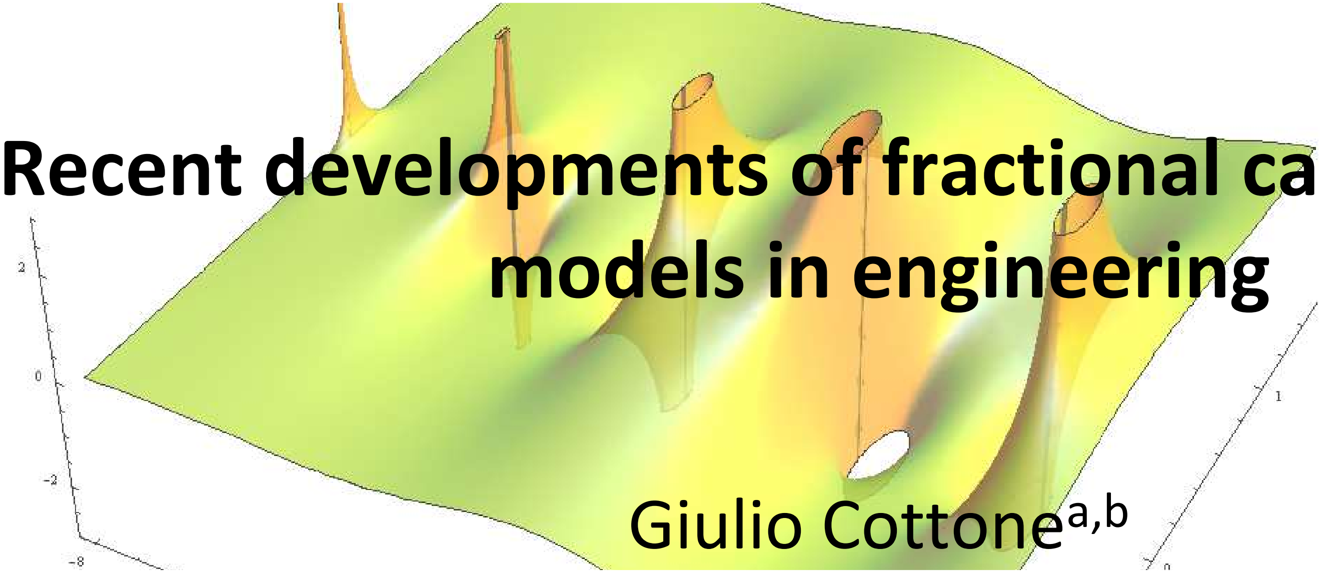
Recent developments of fractional calculus-based models in engineering



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FRACTIONAL CALCULUS: The name of the game: a misnomer!

Fractional operators
have nothing to do with fractions (1/3, 1/2…)!!!

Fractional operators
are derivatives and integrals of real (or complex) order:

$$f(x), \frac{df(x)}{dx}, \frac{d^2 f(x)}{dx^2}, \ldots, \frac{d^j f(x)}{dx^j}, \ldots, j \in N$$

Q: How to define derivatives of order $j = 0.3 + 2.5i$?

Therefore, if someone says:
"I had Fractional Calculus in the elementary school"
there are two possibilities:

1. A probable misunderstanding
2. First effects of the Bologna process

FRACTIONAL CALCULUS? Not exotic!

"What if *n* (in $d^n f \,/\, dx^n$) be ½?"

De L'Hospital

1661 - 1704

Leibniz

1646 - 1716

It will lead to a paradox, from which one day useful consequences will be drawn. (1695)

Leibniz, Euler, Riemann, Liouville, Abel, Feller, Grünwald, Letnikov, Marchaud, Weyl, Caputo, Riesz, Samko

Useful consequences in Mechanics

- Viscoelasticity
- Fracture Mechanics
- Non-local Continuum Mechanics
- Stochastic Dynamics

FRACTIONAL CALCULUS? Never heard! WHY?

- Many definitions and symbols
    - Riemann - Liouville
    - Marchaud
    - Grünwald-Letnikov
    - Riesz
    - Weyl
    - Caputo
    - ...
- Calculations are very hard to be tackled by hand (CAS needed)
- Lack of simple geometrical meaning
- Is it useful?

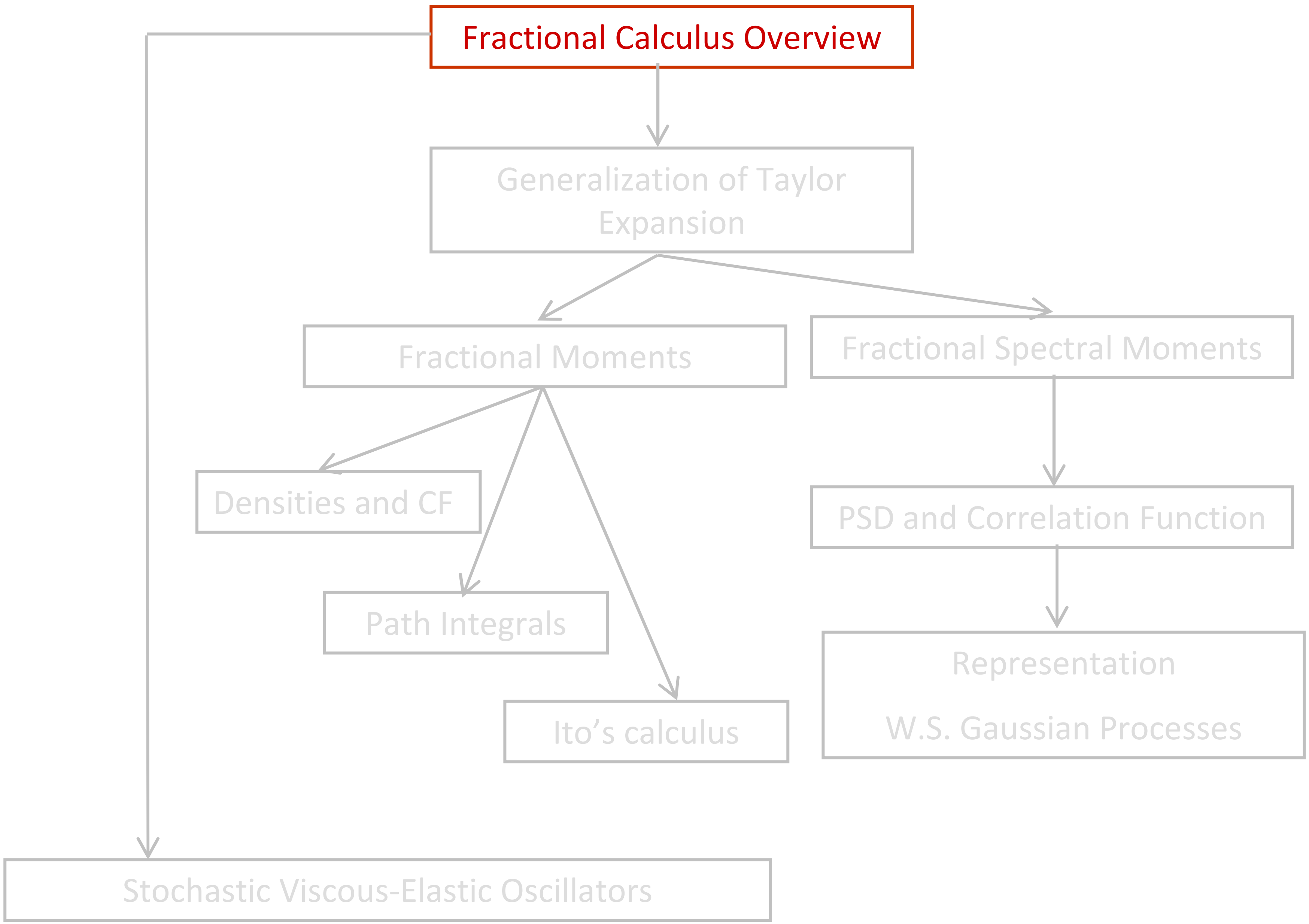
Books:

For engineers: Podlubny

For Physicists: Hilfer

Encyclopedic textbook: Samko, Kilbas and Marichev

OUTLINE



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FRACTIONAL CALCULUS OVERVIEW: WHICH IS THE SCOPE?

- Classical derivatives:

$$f(x), \frac{df(x)}{dx}, \frac{d^2 f(x)}{dx^2}, ..., \frac{d^j f(x)}{dx^j}, ..., j \in N$$

- Fractional derivatives:

$$(D^\alpha f)(x) \qquad \alpha \in \mathbb{C}, \ \mathrm{Re}[\alpha] > 0$$

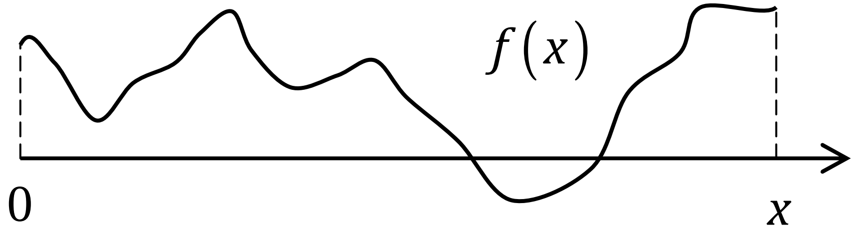
- Classical n-folded integrals:

$$f(x), \int f(x)dx, \int\int f(x)dxdx, ..., \underbrace{\int ... \int f(x)dx...dx}_{n-fold}$$

- Fractional integrals:

$$(I^\alpha f)(x) \qquad \alpha \in \mathbb{C}, \ \mathrm{Re}[\alpha] > 0$$

First step: definition of the Fractional Integral



$$\left(I^1_{0+}f\right)(x)=\int_0^x f(x_1)dx_1$$

$$\left(I^2_{0+}f\right)(x)=\int_0^x\left(\int_0^{x_1} f(x_2)dx_2\right)dx_1$$

...

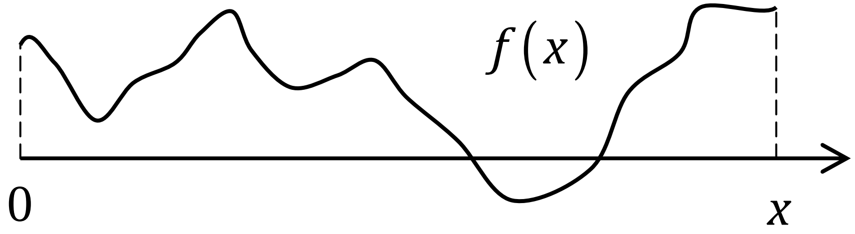
Cauchy formula

$$\left(I^n_{0+}f\right)(x)=\underbrace{\int_0^x dx_1\ldots\int_0^{x_{n-1}} f(x_n)dx_n}_{n-fold}=$$

$$=\frac{1}{(n-1)!}\int_0^x (x-z)^{n-1} f(z)dz$$

We want to extend this formula to non-integer number. HOW?

First step: definition of the Fractional Integral



$$\left(I_{0+}^{1} f\right)(x) = \int_0^x f(x_1)\,dx_1$$

$$\left(I_{0+}^{2} f\right)(x) = \int_0^x \left(\int_0^{x_1} f(x_2)\,dx_2 \right) dx_1$$

...

Cauchy formula

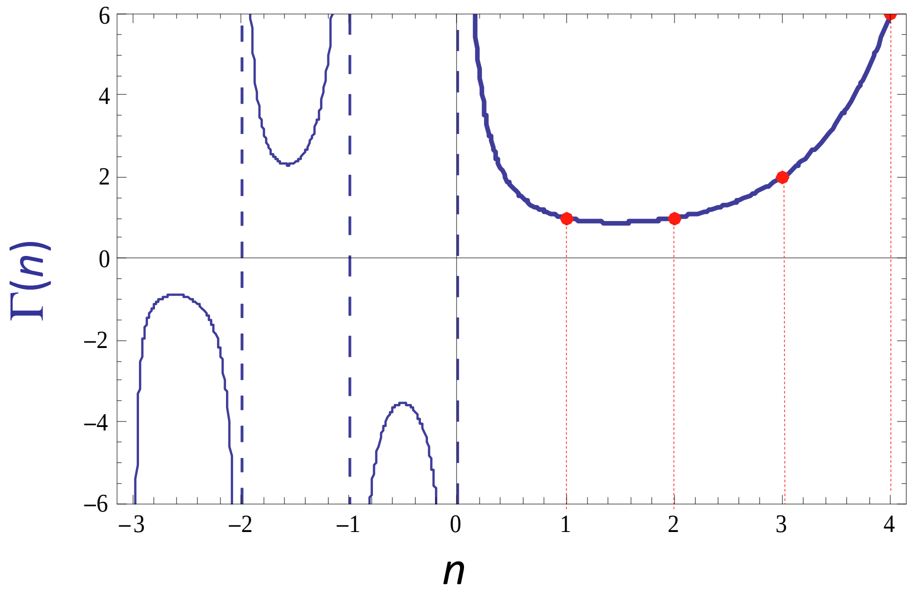
$$\left(I_{0+}^{n} f\right)(x) = \underbrace{\int_0^x dx_1 \ldots \int_0^{x_{n-1}} f(x_n)\,dx_n}_{n-fold} =$$

$$= \frac{1}{(n-1)!} \int_0^x (x-z)^{n-1} f(z)\,dz$$

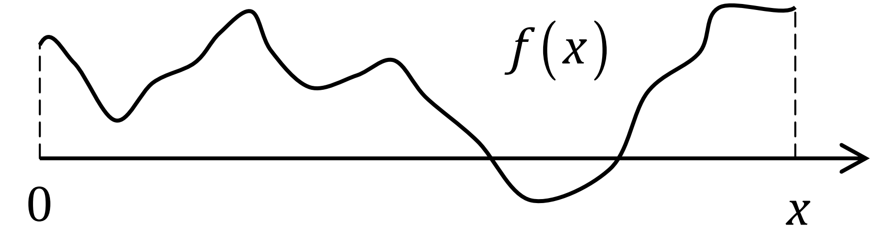
Euler Gamma function

$$\Gamma(\alpha) \overset{def}{=} \int_0^\infty e^{-z} z^{\alpha-1}\,dz$$

$$\Gamma(n) = (n-1)!$$



First step: definition of the Fractional Integral



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$$\left(I_{0+}^{2}f\right)(x)=\int_0^x\left(\int_0^{x_1} f(x_2)\,dx_2\right)dx_1$$

$$\Gamma(n)=(n-1)!$$

...

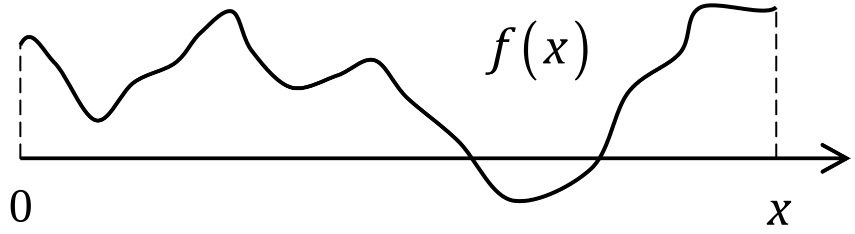
$$\left(I_{0+}^{n}f\right)(x)=\underbrace{\int_0^x dx_1\ldots\int_0^{x_{n-1}} f(x_n)\,dx_n}_{n-fold}=\frac{1}{(n-1)!}\int_0^x (x-z)^{n-1} f(z)\,dz$$

Riemann – Liouville fractional integral

$$\left(I_{0+}^{\alpha}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-z)^{\alpha-1} f(z)\,dz$$

$$\alpha\in\mathbb{C},\quad \mathrm{Re}[\alpha]>0$$

Second step: From the Fractional Integrals to the fractional Derivatives



Riemann – Liouville fractional integral

$$\left(I_{0+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-z)^{\alpha-1} f(z)\,dz$$

$$f(x) = \frac{d}{dx}\left[\left(I_{0+}^{1} f\right)(x)\right]$$

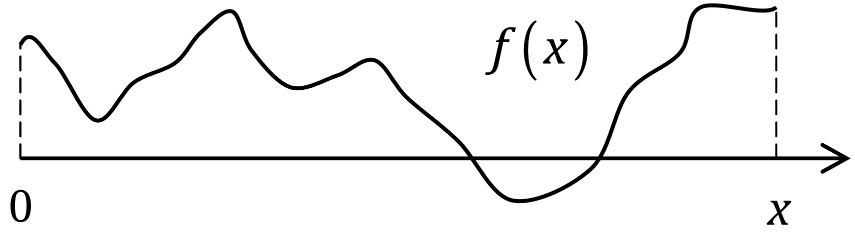
$$\left(D^{1} f\right)(x) = \frac{d}{dx}\left(\frac{d}{dx}\left[\left(I_{0+}^{1} f\right)(x)\right]\right)$$

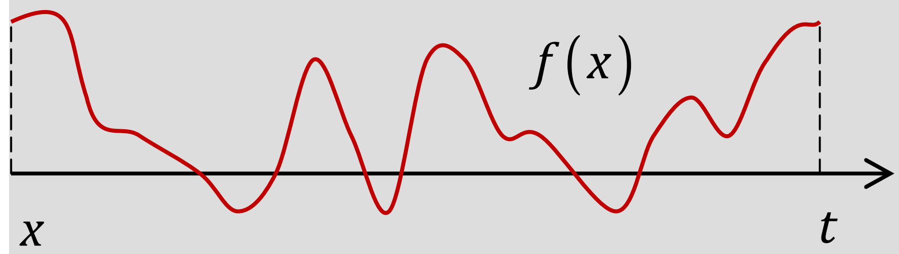
Riemann – Liouville fractional derivative

$$\left(D_{0+}^{\alpha} f\right)(x) = \frac{d^m}{dx^m}\left[\left(I_{0+}^{m-\alpha} f\right)(x)\right]$$

$$m = [\alpha] + 1$$

Second step: From the Fractional Integrals to the fractional Derivatives





Riemann – Liouville fractional integral

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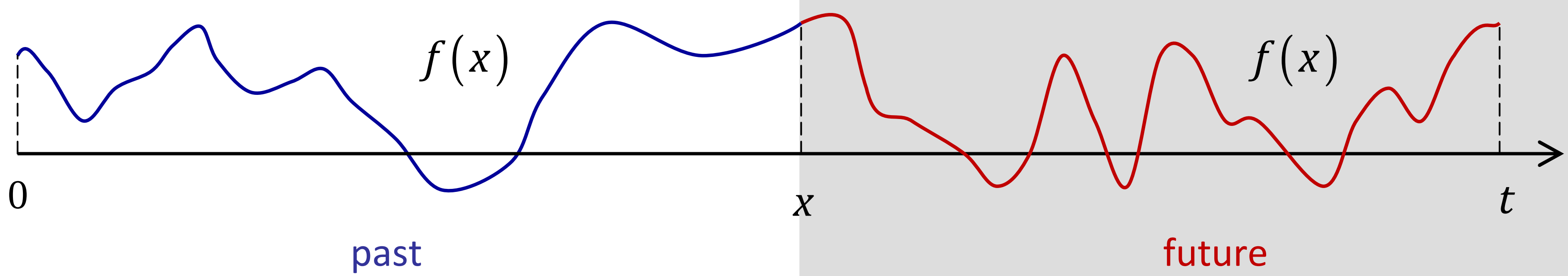
Riemann – Liouville fractional derivative

$$\left(D_{0+}^{\alpha} f\right)(x) = \frac{d^m}{dx^m}\left[\left(I_{0+}^{m-\alpha} f\right)(x)\right]$$

$$m = [\alpha] + 1$$

May we choose another interval?

