Microstructurally-Informed Random Field Description: Case Study on Chaotic Masonry

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Motivation: Analysis of a historic masonry wall

- Key issues
  - Heterogeneity and non-uniformity of the structure
  - Size of constituents not negligible with respect to structural length
  - Random morphology of the material
- Available modeling concepts
  1. Homogenization approaches
  2. Stochastic media theories

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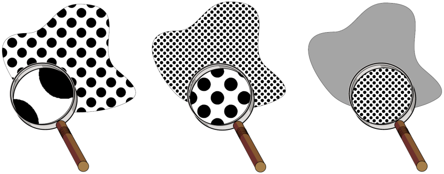
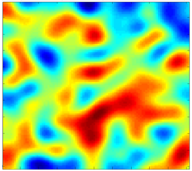
Motivation: Homogenization vs random fields





- Rapidly oscillating coefficients
- Assumes scale separation
- Periodic unit cell (PUC)
- Random materials → SEPUC
- Explicit treatment of heterogeneity (ZEMAN & ŠEJNOHA, MSMSE, 2007)

- Random coefficients
- No a-priory scale separation
- Random field construction
- Randomness is "for free"
- Link with microstructure?

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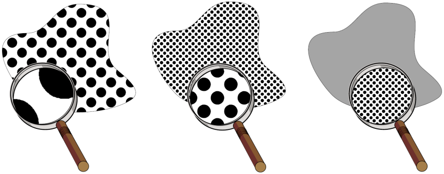

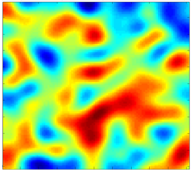

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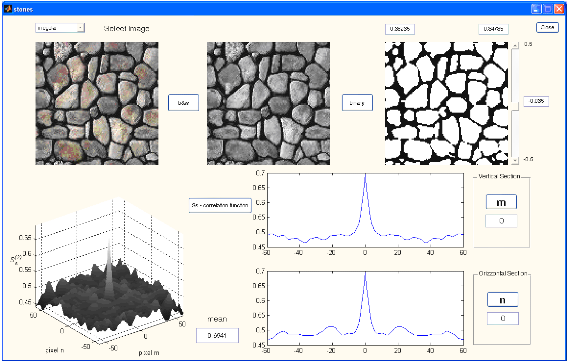
Outline

1. Quantification of random morphology
2. Statistics of structural response
   - Improved perturbation technique
   - Karhunen-Loève expansion
   - Hashin-Shtrikman approach
3. Numerical example
4. Conclusions

Quantification of random morphology

- Elementary geometrical descriptors of random media
  - One-point correlation function $S_s^{(1)} \equiv \gamma_s$ (volume fraction)
  - Two-point correlation function $S_s^{(2)}(m, n)$
- STONES environment (FALSONE & LOMBARDO, 2006)



- Consider a discrete binary random field $\chi_s : \Omega \times \mathcal{S} \to \{0,1\}$

$$\chi_s(\boldsymbol{x},\theta) = \begin{cases} 1 & \text{if } \boldsymbol{x} \in \text{stone}(\theta) \\ 0 & \text{if } \boldsymbol{x} \in \text{mortar}(\theta) \end{cases} \qquad \begin{aligned} \mathbb{E}[\chi_s(\boldsymbol{x})] &= \gamma_s \\ \mathbb{E}[\chi_s(\boldsymbol{x})\chi_s(\boldsymbol{y})] &= S_s^{(2)}(\boldsymbol{x}-\boldsymbol{y}) \end{aligned}$$

- For a general binary field $f : \Omega \times \mathcal{S} \to \mathbb{R}$

$$f(\boldsymbol{x},\theta) = \begin{cases} f^{(s)} & \text{if } \boldsymbol{x} \in \text{stone}(\theta) \\ f^{(m)} & \text{if } \boldsymbol{x} \in \text{mortar}(\theta) \end{cases}$$

- Basic statistical characterization

$$\begin{aligned} \mathbb{E}[f(\boldsymbol{x})] &= \gamma_s f^{(s)} + (1-\gamma_s) f^{(m)} \\ R_{fg}(\boldsymbol{x}-\boldsymbol{y}) &= \left(S_s^{(2)}(\boldsymbol{x}-\boldsymbol{y}) - {\gamma_s}^2\right)\left(f^{(s)} - f^{(m)}\right)\left(g^{(s)} - g^{(m)}\right) \end{aligned}$$

