Convergence Analysis of Fast Fourier Transform Based Method for Modelling of Heterogeneous Materials

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Contents

1 Introduction **1**

2 Setting the initial equations **1**

3 Fourier transform **2**

4 Solution of the differential equation **3**

5 Discretization **5**
Fast Fourier Transform based Method

1 Introduction

In the [?] field [1] of [?] engineering design we come across many complex problems, the mathematical formulation of which is tedious and usually not possible by analytical methods. At such instants we resort to the use of numerical techniques. Here lies the importance of Finite Element Method (FEM), which is a very powerful tool for getting the numerical solution of a wide range of engineering problems. The basic concept is that a body or structure may be divided into smaller elements of finite dimensions called Finite Elements.

Contrary to FEM and mesh-based methods in general there exist situations where some of the meshless methods is convenient to use. This project is based on analysing Fast Fourier Transform based method in one dimension.

The document is structured into following parts. Section 2 provide information about the method and leads to a differential equation that is consequently solved in Section 4 using Fourier transform technique described in Section 3. Finally, Section 5 provides detailed convergence analysis of the method in one dimensional space.

2 Setting the initial equations

This Section provides theoretical approach to the Fast Fourier Transform Based Method. Information about this method are gained from Michel [?]. It is based on analyzing periodically repeating medium with heterogeneities. In this work one dimensional case is considered and it is necessary to note that the cross-section of the structure is set to one and thus it is not considered in the sequel. The medium is composed from basic cell which is characterized with $E(x)$ stiffness defined at the interval $[a, b]$ (a, b are boundary points of the cell) and the boundary condition for strain have to be satisfied. It means that:

$$\varepsilon(a) = \varepsilon(b) \quad \wedge \quad E(a) = E(b)$$

The infinite one dimensional rod is loaded with average strain ε_0. For the needs of the FFT based method the strain along the rod $\varepsilon(x)$ is decomposed into:

$$\varepsilon(x) = \varepsilon_0 + \varepsilon_1(x)$$

where $\varepsilon_1(x)$ is a complement to $\varepsilon(x)$ strain along the rod.

To obtain some information about $\varepsilon_1(x)$ strain, we deduce that:

$$\frac{1}{b-a}\int\limits_a^b (x)dx =_0 \quad \Rightarrow \quad \frac{1}{b-a}\int\limits_a^b {}_0 +_1 (x)dx =_0 \quad \Rightarrow$$

$$\Rightarrow \quad \frac{_0}{b-a}\int\limits_a^b dx + \int\limits_a^b {}_1(x)dx =_0 \quad \Rightarrow \quad \int\limits_a^b {}_1(x)dx = 0,$$

i.e. the overall strain ${}_1(x)$ at the cell equals to zero.

Because of loading just with average strain ${}_0$, the equilibrium state can be described with the following equation:

$$\frac{d}{dx}\Big(E(x)\varepsilon(x)\Big) = 0 \tag{1}$$

For the use of the method the $(E_H - E_H)$ term is added to the equation. The stiffness E_H is an auxiliary value of the analogical homogeneous problem with same strains ε_0, $\varepsilon_1(x)$ and the same boundary condition. Hence Equation (1) follows:

$$\frac{d}{dx}\Big[\Big(E_H - E_H + E(x)\Big)\varepsilon(x)\Big] = 0$$

$$\frac{d}{dx}\Big(E_H\varepsilon(x)\Big) = -\frac{d}{dx}\Big[\Big(E(x) - E_H\Big)\varepsilon(x)\Big] \tag{2}$$

Noticing that $\frac{d}{dx}\big(E_H\varepsilon_0\big) = 0$, Equation (2) leads to:

$$\frac{d}{dx}\Big(E_H\varepsilon_1(x)\Big) = -\frac{d}{dx}\Big[\Big(E(x) - E_H\Big)\varepsilon(x)\Big]$$

The left side of the equation can be interpreted as homogeneous rod deformed with $\varepsilon_1(x)$ strain. Hence, the right side of the equation can be interpreted as generalized load $f(x)$ causing ${}_1(x)$ strain:

$$f(x) = -\frac{d}{dx}\Big[\Big(E(x) - E_H\Big)\varepsilon(x)\Big]$$

Hence, it is necessary to solve following equation:

$$E_H\frac{d\varepsilon_1(x)}{dx} = f(x) \tag{3}$$

that can be classified as a first order linear differential equation with constant coefficients.