The memory of Fractional Integrals and Derivatives



Left Riemann – Liouville fractional integral

$$\left(I_{0+}^{\alpha} f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_{0}^{x}(x-z)^{\alpha-1} f(z)dz$$

Right Riemann – Liouville fractional integral

$$\left(I_{t-}^{\alpha} f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_{x}^{t}(x-z)^{\alpha-1} f(z)dz$$

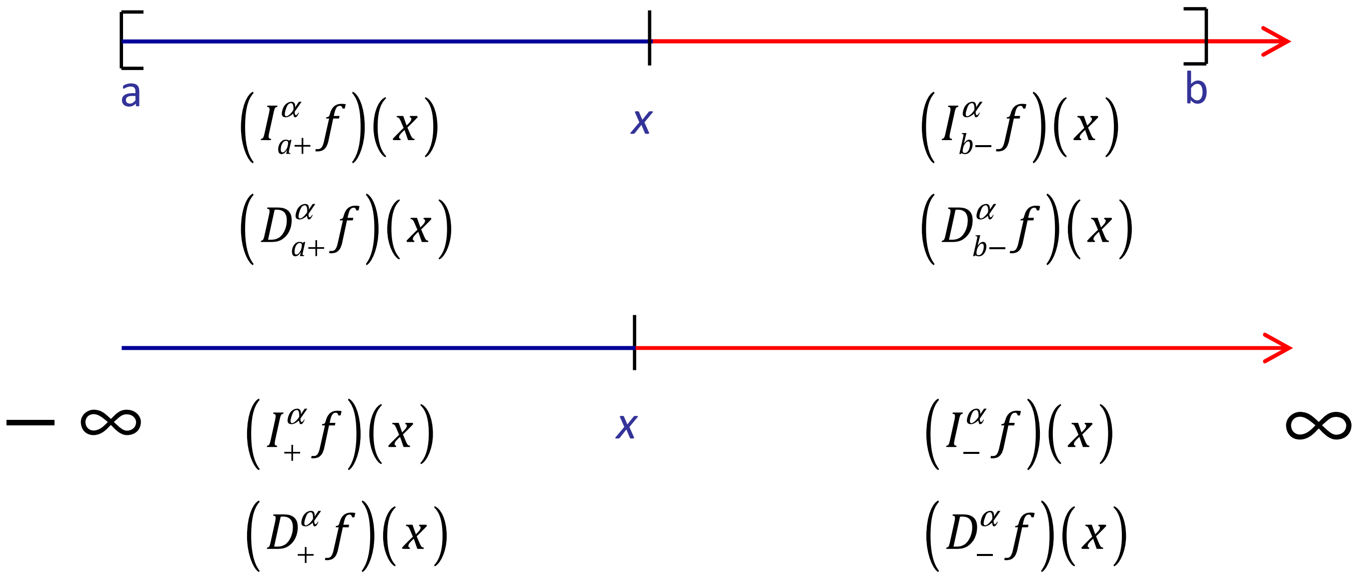
Left Riemann-Liouville fractional derivative

$$\left(D_{0+}^{\alpha} f\right)(x)=\frac{d^{m}}{dx^{m}}\left[\left(I_{0+}^{m-\alpha} f\right)(x)\right]$$

Right Riemann-Liouville fractional derivative

$$\left(D_{t-}^{\alpha} f\right)(x)=(-1)^{m}\frac{d^{m}}{dx^{m}}\left[\left(I_{t-}^{m-\alpha} f\right)(x)\right]$$

Possible intervals for Fractional Derivatives



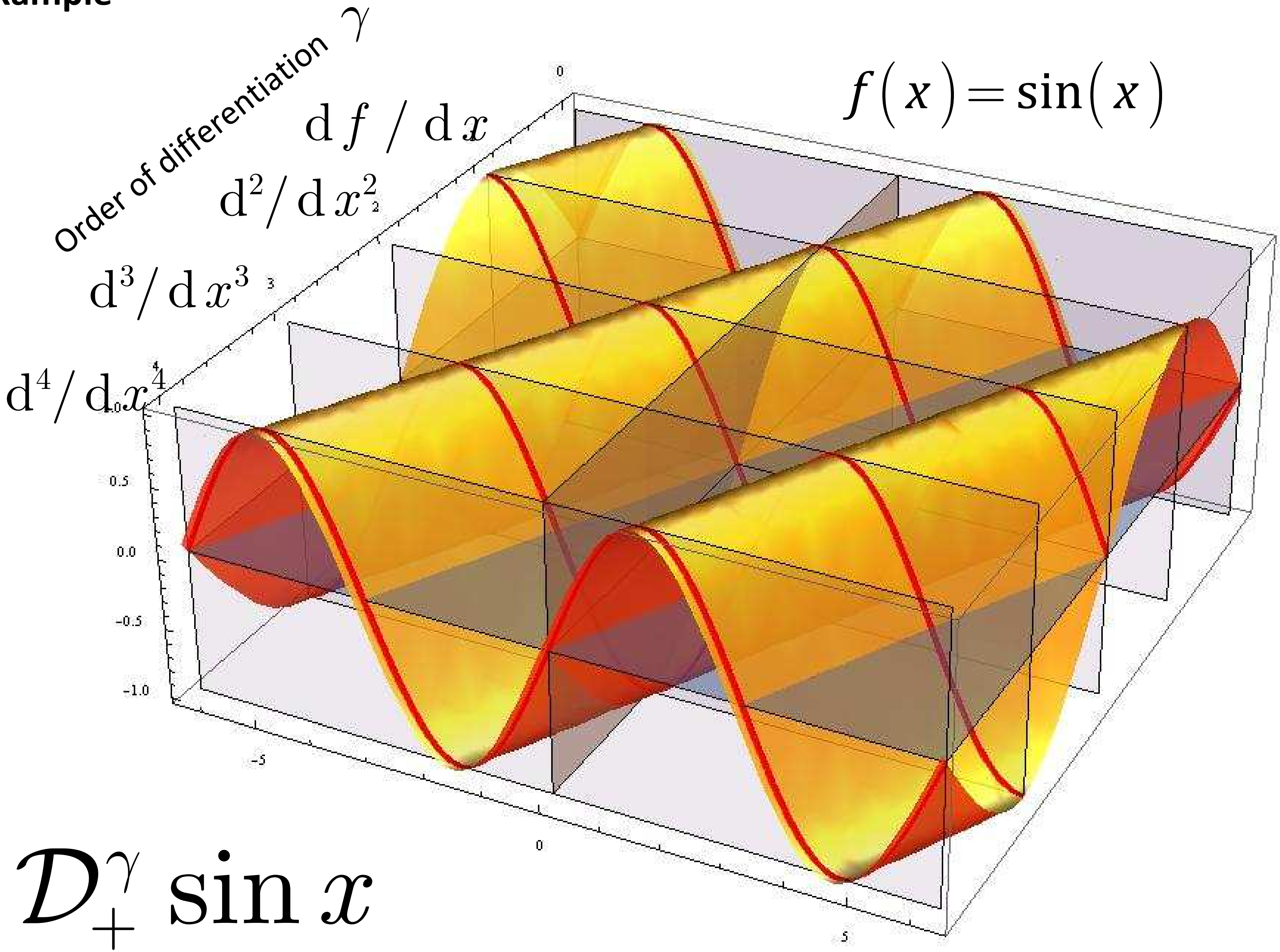
Properties

Composition

$$D_{a\pm}^{\alpha} I_{a\pm}^{\beta} f = D_{a\pm}^{\alpha-\beta} f$$

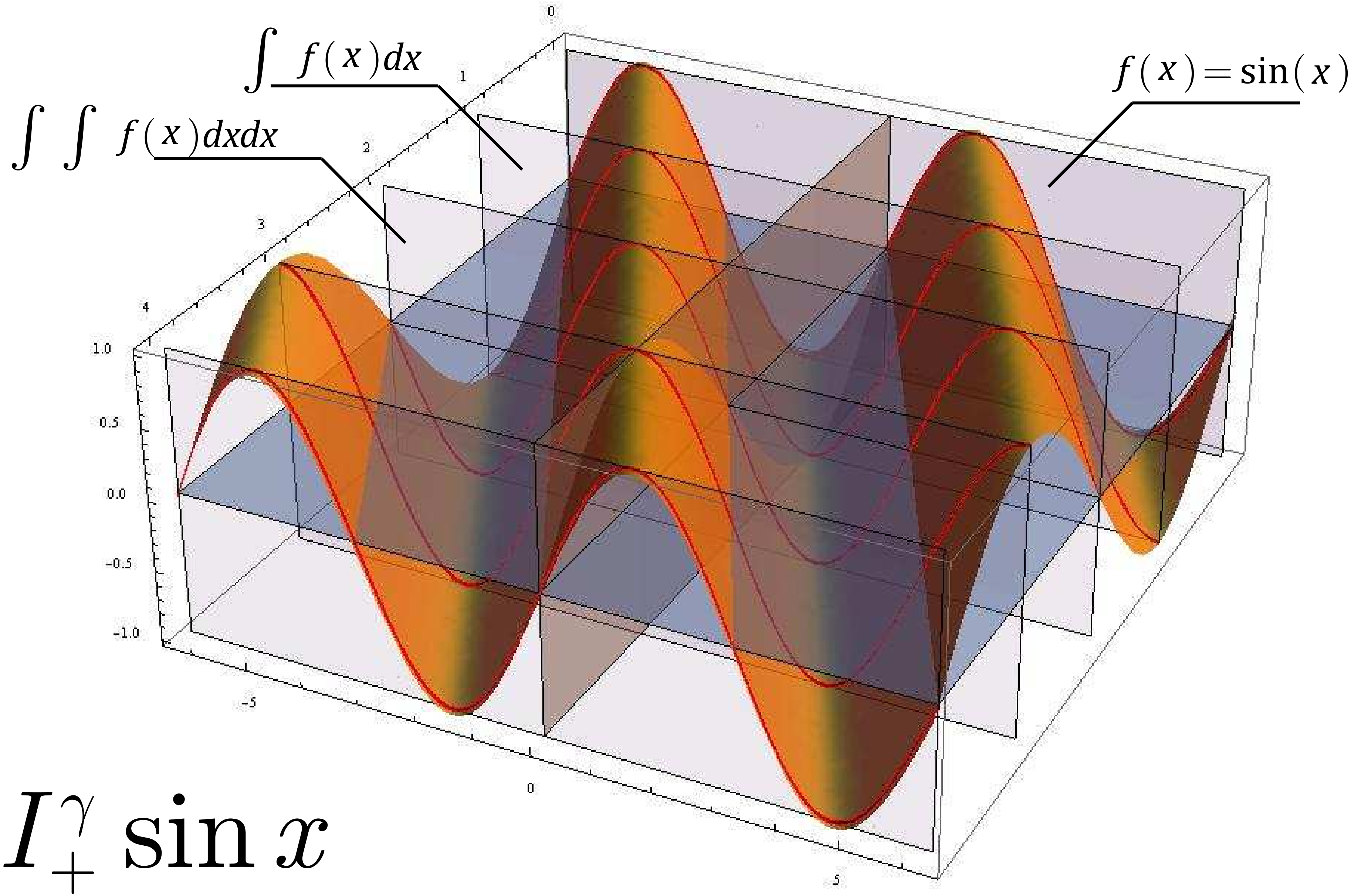
- *Leibnitz‘s rule*
- *Integration by parts*
- *Fourier Transform*
- *Taylor‘s expansion*

Example



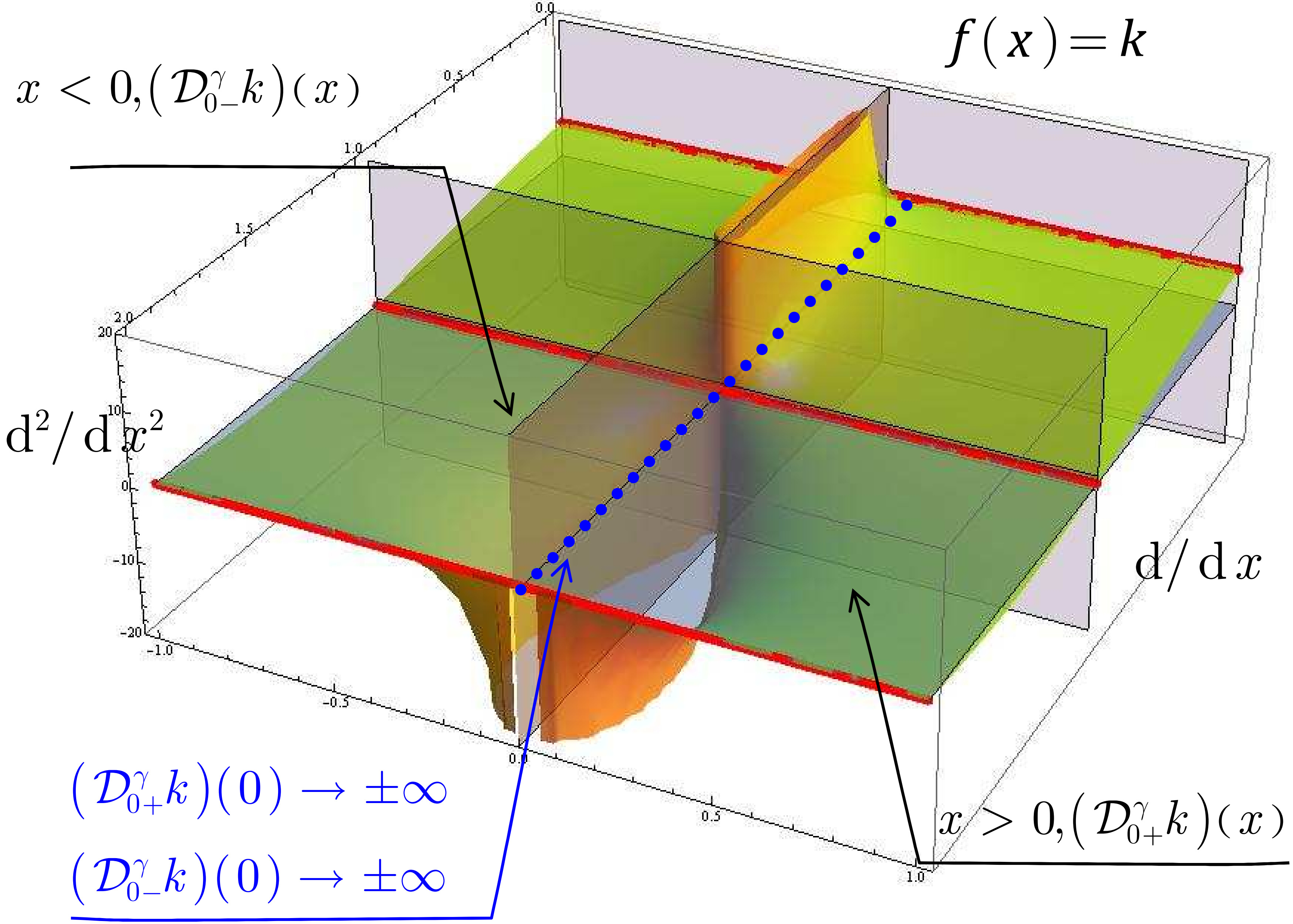
Order of differentiation γ
$\mathrm{d}f / \mathrm{d}x$
$\mathrm{d}^2/\mathrm{d}x^2$
$\mathrm{d}^3/\mathrm{d}x^3$
$\mathrm{d}^4/\mathrm{d}x^4$
$f(x) = \sin(x)$
1.0
0.5
0.0
-0.5
-1.0
-5
0
5
$\mathcal{D}_+^{\gamma} \sin x$

Example



$$I_+^{\gamma} \sin x$$

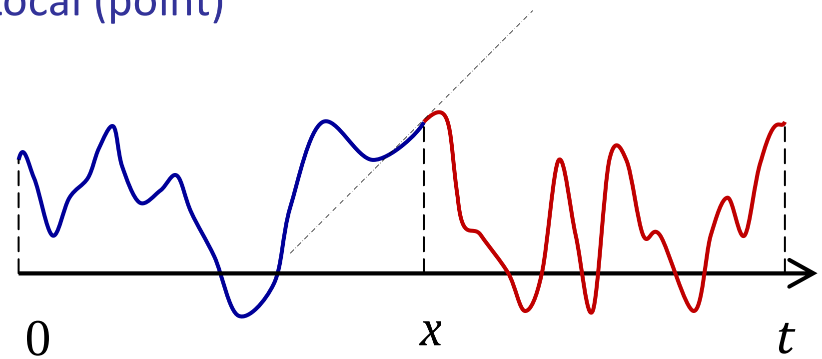
Example



$f(x) = k$
$x < 0, (\mathcal{D}^{\gamma}_{0-}k)(x)$
d^2/dx^2
d/dx
$(\mathcal{D}^{\gamma}_{0+}k)(0) \to \pm\infty$
$(\mathcal{D}^{\gamma}_{0-}k)(0) \to \pm\infty$
$x > 0, (\mathcal{D}^{\gamma}_{0+}k)(x)$

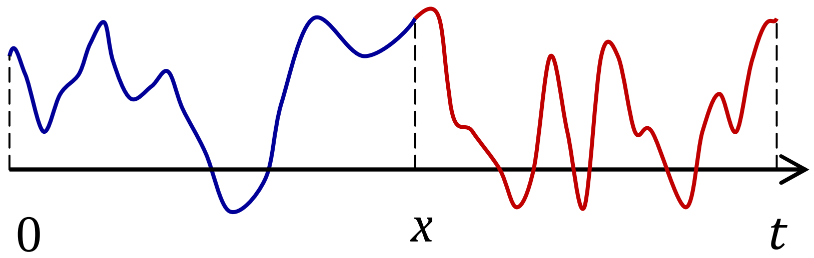
Classical Derivatives **VERSUS** **Fractional Derivatives**

- Integer order
- One definition
- Local (point)



- Taylor series: approximation in a point's neighborhood
- Models via ODE and PDE

- Real or Complex order
- Different definitions
- Generalize the classical derivatives
- Interpolate the classical derivatives
- Global (interval)

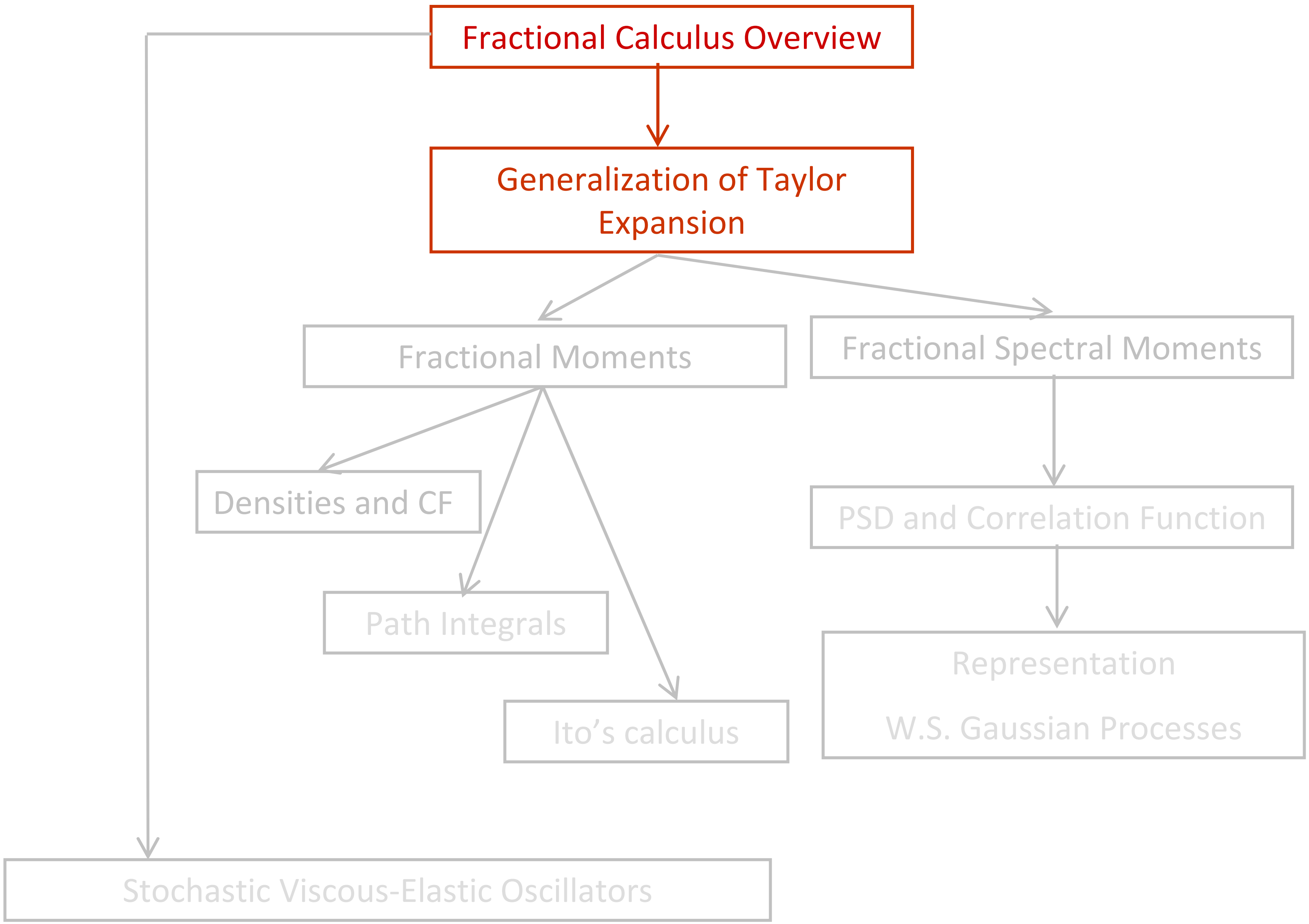


- Generalized Taylor series: approximation in the whole interval
- Models via FDE and PFDE

MODELLING WITH FRACTIONAL DERIVATIVES. WHEN?

