Weak Solution

Galerkin Method

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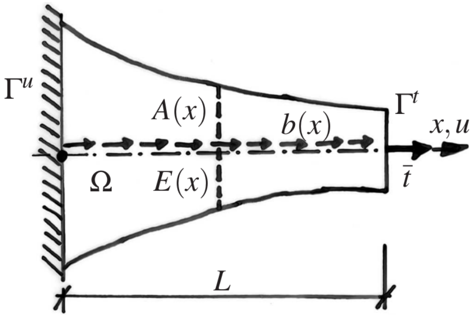
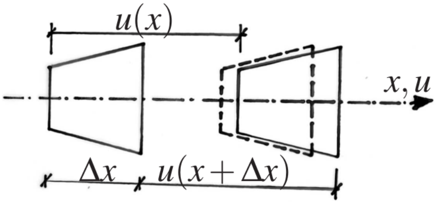
Outline



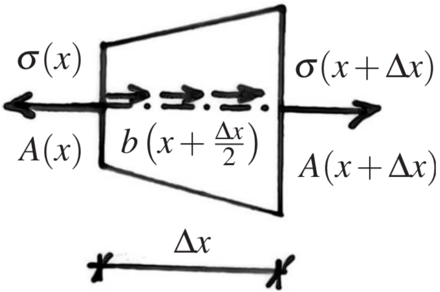
- Let's consider 1D linear elastic problem - beam under axial tension and compression in domain Ω and boundary Γ
- Loading:
  - Continuous volume loading $b(x)$
  - Prescribed displacments on the boundary $\Gamma^u(x = 0)$
  - Prescribed stress on the boundary $\Gamma^t(x = 0)$
- Material property $E(x)$
- Cross-section characteristic - Cross-section area $A(x)$



- Displacement in a given point $u(x)$
- Strain $\varepsilon(x)$, $(x \in \Omega)$

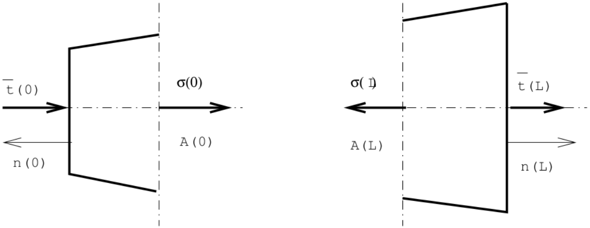
$$\begin{aligned}\varepsilon(x) &= \lim_{\Delta x \to 0} \frac{\Delta\bar{x} - \Delta x}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\Delta x + u(x + \Delta x) - u(x) - \Delta x}{\Delta x} \\ &= \frac{\mathrm{d}u}{\mathrm{d}x}\end{aligned}$$

- Boundary conditions $u(x) = \bar{u}(x)$ for $x \in \Gamma^u$



- In the body: $\Omega (x \in \Omega)$

$$\begin{aligned} \rightarrow -\sigma(x)A(x) + b(x+\frac{\Delta x}{2})\Delta x + \sigma(x+\Delta x)A(x+\Delta x) &= 0 \\ \frac{\sigma(x+\Delta x)A(x+\Delta x) - \sigma(x)A(x)}{\Delta x} + b(x+\frac{\Delta x}{2}) &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}x}(\sigma(x)A(x)) + b(x) &= 0 \end{aligned}$$



- On the boundary: $\Omega(x \in \Omega)$

$$\rightarrow \bar{t}(0) + \sigma(x)A(x) = 0, \qquad \rightarrow \bar{t}(L) - \sigma(x)A(x) = 0$$
$$\sigma(x)A(x)n(x) - \bar{t}(x) = 0, \qquad x \in \Gamma^t$$

- Constitutive equation - Hook's law: $(x \in \Omega)$

$$\sigma(x) = E(x)\varepsilon(x)$$

- Formulation for displacements $(x \in \Omega)$

$$\varepsilon(x) = \frac{\mathrm{d}u}{\mathrm{d}x}(x), \quad \sigma(x) = E(x)\frac{\mathrm{d}u}{\mathrm{d}x}(x), \quad \frac{\mathrm{d}}{\mathrm{d}x}\left(E(x)A(x)\frac{\mathrm{d}u}{\mathrm{d}x}(x)\right) + b(x) = 0$$

We are looking for a solution $u(x)$ smooth enough, which fulfills:

- For $x \in \Omega$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(E(x)A(x)\frac{\mathrm{d}u}{\mathrm{d}x}(x)\right) + b(x) = 0,$$