3 Fourier transform

In order to solve Equation (3) and differential equations in general, it is useful to use some integral transform. The advantage of this approach is that it is possible to express the differential equation as an integral equation.

In this case of periodically distributed heterogeneities, it is convenient to use Fourier transform [?]. It is an operator $\mathcal{F}$ that transform function $f(x)$ into some another function $\hat{f}(t)$ according to following formula:

$$\mathcal{F}\{f(x)\} = \hat{f}(t) = \int\limits_{-\infty}^{\infty} f(x)e^{-ixt}dx$$

It is necessary to note that only absolutely integrable functions are considered for the transform.

The inverse Fourier transform is defined as follows:

$$\mathcal{F}^{-1}\left\{\hat{f}(t)\right\} = f(x) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} f(t)e^{ixt}dt$$

For suitable set of functions it has a following property:

$$\mathcal{F}^{-1}\left\{\mathcal{F}\{f(x)\}\right\} = f(x)$$

The Fourier transform $\mathcal{F}$ is an linear operator meaning:

$$\mathcal{F}\{\alpha f(x) + \beta g(x)\} = \alpha \cdot \mathcal{F}\{f(x)\} + \beta \cdot \mathcal{F}\{g(x)\} \tag{4}$$

The other useful property is transform of function derivative $f'(x)$:

$$\mathcal{F}\{f'(x)\} = it \cdot \mathcal{F}\{f(x)\} \tag{5}$$

The convolution of two function can be expressed as multiplication of Fourier transform:

$$h(x) = f(x) * g(x) = \int\limits_{-\infty}^{\infty} f(x-\xi)g(\xi)d\xi \quad \Longleftrightarrow \quad \mathcal{F}\{h(x)\} = \mathcal{F}\{f(x)\} \cdot \mathcal{F}\{g(x)\} \tag{6}$$

where binary operation $*$ means convolution.

Note that the original equation is posed on interval $[a, b]$. Therefore, we formally extend all functions by zero outside of [a,b] to comply with the Fourier settings.

4 Solution of the differential equation

This section provides solution of Equation (3) using Fourier transform. Hence, it can be written as:

$$E_H{'}_1(x) = f(x) \Longrightarrow \mathcal{F}\{E_H{'}_1(x)\} = \mathcal{F}\{f(x)\}$$

Using linearity (4) and property (5) of fourier transform, we can write:

$$E_H it\mathcal{F}\{_1(x)\} = \mathcal{F}\{f(x)\}$$

and consequently, the Fourier transform of the searched function $_1(x)$ can be written as:

$$\mathcal{F}\{_1(x)\} = \frac{1}{E_H it}\mathcal{F}\{f(x)\} \tag{7}$$

The function $\frac{1}{E_H it}$ from the last equation can be substituted as:

$$\mathcal{F}\{g(x)\} = \frac{1}{E_H it} \tag{8}$$

and thus Equation (7) leads to:

$$\mathcal{F}\{{}_1(x)\} = \mathcal{F}\{g(x)\}\mathcal{F}\{f(x)\}$$

It can be note that the function $g(x)$ is usually called Green function.

Now, it is possible to express multiplication of Fourier transform with the help of convolution theorem (6):

$$ {}_1(x) = \int\limits_{-\infty}^{\infty} g(x-\xi) f(\xi) d\xi$$

This equation can be integrated by parts, thus heading to:

$$ {}_1(x) \overset{P.P.}{=} \int\limits_{-\infty}^{\infty} \frac{\partial g(x-\xi)}{\partial \xi} F(\xi) d\xi + g(x-\xi)F(\xi)\Big|_{-\infty}^{\infty} \tag{9}$$

where $F(x)$ is a function satisfying the condition $F'(x) = f(x)$ and it can be determined as:

$$f(x) = -\frac{d}{dx}\Big[\Big(E(x) - E_H\Big)\varepsilon(x)\Big] \Rightarrow F(x) = \Big(E_H - E(x)\Big)\varepsilon(x)$$

Noting that the second term of Equation (9) is equal to zero due to boundary conditions. Hence, the strain ${}_1(x)$ can be expressed as:

$$ {}_1(x) = g'(x) * F(x)$$

and then its Fourier transform with the help of convolution yields:

$$\mathcal{F}\{{}_1(x)\} = \mathcal{F}\{g'(x)\} \cdot \mathcal{F}\{F(x)\}$$

After expression of Fourier transform of $g'(x)$, it leads to:

$$ {}_1(x) = \mathcal{F}^{-1}\left\{it\mathcal{F}\{g(x)\} \cdot \mathcal{F}\{F(x)\}\right\}$$