- Phenomena with memory
  - Viscoelasticity
  - Nonlinear behaviour
  - Long correlated processes (inverse power-law decay)
  - Long-range interactions (inverse power-law decay)
- Alternative to higher-order models (i.e. gradient non-local theory)
- Failure of Taylor series:
  - Does not catch the necessary information (non-local)
  - Cannot be calculated (not derivable functions)

OUTLINE



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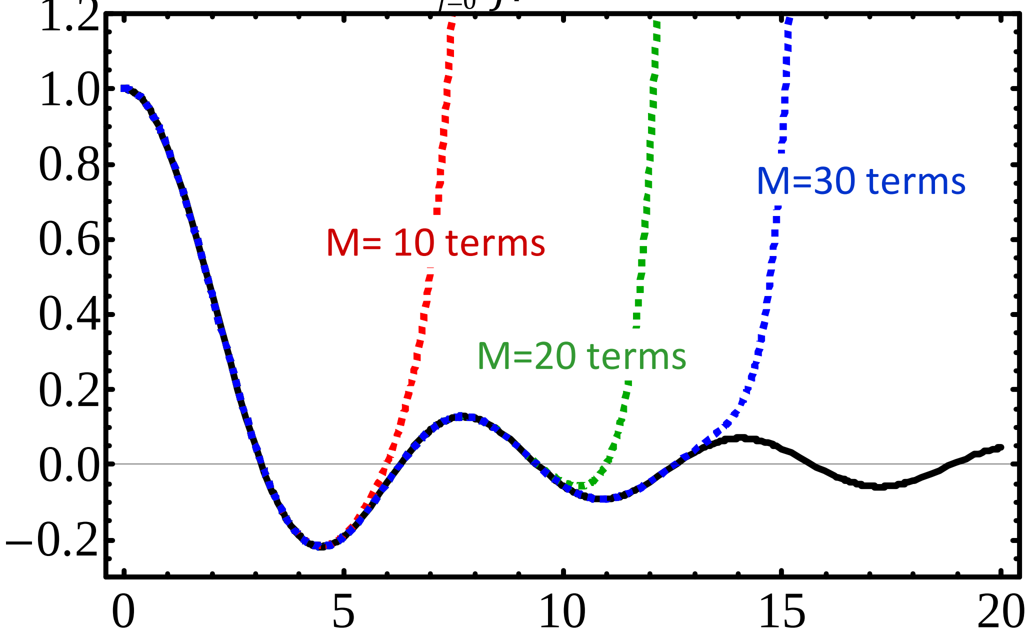
Classical Taylor expansion

$$f(x)=\sum_{j=0}^{\infty}\frac{1}{j!}\left(D^{j}f\right)(0)\,x^{j}$$

$$\left(D^{j}f\right)(0)=\left.\frac{d^{j}f(x)}{dx^{j}}\right|_{x=0}$$

$$f(x)=\sum_{j=0}^{M}\frac{1}{j!}\left(D^{j}f\right)(0)\,x^{j}$$



$$f(x)=\frac{\sin(x)}{x}$$

Generalization of the Taylor expansion by

- Riemann (Hardy proved the asymptotic convergence)

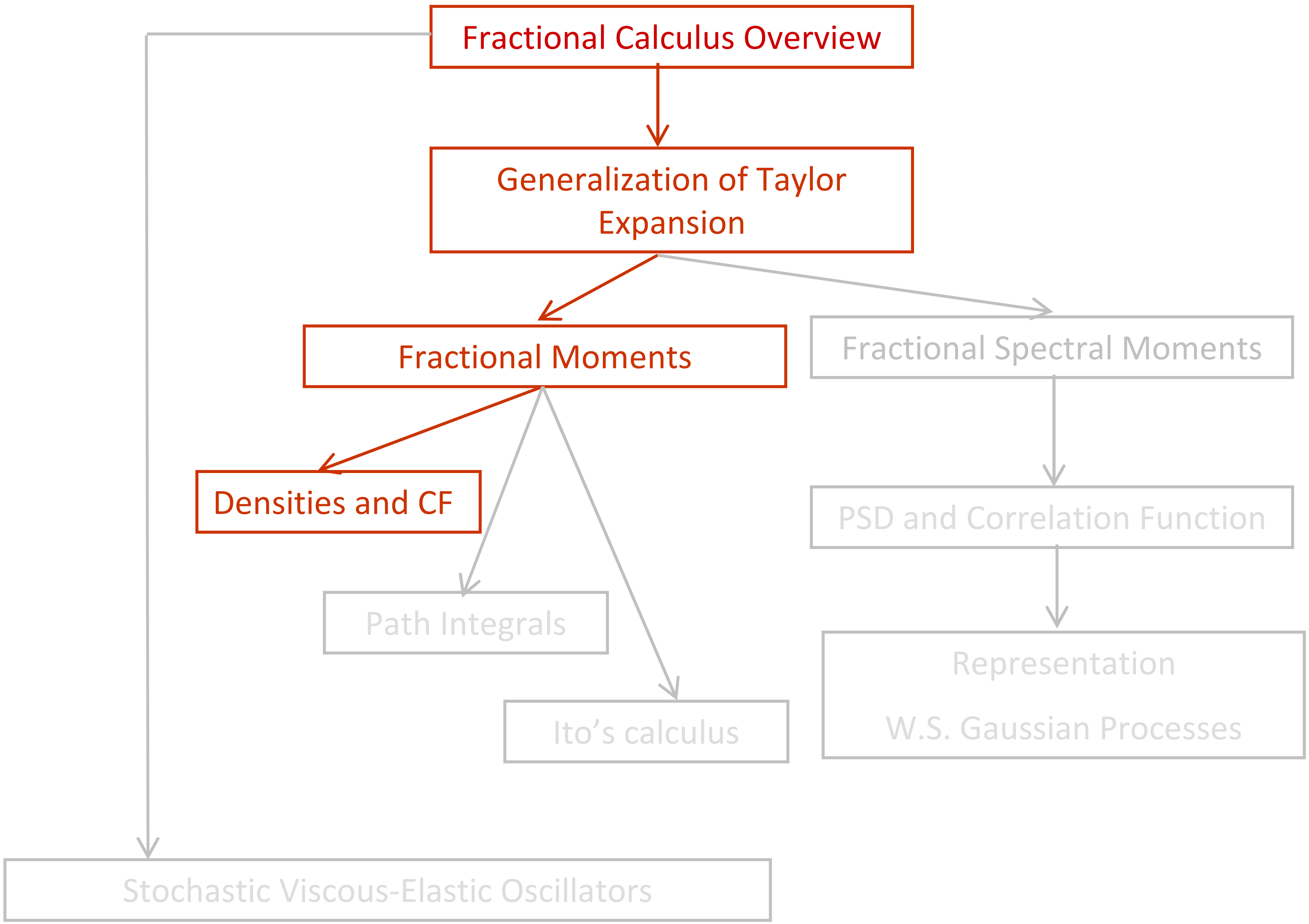
$$f\left(x+h\right)=\sum_{m=-\infty}^{\infty}\frac{h^{m+r}}{\Gamma\left(m+r+1\right)}\left(\mathcal{D}_{+}^{m+r}f\right)\left(x\right)$$ (post. 1876)

- Dzherbashyan and Nerseyan (1958)
- Osler (1972)
- Samko, Kilbas, Marichev* (1993)
- Trujillo, Bonilla, Rivero (1999)
- Jumarie (2006)

Integral Taylor form (Samko et al., Cottone et al.)

$$f\left(\pm\xi\right)=\frac{1}{2\pi i}\int_{\rho-i\infty}^{\rho+i\infty}\Gamma\left(\gamma\right)\left(I_{\pm}^{\gamma}f\right)\left(0\right)\left|\xi\right|^{-\gamma}d\gamma$$

OUTLINE



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Characteristic Function – Integer Moments relation

$$\phi_X(\vartheta) = E\left[\exp(i\vartheta X)\right] = \sum_{j=0}^{\infty} \frac{(i\vartheta)^j \, E\left[X^j\right]}{j!}$$

Generalization of the Taylor expansion

For general function

$$f(\pm\xi) = \frac{1}{2\pi i} \int_{\rho-i\infty}^{\rho+i\infty} \Gamma(\gamma) \left(I_{\pm}^{\gamma} f\right)(0) \, |\xi|^{-\gamma} \, d\gamma$$

For the Fourier transform of the probability density function (characteristic function)

$$\phi_X(\vartheta) = \frac{1}{2\pi i} \int_{\rho-i\infty}^{\rho+i\infty} \Gamma(\gamma) \, E\left[(\mp iX)^{-\gamma}\right] |\vartheta|^{-\gamma} \, d\gamma$$

“Taylor series like” form

$$\phi_X(\vartheta) \cong \frac{\Delta}{2\pi} \sum_{k=-M}^{M} \Gamma(\gamma_k) \, E\left[(-iX)^{-\gamma_k}\right] |\vartheta|^{-\gamma_k} \qquad \gamma_k = \rho + ik\Delta,$$

What are the fractional moments? (keep in mind the misnomer…)

Definition, knowing the density

$$E\left[(\pm iX)^{\mp\gamma}\right] = \int_{-\infty}^{\infty} p(x)(\pm ix)^{\mp\gamma}dx$$

Definition, knowing the characteristic function

$$E\left[(\pm iX)^{\gamma}\right] = \left(D_{\mp}^{\gamma}\phi_X\right)(0) \qquad \mathrm{Re}[\gamma] > 0$$

$$E\left[(\pm iX)^{-\gamma}\right] = \left(I_{\mp}^{\gamma}\phi_X\right)(0)$$

How to calculate fractional moments from data?

$$X_1, X_2, \ldots, X_n$$

$$E\left[(iX)^{0.5+2.3i}\right] = \frac{1}{n}\sum_{j=1}^{n}\left(iX_j\right)^{0.5+2.3i}$$

Not Exotic!

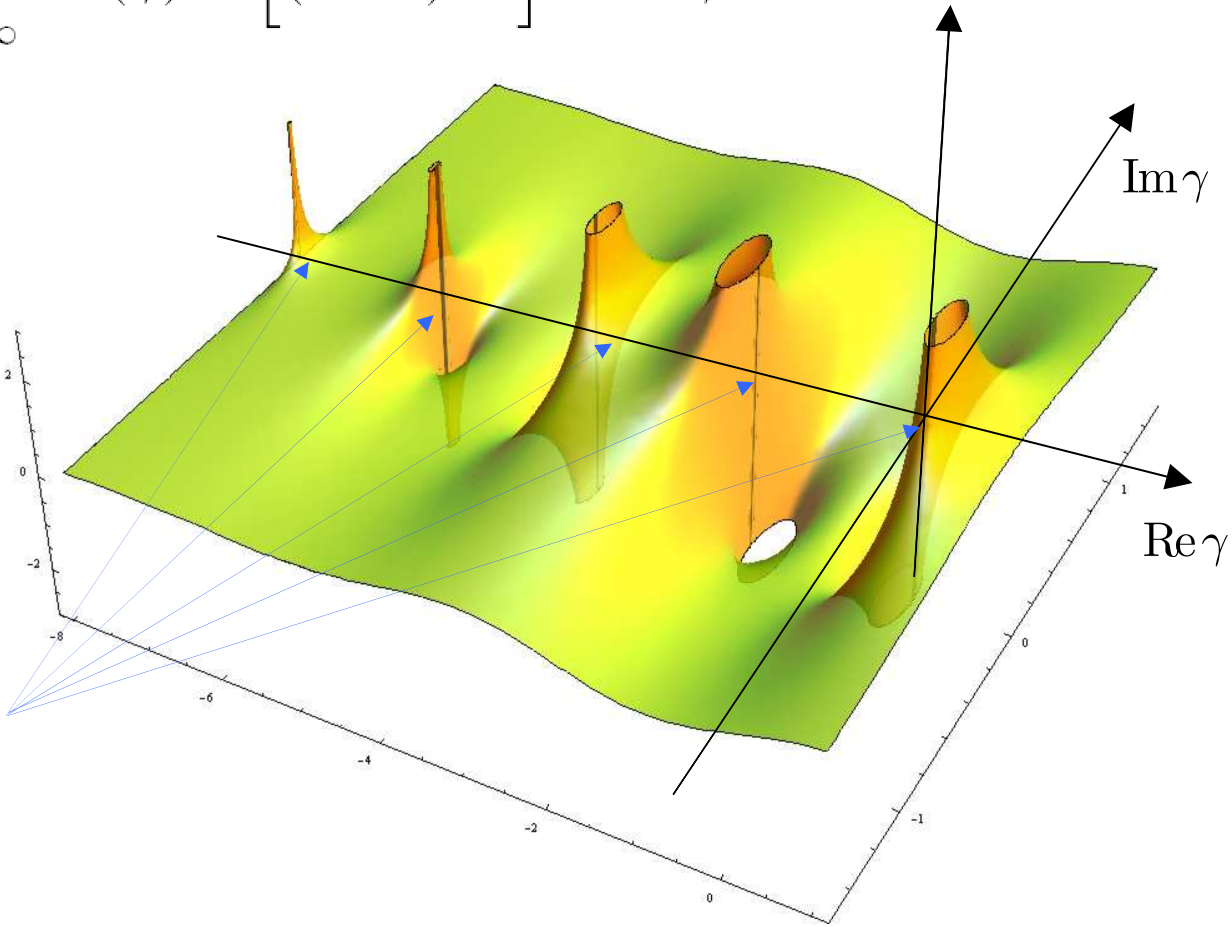
EXAMPLE

Standard Gaussian random variable $\phi_X(\vartheta) = e^{-\frac{\vartheta^2}{2}}$

For a fixed value of ϑ =1.5 we plot the REAL part of the integrand

$$\phi_X(\vartheta) = \frac{1}{2\pi i} \int_{\rho - i\infty}^{\rho + i\infty} \Gamma(\gamma) E\left[(-iX)^{-\gamma}\right] \vartheta^{-\gamma} d\gamma$$

$$Re\left[\Gamma(\gamma) E\left[(-iX)^{-\gamma}\right]^{-\gamma} \bar{\vartheta}^{-\gamma}\right]$$



ISOLATED SINGULARITIES

at **0,-2,-4…**

EXAMPLE Standard Gaussian random variable

In order to evaluate the integral, the value of ρ must be properly selected

$$\phi_X(\vartheta) = \frac{1}{2\pi i}\int_{\rho-i\infty}^{\rho+i\infty} \Gamma(\gamma)\, E\left[(-iX)^{-\gamma}\right] \vartheta^{-\gamma} d\gamma \qquad \gamma = \rho + i\eta$$

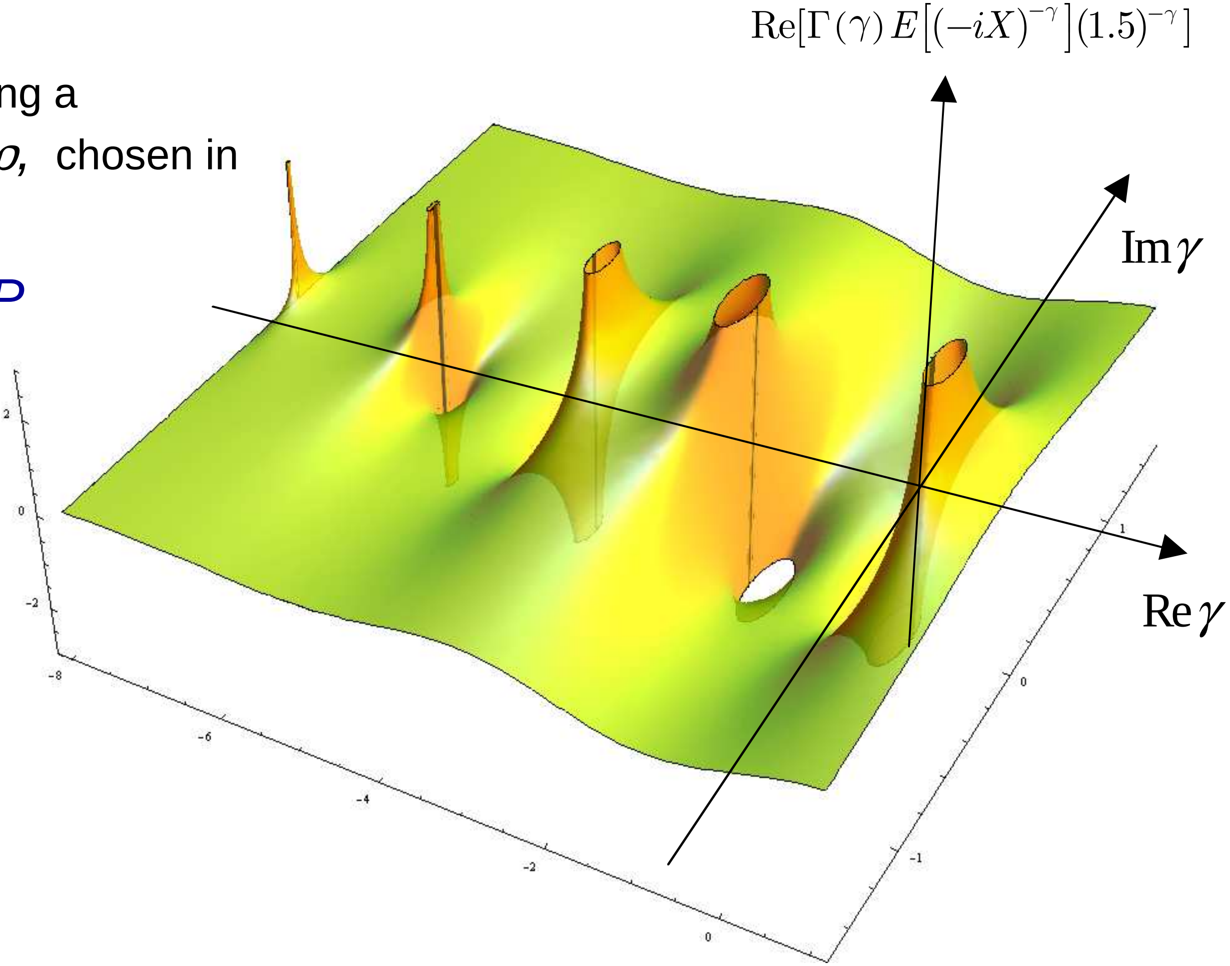
The integral is performed along a imaginary axis with real part ρ, chosen in the

