Finite Element Formulation for One-Dimensional Problems

Axially Loaded Elastic Bar

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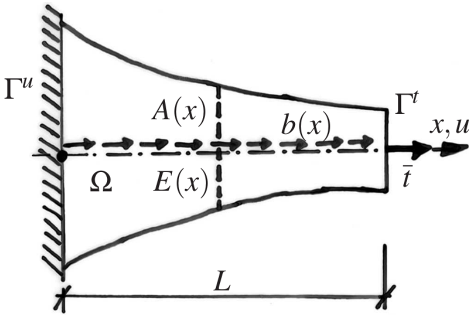
Outline



- Strong form

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(E(x)A(x)\frac{\mathrm{d}u(x)}{\mathrm{d}x}\right) + b(x) = 0$$

- Boundary conditions:

$$\begin{aligned} \bar{u}(x) &= u(x), \qquad x \in \Gamma^u \\ E(x)A(x)\frac{\mathrm{d}u}{\mathrm{d}x}n(x) &= \bar{t}(x), \qquad x \in \Gamma^t \end{aligned}$$

- Weighted residual method:

$$\int_{\Omega} \delta u(x) \left(\frac{\mathrm{d}}{\mathrm{d}x} \left(E(x) A(x) \frac{\mathrm{d}u(x)}{\mathrm{d}x} \right) + b(x) \right) = 0$$

- Integration by parts:

$$\int_{\Gamma} \delta u(x) E(x) A(x) \frac{\mathrm{d}u(x)}{\mathrm{d}x} n(x) \mathrm{d}x - \int_{\Omega} \frac{\mathrm{d}\delta u(x)}{\mathrm{d}x} E(x) A(x) \frac{\mathrm{d}u(x)}{\mathrm{d}x} \mathrm{d}x + \int_{\Omega} \delta u(x) b(x) \mathrm{d}x = 0$$

- Integral on the boundary:

$$\int_{\Gamma^u} \underbrace{\delta u(x)}_{=0} E(x) A(x) \frac{\mathrm{d}u(x)}{\mathrm{d}x} n(x) \mathrm{d}x + \int_{\Gamma^t} \delta u(x) \underbrace{E(x) A(x) \frac{\mathrm{d}u(x)}{\mathrm{d}x} n(x)}_{=\bar{t}} \mathrm{d}x$$

- Weak form - we are looking for such an admissible trial solution u ($u = \bar{u}$ in Γ_u) to be valid:

$$\int_{\Omega} \frac{\mathrm{d}\delta u(x)}{\mathrm{d}x} E(x) A(x) \frac{\mathrm{d}u(x)}{\mathrm{d}x} \mathrm{d}x = \int_{\Omega} \delta u(x) b(x) \mathrm{d}x + \int_{\Gamma^t} \delta u(x), \bar{t} \mathrm{d}x, \qquad \delta u = 0 \in \Gamma_u$$

- The domain is discretized by finite elements and nodes
- The weak form is expressed by an approximation of the trial solution and the weight function in elements. The approximation solution requires C^0 continuity
- The displacement function u is approximated in each element in the following shape:

$$u^e = \boldsymbol{N}^e \boldsymbol{d}^e, \qquad u^e = \bar{u} \quad \text{in} \quad \Gamma_u$$

  where $\boldsymbol{d}^e$ is the vector of nodal displacements.
- The weight function u is approximated in each element in the same way:

$$\delta u^e = \boldsymbol{N}^e \boldsymbol{w}^e, \qquad \delta u^e = 0 \quad \text{in} \quad \Gamma_u$$

  where $\boldsymbol{w}^e$ is the vector of nodal weight function.
- The integral of the weak form is transferred into the sum of integrals in elements:

$$\sum_{e=1}^{n} \left\{ \int_{x_1^e}^{x_2^e} \frac{\mathrm{d}\delta u^e}{\mathrm{d}x} E(x) A(x) \frac{\mathrm{d}u^e}{\mathrm{d}x} \mathrm{d}x - \int_{x_1^e}^{x_2^e} \delta u^e b(x) \mathrm{d}x - (\delta u^e \bar{t})|_{\Gamma_t} \right\} = 0$$

- Derivatives of approximated functions:

$$\frac{\mathrm{d}u^e}{\mathrm{d}x} = \frac{\mathrm{d}\boldsymbol{N}^e}{\mathrm{d}x}\boldsymbol{d}^e = \boldsymbol{B}^e\boldsymbol{d}^e$$
$$\frac{\mathrm{d}\delta u^e}{\mathrm{d}x} = \frac{\mathrm{d}\boldsymbol{N}^e}{\mathrm{d}x}\boldsymbol{w}^e = \boldsymbol{B}^e\boldsymbol{w}^e$$