Now, it possible to use the expression for $\mathcal{F}\{g(x)\}$ from Equation (8) that leads to following equation:

$$ {}_1(x) = \mathcal{F}^{-1}\left\{\frac{1}{E_H} \cdot \mathcal{F}\{F(x)\}\right\}$$

In order to satisfy the zero average conditions, it is necessary to comply with the following equation:

$$\int\limits_a^b {}_1(x)dx = 0 \Rightarrow \int\limits_a^b {}_1(x)e^{-ixt}dx = 0, \quad \text{for } t = 0 \Rightarrow \hat{}_1(0) = 0$$

Hence, it is possible to introduce following function:

$$\hat{K}^{per}(t) = \begin{cases} 0 & \text{for } t = 0 \\ \frac{1}{E_H} & \text{for } t \neq 0 \end{cases}$$

to obtain:

$$ {}_1(x) = \mathcal{F}^{-1}\left\{\hat{K}^{per}(t)\hat{F}(x)\right\} \tag{10}$$

where $F(x) = \Big(E_H - E(x)\Big)(y)$ and $\hat{F}(x) = \mathcal{F}\{F(x)\}$.

5 Discretization

This section provide the discretization of the rod that concerns the process of transferring continuous models and equations into discrete one.

It means that the continuous Fourier transform would became the discrete Fourier transform, hence the algorithms for Fast Fourier transform can be used with the great benefit of shorter time for calculation.

The one dimensional case provide an opportunity to express Equation (10) in analytical form. It can be written:

$$
{}_1(x) = \mathcal{F}^{-1}\left\{\hat{K}^{per}(t)\hat{F}(x)\right\} \quad \Rightarrow \quad {}_1(x) = \int\limits_a^b K^{per}(x-\xi)F(\xi)d\xi \quad \Rightarrow
$$

$$
\Rightarrow \quad {}_1(x) = \int\limits_a^b \frac{\delta(x-\xi)}{E_H}\Big(E_H - E(\xi)\Big)(\xi)d\xi - \frac{1}{b-a}\int\limits_a^b \frac{E_H - E(\xi)}{E_H}\varepsilon(\xi)d\xi \tag{11}
$$

where the $\delta(x-\xi)$ is Dirac distribution satisfying the property that

$$
\int\limits_a^b \delta(x-\xi)f(\xi)d\xi = f(x)
$$

and the second term of Equation (11) satisfies the boundary condition that $\int_a^b {}_1(x) = 0$ and it has the same meaning as the property of $\hat{K}^{per}$ function saying that $\hat{K}^{per}(0) = 0$.

Hence, Equation (11) can be written in following form:

$$
\varepsilon_1(x) = \frac{E_H - E(x)}{E_H}\varepsilon(x) - \frac{1}{b-a}\int\limits_a^b \frac{E_H - E(\xi)}{E_H}\varepsilon(\xi)d\xi \tag{12}
$$

For the sake of following analysis of the method, it is necessary to provide some other emendations. Firstly we have to discretize the interval $[a, b]$ into N nodes that change the integral over continuous field into the sum. It is necessary to note that the discretization leads to a loss in exactness over the continuous field. The first point is set at the beginning of the interval ($x_1 = a$) and the last point at the end of the interval ($x_N = b$). The rest of the nodes are regularly distributed into the interval, so the difference between coordinates of two adjacent points is $h = \frac{b-a}{N-1}$. This discretization change Equation (12) to:

$$
\varepsilon_1(x_j) = \frac{E_H - E(x_j)}{E_H}\varepsilon(x_j) - \frac{1}{N}\sum_{i=1}^{N}\frac{E_H - E(x_i)}{E_H}\varepsilon(x_i), \quad \text{for } j = 1, 2, \ldots, N \tag{13}
$$

The second modification has to be done with the stiffness E_H as it is auxiliary value of imaginary case of homogeneous rod it cannot be known until the strain of the rod is calculated. Hence the E_H stiffness is estimated with some reference stiffness E_{ref}. The inaccurate estimation of this E_{ref} value leads to inexact solution of the strain thus the iteration approach to solution has to be applied. The appropriate setting of the E_{ref} value is the main point of the efficiency or even successfulness of the iteration method.