FUNDAMENTAL STRIP

$$0 < \rho < 1$$

In particular we select

$$\rho = 1/2$$

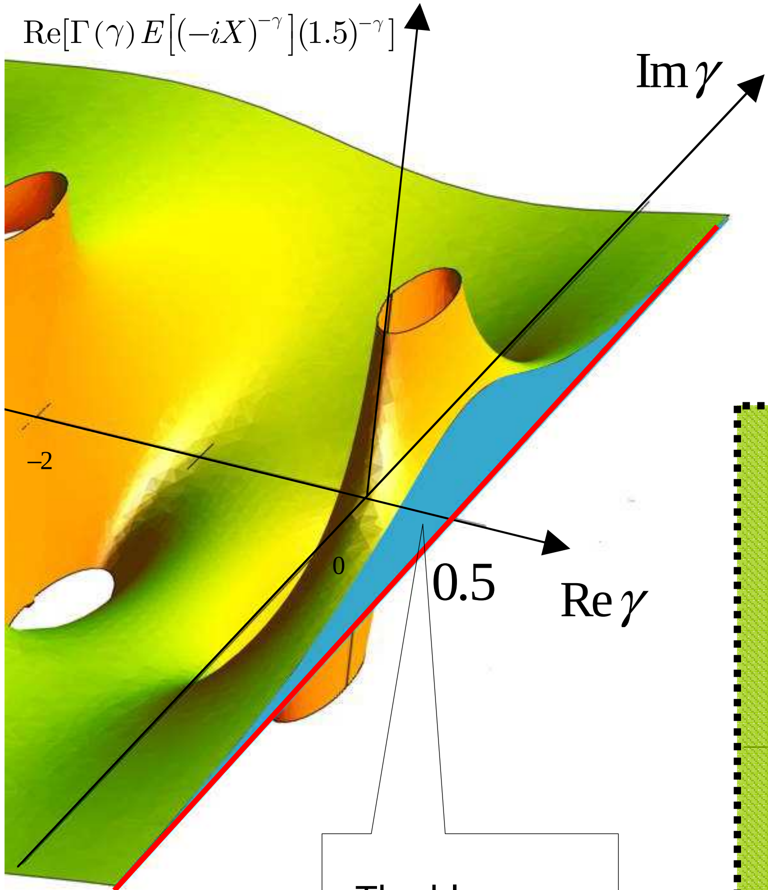


EXAMPLE Standard Gaussian random variable

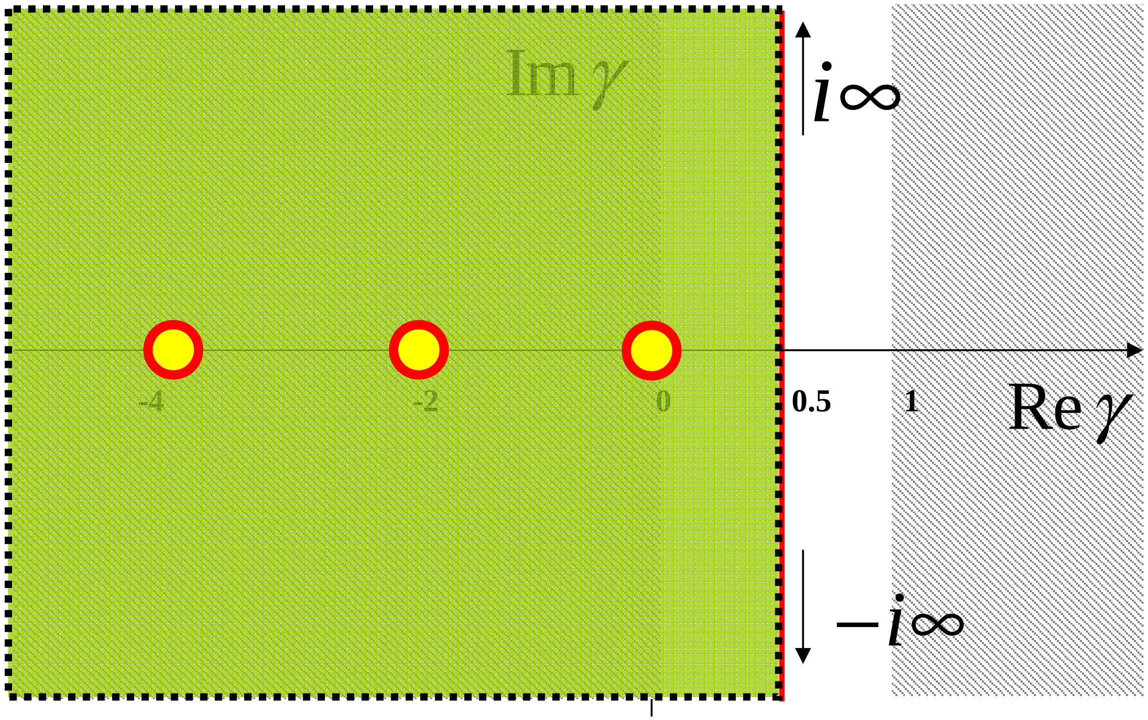
$$\phi_X(\vartheta) = \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} \Gamma(\gamma)\, E\left[(-iX)^{-\gamma}\right] \vartheta^{-\gamma} d\gamma$$ = *RESIDUE THEOREM*

$$= \sum_{n=-\infty}^{0} Res\left\{\Gamma(n)\, E\left[(-iX)^{-n}\right] \vartheta^{-n}\right\} =$$

$$= \sum_{n=0}^{\infty} \frac{(i\vartheta)^n}{n!} E[X^n]$$

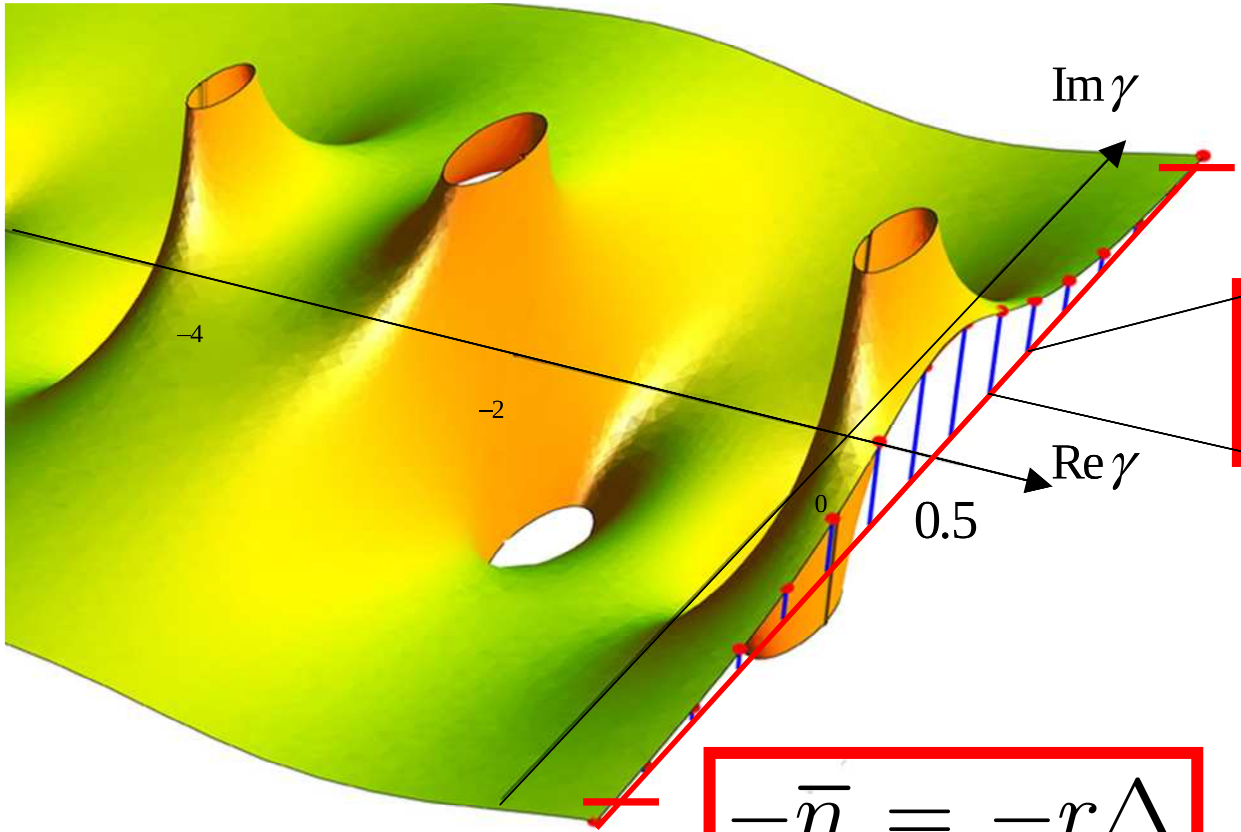


The blue area has to be evaluated



EXAMPLE Standard Gaussian random variable

$$\phi_X(\vartheta) = \frac{1}{2\pi i}\int_{0.5-\bar{\eta}}^{0.5+\bar{\eta}} \Gamma(\gamma)\, E\left[(-iX)^{-\gamma}\right] \vartheta^{-\gamma} d\gamma$$



$$\bar{\eta} = r\Delta$$

$step\ \Delta$

$$-\bar{\eta} = -r\Delta$$

Finite number of COMPX MOM

(2r + 1)

$$\phi_X(\vartheta) = \frac{\Delta}{2\pi}\sum_{k=-r}^{r} \Gamma(0.5 + ik\Delta)\, E\left[(-iX)^{-(0.5+ik\Delta)}\right] (\vartheta)^{-(0.5+ik\Delta)}$$

Comparison between the integer and fractional moments series

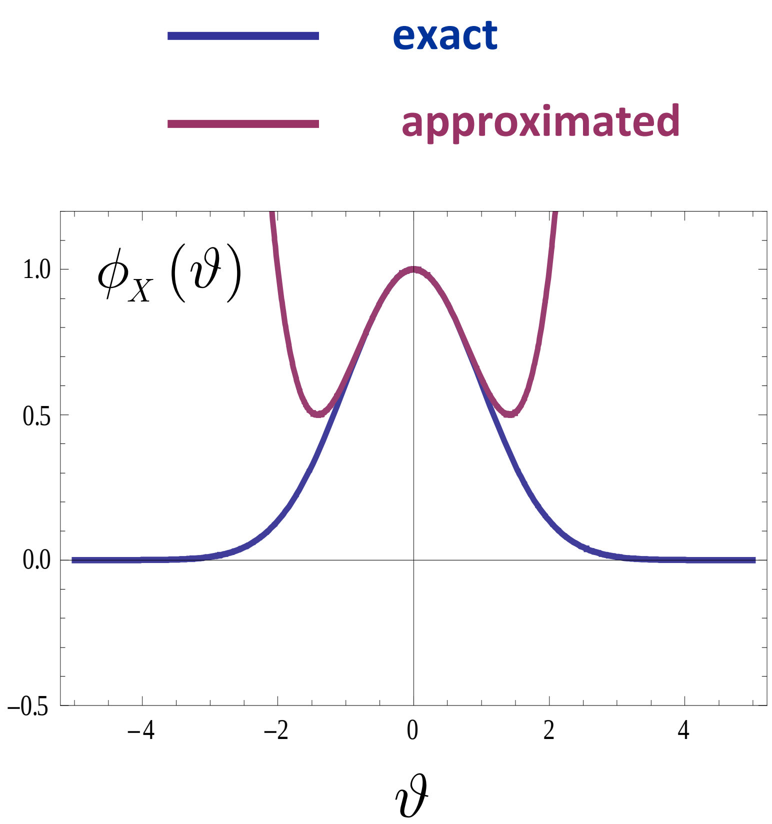
$$\phi_X(\vartheta) = \exp\left(-\vartheta^2 / 2\right)$$

$$\phi_X(\vartheta) \approx \sum_{n=0}^{4} \frac{(i\vartheta)^n}{n!} E[X^n]$$

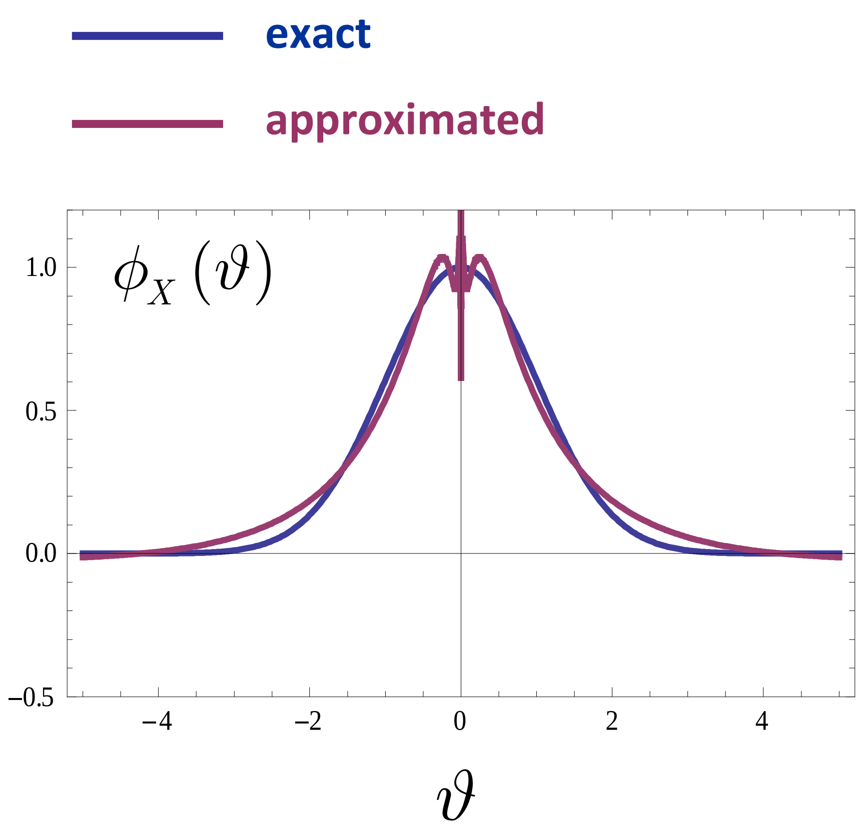
$$\phi_X(\vartheta) = \frac{\Delta}{2\pi} \sum_{k=-2}^{2} \Gamma(\gamma_k) E\left[(-iX)^{-\gamma_k}\right] (\vartheta)^{-\gamma_k}$$

$$\gamma_k = 1/2 + ik\Delta, \qquad \Delta = 0.4$$

5 INTEGER MOMENTS



5 FRACTIONAL MOMENTS



Comparison between the integer and fractional moments series

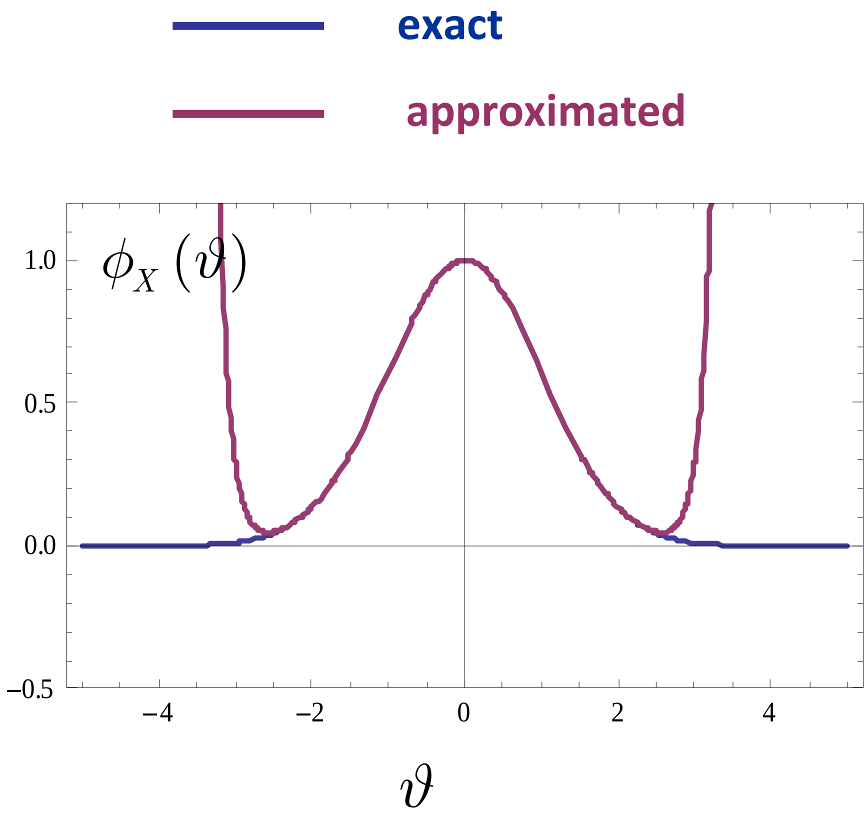
$$\phi_X(\vartheta) = \exp\left(-\vartheta^2/2\right)$$

$$\phi_X(\vartheta) \approx \sum_{n=0}^{20} \frac{(i\vartheta)^n}{n!} E\left[X^n\right]$$