- Rigorous microstructure-based description

- Elasticity problem on Ω with stochastic material data

$$\mathbf{C}(\boldsymbol{x},\theta) = \chi_s(\boldsymbol{x},\theta)\mathbf{C}^{(s)} + (1-\chi_s(\boldsymbol{x},\theta))\mathbf{C}^{(m)}$$

- Discretization of Ω by N_e elements ($2h \lessapprox$ correlation length)
- Midpoint discretization of random field:

$$\boldsymbol{\chi}_s^h(\theta) = \left[\chi_{s,1}^h(\theta), \chi_{s,2}^h(\theta)\ldots,\chi_{s,N_e}^h(\theta)\right]^{\mathsf{T}} \in \{0,1\}^{N_e}$$

- Discretized FEM equations: $\mathbf{K}(\boldsymbol{\chi}_s^h(\theta))\boldsymbol{u}^h(\theta) = \boldsymbol{F}$
- Elementary statistics of displacements

$$\mathbf{A}\,\mathbb{E}[\boldsymbol{u}^h] = \boldsymbol{F}$$

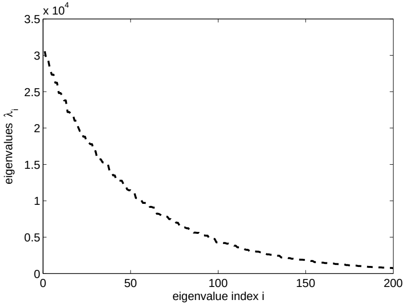
where

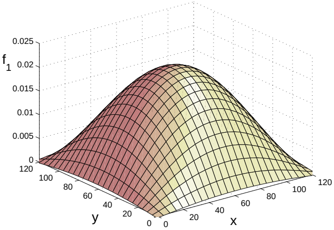
$$\mathbf{A} = \mathbf{K}_0 - \sum_{i=1}^{N_e}\sum_{j=1}^{N_e}\mathbf{K}_i'\,(\mathbf{K}_0)^{-1}\,\mathbf{K}_j'\,\mathbb{E}[\chi_{s,i}\chi_{s,j}]$$

- Karhunen-Loève expansion of a stationary random field $g(\boldsymbol{x},\theta)$

$$g(\boldsymbol{x},\theta) = \mathbb{E}[g] + \sum_{i=1}^{M} \sqrt{\lambda_i} f_i(\boldsymbol{x}) \xi(\theta),$$

$$\int_\Omega R_g(\boldsymbol{x}-\boldsymbol{y}) f_i(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y} = \lambda_i f_i(\boldsymbol{x}) \qquad \xi_i(\theta) = \frac{1}{\sqrt{\lambda_i}} \int_\Omega (g(\boldsymbol{x},\theta) - \mathbb{E}[g]) f_i(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}$$





0.025
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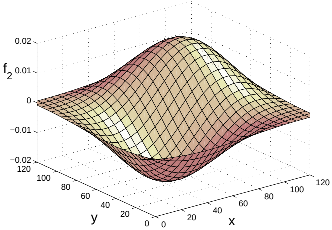
120
x

$i = 1$



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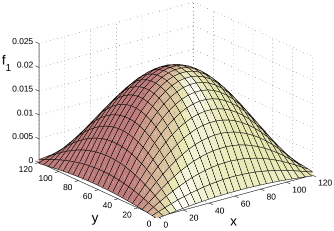
$i = 2$



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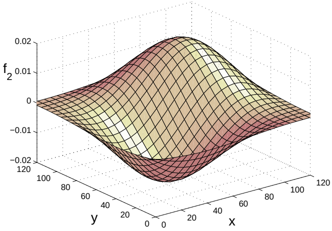
$i = 12$



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y
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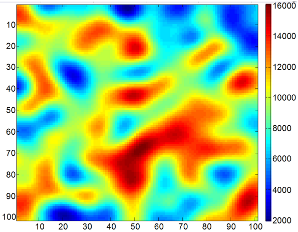
120
x

