Low-rank response surface with application in numerical aerodynamic

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Outline

Overview

Modelling of free stream turbulence

Numerics

Uncertainties in geometry

Post-processing in low-rank data format

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Problem setup

Stationar Navier-Stokes equation:

$$v \cdot \nabla v - \frac{1}{Re} \nabla^2 v + \nabla p = g, \quad \text{and } \nabla \cdot v = 0.$$

+ b.c. and Wilcox-k-w turbulence model
domain: RAE-2822 airfoil

TAU-solver has more than 300 parameters!
Many of them are or can be uncertain!
What does it mean for the solution ?

Overview of uncertainties

Uncertain Input:

1. Parameters and variables (α, Ma, Re, ...)
2. Geometry of airfoil
3. Parameters of a turbulence model

Uncertain solution:

1. statistical moments of $(v, p)^T$
2. exceedance probabilities $P(v > v^*)$ in each point x
3. probability density functions of u
4. position of shock.

Our Aims

1. Sparse representation of the input data (random fields)

2. Computing process in a reasonable time

3. Use the deterministic solver as a black box

4. A low-rank tensor data format for the solution

5. Efficient postprocessing in the low-rank tensor format

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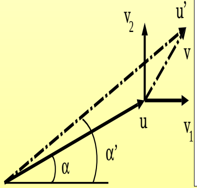
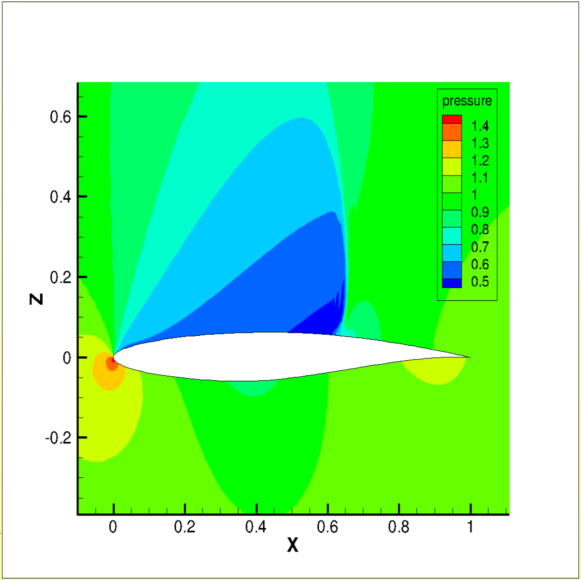
Modelling of uncertainties in free stream turbulence

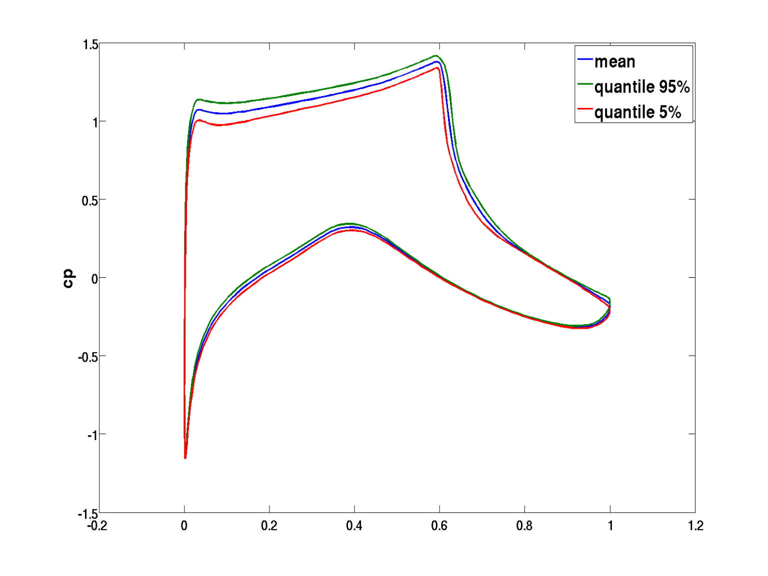




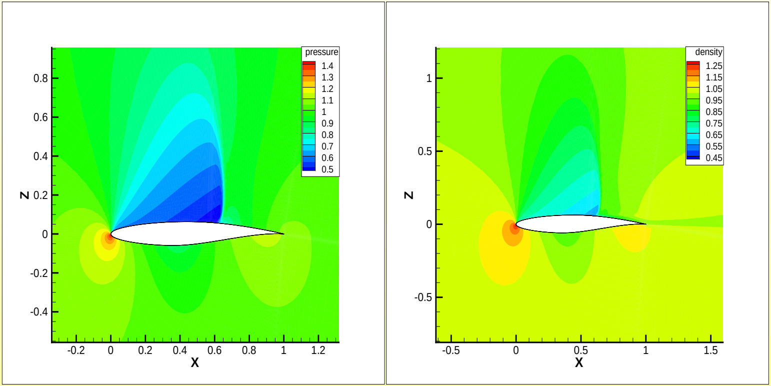
Random vectors $\boldsymbol{v}_1(\theta)$ and $\boldsymbol{v}_2(\theta)$ model free stream turbulence