- Introducing approximations into the previous weak form gives:

$$\sum_{e=1}^{n}\boldsymbol{w}^{e\mathrm{T}}\left\{\underbrace{\int_{x_1^e}^{x_2^e}\boldsymbol{B}^{e\mathrm{T}}EA\boldsymbol{B}^e\mathrm{d}x}_{\boldsymbol{K}^e}\,\boldsymbol{d}^e - \underbrace{\int_{x_1^e}^{x_2^e}\boldsymbol{N}^{e\mathrm{T}}b\mathrm{d}x}_{\boldsymbol{f}_\Omega} - \underbrace{\left(\boldsymbol{N}^{e\mathrm{T}}\bar{t}\right)|_{\Gamma_t}}_{\boldsymbol{f}_{\Gamma_t}}\right\} = 0$$

- If local vectors $\boldsymbol{d}^e$, $\boldsymbol{w}^e$ are expanded into global vectors of nodal values $\boldsymbol{d}$, $\boldsymbol{w}$, we can write:

$$\boldsymbol{w}^{\mathrm{T}}\left(\sum_{e=1}^{n}\tilde{\boldsymbol{K}}^{e}\boldsymbol{d}-\sum_{e=1}^{n}\tilde{\boldsymbol{f}}^{e}\right)=0, \qquad \forall \boldsymbol{w}, \quad w=0\in\Gamma_u$$

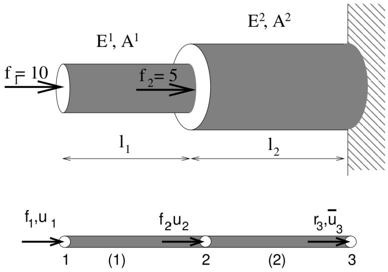
$$\boldsymbol{w}^{\mathrm{T}}\left(\boldsymbol{K}\boldsymbol{d}-\boldsymbol{f}\right)=0$$

- Finaly, we have:

$$\boldsymbol{K}\boldsymbol{d}-\boldsymbol{f}=0$$

- Note: Residuum:

$$\boldsymbol{R}=\boldsymbol{K}\boldsymbol{d}-\boldsymbol{f}$$



$$\boldsymbol{w}^{\mathrm{T}}\boldsymbol{R} = 0, \quad \forall \boldsymbol{w}, \quad \text{except} \quad w = 0 \in \Gamma_u$$

$$\begin{aligned} w_1 R_1 + w_2 R_2 &= 0, \qquad w_1, w_2 \implies R_1 = R_2 = 0 \\ R_3 &\neq 0, \qquad w_3 = 0 \end{aligned}$$

$$\boldsymbol{R} = \left\{ \begin{array}{c} 0 \\ 0 \\ R_3 \end{array} \right\} = \left[\begin{array}{ccc} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{array} \right] \left\{ \begin{array}{c} u_1 \\ u_2 \\ \bar{u}_3 \end{array} \right\} - \left\{ \begin{array}{c} f_1 \\ f_2 \\ f_3 \end{array} \right\}$$

$$\left[\begin{array}{ccc} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{array} \right] \left\{ \begin{array}{c} u_1 \\ u_2 \\ \bar{u}_3 \end{array} \right\} = \left\{ \begin{array}{c} f_1 \\ f_2 \\ f_3 + R_3 \end{array} \right\}$$

Stiffness matrix of an element with linear approximation functions:

- Matrix of interpolation functions:

$$\boldsymbol{N}^e = \frac{1}{l^e}\left[x_2^e - x, x - x_1^e\right]$$

- Geometric matrix:

$$\boldsymbol{B}^e = \frac{\mathrm{d}\boldsymbol{N}^e}{\mathrm{d}x} = \frac{1}{l^e}\left[-1, 1\right]$$

- Introducing the geometric matrix into the element stiffness matrix gives:

$$\begin{aligned}\boldsymbol{K}^e &= \int_{x_1^e}^{x_2^e} \boldsymbol{B}^{e\mathrm{T}} EA \boldsymbol{B}^e \mathrm{d}x = \int_{x_1^e}^{x_2^e} \frac{1}{l^e}\begin{bmatrix} -1 \\ 1 \end{bmatrix} EA \frac{1}{l^e}\left[-1, 1\right] \mathrm{d}x \\ \boldsymbol{K}^e &= \frac{EA}{l^e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\end{aligned}$$

Loading vector in an element with linear approximation functions:

- Loading vector:

$$\boldsymbol{f}_{\Omega}^{e} = \int_{x_1^e}^{x_2^e} \boldsymbol{N}^{e\mathrm{T}} b(x)\mathrm{d}x$$