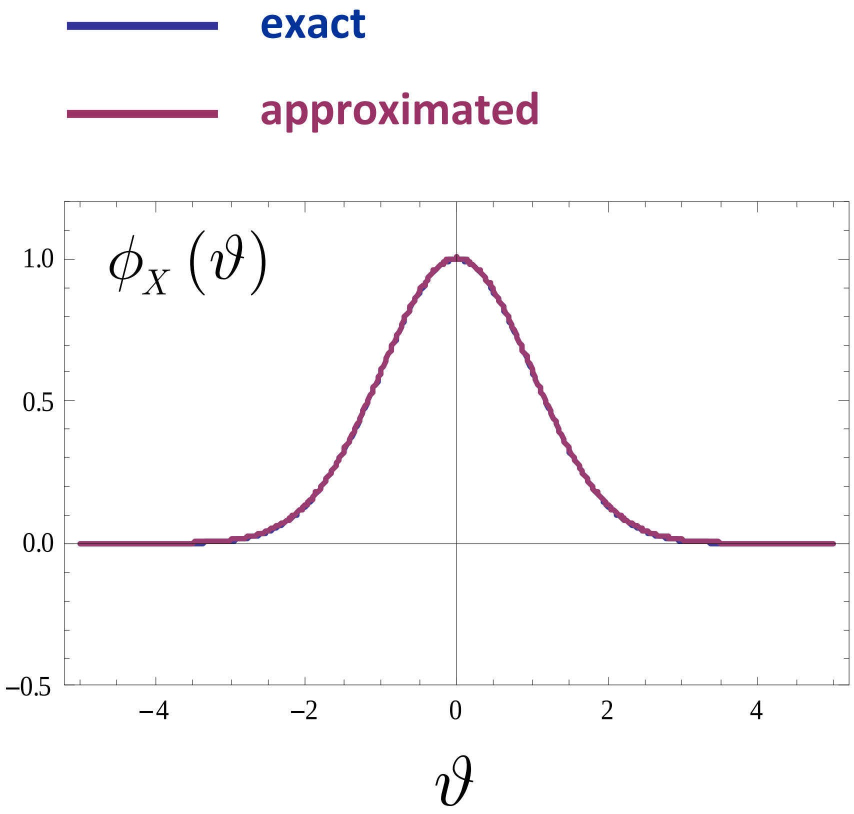
$$\phi_X(\vartheta) = \frac{\Delta}{2\pi} \sum_{k=-10}^{10} \Gamma(\gamma_k) E\left[(-iX)^{-\gamma_k}\right] (\vartheta)^{-\gamma_k}$$

$$\gamma_k = 1/2 + ik\Delta, \qquad \Delta = 0.4$$

21 INTEGER MOMENTS



21 FRACTIONAL MOMENTS



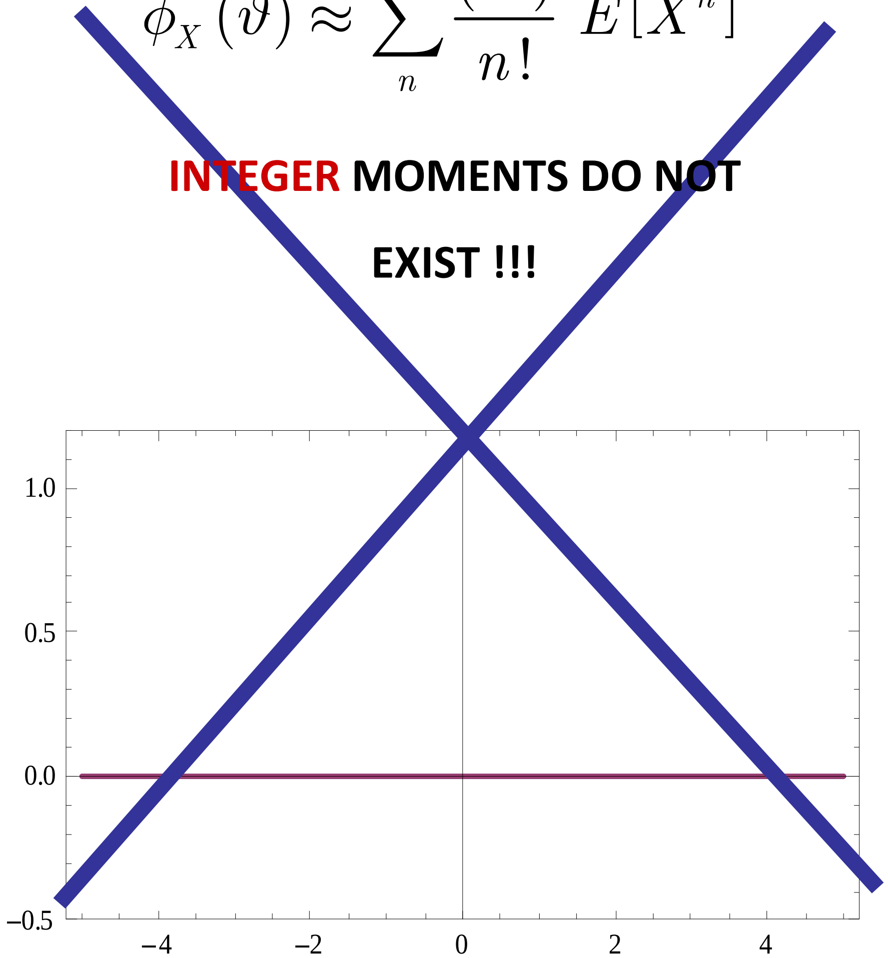
Lévy – Smirnov distribution

REAL PART

$$\phi_X(\vartheta) = \mathrm{Re}\left[e^{-|\vartheta|^{1/2}(1-i\,\mathrm{Sign}(\vartheta))}\right]$$

$$\phi_X(\vartheta) \approx \sum_n \frac{(i\vartheta)^n}{n!} E[X^n]$$

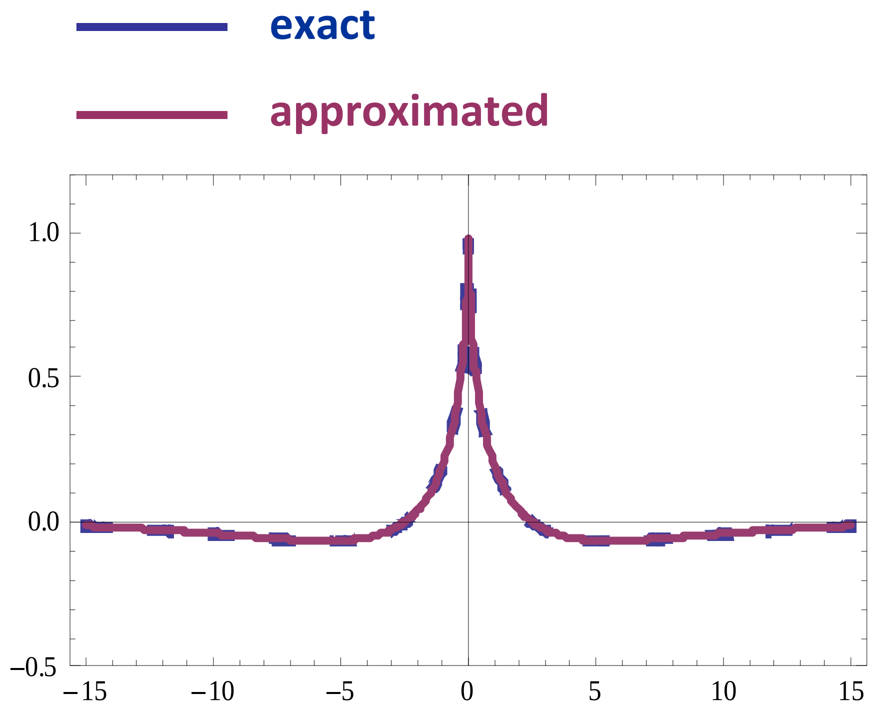
INTEGER MOMENTS DO NOT EXIST !!!



$$\phi_X(\vartheta) = \frac{\Delta}{2\pi} \sum_{k=-10}^{10} \Gamma(\gamma_k) E\left[(-iX)^{-(\gamma_k)}\right](\vartheta)^{-\gamma_k}$$

$$\gamma_k = 1/2 + ik\Delta, \qquad \Delta = 0.4$$

21 FRACTIONAL MOMENTS



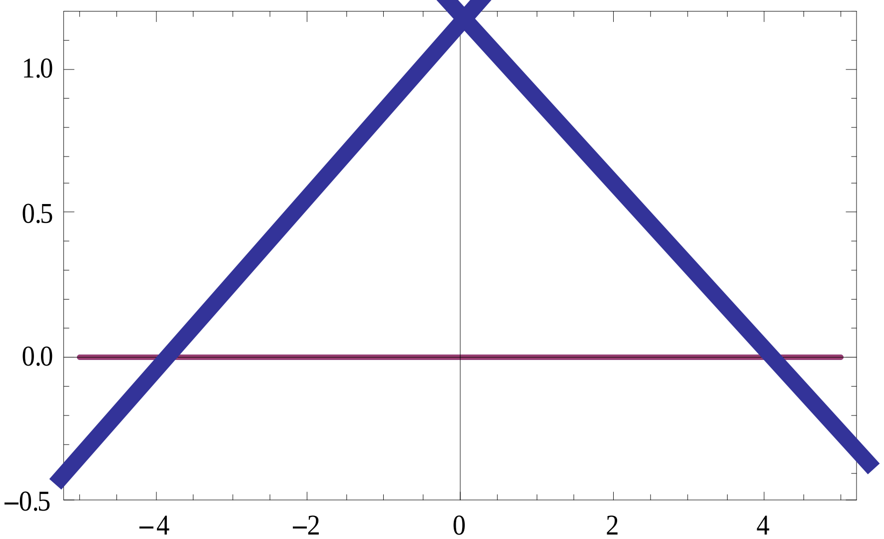
Lévy – Smirnov distribution

IMAGINARY PART

$$\phi_X(\vartheta) = \mathrm{Im}\left[e^{-|\vartheta|^{1/2}(1-i\,\mathrm{Sign}(\vartheta))}\right]$$

$$\phi_X(\vartheta) \approx \sum_{n=0}^{4} \frac{(i\vartheta)^n}{n!} E[X^n]$$

INTEGER MOMENTS DO NOT EXIST !!!



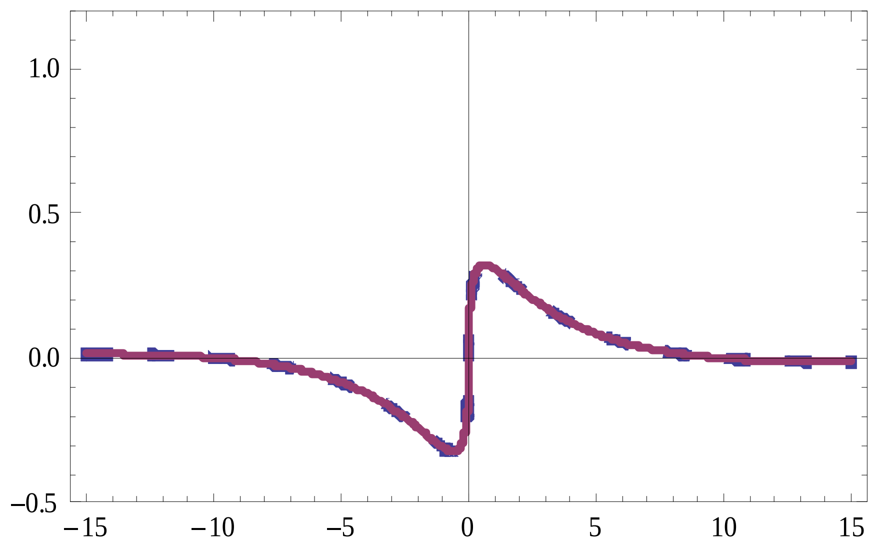
$$\phi_X(\vartheta) = \frac{\Delta}{2\pi} \sum_{k=-2}^{2} \Gamma(\gamma_k)\, E\left[(-iX)^{-(\gamma_k)}\right] (\vartheta)^{-\gamma_k}$$

$$\gamma_k = 1/2 + ik\Delta, \qquad \Delta = 0.4$$

21 FRACTIONAL MOMENTS

exact

approximated



Density by complex moments

In the case of integer moments the well-known expression are valid

$$\phi_X(\vartheta) = \sum_{j=0}^{\infty} \frac{(i\vartheta)^j E[X^j]}{j!} \quad \xrightarrow{\textbf{FT}} \quad p_X(x) = \sum_{j=0}^{\infty} (-1)^j \frac{E[X^j]}{j!} \frac{d^j \delta(x)}{dx^j}$$

In the case of fractional moments of complex order

$$\phi_X(\pm\vartheta) = \frac{1}{2\pi i} \int_{\rho - i\infty}^{\rho + i\infty} \Gamma(\gamma) E\left[(\mp iX)^{-\gamma}\right] |\vartheta|^{-\gamma} d\gamma$$

By making Inverse Fourier transform

$$p_X(x) = \frac{1}{(2\pi)^2 i} \int_{\rho - i\infty}^{\rho + i\infty} \Gamma(\gamma)\Gamma(1-\gamma) \times$$
$$\times \left\{ E\left[(-iX)^{-\gamma}\right](ix)^{\gamma-1} + E\left[(iX)^{-\gamma}\right](-ix)^{\gamma-1} \right\} \mathrm{d}\gamma$$

As the CF has symmetry properties, it simplifies

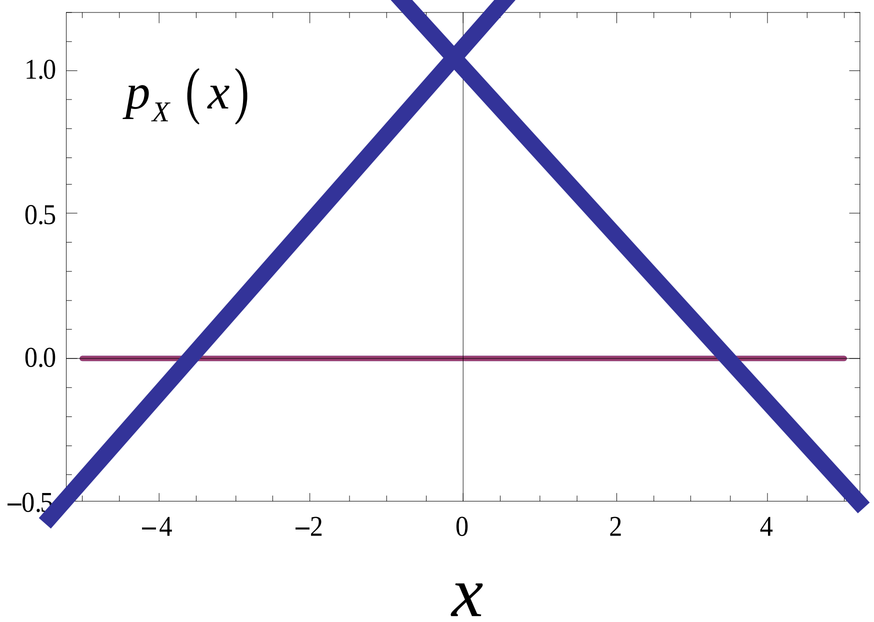
$$p_X(x) = \frac{1}{2\pi^2 i} \mathrm{Re}\left\{ \int_{\rho - i\infty}^{\rho + i\infty} \Gamma(\gamma)\Gamma(1-\gamma) E\left[(-iX)^{-\gamma}\right](ix)^{\gamma-1} \mathrm{d}\gamma \right\}$$

PDF Lévy-Smirnov random variable

$$p_X(x) = (1/2\pi)^{1/2}\,(x)^{-3/2}\,e^{-\frac{1}{2x}}$$

Integer moments do not exist!

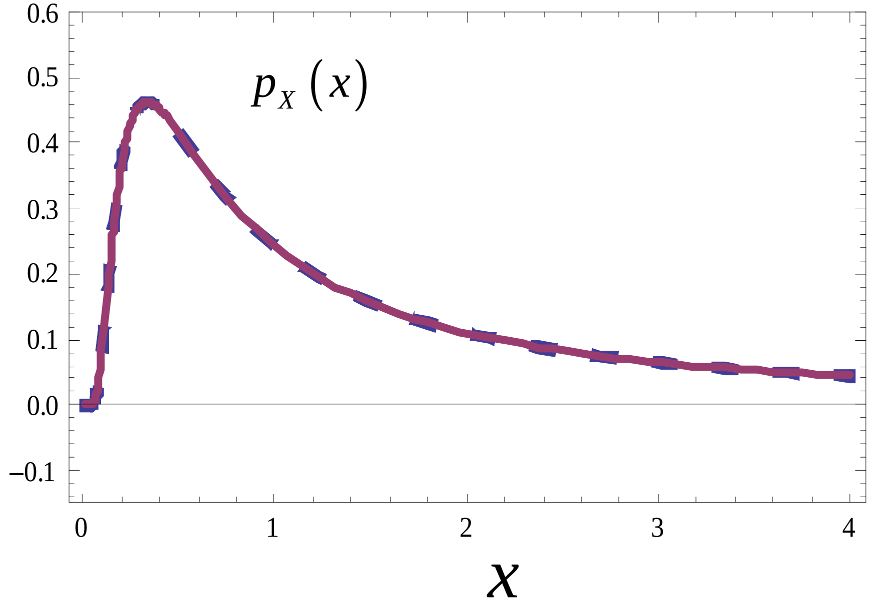
$$p_X(x) = \sum_{j=0}^{\infty} (-1)^j \frac{E[X^j]}{j!} \frac{d^j \delta(x)}{dx^j}$$



21 FRACTIONAL MOMENTS

exact

approximated



Bivariate Gaussian Random Vector

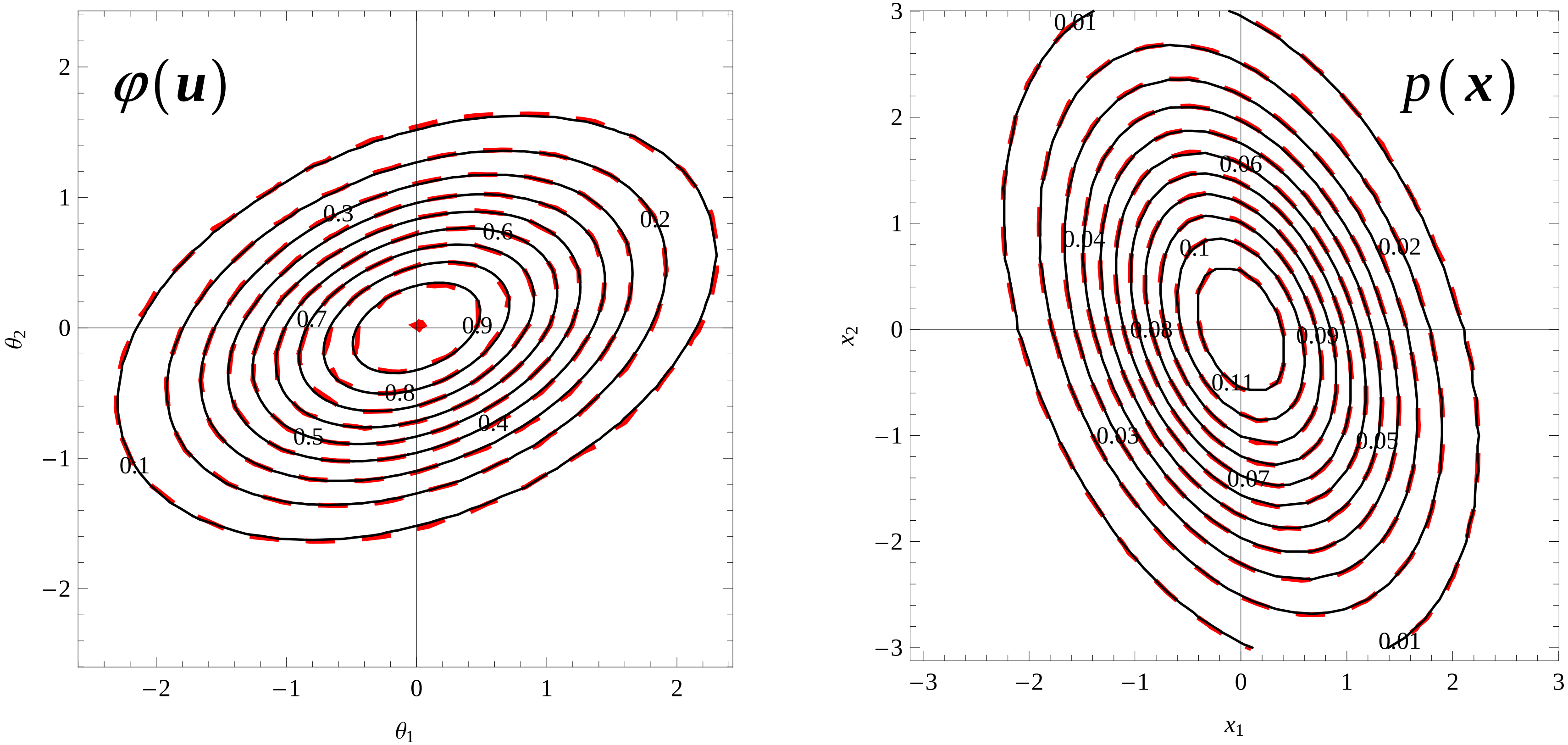
$$p(\boldsymbol{x}) = \left[(2\pi)^2 \det(\boldsymbol{\Sigma})\right]^{-1/2} \exp\left[-\frac{1}{2}\boldsymbol{x}^T(\boldsymbol{\Sigma})^{-1}\boldsymbol{x}\right] \qquad \varphi(\boldsymbol{u}) = \exp\left[-\frac{1}{2}\boldsymbol{u}^T\boldsymbol{\Sigma}\boldsymbol{u}\right]$$

$$\boldsymbol{\Sigma} = \begin{pmatrix} 1 & .5 \\ .5 & 2 \end{pmatrix}; \ \mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

exact

Approximated

;