Examples

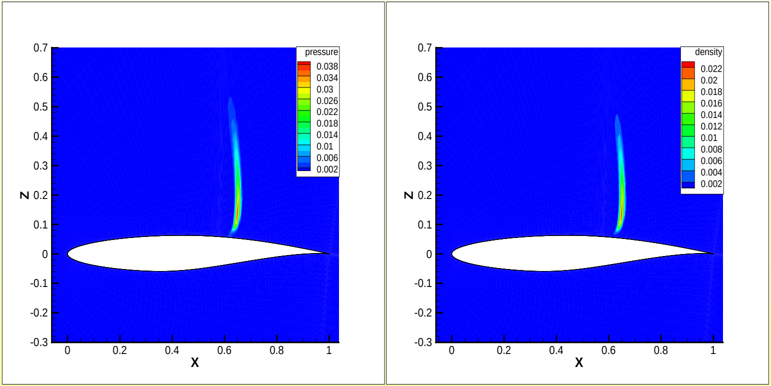
5% and 95% quantiles for cp from 500 MC realisations.



Mean of the pressure and density (from MC)



Variance of the pressure and density (from MC)



Polynomial Chaos Expansion of $p(x, \boldsymbol{\theta})$

Represent $p(x_\ell, \boldsymbol{\theta})$, $x_\ell \in \mathcal{G}$, $\ell = 1..n$, in a Hermitian basis $H_\beta(\boldsymbol{\theta})$, $\beta \in \mathcal{J}$:

$$\rho(x_\ell, \boldsymbol{\theta}) \approx \sum_{\beta \in \mathcal{J}_{M,p}} H_\beta(\boldsymbol{\theta}) \rho_\beta(x_\ell), \tag{1}$$

$$\rho_\beta(x_\ell) = \frac{1}{\beta!} \int_\Theta H_\beta(\boldsymbol{\theta}) \rho(x_\ell, \boldsymbol{\theta}) \, \mathbb{P}(\mathrm{d}\boldsymbol{\theta}) \quad \approx \frac{1}{\beta!} \sum_{i=1}^{n_q} H_\beta(\boldsymbol{\theta}_i) \rho(x_\ell, \boldsymbol{\theta}_i) w_i,$$

Further compression

$$\rho_\beta(x_\ell) \approx \sum_{k=1}^{r} \bigotimes_{j=1}^{M} p_k^{(\beta_j)}(x_\ell)$$

.

Sparse Gauss-Hermite Quadratures

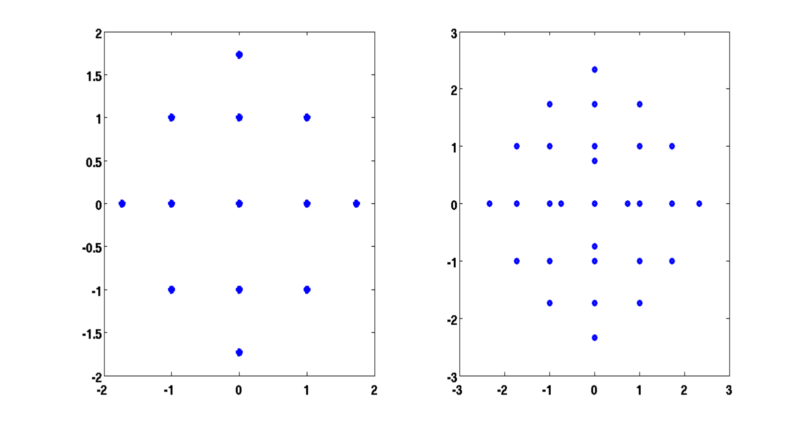


Figure: Sparse Gauss-Hermite grids of order 2 (13 points) and 3 (29 points).

