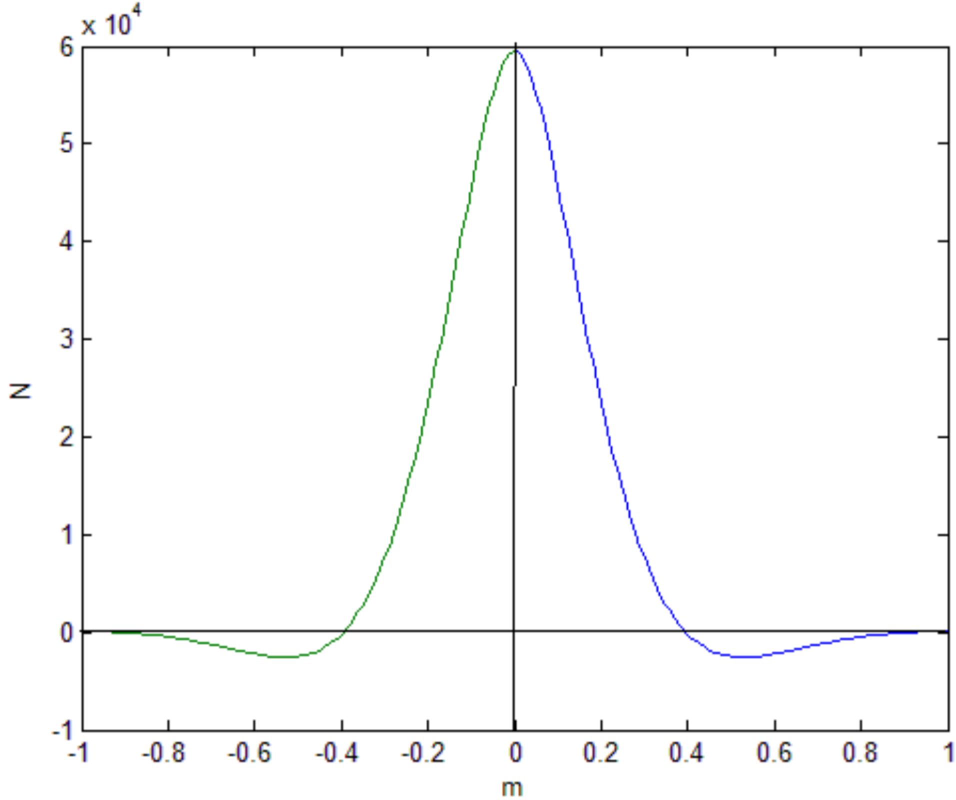


0.04
0.02
0
-0.02
-0.04
-0.06
-0.08
-0.1
-0.12
-0.14
rad
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
m

Ohybový moment:



Posouvající síla:



Diferenciální rovnice prutu dvojparametrického pasternakovského pružného podkladu:

$$E \cdot I_y \cdot \frac{d^4}{dx^4} w(x) - \frac{C_2}{b} \cdot \frac{d^2}{dx^2} w(x) + b \cdot C_1\ w(x) = p_0 + p_1 \cdot x$$

$$w = w_H + w_P$$

$$w(x) = K_1 \cdot e^{\frac{x \cdot \sqrt{\frac{C_2 - \sqrt{{C_2}^2 - 4 \cdot C_1 \cdot E \cdot I_y}}{E \cdot I_y}}}{\sqrt{2}}} + K_2 \cdot e^{-\frac{x \cdot \sqrt{\frac{C_2 - \sqrt{{C_2}^2 - 4 \cdot C_1 \cdot E \cdot I_y}}{E \cdot I_y}}}{\sqrt{2}}} +$$

$$+ K_3 \cdot e^{\frac{x \cdot \sqrt{\frac{C_2 + \sqrt{{C_2}^2 - 4 \cdot C_1 \cdot E \cdot I_y}}{E \cdot I_y}}}{\sqrt{2}}} + K_4 \cdot e^{-\frac{x \cdot \sqrt{\frac{C_2 + \sqrt{{C_2}^2 - 4 \cdot C_1 \cdot E \cdot I_y}}{E \cdot I_y}}}{\sqrt{2}}} + \frac{p_0 + p_1 \cdot x}{C_1}$$

Dosazení konstant:

$$w(x) = K_1 \cdot e^{2.09205 \cdot x} + K_2 \cdot e^{-2.09205 \cdot x} + K_3 \cdot e^{106.884 \cdot x} + K_4 \cdot e^{-106.884 \cdot x}$$