Bivariate Cauchy Random Vector

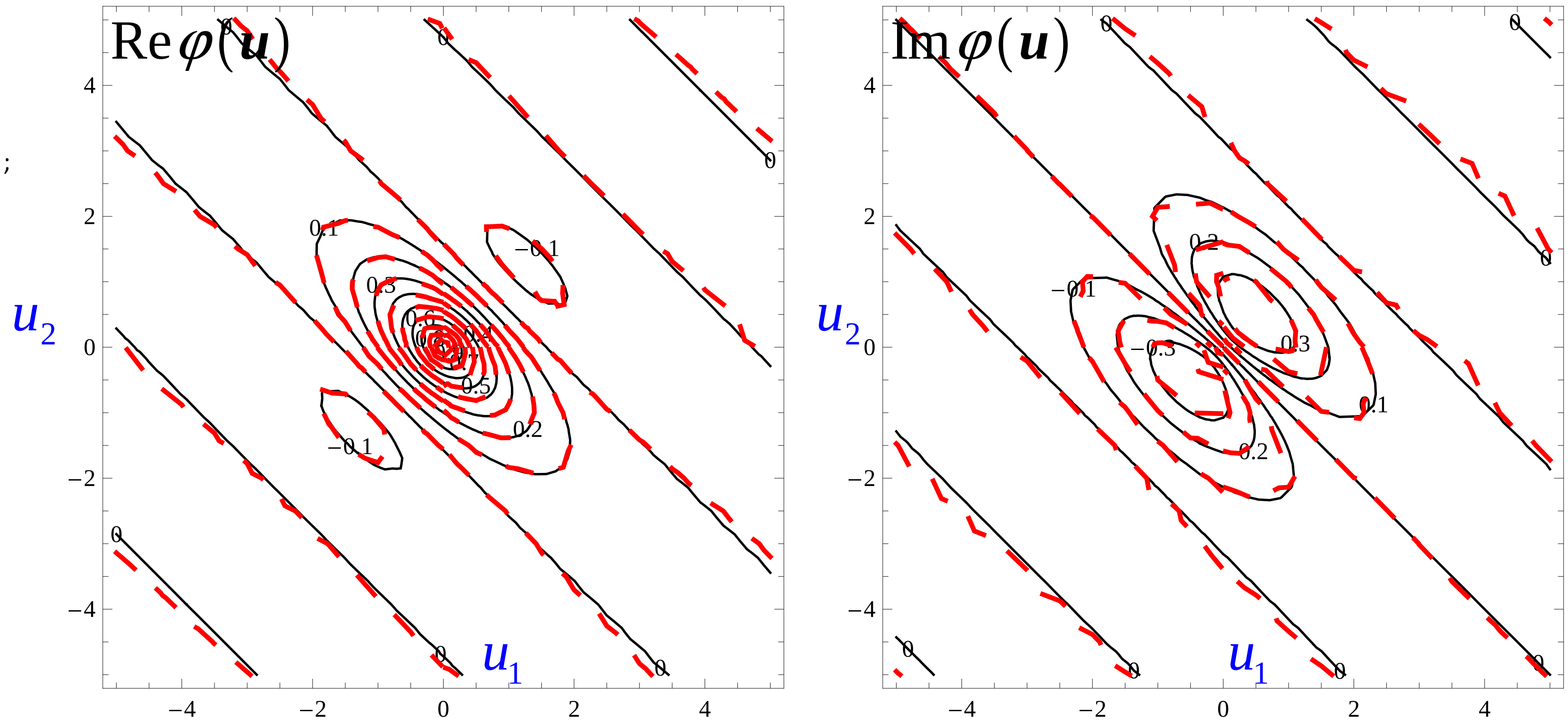
$$p(\boldsymbol{x}) = \frac{c\det(\boldsymbol{\Sigma})^{-1/2}}{\left(1+(\boldsymbol{x}-\boldsymbol{\mu})^T\,\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right)^{3/2}}$$

$$\varphi(\boldsymbol{u}) = \exp\left(-\boldsymbol{u}^T\boldsymbol{\Sigma}\boldsymbol{u}\right)^{1/2} + i\boldsymbol{u}^T\boldsymbol{\mu}$$

$$\Sigma = \begin{pmatrix} 1 & 0.2 \\ 0.2 & 1 \end{pmatrix};\ \mu = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

exact

approximated



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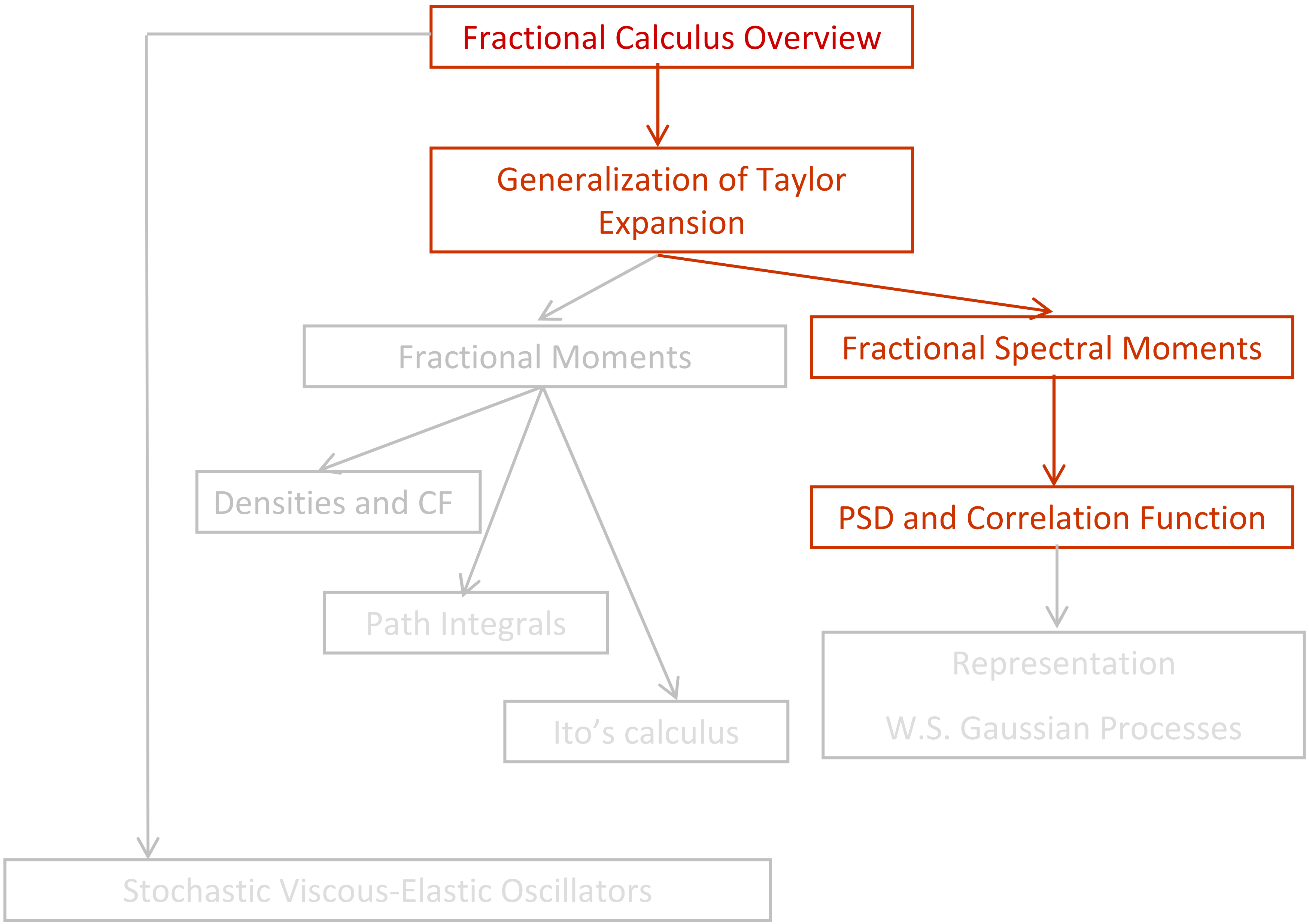
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Application of Fractional calculus to the representation of Stochastic processes

Define the analytical process

$$X(t) = \left[Y(t) + \mathrm{i}\hat{Y}(t)\right] / \sqrt{2}$$

Hilbert transform

$$\hat{Y}(t) = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{Y(\rho)}{t-\rho}\mathrm{d}\rho$$

Correlation Function of *X*(*t*)

$$R_X(\tau) = R_Y(\tau) + \mathrm{i}\hat{R}_Y(\tau)$$

One-sided Power Spectral Density

$$S_X(\omega) = 2U(\omega)S_Y(\omega)$$

Spectral moments of the analytical process (Vanmarke)

$$\lambda_X^j = \int_0^\infty 2U(\omega)\, S_Y(\omega)\, \omega^j \mathrm{d}\omega = \int_0^\infty S_X(\omega)\, \omega^j \mathrm{d}\omega$$

$$\lambda_X^{2n} = E\left[\frac{d^n X(t)}{dt^n} \frac{d^n X^*(t)}{dt^n}\right]$$

$$\mathrm{i}\lambda_X^{2n+1} = E\left[\frac{d^{n-1} X(t)}{dt^{n-1}} \frac{d^n X^*(t)}{dt^n}\right]$$

Fractional spectral moments of the analytical process

$$\Lambda_X(\gamma) = \int_0^\infty S_X(\omega)\, \omega^\gamma \mathrm{d}\omega \qquad \gamma = \rho + i\eta$$

↘ Mellin transform of $S_X(\omega)$

INVERTIBLE

Two important results:

$$(I^{\gamma}R_Y)(0) = \int_0^{\infty} \omega^{-\gamma} S_X(\omega)\,\mathrm{d}\omega = \Lambda_X(-\gamma)$$

$$(\mathcal{D}^{\gamma}R_Y)(0) = \int_0^{\infty} \omega^{\gamma} S_X(\omega)\,\mathrm{d}\omega = \Lambda_X(\gamma)$$

Correlation function

$$R_Y(s) = \frac{1}{2\pi \mathrm{i}} \int_{\rho-\mathrm{i}\infty}^{\rho+\mathrm{i}\infty} \nu(\gamma)\,\Lambda_X(-\gamma)\,|s|^{-\gamma}\,\mathrm{d}\gamma$$

Power Spectral Density

$$S_Y(\omega) = \frac{1}{4\pi \mathrm{i}} \int_{\rho-\mathrm{i}\infty}^{\rho+\mathrm{i}\infty} \Lambda_X(-\gamma)\,|\omega|^{\gamma-1}\,\mathrm{d}\gamma$$

Exact Form

Approximate Form

$$R_Y(s) \cong \frac{\Delta\eta}{2\pi} \sum_{k=-m}^{m} \nu(\gamma_k)\,\Lambda_X(-\gamma_k)\,|s|^{-\gamma_k}$$

$$S_Y(\omega) \cong \frac{\Delta\eta}{4\pi} \sum_{k=-m}^{m} \Lambda_X(-\gamma_k)\,|\omega|^{\gamma_k-1}$$

Davenport's spectrum

$$S_V(\omega) = \frac{4\pi k_0 V_{ref}^2}{|\omega|} \frac{q(\omega)^2}{\left(1+q(\omega)^2\right)^{4/3}}$$

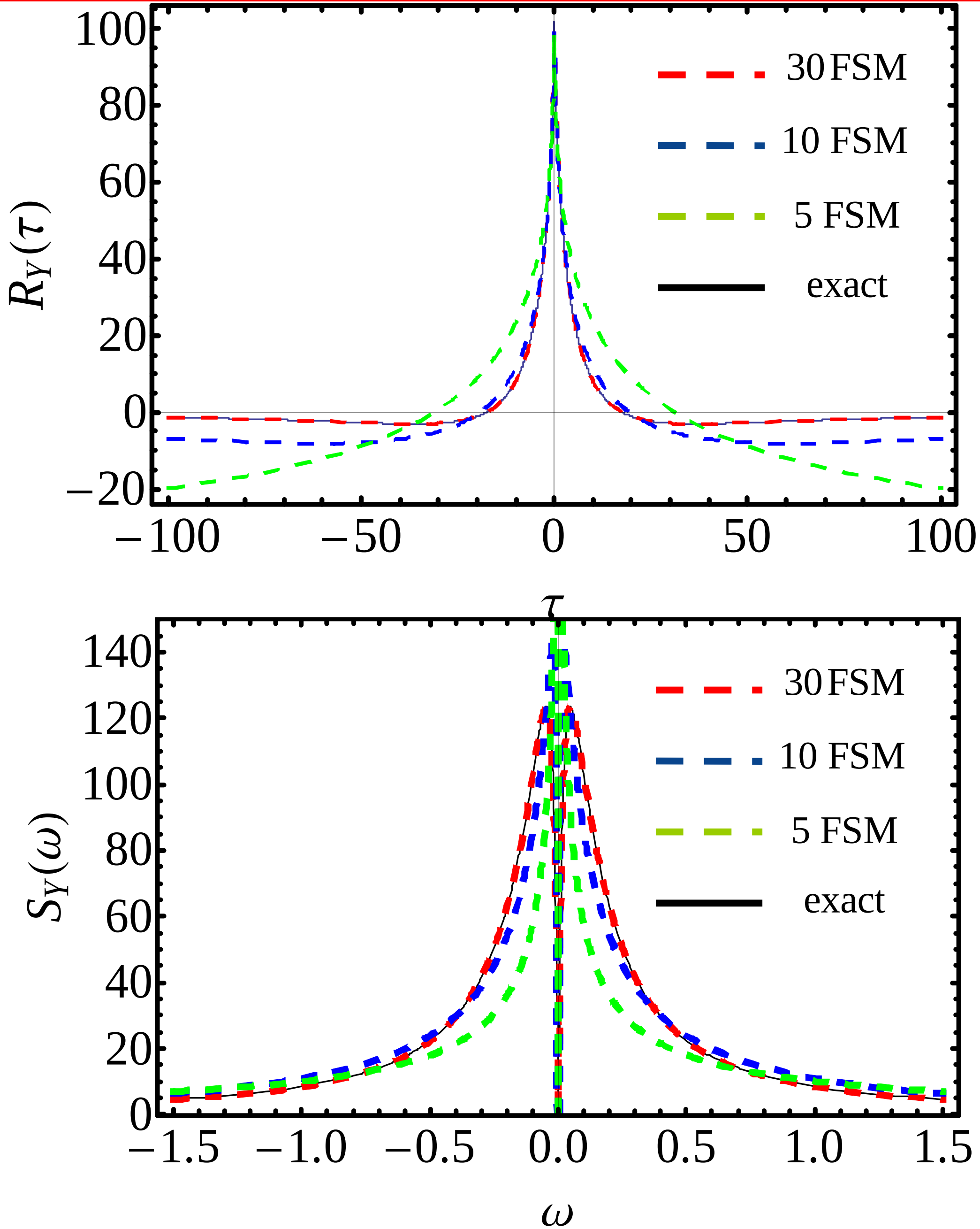
V_{ref}: mean wind speed at the reference level

k_0: roughness characteristic of the analyzed site

$q(\omega) = 1200\,\omega / (2\pi V_{ref})$

$$R_Y(s) \cong \frac{\Delta\eta}{2\pi} \sum_{k=-m}^{m} \nu(\gamma_k)\, \Lambda_X(-\gamma_k)\, |s|^{-\gamma_k}$$

$$S_Y(\omega) \cong \frac{\Delta\eta}{4\pi} \sum_{k=-m}^{m} \Lambda_X(-\gamma_k)\, |\omega|^{\gamma_k - 1}$$



Kaimal's spectrum

$$S_V(\omega) = \frac{200 v^* z_0}{4\pi v}\left(\frac{1}{1+50|\omega| z_0/(2\pi v)}\right)^{5/3}$$

z_0: height of the measured mean wind speed

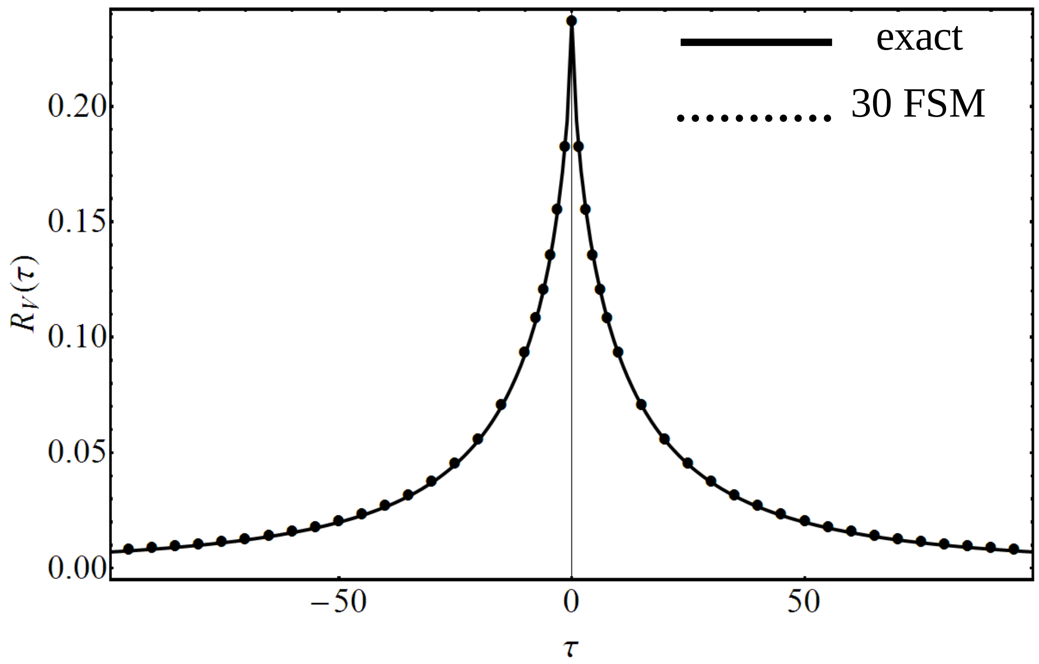
v : mean wind speed

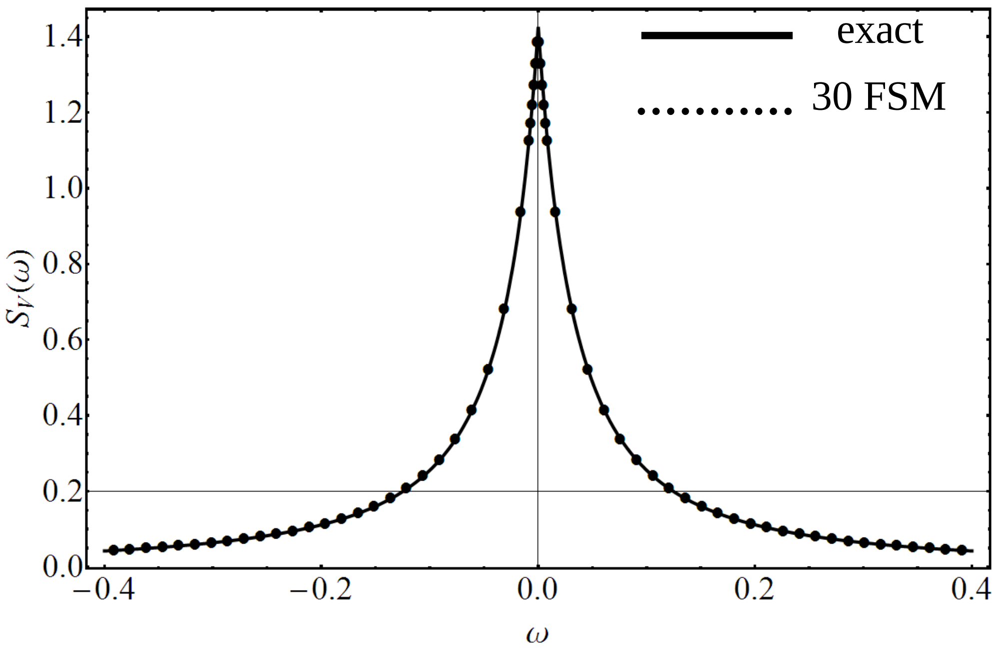
v^*: $0.4v/ln(z_0/k)$

k : roughness coefficient related to the exposed site

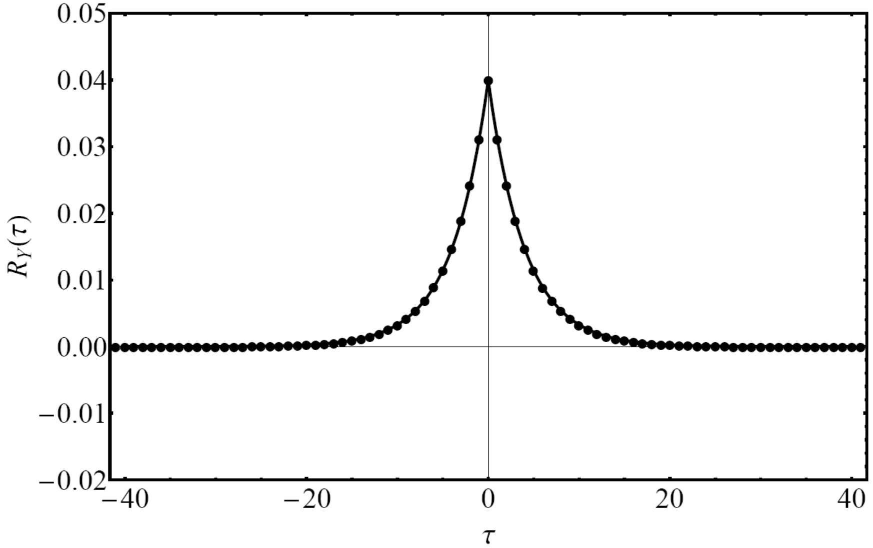
$$R_Y(s) \cong \frac{\Delta\eta}{2\pi}\sum_{k=-m}^{m} \nu(\gamma_k)\Lambda_X(-\gamma_k)|s|^{-\gamma_k}$$

$$S_Y(\omega) \cong \frac{\Delta\eta}{4\pi}\sum_{k=-m}^{m} \Lambda_X(-\gamma_k)|\omega|^{\gamma_k-1}$$