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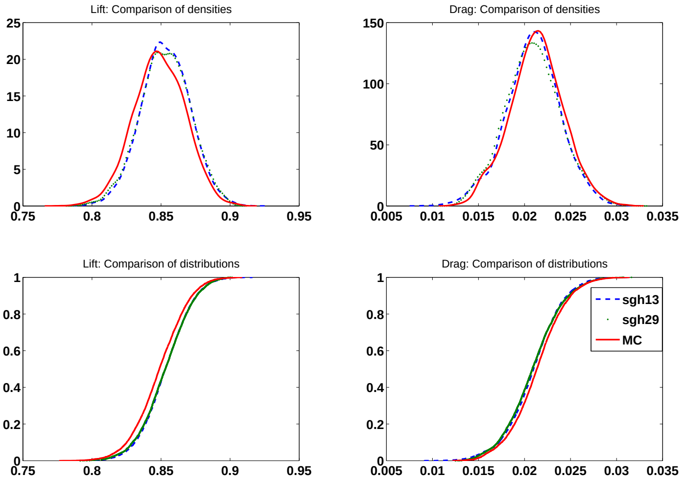
Sparse Gauss-Hermite grid, 13 points.

Assume that RVs α and *Ma* are Gaussian with

	mean	st. dev. σ	σ/mean
α	2.79	0.1	0.036
Ma	0.734	0.005	0.007

Then uncertainties in the solution lift *CL* and drag *CD* are

CL	0.853	0.0174	0.02
CD	0.0206	0.003	0.146



Density functions and distribution functions of *CL* and *CD*. PCE is of order 1 and depends on two RVs. Comparison is done with 6360 MC simulations.

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Uncertainties in geometry

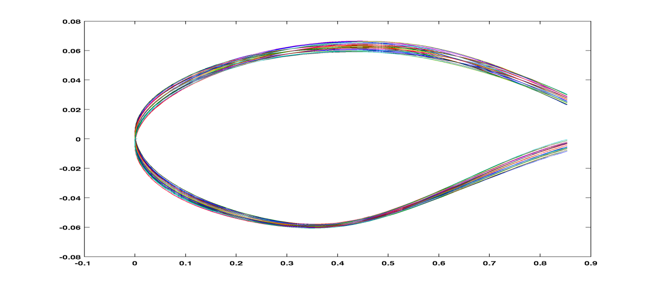
Random boundary perturbations:
$\partial D_\varepsilon(\omega) = \{x + \varepsilon\kappa(x,\omega)\mathbf{n}(x) \,:\, x \in \partial D\}$.
where $\kappa(x,\omega)$ is a random field.

How to generate geometry with uncertainties ?
Algorithm:

1. Assume cov. function $\mathrm{cov}(x,y)$ for random field $\kappa(x,\omega)$ given
2. Compute $C_{ij} := \mathrm{cov}(x_i,x_j)$ for all grid points (in a sparse format!)
3. Solve eigenproblem $C\phi_i = \lambda_i\phi_i$
4. Then $\kappa(x,\omega) \approx \sum_{i=1}^{m}\sqrt{\lambda_i}\phi_i\xi_i(\omega)$, where $\xi_i(\omega)$ are uncorrelated random variables.

Sparse approximation of dense matrix C is done in [Khoromskij, Litvinenko, Matthies, 2009]

21 realisations of RAE-2822 airfoil (expanded along y)



Covariance function is of Gaussian type
$\mathrm{cov}(\rho) = 10^{-5}\exp(-\sum_{i=1}^{2}(x_i - y_i)^2/\ell_i^2)$.
For each airfoil solve the deterministic problem.

Uncertainties in geometry

	mean	st. dev. σ	σ/mean
CL	0.8552	0.0049	0.0058
CD	0.0183	0.00012	0.0065

PCE of order 1 with 3 random variables and sparse Gauss-Hermite grid with 21 points were used.

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Low-rank approximation of the solution

Let $\mathbf{W} := [\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_Z]$, where $\mathbf{v}_i$ are solution vectors.
Given tSVD $\mathbf{W}_k = \mathbf{U}_k \Sigma_k V_k^T \approx \mathbf{W} \in \mathbb{R}^{n \times Z}$.
How to compute tSVD of $[\mathbf{W}_k, v_{Z+1}, ..., v_{Z+R}]$ with a linear complexity ? (M. Brand, 2006)

$$\overline{v} = \frac{1}{Z}\sum_{i=1}^{Z} v_i = \frac{1}{Z}\sum_{i=1}^{Z} \mathbf{A} \cdot b_i = \mathbf{A}\overline{b}, \tag{2}$$

$$C = \frac{1}{Z-1}\mathbf{W}_c \mathbf{W}_c^T = \frac{1}{Z-1}\mathbf{U}\Sigma\mathbf{V}^T\mathbf{V}\Sigma^T\mathbf{U}^T = \frac{1}{Z-1}\mathbf{U}\Sigma\Sigma^T\mathbf{U}^T. \tag{3}$$

Diagonal of C can be computed with the complexity $\mathcal{O}(k^2(Z+n))$.

