Spatial Reliability Assessment of Deteriorating Concrete Slabs with Inspection Data

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• Joint probability density

\[ f_X(x) , \]
\[ X = [X_1, X_2, \ldots, X_n]^T \]

• Failure domain

\[ \Omega_F = \{ R(x) - S(x) \leq 0 \} \]
\[ = \{ g(x) \leq 0 \} \]

• Failure probability

\[ \Pr(F) = \int_{\Omega_F} f_X(x) \, dx \]
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How can we get Information?

Information $\mathcal{I}$ from

- Assessment
- Measurement
- Inspection
- Monitoring

$\rightarrow$ Expressed through mathematical functions
Information described by Domains $\Omega$

- Information $Z$ given by
  $$\Omega_Z = \{ h(x) \leq 0 \}$$
  *(Inequality Type)*

- Information probability
  $$\Pr(Z) = \int_{\Omega_Z} f_X(x) \, dx$$
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Updated Failure Probability

A-Posteriori Failure Probability

• Conditional probability

\[ \Pr(F|Z) = \frac{\Pr(F \cap Z)}{\Pr(Z)} \]

• Intersection probability

\[ \Pr(F \cap Z) = \int_{\Omega_F \cap \Omega_Z} f_X(x) \, dx \]

• Probability update

\[ \Pr(F|Z) = \frac{\int_{\Omega_F \cap \Omega_Z} f_X(x) \, dx}{\int_{\Omega_Z} f_X(x) \, dx} \]
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Measurement: Equality Information

Mathematical Description

• Measurement of system characteristic $S(X)$

$$h(X, \epsilon_M) = s(X) - s_M + \epsilon_M$$

• Information $Z$ given by

$$\Omega_Z = \{h(X, \epsilon_M) = 0\}$$

(Equality Type)
• Measurement of system characteristic \( S(X) \)

\[
h(X, \epsilon_M) = s(X) - s_M + \epsilon_M
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• Information \( Z \) given by

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(Equality Type)
Measurement: Equality Information

How to Update?

• Information probability

\[ \Pr(Z) = \int_{\Omega_Z} f_X(x) \, dx = 0 \]

• Intersection probability

\[ \Pr(F \cap Z) = \int_{\Omega_F \cap \Omega_Z} f_X(x) \, dx = 0 \leq \Pr(Z) \]

• Probability update

\[ \Pr(F|Z) = \frac{0}{0} = ? \]
Measurement: Equality Information

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\[ = 0 \leq \Pr(Z) \]

- Probability update

\[ \Pr(F|Z) = 0 \]

\[ = 0 \]
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Summary on Information

What about the Update?

Résumé

• **Inequality** type information
  → any SR method **suitable** for probability update

• **Equality** type information
  → common SR methods **not suitable** for probability update
Introduction of Likelihood

- Likelihood describes information $Z$

$$\mathcal{L}(x|Z) = a \cdot \Pr(Z|X = x) \propto \Pr(Z|X = x)$$

- Express likelihood by error $\epsilon_M$

$$\mathcal{L}(x|Z) = f_{\epsilon_M}(s_M - s(x))$$
Introduction of Likelihood

- Likelihood describes information $Z$

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- Express likelihood by error $\epsilon_M$

  $$\mathcal{L}(x|Z) = f_{\epsilon_M}(s_M - s(x))$$
Preparations are necessary

- Introduce standard uniform RV $U$
  
  $$f_U(u) = 1 \quad \forall \quad u \in [0; 1]$$

- Introduce constant $c$
  
  $$0 \leq c \cdot \mathcal{L}(x|Z) \leq 1$$

- Express likelihood
  
  $$\mathcal{L}(x|Z) = \frac{1}{c} \cdot \Pr(U \leq c\mathcal{L}(x|Z))$$
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\[ \mathcal{L}(x|Z) = \frac{1}{c} \cdot \Pr(U \leq c\mathcal{L}(x|Z)) \]
Preparations have to be made

- Total probability theorem

\[
\Pr(Z) = \int_{\Omega_X} \Pr(Z|X = x) f_X(x) \, dx
\]

\[
= \int_{\Omega_X} \frac{1}{ac} \cdot \Pr(U \leq c\mathcal{L}(x|Z)) f_X(x) \, dx
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- Information domain

\[
\Omega_Z = \{ U - c\mathcal{L}(x|Z) \leq 0 \} 
\]
Preparations have to be made

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\Pr(Z) = \int_{\Omega_X} \Pr(Z|X = x)f_X(x) \, dx \\
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\Omega_Z = \{U - cL(x|Z) \leq 0\}
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Update can be performed now!