- In the case of linear volume loading $b(x)$, the loading can be expressed by approximation functions:

$$b(x) = \boldsymbol{N}^{e}\boldsymbol{b}^{e}$$

- The introduction of the previous equation into the loading vector integral leads to the following relationship:

$$\begin{aligned}
\boldsymbol{f}_{\Omega}^{e} &= \int_{x_1^e}^{x_2^e} \boldsymbol{N}^{e\mathrm{T}} \boldsymbol{N}^{e} \mathrm{d}x \boldsymbol{b}^{e} \\
&= \frac{1}{l^e}^{2} \int_{x_1^e}^{x_2^e} \left[\begin{array}{cc} (x_2^e - x)^2 & (x_2^e - x)(x - x_1^e) \\ (x_2^e - x)(x - x_1^e) & (x - x_1^e)^2 \end{array} \right] \mathrm{d}x \boldsymbol{b}^{e} \\
&= \frac{l^e}{6} \left[\begin{array}{cc} 2 & 1 \\ 1 & 2 \end{array} \right] \left\{ \begin{array}{c} b_1^e \\ b_2^e \end{array} \right\}
\end{aligned}$$

Element with Linear Approximation in Natural Coordinates:

- The matrix of interpolation functions in the natural coordinate system is given by this expression:

$$\boldsymbol{N}^e = \left[\frac{1}{2}(1-\xi), \frac{1}{2}(1+\xi)\right]$$

- Derivatives of shape functions with respect to x are needed for the geometric matrix $\boldsymbol{B}^e$ calculation. The derivative of a compound function reads:

$$\frac{\mathrm{d}f}{\mathrm{d}\xi} = \frac{\mathrm{d}f}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}\xi}$$

- The inverse relation is:

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \left(\frac{\mathrm{d}x}{\mathrm{d}\xi}\right)^{-1}\frac{\mathrm{d}f}{\mathrm{d}\xi}$$

- The dependence of x on ξ can be taken from the isoparametric element definition:

$$x(\xi) = \boldsymbol{N}^e(\xi)\boldsymbol{x}^e, \qquad \mathrm{d}x(\xi) = \frac{\mathrm{d}\boldsymbol{N}^e}{\mathrm{d}\xi}\boldsymbol{x}^e\mathrm{d}\xi = J\mathrm{d}\xi$$

- In the case of the linear approximation:

$$\begin{aligned} x(\xi) &= \frac{1}{2}(1-\xi)\cdot x_1^e + \frac{1}{2}(1+\xi)\cdot x_2^e \\ \mathrm{d}x(\xi) &= \frac{1}{2}(x_2^e - x_1^e)\mathrm{d}\xi = \frac{l^e}{2}\mathrm{d}\xi \end{aligned}$$

- Introducing the previous equations into the element stiffness matrix:

$$\begin{aligned} \boldsymbol{K}^e &= \int_{-1}^{1} \boldsymbol{B}^{e\mathrm{T}} EA \boldsymbol{B}^e J \mathrm{d}\xi = \int_{-1}^{1} \begin{bmatrix} -1/l^e \\ 1/l^e \end{bmatrix} EA \left[\frac{-1}{l^e}, \frac{1}{l^e}\right] \frac{l_e}{2}\mathrm{d}\xi \\ \boldsymbol{K}^e &= \frac{EA}{l^e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{aligned}$$

Element with Quadratic Approximation in Natural Coordinates:

- Matrix of interpolation functions in natural coordinate system:

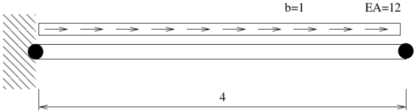
$$\boldsymbol{N}^e = \left[-\frac{1}{2}(1-\xi)+\frac{1}{2}(1-\xi)^2, (1-\xi^2), \frac{1}{2}(1+\xi)-\frac{1}{2}(1-\xi^2)\right] \tag{1}$$

- Approximation of coordinates:

$$\begin{aligned} x(\xi) &= \boldsymbol{N}^e(\xi)\boldsymbol{x}^e = \left(-\frac{1}{2}\xi+\frac{1}{2}\xi^2\right)\cdot x_1^e + (1-\xi^2)\cdot x_2^e + \left(\frac{1}{2}\xi+\frac{1}{2}\xi^2\right)\cdot x_3^e \\ J &= \frac{\mathrm{d}x}{\mathrm{d}\xi} = \left(-\frac{1}{2}+\xi\right)\cdot x_1^e - 2\xi\cdot x_2^e + \left(\frac{1}{2}+\xi\right)\cdot x_3^e \end{aligned}$$