Using Equation (13) the $(k+1)^{\text{th}}$ iteration of the algorithm can be written as:

$$\varepsilon_1(x_j, k+1) = \frac{E_{\text{ref}} - E(x_j)}{E_{\text{ref}}}\varepsilon(x_j, k) - \frac{1}{N}\sum_{i=1}^{N}\frac{E_{\text{ref}} - E(x_i)}{E_{\text{ref}}}\varepsilon(x_i, k), \quad \text{for } j = 1, 2, \ldots, N \tag{14}$$

It is necessary to note that the initial strain is set to zero:

$$\varepsilon_1(x, 0) = 0$$

Convergence study of FFT based method This section provides analytical solution of heterogeneous rod using the FFT based method. It means that the studied sequence $\varepsilon_1(x, k)$ defined with recurrence formula is being expressed directly using initial strain $\varepsilon_1(x, 0)$. The convergence of the sequence with regard to E_{ref} parameter is also discussed here.

A binary heterogeneous rod with two stiffnesses E_0 and $E_0(1+p)$, $p \in (-1, \infty\rangle$ is considered. A arbitrary node with coordinate x and stiffness E_0 is marked as x_{E_0} and analogically with stiffness $E_0(1+p)$ as $x_{E_0(1+p)}$. The periodic cell is discretised using N nodes and we can assume that stiffness $E_0(1+p)$ occurs in m cases. Hence, it is obvious that stiffness E_0 occurs in $(N-m)$ cases. All used variables related to both sequences are shown in Table 1. The definition domain of these variables are provided in Table 2 (noting that the set $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$).

Table 1: Sequences and related variables

studied sequence	coordinate	iteration	stiffness $E(x)$	nodes	frequency	volume fraction
$\varepsilon_1(x_{E_0}, k)$	x_{E_0}	k	E_0	N	$N-m$	$\frac{N-m}{N} = 1-c$
$\varepsilon_1(x_{E_0(1+p)}, k)$	$x_{E_0(1+p)}$	k	$E_0(1+p)$	N	m	$\frac{m}{N} = c$

Table 2: Definition scope of variables

variables	E_0, $E_0(1+p)$, E_{ref}	p	x	N	m	$(N-m)$	c	k
def. domain	$\mathbb{R}^+$	$(-1, \infty\rangle$	$\langle a, b\rangle$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$	$\langle 0, 1\rangle$	$\mathbb{N}_0$

Now, we can take a look at ε_1 strain in the characteristic node x_{E_0}. Using Equation (14), the recurrence relation for the strain ε_1 as the k^{th} iteration can be written as follows:

$$\begin{aligned}\varepsilon_1^{k+1}(x_{E_0}, k+1) &= \frac{E_{\text{ref}} - E_0}{E_{\text{ref}}}\varepsilon^k(x_{E_0}, k) - \\ &- \frac{1}{N}\left[(N-m)\frac{E_{\text{ref}} - E_0}{E_{\text{ref}}}\varepsilon^k(x_{E_0}, k) + m\frac{E_{\text{ref}} - E_0(1+p)}{E_{\text{ref}}}\varepsilon^k(x_{E_0(1+p)}, k)\right]\end{aligned} \tag{15}$$

Subsequently, we can use condition that average strain of $\varepsilon_1(x, k)$ is equal to zero, yielding:

$$\sum_{i=1}^{N}\varepsilon_1(x_i, k) = 0 \quad \Rightarrow \quad (N-m)\varepsilon_1(x_{E_0}, k) + m\varepsilon_1(x_{E_0(1+p)}, k) = 0 \quad \Leftrightarrow \tag{16}$$

$$\Leftrightarrow \quad \varepsilon_1(x_{E_0(1+p)}, k) = \left(1 - \frac{N}{m}\right) \varepsilon_1(x_{E_0}, k) \tag{17}$$

It is necessary to note that even the initial strain $\varepsilon_1(x, 0)$ has to satisfy condition (17).