**Total probability theorem**

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**Intersection probability**

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Pr(F \cap Z) = \frac{1}{ac} \int_{\Omega_{F \cap \Omega_Z}} f_X(x) \, du \, dx
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**Conditional probability**

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Pr(F|Z) = \frac{Pr(F \cap Z)}{Pr(Z)} = \frac{\int_{\Omega_{F \cap \Omega_Z}} f_X(x) \, du \, dx}{\int_{\Omega_Z} f_X(x) \, du \, dx}
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Probability Update
Integration of $f_X(x)$ by Importance Sampling

• Importance sampling solution

$$
Pr(F|Z) \approx \frac{\sum_{i=1}^{n_S} I[h_e(x_i, u_i) \leq 0]I[g(x_i) \leq 0] \frac{f_x(x_i)}{\psi(x_i, u_i)}}{\sum_{i=1}^{n_S} I[h_e(x_i, u_i) \leq 0] \frac{f_x(x_i)}{\psi(x_i, u_i)}}
$$

• Optimal sampling density [Straub, 2010]

$$
\psi(x, u) = \psi_1(x) \cdot \psi_2(u|x) = \psi_1(x) \cdot \frac{1}{c \cdot \mathcal{L}(x|Z)}
$$

(Remember: $0 \leq c\mathcal{L}(x|Z) \leq 1$)
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Application to Corrosion

Fick’s law of diffusion (1D)

\[
\frac{dC(z,t)}{dt} = D \frac{\partial^2 C(z,t)}{\partial z^2}
\]

Chloride concentration

\[
C(z,t) = C_S \left( 1 - \text{erf} \left( \frac{z}{2\sqrt{D \cdot t}} \right) \right)
\]

Failure domain

\[
\Omega_F = \{ C_{\text{crit}} - C(W,t) \leq 0 \} 
\]
Application to Corrosion

**Diffusion Model**

- **Fick’s law of diffusion (1D)**

  \[
  \frac{dC(z,t)}{dt} = D \frac{\partial^2 C(z,t)}{\partial z^2}
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- **Chloride concentration**

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  C(z,t) = C_S \left( 1 - \text{erf} \left( \frac{z}{2\sqrt{D \cdot t}} \right) \right)
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Diffusion Model

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# Application to Corrosion

## Model Parameters

<table>
<thead>
<tr>
<th>RV</th>
<th>Dimension</th>
<th>Distribution</th>
<th>Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W$</td>
<td>[mm]</td>
<td>LogNormal</td>
<td>$\mu_W = 40.0$  [\sigma_W = 8.0]</td>
</tr>
<tr>
<td>$D$</td>
<td>[mm$^2$/yr]</td>
<td>LogNormal</td>
<td>$\mu_D = 20.0$  [\sigma_D = 10.0]</td>
</tr>
<tr>
<td>$C_S$</td>
<td>[m.-% cem.]</td>
<td>Normal</td>
<td>$\mu_{C_S} = 3.1$  [\sigma_{C_S} = 1.23]</td>
</tr>
<tr>
<td>$C_{\text{crit}}$</td>
<td>[m.-% cem.]</td>
<td>Normal</td>
<td>$\mu_{C_{\text{crit}}} = 0.8$  [\sigma_{C_{\text{crit}}} = 0.1]</td>
</tr>
</tbody>
</table>

Table: The Random variables of the corrosion model.
Application to Corrosion

A-Priori Failure Probability

Figure: A-priori corrosion probability of the reinforcement.
• Measurement of cover depth

\[ \mathbf{w}_M = [w_1, w_2, \ldots, w_{800}] \]

• Likelihood of measurement

\[ \mathcal{L}(w|w_M) = f_{w_M|w}(w_M|w) \]

• Bayesian update

\[ f''_W(w) \propto \mathcal{L}(w|w_M)f_W(w) \]
Application to Corrosion

Cover Depth Update

• Measurement of cover depth

\[ \mathbf{w}_M = [w_1, w_2, \ldots, w_{800}] \]

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Application to Corrosion

Chloride Concentration Update

• Measurement of chloride concentration

\[ c_{ZM,j}(X,X,t) = C_{S,j} \cdot \left( 1 - \text{erf} \left( \frac{z_M}{\sqrt{4Dt}} \right) \right) \]

• Likelihood (Error \( \epsilon \sim N(0, \sigma_{\epsilon}) \))

\[ L_j(x) = \frac{1}{\sigma_{\epsilon} \sqrt{2\pi}} \exp \left( -\frac{1}{2} \left( \frac{c_{ZM}(x,t) - c_{M,j}(z_M,t)}{\sigma_{\epsilon}} \right)^2 \right) \]
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Figure: Corrosion probability \((t = 15 \text{ [yr]})\) conditional on measurement results (cover depth & concentration).
Application to Corrosion

A-Posteriori Failure Probability

Figure: Corrosion probability \((t = 15 \text{ [yr]})\) conditional on measurement results (cover depth & concentration).
To Conclude…

What was learned

• Efficient method available for updating failure probability using equality information
• Several different information can be taken into account
Some Literature

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*Lecture Notes, Technische Universität München*, Munich, 2011.

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Thank you for your attention!

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