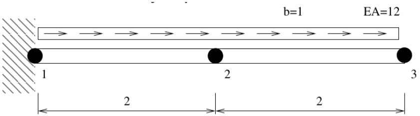
$$\begin{aligned} \boldsymbol{K}^e &= \int_{-1}^{1} \boldsymbol{B}^{e\mathrm{T}} EA \boldsymbol{B}^e J \mathrm{d}\xi \\ \boldsymbol{f}^e &= \int_{-1}^{1} \boldsymbol{N}^{e\mathrm{T}} b(\xi) J \mathrm{d}\xi \end{aligned}$$

Example:



- Analytical solution - Strong form:
- Integration:
- Integration constant from boundary conditions:
- Solution:

Example:



- Solution with one quadratic element:

$$\boldsymbol{x}^e = \{0; 2; 4\}^{\mathrm{T}}$$

$$\boldsymbol{N}^e = \left[\frac{1}{2}(1-\xi) - \frac{1}{2}(1-\xi)^2, (1-\xi^2), \frac{1}{2}(1+\xi) - \frac{1}{2}(1-\xi^2)\right]$$

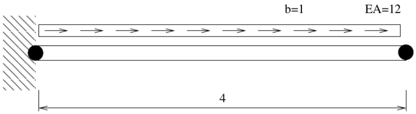
$$x = \boldsymbol{N}^e \boldsymbol{x}^e = 2 + 2 \cdot \xi \implies \frac{\mathrm{d}x}{\mathrm{d}\xi} = 2, \quad J = 2$$

$$\frac{\mathrm{d}\boldsymbol{N}^e}{\mathrm{d}\xi} = \left[-\frac{1}{2} + \xi, -2\xi, \frac{1}{2} + \xi\right]$$

$$\boldsymbol{B}^e = \frac{\mathrm{d}\boldsymbol{N}^e}{\mathrm{d}\xi}\frac{\mathrm{d}\xi}{\mathrm{d}x} = \left[-\frac{1}{4} + \frac{1}{2}\xi, -\xi, \frac{1}{4} + \frac{1}{2}\xi\right]$$

$$
\begin{aligned}
\boldsymbol{K}^e &= \int_{-1}^{1} \boldsymbol{B}^{e\mathrm{T}} EA \boldsymbol{B}^e J \mathrm{d}\xi = \frac{EA}{12} \begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} \\
\boldsymbol{f}^e &= \int_{-1}^{1} \boldsymbol{N}^{e\mathrm{T}} b J \mathrm{d}\xi = \{2/3; 8/3; 2/3\}^{\mathrm{T}} \\
\boldsymbol{f} &= \{8/3; 2/3\}^{\mathrm{T}} \\
\boldsymbol{K} &= \begin{bmatrix} 16 & -8 \\ -8 & 7 \end{bmatrix} \\
\boldsymbol{d} &= \boldsymbol{K}^{-1}\boldsymbol{f} = \{1/2; 2/3\}^{\mathrm{T}} \\
\varepsilon &= \frac{\mathrm{d}u}{\mathrm{d}x} = \boldsymbol{B}^e \boldsymbol{d}^e = \boldsymbol{B}\{0, 1/2; 2/3\}^{\mathrm{T}} = 1/6 - 1/6 \cdot \xi \\
R_1 &= \boldsymbol{K}^e(1)\boldsymbol{d}^e - \boldsymbol{f}^e(1) = \{7; -8; 1\}\{0; 1/2; 2/3\}^{\mathrm{T}} - 2/3 = -4
\end{aligned} \tag{2}
$$

Example:



- Solution with one linear element:

$$\begin{aligned}\boldsymbol{K}^e &= \frac{EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} \\ \boldsymbol{f}^e &= \int_0^4 \boldsymbol{N}^{e\mathrm{T}} b \mathrm{d}x = \{2;2\}^\mathrm{T}\end{aligned} \tag{3}$$

- System of equations:

$$\boldsymbol{d} = \boldsymbol{K}^{-1}\boldsymbol{f} \implies \left\{ \begin{array}{c} u_1 \\ u_2 \end{array} \right\} = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}^{-1} \left\{ \begin{array}{c} 2 \\ 2 \end{array} \right\} = \left\{ \begin{array}{c} 0 \\ 2/3 \end{array} \right\} \tag{4}$$

References

- English course of "Numerical analysis of structures" by J. Zeman (jan.zeman@fsv.cvut.cz)
- Czech course of "Numerická analýza konstrukcí" (Numerical analysis of structures) by B. Patzák (borek.patzak@fsv.cvut.cz)
- J. Fish and T. Belytschko: A First Course in Finite Elements, John Wiley & Sons, 2007