Finite Element Formulation for One-Dimensional Non-stationary Heat Conduction Problems

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Outline

Transport equation:

- Fourier's law: heat flux in a material point $x \in \Omega$

$$\boldsymbol{q}(x,t) = -\lambda(\boldsymbol{x})\,\mathrm{grad}T(\boldsymbol{x},t)$$

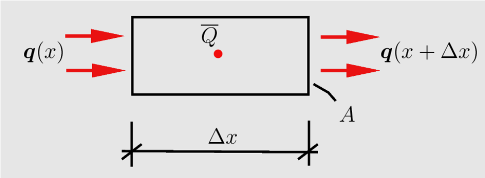
Balance equation:

- Energy conservation equation in a volume element Ω

$$-\mathrm{div}\,(-\lambda(\boldsymbol{x})\,\mathrm{grad}T(\boldsymbol{x},t)) = \rho(x)C(x)\,\frac{\partial T(\boldsymbol{x},t)}{\partial t}, \qquad \boldsymbol{x}, \in \Omega$$

where $\lambda(\boldsymbol{x})$ is the conductivity coefficient [W·m^{-1}K^{-1}], $\rho(\boldsymbol{x})$ [kg·m^{-3}] is the volume weight, and $C(\boldsymbol{x})$ [J·kg^{-1}·K^{-1}] is the specific heat (capacity).

Derivation of non-stationary heat transfer equation:



- For 1D problem - partial differential equation:

$$\frac{\partial}{\partial x}\left(\lambda(x)\frac{\partial T(x,t)}{\partial x}\right)+\overline{Q}(x,t)=\rho(x)C(x)\,\frac{\partial T(x,t)}{\partial t}$$

- Initial condition:

$$T(x,t=0)=T_0(x,t),\qquad x\in\Omega$$

Boundary conditions:

- Dirichlet b.c. - prescribed temperature on the boundary:

$$T(x,t) = \overline{T}(x,t) \qquad x \in \Gamma_T$$

- Neumann b.c - prescribed flux on the boundary:

$$q(x,t) = \overline{q}(x,t) \qquad x \in \Gamma_{qp}$$

- Cauchy - Newton b.c. - heat transfer on the boundary:

$$q(x,t) = \alpha(x)\left(T(x,t) - T_\infty(x,t)\right) \qquad x \in \Gamma_{qc}$$

$\alpha(x)$ is the heat transfer coefficient [$\text{Wm}^{-2}\text{K}^{-1}$], $T_\infty(x,t)$ is the ambient temperature

- Non-linear b.c. (Newton) - heat radiation on the boundary:

$$q(x,t) = \varepsilon(x,t)\sigma(x,t)\left(T^4(x,t) - T_\infty^4(x,t)\right) \qquad x \in \Gamma_{qr}$$

where $\varepsilon(x,t)$ is the rate of the surface radiation related to a black body radiation $(0 < \varepsilon < 1)$, $\sigma(x,t) = 5,67 \cdot 10^{-8}$ $\text{Wm}^{-2}\text{K}^{-4}$ is the Stefan-Boltzmann constant, and $T_\infty(x,t)$ is the ambient temperature (temperature of a radiator).

Weighted residual method (Galerkin method):

- We are looking for a solution $T(x,t)$ smooth enough, which fulfills $x \in \Omega$:

$$\frac{\partial}{\partial x}\left(\lambda(x)\frac{\partial T(x,t)}{\partial x}\right) + \overline{Q}(x,t) - \rho(x)C(x)\,\frac{\partial T(x,t)}{\partial t} = 0$$

- Initial condition: for $t = 0$:

$$T(x,t=0) = T_0(x,t), \qquad x \in \Omega$$

- for $x \in \Gamma_T$:

$$T(x,t) = \overline{T}(x,t)$$

- for $x \in \Gamma_q$:

$$q(x,t) = -\lambda(x)\frac{\partial T(x,t)}{\partial x}\boldsymbol{n}(x,t) = \overline{q}(x,t),$$

  where:
  - for $x \in \Gamma_{qp}$: $\overline{q}(x,t)$ is prescribed
  - for $x \in \Gamma_{qc}$: $\overline{q}(x,t) = \alpha(x)\left(T(x,t) - T_\infty(x,t)\right)$
  - for $x \in \Gamma_{qr}$: $\overline{q}(x,t) = \varepsilon(x,t)\sigma(x,t)\left(T^4(x,t) - T^4_\infty(x,t)\right)$ (it is avoided in the derivation)