$i = 24$

Statistics of structural response

Karhunen-Loève expansion

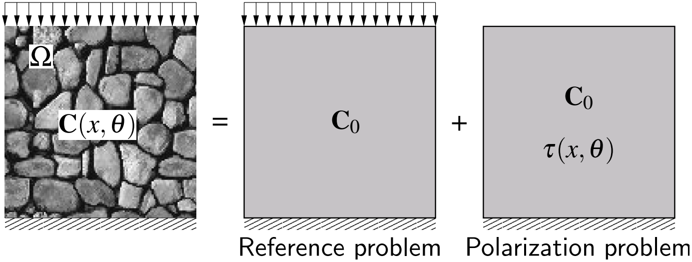
- Spatial dimensional $\boldsymbol{x}$ is accurately captured
- Stochastic dimension θ: Truncated Gaussian variables



- Evaluation of statistics: Monte-Carlo integration

$$\mathbb{E}[\boldsymbol{u}] \approx \frac{1}{N_s} \sum_{s=1}^{N_s} \boldsymbol{u}^h(\theta_i)$$

with $N_s = 10,000$



- Reference body Ω with stiffness $\mathbf{C}_0$
- Stress equivalence condition

$$\boldsymbol{\sigma}(\boldsymbol{x},\theta) = \mathbf{C}(\boldsymbol{x},\theta)\boldsymbol{\varepsilon}(\boldsymbol{x},\theta) = \mathbf{C}_0\boldsymbol{\varepsilon}(\boldsymbol{x},\theta) + \boldsymbol{\tau}(\boldsymbol{x},\theta)$$

- Approximate solution of reference problem using FEM

$$\varepsilon_0^h(\boldsymbol{x}),\ \mathbf{G}_0^h(\boldsymbol{x},\boldsymbol{y}) = \mathbf{N}_u^h(\boldsymbol{x})(\mathbf{K}_0^h)^{-1}\mathbf{B}^{h\mathsf{T}}(\boldsymbol{y}),\ \boldsymbol{\Gamma}_0^h(\boldsymbol{x},\boldsymbol{y}) = \mathbf{B}^h(\boldsymbol{x})(\mathbf{K}_0^h)^{-1}\mathbf{B}^{h\mathsf{T}}(\boldsymbol{y})$$

Statistics of structural response

Hashin-Shtrikman approach: Polarization problem

- Polarization stress: $\boldsymbol{\tau}^h(\boldsymbol{x},\theta) = \mathbf{N}^h(\boldsymbol{x})\left(\chi_1(\boldsymbol{x},\theta)\boldsymbol{d}_1^h + \chi_2(\boldsymbol{x},\theta)\boldsymbol{d}_2^h\right)$
- Discretized version of H-S variational principles

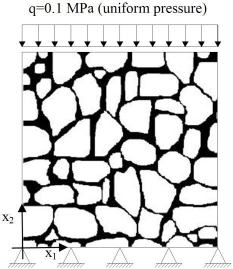
$$\mathbf{K}_i^h \boldsymbol{d}_i^h + \sum_j \mathbf{K}_{ij}^h \boldsymbol{d}_j^h = \boldsymbol{R}_i^h$$

$$\begin{aligned}
\mathbf{K}_i^h &= \int_\Omega \gamma_i(\boldsymbol{x})\mathbf{N}^{h\mathsf{T}}(\boldsymbol{x})\left[\mathbf{C}_i - \mathbf{C}_0\right]^{-1}\mathbf{N}^h(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\
\mathbf{K}_{ij}^h &= \int_\Omega \mathbf{N}^{h\mathsf{T}}(\boldsymbol{x})\int_\Omega S_{ij}^{(2)}(\boldsymbol{x}-\boldsymbol{y})\boldsymbol{\Gamma}_0^h(\boldsymbol{x},\boldsymbol{y})\mathbf{N}^h(\boldsymbol{y})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y} \\
\boldsymbol{R}_i^h &= \int_\Omega \gamma_i(\boldsymbol{x})\mathbf{N}^{h\mathsf{T}}(\boldsymbol{x})\boldsymbol{\varepsilon}_0^h(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}
\end{aligned}$$

- Statistics of the solution:

$$\mathbb{E}[\boldsymbol{u}^h(\boldsymbol{x})] = \boldsymbol{u}_0^h(\boldsymbol{x}) - \int_\Omega \mathbf{G}_0^h(\boldsymbol{x},\boldsymbol{y})\mathbf{N}^h(\boldsymbol{y})\left(\gamma_1\boldsymbol{d}_1^h + \gamma_2\boldsymbol{d}_2^h\right)\,\mathrm{d}\boldsymbol{y}$$