Nekonečně dlouhý nosník uprostřed zatížený jedním momentem:

$$w(\infty) = 0$$

$$\varphi(\infty) = 0 \qquad K_1 = K_3 = 0$$

$$w(0) = 0 \qquad 0 = K_2 + K_4 \qquad K_4 = -K_2$$

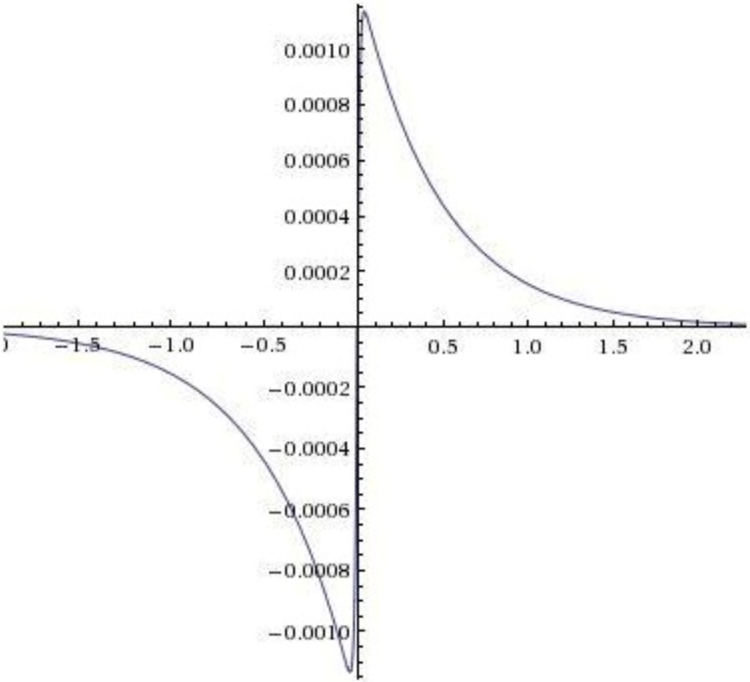
$$M(0) = \frac{M}{2} \qquad w''(x) = -E \cdot I_y \cdot \left(4.37667 \cdot K_2 \cdot e^{-2.09205 \cdot x} + 11424.2 \cdot K_4 \cdot e^{-106.884 \cdot x}\right)$$

$$\frac{100}{7} = -4.37667 \cdot K_4 + 11424.2 \cdot K_4$$

$$K_4 = -0.001250957$$

$$w(x) = 0.00150957 \cdot e^{-2.09205 \cdot x} - 0.001250957 \cdot e^{-106.884 \cdot x}$$

Průhyb:


0.0010
0.0008
0.0006
0.0004
0.0002
−1.5
−1.0
−0.5
0.5
1.0
1.5
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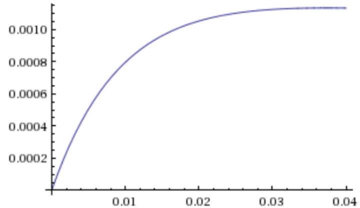
−0.0002
−0.0004
−0.0006
−0.0008
−0.0010


0.0010
0.0008
0.0006
0.0004
0.0002
0.01
0.02
0.03
0.04

Nekonečně dlouhý nosník uprostřed zatížený silou:

$w(\infty) = 0$

$\varphi(\infty) = 0$ $\qquad K_1 = K_3 = 0$

$\varphi(0) = 0$ $\qquad K_2 = -51.07143711 \cdot K_4$

$V(0) = -\dfrac{F}{2}$ $\qquad w'''(x) = \left(6409.35 \cdot K_2 \cdot e^{-2.09205 \cdot x} + 8.54745 \cdot 10^8 \cdot K_4 \cdot e^{-106.884 \cdot x}\right)$

$$-10000 = -327334.7154 \cdot K_4 + 8.54745 \cdot 10^8 \cdot K_4$$

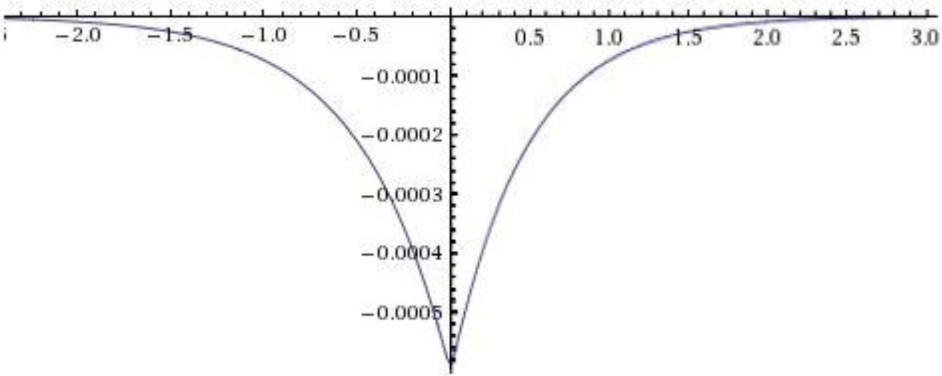
$$K_2 = 0.000597733$$

$$K_4 = -0.000011703$$

$$w(x) = 0.000597733 \cdot e^{-2.09205 \cdot x} - 0.000011703 \cdot e^{-106.884 \cdot x}$$

Průhyb:

$w(0) = 0.00058\ m$



Použitá literatura:

[1] Chobot, K., Drahoňovský, Z., Hájek, V.: *Statika stavebních konstrukcí III*, SNTL, Praha 1985

[2] Novák, O., Hořejší, J. a kol.: *Statika stavebních konstrukcí*, SNTL, Praha 1973