Accuracy of the mean value

Let $\mathbb{1} = (1, ..., 1)^T$.

Lemma

Let $\|\mathbf{W} - \mathbf{W}_k\| \leq \varepsilon$. Let $\overline{\mathbf{v}}_k$ be a rank-k approximation of the mean $\overline{\mathbf{v}}$, then $\|\overline{\mathbf{v}} - \overline{\mathbf{v}}_k\| \leq \varepsilon$.

Proof: Since $\overline{\mathbf{v}} = \frac{1}{n}\mathbf{W}\mathbb{1}$ and $\overline{\mathbf{v}}_k = \frac{1}{n}\mathbf{W}_k\mathbb{1}$, then

$$\|\overline{\mathbf{v}} - \overline{\mathbf{v}}_k\| = \frac{1}{n}\|(\mathbf{W} - \mathbf{W}_k)\mathbb{1}\| \leq \frac{1}{n}\|(\mathbf{W} - \mathbf{W}_k)\| \cdot \|\mathbb{1}\| \leq \varepsilon.$$

i.e. the mean can be computed in rank-k data format with ε accuracy.

Accuracy of the variance value

Lemma

Let $\|\mathbf{W} - \mathbf{W}_k\| \leq \varepsilon$*, then* $\|\mathbf{W}_c - (\mathbf{W}_c)_k\| \leq \varepsilon$*.*

Proof:

$$\|\mathbf{W}_c - (\mathbf{W}_c)_k\| \leq \|\mathbf{W} - \mathbf{W}_k\| \cdot \|I - \frac{1}{n} \cdot \mathbb{1} \cdot \mathbb{1}^T\| \leq \|\mathbf{W} - \mathbf{W}_k\| \cdot 1 \leq \varepsilon,$$

Lemma

Let $\|\mathbf{W}_c - (\mathbf{W}_c)_k\| \leq \varepsilon$*, then* $\|C - C_k\| \leq \frac{1}{Z-1}\varepsilon^2$*.*

Proof:

$$\|C - C_k\| \leq \frac{1}{Z-1}\|\mathbf{W}_c\mathbf{W}_c^T - (\mathbf{W}_c)_k(\mathbf{W}_c)_k^T\| \quad (4)$$

$$= \frac{1}{Z-1}\|\mathbf{U}\Sigma\mathbf{V}^T\mathbf{V}\Sigma^T\mathbf{U}^T - \mathbf{U}_k\Sigma_k\mathbf{V}_k^T\mathbf{V}_k\Sigma_k^T\mathbf{U}_k^T\| \quad (5)$$

$$= \|\mathbf{U}\Sigma\Sigma^T\mathbf{U}^T - \mathbf{U}_k\Sigma_k\Sigma_k^T\mathbf{U}_k^T\| \leq \frac{1}{Z-1}\varepsilon^2. \quad (6)$$

Relative errors of rank-k approximations

k	press.	density	tke	ev	xv	zv	memory, MB
10	1.9e-2	1.9e-2	4.0e-3	1.4e-3	1.1e-2	1.3e-2	21
20	1.4e-2	1.3e-2	5.9e-3	4.1e-4	9.7e-3	1.1e-2	42
50	5.3e-3	5.1e-3	1.5e-4	7.7e-5	3.4e-3	4.8e-3	104

Table: each matrix $\in \mathbb{R}^{260000 \times 600}$. Dense matrix format costs 1.25 GB.

Conclusion: already with a small rank a good accuracy can be achieved. Why?

Comparison of computing times

rank k	Update time, sec.	SVD time, sec.
10	107	1537
20	150	2084
50	228	8236

Table: Computing times of rank-k approximations of $W \in \mathbb{R}^{260000 \times 600}$.

Literature

1. A.Litvinenko, H. G. Matthies, *Sparse Data Representation of Random Fields*, PAMM, 2009.
2. B.N. Khoromskij, A.Litvinenko, H. G. Matthies, *Application of hierarchical matrices for computing the Karhunen-Loève expansion*, Springer, Computing, 84:49-67, 2009.
3. B.N. Khoromskij, A.Litvinenko, *Data Sparse Computation of the Karhunen-Loève Expansion*, AIP Conference Proceedings, 1048-1, pp. 311-314, 2008.
4. H. G. Matthies, *Uncertainty Quantification with Stochastic Finite Elements*, Encyclopedia of Computational Mechanics, Wiley, 2007.

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Elmar Zander:
A Malab/Octave toolbox for stochastic Galerkin methods
(KLE, PCE, sparse grids, tensors, many examples etc)

http://ezander.github.com/sglib/