Finite Element Formulation for One-Dimensional Heat Conduction Problem

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132NAST - Numerical analysis of structures
26 October 2022

Outline

- Examples of physical problems:

Phenomenon	Characteristic variable
Linear statics	Displacements
Shear deplanation	Deplanation function
Membrane problems	Deflection
Electrostatics	El. potential
Diffusion	Concentration
Water flow	Hydraulic head
Moisture transport	Relative humidity
Heat transfer	Temperature

Heat transfer:
- function of temperature distribution $T(x)$ [K]
- solution procedure:

Differential equation + boundary conditions
→ weak form → FEM discretization

Basic terms and quantities:

- The heat $Q(x)$ [J] (thermal energy) is a part of intrinsic energy, which a body receives and gives by the thermal exchange with another body (expresses the change of a state).
- Temperature $T(x)$ stands for a body state [K].
- The exchange depends on the temperature difference, not on the temperature!
- Temperature gradient:

$$\nabla T(x) = \mathrm{grad} T(x) = \lim_{\Delta x \to 0} \frac{T(x+\Delta x) - T(x)}{\Delta x} = \frac{\mathrm{d}T}{\mathrm{d}x}(x)$$

- Heat flux $\boldsymbol{q}_n(x)$ is an amount of heat energy, which is transferred through a unit area A [1m^2] with outer normal $\boldsymbol{n}$ per time unit t [s]:

$$\boldsymbol{q}_n(x) = \frac{Q(x)}{A \cdot t}\boldsymbol{n}(x), \qquad [\mathrm{Wm}^{-2}]$$

Note.: Heat flux is independent of time in a steady state

Transport equation:

- Fourier's law: heat flux in a material point $x \in \Omega$

$$\boldsymbol{q}(x) = -\lambda(x)\,\mathrm{grad}T(x)$$

where $\lambda(x)$ is the conductivity coefficient [Wm^{-1}K^{-1}]

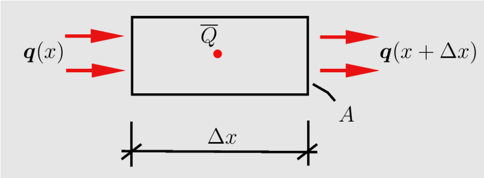
Balance equation:

- Energy conservation equation in a volume element Ω

$$\mathrm{div}\left(-\lambda(x)\,\mathrm{grad}T(x)\right) = 0$$

Derivation of heat transfer equation:

- During heat transfer, the heat flux "flows" throughout the volume element. Balance of energy requires the change of thermal energy generated in the volume element to be equal to gained thermal energy - temperature, and the energy is constant in the volume element in a steady state



- Inside the body ($x \in \Omega$):

$$q(x)A(x) + \overline{Q}(x)\left(x + \frac{\Delta x}{2}\right)\Delta x A\left(x + \frac{\Delta x}{2}\right) = q(x+\Delta x)A(x+\Delta x),$$

where $\overline{Q}(x)$ denotes the heat source [$\mathrm{Jm}^{-3}\mathrm{s}^{-1}$] (positive when the heat is generated and negative when the heat is removed).

Derivation of heat transfer equation:

- The limit $\Delta x \to 0$:

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(q(x)\right) A(x) + \overline{Q}(x)A(x) = 0$$

- Assuming the constant cross-section area $A(x)$, the previous equation is simplified:

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(q(x)\right) + \overline{Q}(x) = 0, \qquad x \in \Omega$$

- Introducing Fourier's law into the previous equation, we obtain the second-order differential ordinary equation (strong form):

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\lambda(x)\frac{\mathrm{d}T(x)}{\mathrm{d}x}\right) + \overline{Q}(x) = 0$$

Note.: Thermal conductivity coefficient $\lambda(x)$ is assumed constant, but generally, it can be a function of temperature T ($\lambda(T, x)$).