Weighted residual method (Galerkin method):

- For an arbitrary weight function δT so that $\delta T(x,t) = 0$ for $x \in \Gamma_T$, it holds:

$$\int_\Omega \left\{ \delta T(x,t) \left(\frac{\partial}{\partial x} \left(\lambda(x) \frac{\partial T(x,t)}{\partial x} \right) + \overline{Q}(x,t) - \rho(x) C(x) \, \frac{\partial T(x,t)}{\partial t} \right) \mathrm{d}\Omega \right\} = 0, \qquad x \in \Omega$$

- Integration by parts, we get:

$$\int_\Gamma \delta T(x,t) \lambda(x) \frac{\mathrm{d}T(x,t)}{\mathrm{d}x} \boldsymbol{n}(x,t) \mathrm{d}\Gamma \quad - \quad \int_\Omega \frac{\mathrm{d}\delta T(x,t)}{\mathrm{d}x} \lambda(x) \frac{\mathrm{d}T(x,t)}{\mathrm{d}x} \mathrm{d}\Omega + \int_\Omega \delta T(x,t) \overline{Q}(x,t) \mathrm{d}\Omega$$
$$- \quad \int_\Omega \delta T(x,t) \rho(x) C(x) \, \frac{\partial T(x,t)}{\partial t} \mathrm{d}\Omega = 0$$

where the integral on the boundary:

$$\int_\Gamma \delta T(x,t) \lambda(x) \frac{\mathrm{d}T(x,t)}{\mathrm{d}x} \boldsymbol{n}(x,t) \mathrm{d}\Gamma \quad = \quad \int_{\Gamma_T} \overbrace{\delta T(x,t)}^{=0} \lambda(x) \frac{\mathrm{d}T(x,t)}{\mathrm{d}x} \boldsymbol{n}(x,t) \mathrm{d}\Gamma$$
$$+ \quad \int_{\Gamma_q} \delta T(x,t) \underbrace{\lambda(x) \frac{\mathrm{d}T(x,t)}{\mathrm{d}x} \boldsymbol{n}(x,t)}_{=-q(x,t)} \mathrm{d}\Gamma$$

It follows:

$$\int_{\Gamma_q} \delta T(x,t) q(x,t) \mathrm{d}\Gamma = \int_{\Gamma_{qp}} \delta T(x,t) \overline{q}(x,t) \mathrm{d}\Gamma + \int_{\Gamma_{qc}} \delta T(x,t) \alpha(x) \left(T(x,t) - T_\infty(x,t)\right) \mathrm{d}\Gamma$$

- Weak form:

$$\int_\Omega \frac{\partial \delta T(x,t)}{\partial x} \lambda \frac{\partial T(x,t)}{\partial x} \mathrm{d}\Omega + \int_{\Gamma_{qc}} \delta T(x,t) \alpha T(x,t) \mathrm{d}\Gamma + \int_\Omega \delta T(x,t) \rho C \ \frac{\partial T(x,t)}{\partial t} \mathrm{d}\Omega =$$

$$= \int_{\Gamma_{qp}} \delta T(x,t) \overline{q}(x,t) \mathrm{d}\Gamma + \int_{\Gamma_{qc}} \delta T(x,t) \alpha T_\infty(x,t) \mathrm{d}\Gamma + \int_\Omega \delta T(x,t) \overline{Q}(x,t) \mathrm{d}\Omega,$$

We are looking for such an admissible trial solution $T(x,t)$ which satisfies the above formulation.

Finite Element Method:

- The temperature function T is approximated in each element in the following shape:

$$T^e(x,t) \approx \boldsymbol{N}^e(x)\boldsymbol{r}^e(t),\ \mathrm{grad}T^e(x,t) \approx \boldsymbol{B}^e(x)\boldsymbol{r}^e(t),\ \delta T^e(x,t) \approx \boldsymbol{N}^e(x)\boldsymbol{w}^e(t),\ \mathrm{grad}\delta T^e(x,t) \approx \boldsymbol{B}^e(x)\boldsymbol{w}^e(t)$$