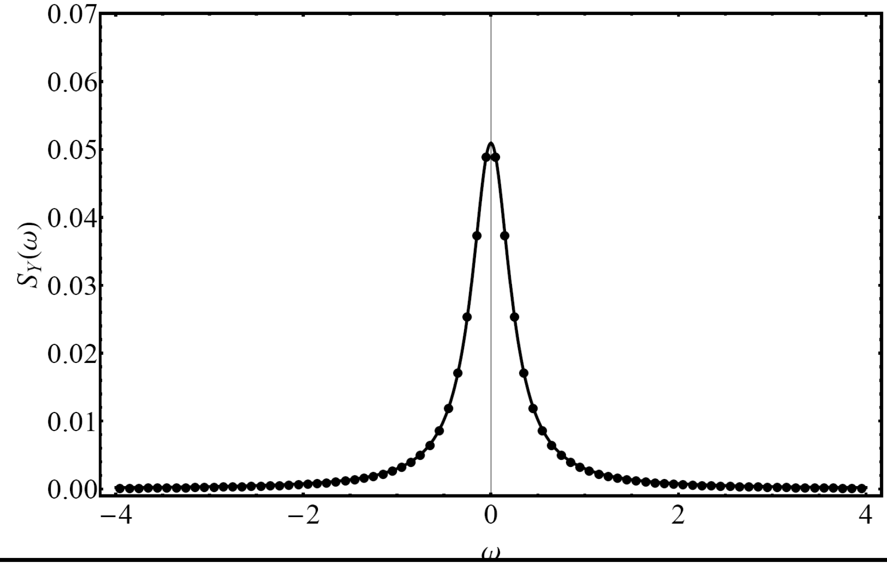




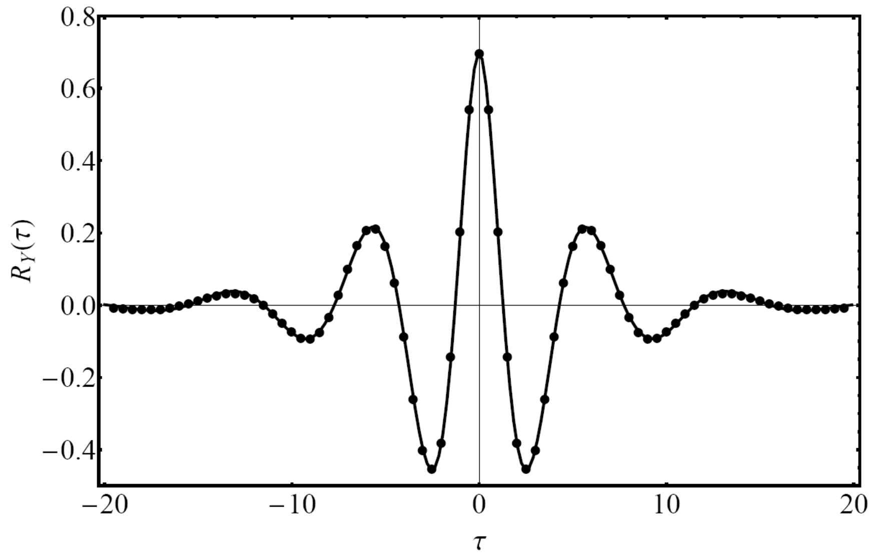
Exp -Correlation Function



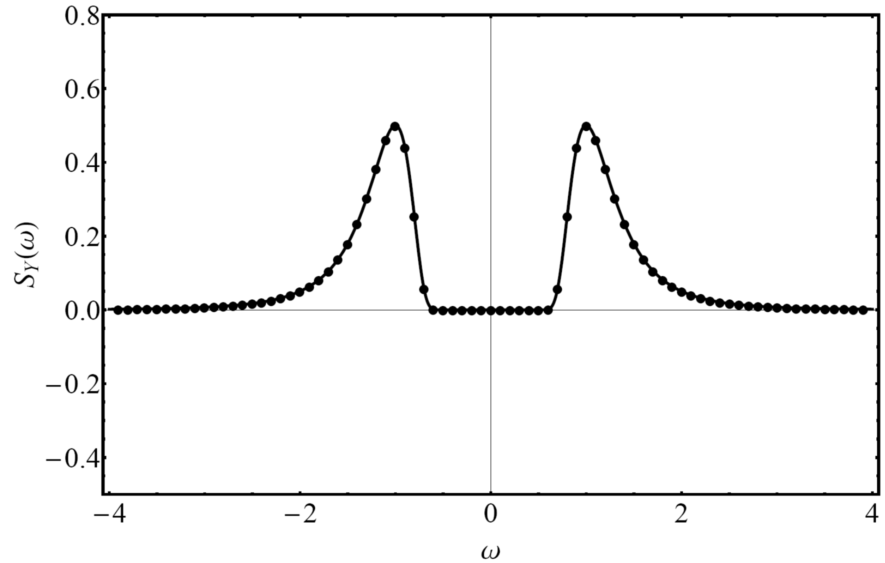
Power Spectral Density



PM -Correlation Function

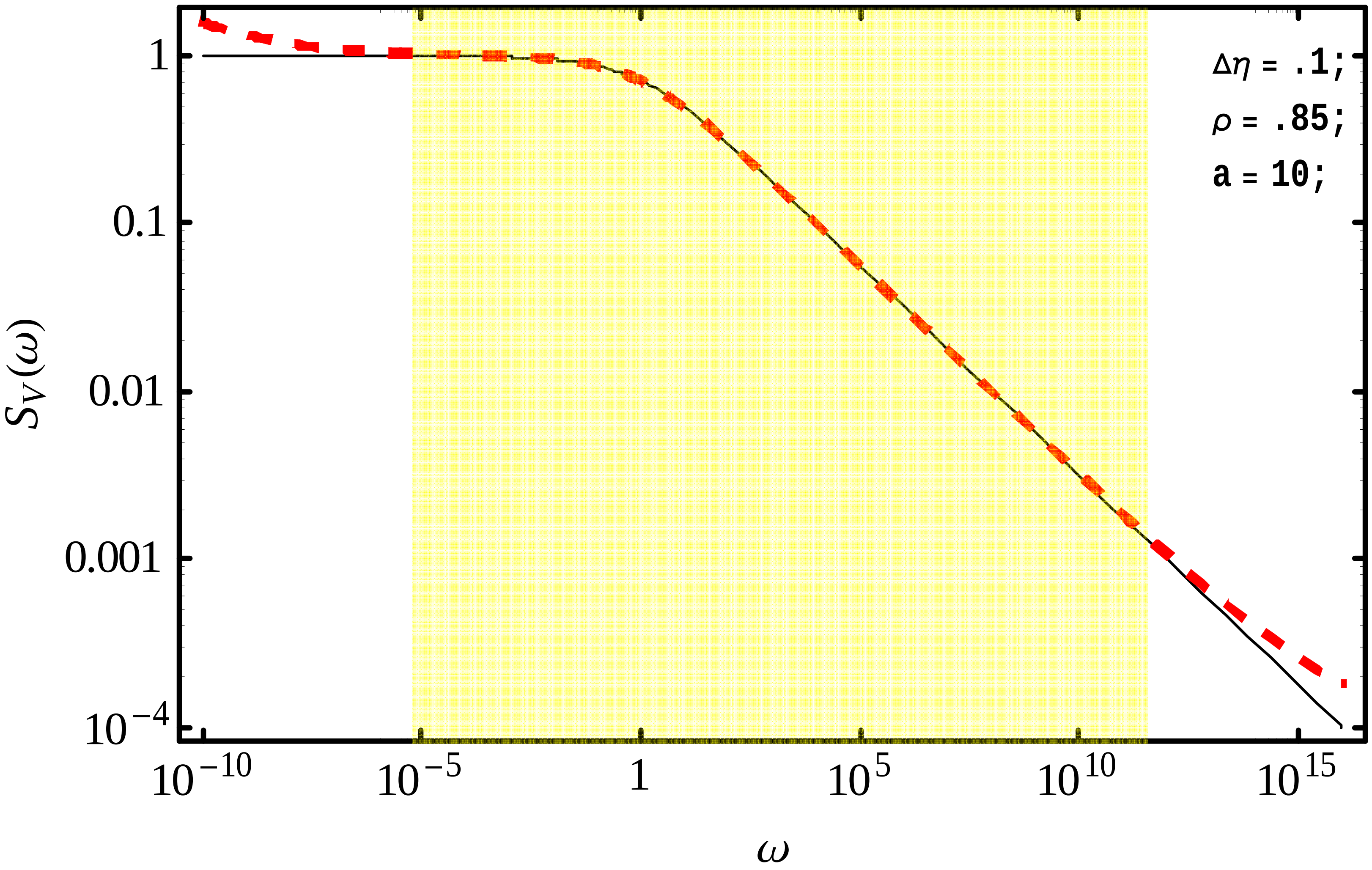


Power Spectral Density



Long-Term Correlation Function

$$S_V(\omega) = \frac{1}{1+|\omega|^{1/2}} \qquad R(\tau) \simeq \tau^{-1/2}$$



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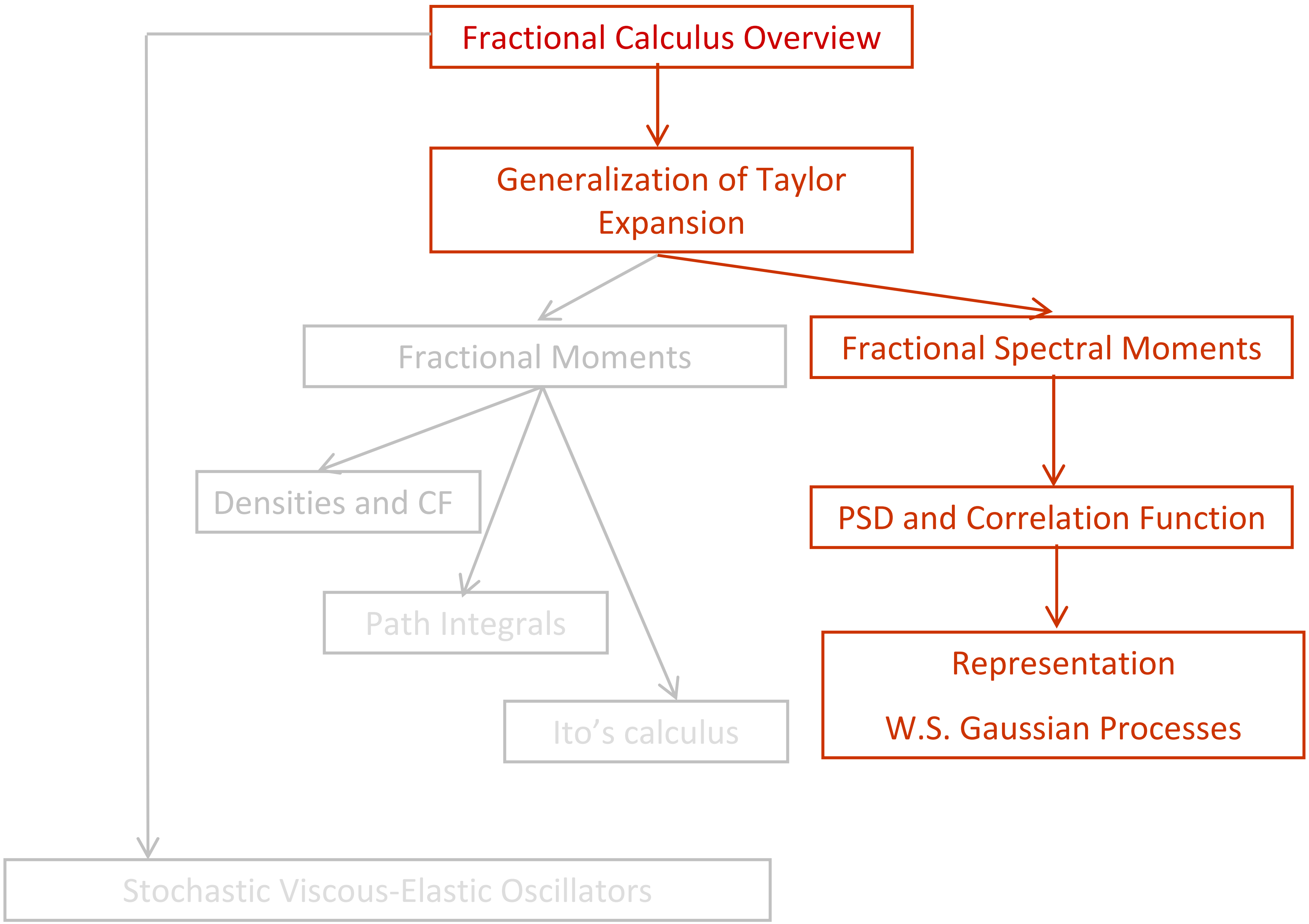

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Generation of weakly stationary Gaussian coloured noises (H-FSM)

GOAL: Process F(*t*) with target $S_F(\omega)$ as output of linear system

$$\mathcal{L}(F(t)) = W(t)$$

→ **Gaussian White Noise**

In the frequency domain the Power Spectral Density of the process *F*(*t*) can be obtained as:

$$S_F(\omega) = \left|H(\omega)\right|^2 S_W(\omega) = \frac{q}{2\pi}\left|H(\omega)\right|^2$$

The transfer function is the Fourier transform of the Impulse response function *h*(*t*)

Generation of weakly stationary Gaussian coloured noises (H-FSM)

Fractional Spectral Moments of the transfer function:

$$\Pi_H(\gamma) \overset{def}{=} 2\int_0^{\infty} |\omega|^{\gamma} H(\omega)\,\mathrm{d}\omega, \quad \mathrm{Re}\gamma > 0$$

H-FSM are Riesz fractional integrals and derivatives of the impulse response function *h(t),* evaluated in zero.

By inverse Mellin transform:

$$H(\omega) = \frac{1}{4\pi i}\int_{\rho-i\infty}^{\rho+i\infty} \Pi_H(-\gamma)|\omega|^{\gamma-1}\,\mathrm{d}\gamma, \qquad \gamma = \rho + i\eta, \quad 0 < \rho < 1$$

Approximating by truncated series:

$$H(\omega) \cong \frac{\Delta\eta}{4\pi}\sum_{k=-m}^{m} \Pi_H(-\gamma_k)|\omega|^{\gamma_k - 1} \qquad \gamma_k = \rho + ik\Delta\eta$$

Generation of weakly stationary Gaussian coloured noises (H-FSM)

From System's Linearity and Inverse Fourier it follows

$$F(t) = \frac{1}{4\pi i} \int_{\rho - i\infty}^{\rho + i\infty} \Pi_H(-\gamma)\left(I^{1-\gamma}W\right)(t)\,\mathrm{d}\gamma$$

Fractional moments of the system's transfer function

Fractional Brownian motions

Riesz Fractional Integral

$$\left(I^{\gamma}W\right)(t) \propto \left(I_-^{\gamma}W + I_+^{\gamma}W\right)(t)$$

Discrete form

$$F(t) = \frac{\Delta\eta}{4\pi} \sum_{k=-m}^{m} \Pi_H(-\gamma_k)\left(I^{1-\gamma_k}W\right)(t)$$