Boundary conditions:

- Dirichlet b.c. - prescribed temperature on the boundary:

$$T(x) = \overline{T}(x), \qquad x \in \Gamma_T$$

- Neumann b.c - prescribed flux on the boundary:

$$q(x) = \overline{q}(x), \qquad x \in \Gamma_{qp}$$

- Cauchy - Newton b.c. - heat transfer on the boundary:

$$q(x) = \alpha(x)\left(T(x) - T_\infty(x)\right), \qquad x \in \Gamma_{qc}$$

$\alpha(x)$ is the heat transfer coefficient [Wm^{-2}K^{-1}], $T_\infty(x)$ is the ambient temperature

- Non-linear b.c. (Newton) - heat radiation on the boundary:

$$q(x) = \varepsilon(x)\sigma(x)\left(T^4(x) - T^4_\infty(x)\right), \qquad x \in \Gamma_{qr},$$

where $\varepsilon(x)$ is the rate of the surface radiation related to a black body radiation $(0 < \varepsilon < 1)$, $\sigma(x) = 5,67 \cdot 10^{-8}$ Wm^{-2}K^{-4} is the Stefan-Boltzmann constant, and $T_\infty(x)$ is the ambient temperature (temperature of a radiator)

Weighted residual method (Galerkin method):

- We are looking for a solution $T(x)$ smooth enough, which fulfills $x \in \Omega$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\lambda(x)\frac{\mathrm{d}T(x)}{\mathrm{d}x}\right) + \overline{Q}(x) = 0$$

- for $x \in \Gamma_T$:

$$T(x) = \overline{T}(x)$$

- for $x \in \Gamma_q$:

$$q(x) = -\lambda(x)\frac{\mathrm{d}T(x)}{\mathrm{d}x}\boldsymbol{n}(x) = \overline{q}(x),$$

where:

  - for $x \in \Gamma_{qp}$: $\overline{q}(x)$ is prescribed
  - for $x \in \Gamma_{qc}$: $\overline{q}(x) = \alpha(x)\left(T(x) - T_\infty(x)\right)$
  - for $x \in \Gamma_{qr}$: $\overline{q}(x) = \varepsilon(x)\sigma(x)\left(T^4(x) - T^4_\infty(x)\right)$ (it is avoided in the derivation)

Weighted residual method (Galerkin method):

- For an arbitrary weight function δT so that $\delta T(x) = 0$ for $x \in \Gamma_T$, it holds:

$$\int_\Omega \left\{ \delta T(x) \left(\frac{\mathrm{d}}{\mathrm{d}x} \left(\lambda(x) \frac{\mathrm{d}T(x)}{\mathrm{d}x} \right) + \overline{Q}(x) \right) \right\} \mathrm{d}\Omega = 0, \qquad x \in \Omega$$

- Integration by parts, we get:

$$\int_\Gamma \delta T(x) \lambda(x) \frac{\mathrm{d}T(x)}{\mathrm{d}x} \boldsymbol{n}(x) \mathrm{d}\Gamma - \int_\Omega \frac{\mathrm{d}\delta T(x)}{\mathrm{d}x} \lambda(x) \frac{\mathrm{d}T(x)}{\mathrm{d}x} \mathrm{d}\Omega + \int_\Omega \delta T(x) \overline{Q}(x) \mathrm{d}\Omega = 0$$

where the integral on the boundary:

$$\int_\Gamma \delta T(x) \lambda(x) \frac{\mathrm{d}T(x)}{\mathrm{d}x} \boldsymbol{n}(x) \mathrm{d}\Gamma = \int_{\Gamma_T} \overbrace{\delta T(x)}^{=0} \lambda(x) \frac{\mathrm{d}T(x)}{\mathrm{d}x} \boldsymbol{n}(x) \mathrm{d}\Gamma + \int_{\Gamma_q} \delta T(x) \underbrace{\lambda(x) \frac{\mathrm{d}T(x)}{\mathrm{d}x} \boldsymbol{n}(x)}_{=-q(x)} \mathrm{d}\Gamma$$

It follows:

$$\int_{\Gamma_q} \delta T(x) q(x) \mathrm{d}\Gamma = \int_{\Gamma_{qp}} \delta T(x) \overline{q}(x) \mathrm{d}\Gamma + \int_{\Gamma_{qc}} \delta T(x) \alpha(x) \left(T(x) - T_\infty(x) \right) \mathrm{d}\Gamma$$

- Weak form:

$$\int_\Omega \frac{\mathrm{d}\delta T(x)}{\mathrm{d}x} \lambda(x) \frac{\mathrm{d}T(x)}{\mathrm{d}x} \mathrm{d}\Omega + \int_{\Gamma_{qc}} \delta T(x) \alpha(x) T(x) \mathrm{d}\Gamma = \int_{\Gamma_{qp}} \delta T(x) \overline{q}(x) \mathrm{d}\Gamma + \\ + \int_{\Gamma_{qc}} \delta T(x) \alpha(x) T_\infty(x) \mathrm{d}\Gamma + \int_\Omega \delta T(x) \overline{Q}(x) \mathrm{d}\Omega,$$

We are looking for such an admissible trial solution $T(x)$ which satisfies the above formulation.