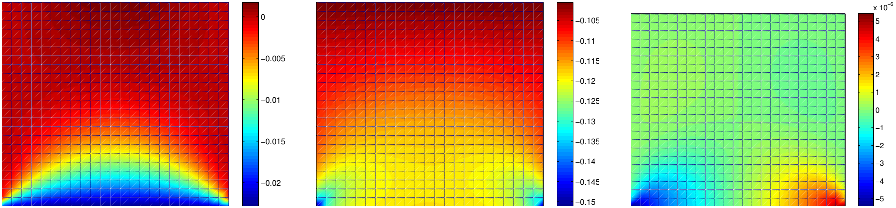
Numerical example



Phase	E [MPa]	ν [-]
Stone	12 500	0.2
Mortar	1 200	0.3

Q1/(Q0) elements

Improved perturbation method (25×25 elements)



$\mathbb{E}[\sigma_{xx}]$ [MPa] $\mathbb{E}[\sigma_{yy}]$ [MPa] $\mathbb{E}[\sigma_{xy}]$ [MPa]

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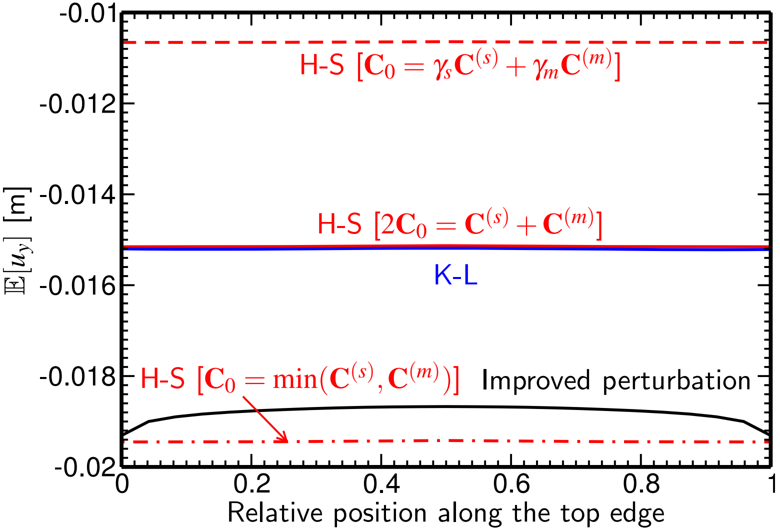

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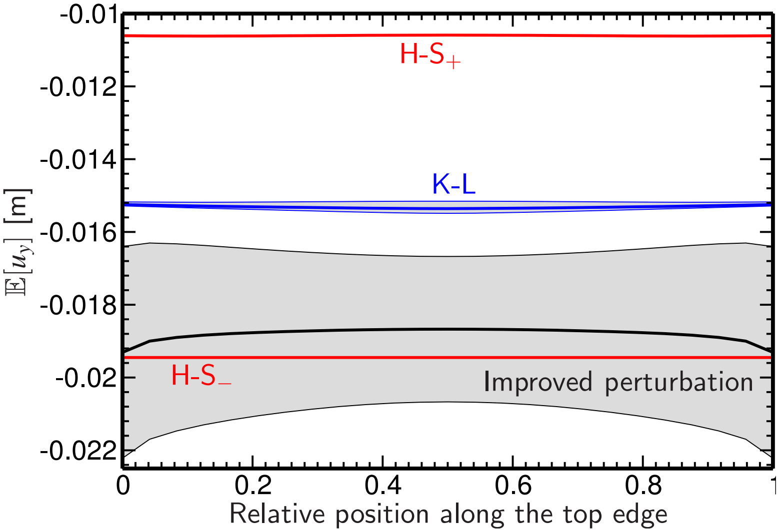

Conclusions

- Spatial correlation of a random field can be directly obtained from samples of microstructure
- Improved perturbation method seems to fully utilize second-order data, but it inconsistent with the H-S bounds
- Karhunen-Loeve expansion needs a large number of terms for realistic structures
- Results of Hashin-Shtrikman method strongly depend on the choice of reference media
- Need for a rational construction of non-Gaussian fields with prescribed correlation structure

Additional info available at
`http://www.arxiv.org/abs/0811.0972`