- Introducing approximations of trial solution and weight function into the weak form (for all $\boldsymbol{w}^e(t)$ that $\boldsymbol{w}^e(t) = 0$ on Γ_T), we obtain the following equation:

$$\sum_{e=1}^{n} \boldsymbol{w}^{e\mathrm{T}}(t) \left\{ \overbrace{\int_{\Omega^e} \boldsymbol{B}^{e\mathrm{T}}(x)\lambda^e \boldsymbol{B}^e(x)\mathrm{d}\Omega}^{\boldsymbol{K}^e_\Omega}\, \boldsymbol{r}^e + \overbrace{\int_{\Gamma^e} \boldsymbol{N}^{e\mathrm{T}}(x)\alpha^e \boldsymbol{N}^e(x)\mathrm{d}\Gamma}^{\boldsymbol{K}^e_\Gamma}\, \boldsymbol{r}^e + \right.$$

$$+ \overbrace{\int_{\Omega^e} \boldsymbol{N}^{e\mathrm{T}}(x)\rho C \boldsymbol{N}^e(x)\mathrm{d}\Omega}^{\boldsymbol{C}^e_\Omega} \frac{\mathrm{d}\boldsymbol{r}^e}{\mathrm{d}t} - \overbrace{\int_{\Gamma^e} \boldsymbol{N}^{e\mathrm{T}}(x)\alpha^e \boldsymbol{N}^e(x)\mathrm{d}\Gamma \boldsymbol{T}^e_0}^{\boldsymbol{f}^e_{\Gamma_c}} +$$

$$\left. + \overbrace{\int_{\Gamma^e} \boldsymbol{N}^{e\mathrm{T}}(x)\boldsymbol{N}^e(x)\mathrm{d}\Gamma \overline{\boldsymbol{q}}^e}^{-\boldsymbol{f}^e_{\Gamma_p}} - \overbrace{\int_{\Omega^e} \boldsymbol{N}^{e\mathrm{T}}(x)\boldsymbol{N}^e(x)\mathrm{d}\Omega \overline{\boldsymbol{Q}}^e}^{\boldsymbol{f}^e_\Omega} \right\} = 0$$

Finite Element Method:

- If local vectors $\boldsymbol{r}^e$, $\boldsymbol{w}^e$ are expanded into global vectors of nodal values $\boldsymbol{r}$, $\boldsymbol{w}$, we can write:

$$\boldsymbol{w}^{\mathrm{T}}\left(\sum_{e=1}^{n}\hat{\boldsymbol{K}}^{e}\boldsymbol{r}+\sum_{e=1}^{n}\hat{\boldsymbol{C}}^{e}\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}-\sum_{e=1}^{n}\hat{\boldsymbol{f}}^{e}\right)=0$$

- Finaly, we have:

$$\boldsymbol{K}\boldsymbol{r}+\boldsymbol{C}\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}=\boldsymbol{f}$$

- Note: Two basic approaches of weak form solution:
  - Discretization of the whole "time-space" domain - time-spatial finite elements
  - **Separate spatial discretization and time disretization (time integration)**

Time integration:

$$\boldsymbol{K}\boldsymbol{r} + \boldsymbol{C}\dot{\boldsymbol{r}} = \boldsymbol{f} \tag{1}$$

- Time integration - we consider time intervals, where $\Delta t = t_i - t_{i-1}$
- Known solution in previous time t_{i-1}: $\boldsymbol{r}_{i-1}$
- Linear approximation of trial solution in time $\boldsymbol{r}$:

$$\boldsymbol{r}(t) = \tau\boldsymbol{r}_i + (1-\tau)\boldsymbol{r}_{i-1}, \tag{2}$$

  where $\tau = (t - t_{i-1})/\Delta t$
- The same linear approximation of right-hand side vector $\boldsymbol{f}$:

$$\boldsymbol{f}(t) = \tau\boldsymbol{f}_i + (1-\tau)\boldsymbol{f}_{i-1}, \tag{3}$$

- Time derivative of the vector of nodal temperatures

$$\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \frac{1}{\Delta t}(\boldsymbol{r}_i - \boldsymbol{r}_{i-1}) \tag{4}$$

- Introduction of equations (2), (3), and (4) into the final system of equations (1), we obtain the system of linear algebraic equations:

$$\left[\boldsymbol{K}\tau + \frac{\boldsymbol{C}}{\Delta t}\right]\boldsymbol{r}_i = \boldsymbol{f}_{i-1}(1-\tau) + \boldsymbol{f}_i\tau + \left[\frac{\boldsymbol{C}}{\Delta t} - \boldsymbol{K}(1-\tau)\right]\boldsymbol{r}_{i-1}$$

References

- English course of "Numerical analysis of structures" by J. Zeman (jan.zeman@fsv.cvut.cz)
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- J. Fish and T. Belytschko: A First Course in Finite Elements, John Wiley & Sons, 2007