Finite Element Method:

- The domain Ω is discretized by finite elements and nodes
- The weak form is expressed by an approximation of the trial solution and the weight function in elements. The approximation solution requires C^0 continuity
- The temperature function T is approximated in each element in the following shape:

$$T^e(x) \approx \boldsymbol{N}^e(x)\boldsymbol{r}^e, \quad \mathrm{grad}T^e(x) \approx \boldsymbol{B}^e(x)\boldsymbol{r}^e, \quad \delta T^e(x) \approx \boldsymbol{N}^e(x)\boldsymbol{w}^e, \quad \mathrm{grad}\delta T^e(x) \approx \boldsymbol{B}^e(x)\boldsymbol{w}^e$$

- Introducing approximations of trial solution and weight function into the weak form (for all $\boldsymbol{w}^e$ that $\boldsymbol{w}^e = 0$ on Γ_T), we obtain the following equation, where the integrals of the weak form are transferred into the sum of integrals in elements:

$$\sum_{e=1}^{n} \boldsymbol{w}^{e\mathrm{T}} \Bigg\{ \overbrace{\int_{\Omega^e} \boldsymbol{B}^{e\mathrm{T}}(x)\lambda^e(x)\boldsymbol{B}^e(x)\mathrm{d}\Omega}^{\boldsymbol{K}^e_\Omega}\, \boldsymbol{r}^e + \overbrace{\int_{\Gamma^e} \boldsymbol{N}^{e\mathrm{T}}(x)\alpha^e(x)\boldsymbol{N}^e(x)\mathrm{d}\Gamma}^{\boldsymbol{K}^e_\Gamma}\, \boldsymbol{r}^e +$$

$$- \overbrace{\int_{\Gamma^e} \boldsymbol{N}^{e\mathrm{T}}(x)\alpha^e(x)\boldsymbol{N}^e(x)\mathrm{d}\Gamma \boldsymbol{T}^e_0}^{\boldsymbol{f}^e_{\Gamma_c}} + \overbrace{\int_{\Gamma^e} \boldsymbol{N}^{e\mathrm{T}}(x)\boldsymbol{N}^e(x)\mathrm{d}\Gamma \overline{\boldsymbol{q}}^e}^{-\boldsymbol{f}^e_{\Gamma_p}} - \overbrace{\int_{\Omega^e} \boldsymbol{N}^{e\mathrm{T}}(x)\boldsymbol{N}^e(x)\mathrm{d}\Omega \overline{\boldsymbol{Q}}^e}^{\boldsymbol{f}^e_\Omega} \Bigg\} = 0$$

Finite Element Method:

- If local vectors $\boldsymbol{r}^e$, $\boldsymbol{w}^e$ are expanded into global vectors of nodal values $\boldsymbol{r}$, $\boldsymbol{w}$, we can write:

$$\boldsymbol{w}^{\mathrm{T}}\left(\sum_{e=1}^{n}\hat{\boldsymbol{K}}^{e}\boldsymbol{r}-\sum_{e=1}^{n}\hat{\boldsymbol{f}}^{e}\right)=0$$

- Finaly, we have:

$$\boldsymbol{K}\boldsymbol{r}=\boldsymbol{f}$$

- Decomposition of the conductivity matrix according to Dirichlet b.c. $(\boldsymbol{r}_d)$:

$$\begin{bmatrix}\boldsymbol{K}_{TT} & \boldsymbol{K}_{Td}\\ \boldsymbol{K}_{dT} & \boldsymbol{K}_{dd}\end{bmatrix}\begin{bmatrix}\boldsymbol{r}_T\\ \boldsymbol{r}_d\end{bmatrix}=\begin{bmatrix}\boldsymbol{f}_T\\ \boldsymbol{f}_d\end{bmatrix}+\begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{R}\end{bmatrix},$$

where

$$\boldsymbol{K}_{TT}\boldsymbol{r}_T=\boldsymbol{f}_T-\boldsymbol{K}_{Td}\boldsymbol{r}_d$$

and residuum vector - vector of nodal heat fluxes on the boundary:

$$\boldsymbol{R}=\boldsymbol{K}_{dT}\boldsymbol{r}_T+\boldsymbol{K}_{dd}\boldsymbol{r}_d-\boldsymbol{f}_d$$

References

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