- For $x \in \Gamma^u$:

$$\bar{u}(x) = u(x),$$

- For $x \in \Gamma^t$:

$$E(x)A(x)\frac{\mathrm{d}u}{\mathrm{d}x}n(x) = \bar{t}(x),$$

PRELIMINARY

- Per partes integration (Green's theorem)

$$\int_0^L \frac{\mathrm{d}f}{\mathrm{d}x}(x)g(x)\mathrm{d}x = [f(x)g(x)]_0^L - \int_0^L f(x)\frac{\mathrm{d}g}{\mathrm{d}x}(x)\mathrm{d}x = 0$$

$$\int_\Omega \frac{\mathrm{d}f}{\mathrm{d}x}(x)g(x)\mathrm{d}x = \int_\Gamma f(x)g(x)\mathrm{d}x - \int_\Omega f(x)\frac{\mathrm{d}g}{\mathrm{d}x}(x)\mathrm{d}x = 0$$

- The definition of solution residuum for the given function $v(x)$

$$\begin{aligned} x \in \Omega \quad &: \quad r(v(x)) = \frac{\mathrm{d}}{\mathrm{d}x}\left(E(x)A(x)\frac{\mathrm{d}v}{\mathrm{d}x}(x)\right) + b(x) \\ x \in \Gamma^u \quad &: \quad r(v(x)) = \bar{u}(x) - v(x) \\ x \in \Gamma^t \quad &: \quad r(v(x)) = \bar{t}(x) - E(x)A(x)\frac{\mathrm{d}v}{\mathrm{d}x}(x)n(x) \end{aligned}$$

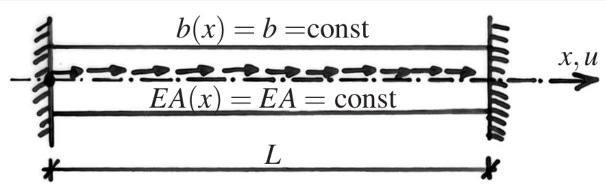
- For $v(x) \equiv u(x)$ we have $r(x) \equiv 0$

Weighted residual method

- Is $v(x)$ the solution?
- The main idea of the weighted residual method: We choose an arbitrary function $\delta u(x)$ (weight, test function) and calculate:

$$\int_{\Omega} \delta u(x) r(v(x)) \mathrm{d}x + \int_{\Gamma} \delta u(x) r(v(x)) \mathrm{d}x$$

- If the value of the integrals is zero for every weight functions $\delta u(x)$, then $v(x)$ is the solution of the problem.
- Example



Weighted residual method

- The solution $u(x)$ has to satisfy:

$$\int_\Omega \delta u(x) \left(\frac{\mathrm{d}}{\mathrm{d}x} (E(x)A(x)\frac{\mathrm{d}u}{\mathrm{d}x}(x)) + b(x) \right) = 0,$$

- Integration by parts:

$$\int_\Gamma \delta u(x) E(x)A(x)\frac{\mathrm{d}u}{\mathrm{d}x}(x)n(x)\mathrm{d}x - \int_\Omega \frac{\mathrm{d}\delta u}{\mathrm{d}x} E(x)A(x)\frac{\mathrm{d}u}{\mathrm{d}x}(x)\mathrm{d}x + \int_\Omega \delta u(x) b(x)\mathrm{d}x = 0$$

- Integrals on the boundary:

$$\int_{\Gamma^u} \underbrace{\delta u(x)}_{=0} E(x)A(x)\frac{\mathrm{d}u}{\mathrm{d}x}(x)n(x)\mathrm{d}x + \int_{\Gamma^t} \delta u(x) \underbrace{E(x)A(x)\frac{\mathrm{d}u}{\mathrm{d}x}(x)n(x)}_{=\bar{t}} \mathrm{d}x$$

Weighted residual method

- We find $u(x)$, $u(x) = \bar{u}(x)$ for $x \in \Gamma^u$, such that:

$$\int_\Omega \frac{\mathrm{d}\delta u}{\mathrm{d}x} E(x) A(x) \frac{\mathrm{d}u}{\mathrm{d}x}(x)\mathrm{d}x = \int_\Omega \delta u(x) b(x) \mathrm{d}x + \int_{\Gamma^t} \delta u(x) \bar{t} \mathrm{d}x$$

- For an arbitrary $\delta u(x)$, where $\delta u(x) = 0$ for $x \in \Gamma^u$. Such function $u(x)$ is called the weak solution of the problem or trial solution
- Why weak solution?
  - $u(x)$ has to be continuous (C^0) and integrable
  - Strong solution (solution of differential equation) $\Rightarrow$ weak solution
  - It allows a flexible numerical solution
- For the numerical application, we need:
  - Suitable expression of the trial solution and the weighted function
  - Suitable numerical method for calculation of integrals

Lagrange's variational principle of the minimum of potential energy

Optional topic

- From all kinematically admissible states, the right is that one which gives the minimum value to the total potential energy.