It can be noticed that both Equation (15) and (17) can be rewritten using

$$c = \frac{m}{N} \tag{18}$$

where $c \in \langle 0, 1 \rangle$. In particular, substituting (18) into (15) and using (17) leads to:

$$\begin{aligned} \varepsilon_1(x_{E_0}, k+1) = \frac{E_{\text{ref}} - E_0}{E_{\text{ref}}} \big[\varepsilon_0 + \varepsilon_1(x_{E_0}, k)\big] - (1-c) \frac{E_{\text{ref}} - E_0}{E_{\text{ref}}} \big[\varepsilon_0 + \varepsilon_1(x_{E_0}, k)\big] + \\ + c \cdot \frac{E_{\text{ref}} - E_0(1+p)}{E_{\text{ref}}} \left[\varepsilon_0 + \frac{1-c}{c} \varepsilon_1(x_{E_0}, k)\right] \end{aligned} \tag{19}$$

In the following text, $\varepsilon_1(x_{E_0}, k)$ is being abbreviated to $\varepsilon_1(k)$. Hence, after several algebraic emendations, Equation (19) leads to:

$$\varepsilon_1(k+1) = \frac{E_{\text{ref}} - E_0(1+p-cp)}{E_{\text{ref}}} \cdot \varepsilon_1(k) + \frac{E_0 \varepsilon_0 c p}{E_{\text{ref}}} \tag{20}$$

After following substitution:

$$a = \frac{E_{\text{ref}} - E_0(1+p-cp)}{E_{\text{ref}}}, \quad b = \frac{E_0 \varepsilon_0 c p}{E_{\text{ref}}} \tag{21}$$

Equation (20) can be written as:

$$\varepsilon_1(k+1) = a \cdot \varepsilon_1(k) + b \tag{22}$$

and this equation can be classified as linear inhomogeneous recurrence relation with constant coefficients. To solve this equation, we first convert the recurrence relation into homogeneous form. We start with writing a formula for the following member of the sequence:

$$\varepsilon_1(k+2) = a \cdot \varepsilon_1(k+1) + b \tag{23}$$

Subtraction of Equation (22) and (23) leads to homogeneous form of linear recurrence relation:

$$\varepsilon_1(k+2) - (a+1)\varepsilon_1(k+1) + a\varepsilon_1(k) = 0 \tag{24}$$

Now, we are going to find solution in the following form:

$$\varepsilon_1(k) = rt^k \tag{25}$$

where $r \in \mathbb{R}$ and $t \in \mathbb{C}$.

If the assumption about solution written in Equation (25) is correct it has to satisfy the recurrence formula from Equation (24). Thus, it leads to:

$$rt^{k+2} - (a+1) \cdot rt^{k+1} + a \cdot rt^k = 0 \quad \Rightarrow \quad rt^k\big(t^2 - (a+1)t + a\big) = 0$$

In order to find nontrivial solution the variables c and t has to be nonzero, thus the solution leads to a problem of finding roots of quadratic equation:

$$t^2 - (a+1)t + a = 0$$

which are:

$$t_1 = 1, \quad t_2 = a \tag{26}$$

It means that both $\varepsilon_1(k) = r_1 t_1^k$ and $\varepsilon_1(k) = r_2 t_2^k$, $r_1, r_2 \in \mathbb{R}$ are solution of the recurrence Equation (24). It can be shown that even the sum

$$\varepsilon_1(k) = r_1 t_1^k + r_2 t_2^k, \quad r_1, r_2 \in \mathbb{R} \tag{27}$$

is solution of that recurrence equation as we can write:

$$\begin{aligned}
&\varepsilon_1(k+2) - \big(a+1\big) \cdot \varepsilon_1(k+1) + a \cdot \varepsilon_1(k) = \\
&= \big(r_1 t_1^{k+2} + r_2 t_2^{k+2}\big) + \big(r_1 t_1^{k+1} + r_2 t_2^{k+1}\big) + \big(r_1 t_1^k + r_2 t_2^k\big) = \\
&= r_1 t_1^k \big[t_1^2 - (a+1)t_1 + a\big] + r_2 t_2^k \big[t_2^2 - (a+1)t_2 + a\big] = 0
\end{aligned}$$

In order to determine r_1 and r_2 coefficients, we have to satisfy two linearly independent equations obtained for two members of the studied sequence:

$$\begin{aligned}
\varepsilon_1(0) &= r_1 t_1^0 + r_2 t_2^0 \\
\varepsilon_1(1) &= r_1 t_1^1 + r_2 t_2^1
\end{aligned}
\quad \Longrightarrow \quad
\begin{aligned}
\varepsilon_1(0) &= r_1 + r_2 \\
a \cdot \varepsilon_1(0) + b &= r_1 + r_2 a
\end{aligned}$$

and the solution is following:

$$r_1 = \frac{b}{1-a}, \quad r_2 = \varepsilon_1(0) + \frac{b}{a-1} \tag{28}$$

noting that $1-a > 0$, so we do not divide with zero. Using (27), (26) and (21) in Equation (28), the formula for the searched sequence can be written as:

$$\varepsilon_1(k) = \frac{\varepsilon_0 cp}{1+p-cp} + \left(\varepsilon_1(0) - \frac{\varepsilon_0 cp}{1+p-cp}\right)\left(1 - \frac{E_0(1+p-cp)}{E_{\text{ref}}}\right)^k \tag{29}$$