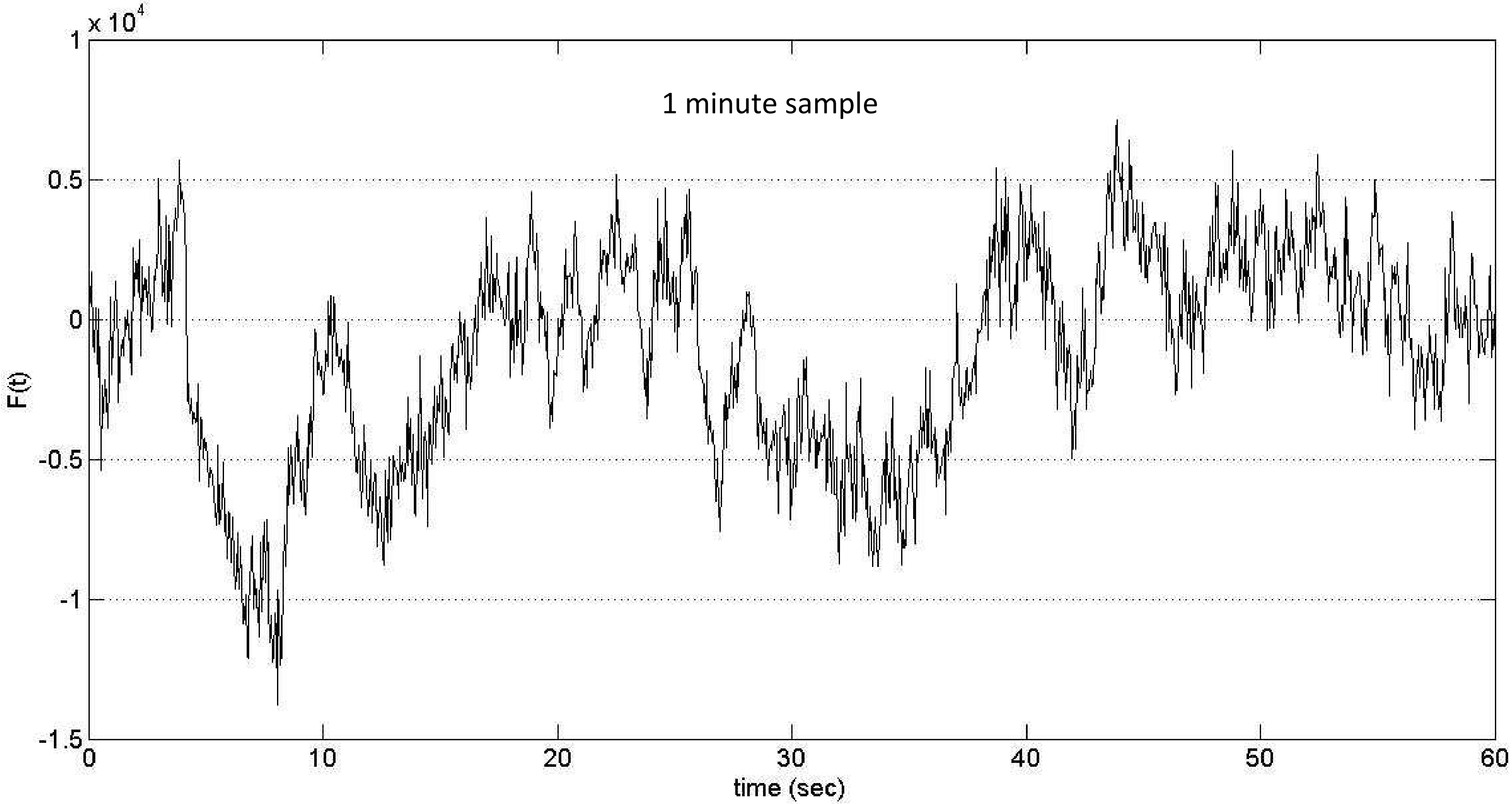
Application to wind loads

$$\mathbf{F} = \sum_{k=-m}^{m} \beta_k \mathbf{B}_{1-\gamma_k}$$

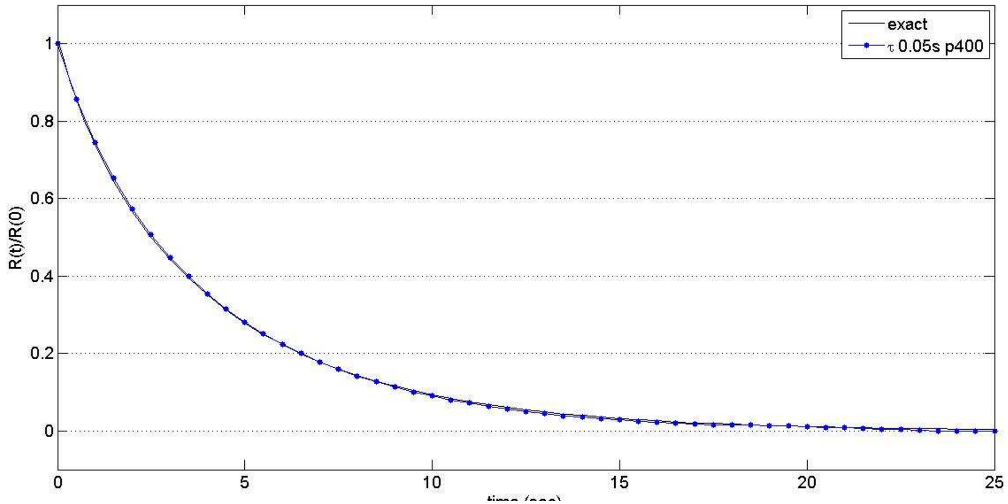
$$\gamma_k = \rho + ik\Delta\eta$$

β_k → H-FSM

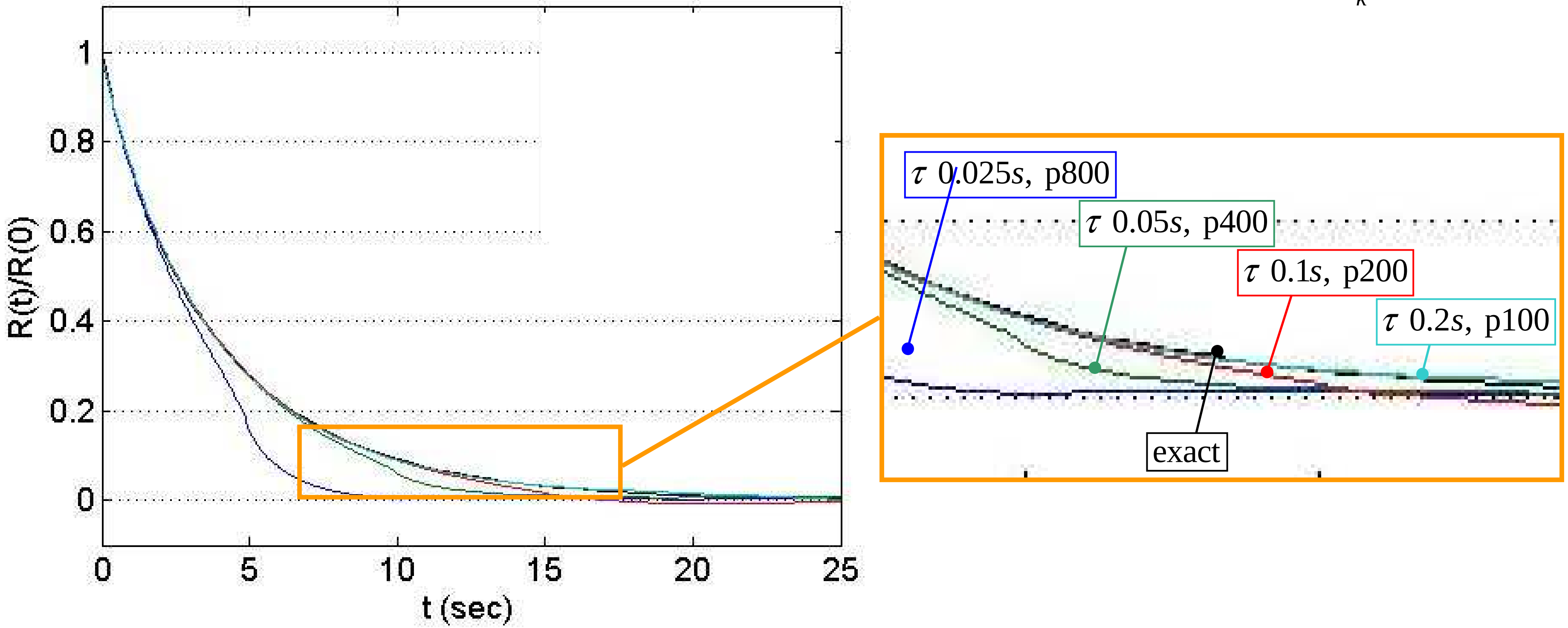
$\mathbf{B}_{1-\gamma_k}$ → fBm



Auto-correlation of generated wind data: Memory Propagation



Autocorrelation function for constant value of t and different non zero coefficients α_k



Extension: Multivariate processes

$$\mathbf{S_V}(\omega) = \begin{bmatrix} S_{V_1V_1}(\omega) & S_{V_1V_2}(\omega) & \dots & S_{V_1V_N}(\omega) \\ S_{V_1V_2}(\omega) & S_{V_2V_2}(\omega) & \dots & \dots \\ \dots & \dots & \dots & \dots \\ S_{V_1V_N}(\omega) & S_{V_2V_N}(\omega) & \dots & S_{V_NV_N}(\omega) \end{bmatrix}$$

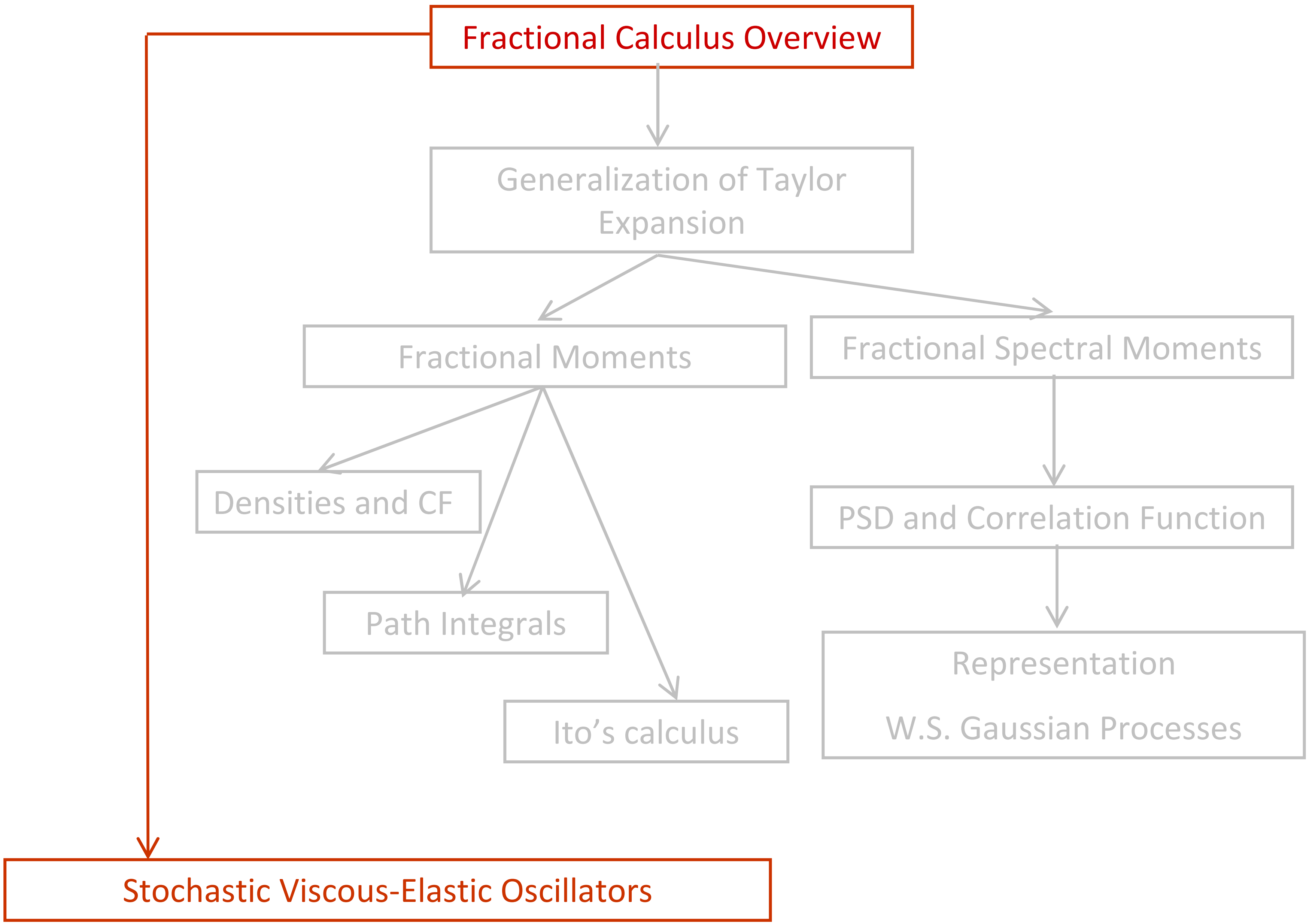
$$\mathbf{S_V}(\omega) = \mathbf{\Psi}(\omega)\,\mathbf{L}^{1/2}(\omega)\,\mathbf{L}^{1/2}(\omega)\,\mathbf{\Psi}^{*T}(\omega)$$

$$\mathbf{H}(\omega) = \mathbf{\Psi}(\omega)\,\mathbf{L}^{1/2}(\omega)$$

$$\mathbf{\Pi}(\gamma) \overset{def}{=} \int_{-\infty}^{\infty} |\omega|^{\gamma}\,\mathbf{H}(\omega)\,\mathrm{d}\omega$$

$$\mathbf{V}(t) = \frac{1}{4\pi i}\int_{\rho-i\infty}^{\rho+i\infty} \mathbf{\Pi}(-\gamma)\left(I^{1-\gamma}\mathbf{W}\right)(t)\,d\gamma$$

OUTLINE



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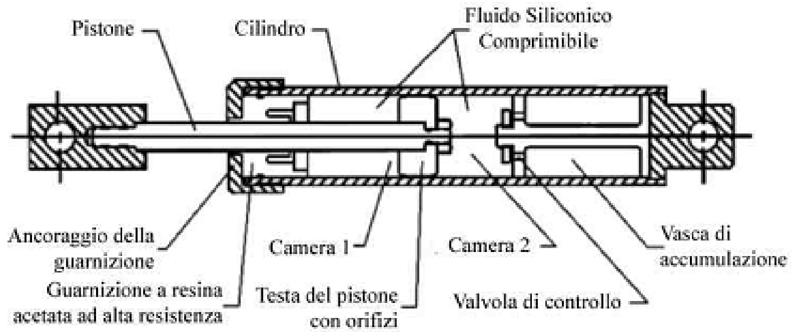
APPLICATION: STOCHASTIC VISCOUS-ELASTIC SYSTEMS



Costitutive equation of viscous-elastic dampers:

$$f\left(\dot{X}(t)\right) = c_d \left|\dot{X}\right|^{\gamma} \operatorname{sign}(\dot{X}) \qquad \gamma \in \mathbb{R}$$

Equation of motion

$$m\ddot{X}(t) + f\left(\dot{X}(t)\right) + k\,X(t) = W_{\alpha}(t)$$

Non-linear Stochastic Differential Equation subjected to Lévy noise

APPLICATION: STOCHASTIC VISCOUS-ELASTIC SYSTEMS

$$m\ddot{X}(t)+c_d\left|\dot{X}(t)\right|^{\gamma}\,sign\left(\dot{X}(t)\right)+k\,X(t)=W_{\alpha}(t)$$

$\gamma \in \mathbb{R}$

$\gamma \neq 0, 2, 4, \ldots$

Spectral Einstein-Smoluchowski equation

$$\frac{\partial\phi_{\mathbf{Z}}(\boldsymbol{\theta},t)}{\partial t}=\vartheta_1\frac{\partial\phi_{\mathbf{Z}}(\boldsymbol{\theta},t)}{\partial\vartheta_2}-\frac{k}{m}\vartheta_2\frac{\partial\phi_{\mathbf{Z}}(\boldsymbol{\theta},t)}{\partial\vartheta_1}+\frac{c_d}{m}\vartheta_2\;{}_{\vartheta_2}H^{-\gamma}\left(\phi_{\mathbf{Z}}(\boldsymbol{\theta},t)\right)-\left(\frac{|\vartheta_2|}{m}\right)^{\alpha}\phi_{\mathbf{Z}}(\boldsymbol{\theta},t)$$

Einstein-Smoluchowski equation

$$\frac{\partial p_{\mathbf{Z}}(\mathbf{z},t)}{\partial t}=-z_2\frac{\partial p_{\mathbf{Z}}(\mathbf{z},t)}{\partial z_1}+\frac{k}{m}z_1\frac{\partial p_{\mathbf{Z}}(\mathbf{z},t)}{\partial z_2}+\frac{c_d}{m}\frac{\partial}{\partial z_2}\left(|z_2|^{\gamma}\,sign(z_2)\,p_{\mathbf{Z}}(\mathbf{z},t)\right)+\frac{1}{m^{\alpha}}\left({}_{z_2}\mathcal{D}^{\alpha}p_{\mathbf{Z}}(\mathbf{z},t)\right)$$

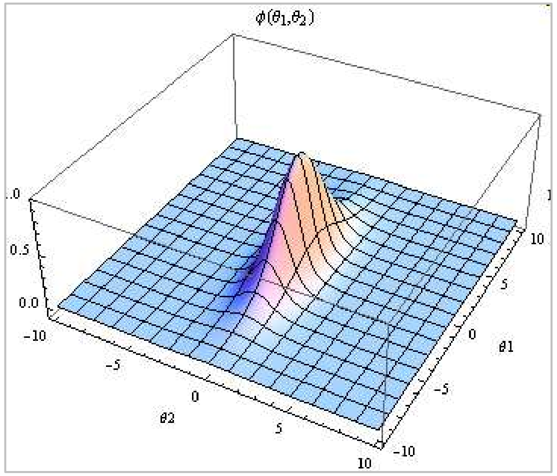
where $\left({}_{z_2}\mathcal{D}^{\alpha}p_{\mathbf{Z}}(\mathbf{z},t)\right)$ is the partial Riesz derivative of the joint PDF

Deterministic Fractional Differential Equations

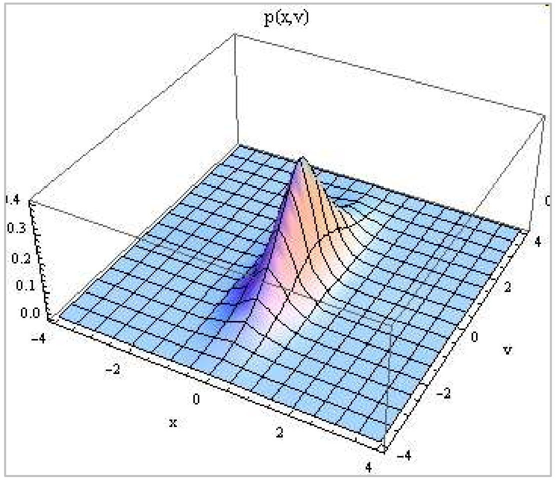
APPLICATION: STOCHASTIC VISCOUS-ELASTIC SYSTEMS

$$\alpha = 2, \quad \gamma = 0.5$$

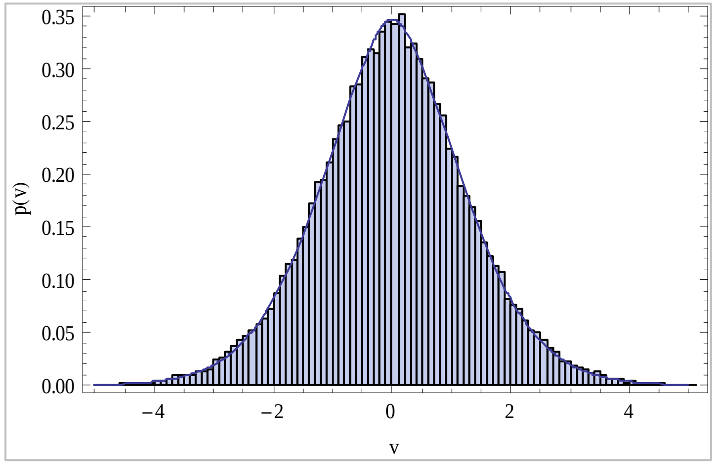
Characteristic function



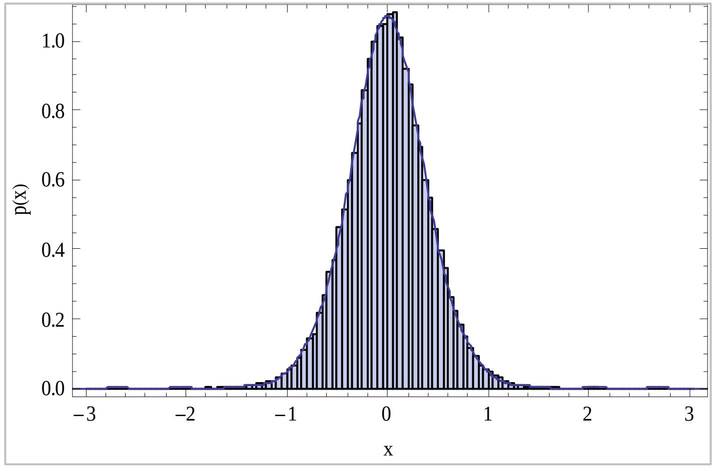
Probability Density function



Marginal probability density function



Marginal probability density function



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Thank you ! *

Giulio Cottone

*any inaccuracy should be ascribed to the effect of German Coffee on Italian brain

Acknowledgments to papers' co-authors:

Giorgio Barone (Tokyo Univ), Salvatore Butera, Mario Di Paola (Univ. Palermo), Ralf Metzler (TU München), Antonina Pirrotta (Univ. Palermo), Katrin Runtemund (TU München), Roberta Santoro (Univ. Messina)

References at: www.researcherid.com/rid/B-3565-2009