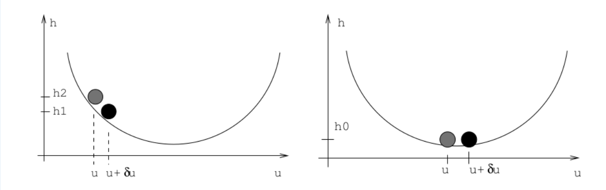
$$\Pi = E_i + E_e = min,$$
$$E_i = \frac{1}{2}\int_\Omega \sigma\varepsilon \mathrm{d}\Omega$$
$$E_e = -\int_\Omega ub\mathrm{d}\Omega - \int_{\Gamma^t} u\bar{t}\mathrm{d}\Omega$$

OPTIONAL TOPIC

- Π is functional (a function of a function). The small change of the function is called its variation, and can be expressed as $\delta u(x) = \xi w(x)$, where $w(x)$ is an arbitrary function and ξ is some small positive number. A change of the functional, which is called its variation, is defined as:

$$\delta\Pi = \Pi(u(x) + \xi w(x)) - \Pi(u(x)) \equiv \Pi(u(x) + \delta u(x)) - \Pi(u(x))$$

- WE are looking for the minimum of the functional Π - the variation must be zero $\delta\Pi = 0$.



Lagrange's variational principle of the minimum of potential energy

Optional topic

- For our problem:

$$
\begin{aligned}
\delta\Pi &= \frac{1}{2}\int EA\left(\frac{\mathrm{d}u}{\mathrm{d}x}+\xi\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2\mathrm{d}x-\frac{1}{2}\int EA\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2\mathrm{d}x\\
&- \int (u+\xi w)\,b\mathrm{d}x-\int (u+\xi w)\,\bar{t}\mathrm{d}\Gamma+\int (u)b\mathrm{d}x+\int (u)\bar{t}\mathrm{d}\Gamma\\
&= \frac{1}{2}\int EA\left(\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2+2\xi\frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}w}{\mathrm{d}x}+\xi^2\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2\right)\mathrm{d}x\\
&- \frac{1}{2}\int EA\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2\mathrm{d}x-\xi\int wb\mathrm{d}x-\xi(w\bar{t})|_\Gamma
\end{aligned}
$$

$$
\delta\Pi = \xi\int EA\frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}w}{\mathrm{d}x}\mathrm{d}x-\xi\int wb\mathrm{d}x-\xi(w\bar{t})|_\Gamma
$$

Lagrange's variational principle of the minimum of potential energy

Optional topic

- WE are looking for the minimum of the functional. The variation must be zero, so $\delta\Pi = 0$. Introducing from previous equations and dividing by ξ, we have:

$$\begin{aligned}\delta\Pi/\xi &= \int EA\frac{\mathrm{d}w}{\mathrm{d}x}\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x - \int wb\mathrm{d}x - (w\bar{t})|_\Gamma = 0\\ \delta\Pi &= \int EA\frac{\mathrm{d}\delta u}{\mathrm{d}x}\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x - \int \delta ub\mathrm{d}x - (\delta u\bar{t})|_\Gamma = 0\end{aligned}$$

- After the introduction of constitutive relations, previous equation is simplified:

$$\delta\Pi = \int A\delta\sigma\varepsilon\mathrm{d}x - \int \delta ub\mathrm{d}x - (\delta u\bar{t})|_\Gamma = 0$$

- The resulting equation above expresses the well-known "Principle of virtual displacements." Note the equivalence between these equations and equations from the beginning of this lecture!

References

- English course of “Numerical analysis of structures” by J. Zeman (jan.zeman@fsv.cvut.cz)
- Czech course of “Numerická analýza konstrukcí” (Numerical analysis of structures) by B. Patzák (borek.patzak@fsv.cvut.cz)
- J. Fish and T. Belytschko: A First Course in Finite Elements, John Wiley & Sons, 2007