Now, we investigate the sequence behaviour in respect to E_{ref} parameter. Firstly, we consider the following situation:

$$\varepsilon_1(0) = \frac{\varepsilon_0 cp}{1+p-cp}$$

it implies:

$$\varepsilon_1(k) = \frac{\varepsilon_0 cp}{1+p-cp}$$

and it is constant sequence regardless the parameter E_{ref}.

Next, we consider that:

$$\varepsilon_1(0) \neq \frac{\varepsilon_0 cp}{1+p-cp}$$

In order to investigate the sequence behaviour in respect to E_{ref} parameter, it is necessary to find a following limit:

$$\lim_{k\to\infty} \varepsilon_1(k)$$

which converge if and only if:

$$-1 < t_2 < 1 \quad \Leftrightarrow \quad -1 < 1 - \frac{E_0(1+p-cp)}{E_{\text{ref}}} < 1 \quad \Leftrightarrow \quad 0 < \frac{E_0(1+p-cp)}{E_{\text{ref}}} < 2 \quad \Leftrightarrow$$

$$\Leftrightarrow \quad \frac{E_0}{2}(1+p-cp) < E_{ref} \tag{30}$$

If the condition written in Equation (30) is satisfied then the sequence converge to:

$$\lim_{k\to\infty} \varepsilon_1(k) = \frac{\varepsilon_0 cp}{1+p-cp} \tag{31}$$

If the condition (30) is not satisfied, ($\lim_{k\to\infty} \varepsilon_1(k)$ does not exist because the following inequality is always fulfilled:

$$t_2 < 1 \quad \Leftrightarrow \quad E_0(1+p-cp) > 0$$

due to the definition domain of the variables (see Table 2). It can be added that, the sequence is absolutely convergent ($\lim_{k\to\infty} |\varepsilon_1(k)| = \infty$) for $t_2 < -1$.

Next, it can be noticed that if the following condition is satisfied

$$1 - \frac{E_0(1+p-cp)}{E_{\mathrm{ref}}} = 0 \quad \Leftrightarrow \quad E_{\mathrm{ref}} = E_0(1+p-cp) \tag{32}$$

then the sequence has special property that:

$$\forall k > 0: \quad \varepsilon_1(k) = \frac{\varepsilon_0 cp}{1+p-cp}$$

Hence, it can be said that Equation (32) describes condition for the optimal convergence of the studied sequence as it is get convergent in the first iteration step.

Now, we can have a look at sequence

$$\varepsilon_1(x_{E_0(1+p)}, k)$$

describing arbitrary point with $E_0(1+p)$ stiffness. The quality of this sequence can be determined from Equation (16) saying that in each iteration step k the average strain of ε_1 is equal to zero. Thus, it implies that both sequences have the same domain of convergence.

For the sake of lucidity, all significant qualities of the both sequences are provided in Table 3.

Table 3: The sequences behaviour in regard to E_{ref} parameter

E_{ref}	$\varepsilon_1(x_{E_0}, k)$	$\varepsilon_1(x_{E_0(1+p)}, k)$
$\frac{E_0}{2}(1+p-cp) < E_{\mathrm{ref}}$	$\lim_{k\to\infty} \varepsilon_1(x_{E_0}, k) = \varepsilon_0 \frac{cp}{1+p-cp}$	$\lim_{k\to\infty} \varepsilon_1(x_{E_0(1+p)}, k) = \varepsilon_0 \frac{p(1-c)}{1+p-cp}$
$\frac{E_0}{2}(1+p-cp) \geq E_{\mathrm{ref}}$	$\lim_{k\to\infty} \varepsilon_1(x_{E_0}, k)$ do not exist	$\lim_{k\to\infty} \varepsilon_1(x_{E_0(1+p)}, k)$ do not exist
$E_0(1+p-cp) = E_{\mathrm{ref}}$	$\forall k > 0: \quad \varepsilon_1(k) = \varepsilon_0 \frac{cp}{1+p-cp}$	$\forall k > 0: \quad \varepsilon_1(k) = \frac{p(1-c)}{1+p-cp}$

References

[1] Wikipedia, *Dirac delta function [online]*, 2008.