Finite Element Formulation for Two-Dimensional Heat Conduction Problem

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Outline

- Stationary 2D problem:
- The derivation similar to the 1D problem - extension to 2D (x, y)
- Heat transfer (conduction):
  - function of temperature distribution $T(x, y)$ [K]
  - solution procedure:

  Differential equation + boundary conditions
  → weak form → FEM discretization

- Temperature gradient:

$$\nabla T(x,y) = \mathrm{grad} T(x,y) = \left[\frac{\partial T(x,y)}{\partial x}, \frac{\partial T(x,y)}{\partial y}\right]^{\mathrm{T}}$$

- Heat flux $\boldsymbol{q}_n(x, y)$ is an amount of heat energy, which is transferred through a unit area A [1m^2] with outer normal $\boldsymbol{n}$ per time unit t [s]:

$$\boldsymbol{q}_n(x,y) = \frac{Q(x,y)}{A \cdot t}\boldsymbol{n}(x,y), \qquad [\mathrm{Wm}^{-2}]$$

- Heat flux inside a body ($x, y \in \Omega$) can be decomposed into two directions:

$$\boldsymbol{q} = [q_x(x,y), q_y(x,y)]^{\mathrm{T}}$$

Transport equation:

- Fourier's law: heat flux in an internal body point (material point) $x, y \in \Omega$

$$\boldsymbol{q}(x,y) = -\boldsymbol{\lambda}(x,y)\,\boldsymbol{\nabla} T(x,y),$$

where $\boldsymbol{\lambda}(x,y)$ is the matrix of heat conduction coefficients [$\mathrm{Wm}^{-1}\mathrm{K}^{-1}$]

Balance equation:

- Energy balance in a volume element Ω

$$\boldsymbol{\nabla}^{\mathrm{T}}\left(-\boldsymbol{\lambda}(\boldsymbol{x})\,\boldsymbol{\nabla} T(\boldsymbol{x})\right) = 0,$$

where $\boldsymbol{x} = (x, y)$

Derivation of heat transfer equation:

- Inside a body ($\boldsymbol{x} \in \Omega$):

$$
\begin{aligned}
&q_x(x, y)\Delta y - q_x(x + \Delta x, y)\Delta y && (\rightarrow x)\\
&+q_y(x, y)\Delta x - q_y(x, y + \Delta y)\Delta x && (\uparrow y)\\
&+\overline{Q}\left(x + \frac{\Delta x}{2}, y + \frac{\Delta y}{2}\right)\Delta x \Delta y = 0.
\end{aligned}
$$

- Dividing by $\Delta x \Delta y$, and for the limit limitním přechodem $\Delta x \rightarrow 0$ a $\Delta y \rightarrow 0$, it follows

$$
-\frac{\partial q_x}{\partial x}(x, y) - \frac{\partial q_y}{\partial y}(x, y) + \overline{Q}(x, y) = 0,
$$

- which can be written in matrix form

$$
-\begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix} \begin{bmatrix} q_x(x, y) \\ q_y(x, y) \end{bmatrix} + \overline{Q}(x, y) = 0
$$

and:

$$
-\boldsymbol{\nabla}^{\mathrm{T}} \boldsymbol{q}(\boldsymbol{x}) + \overline{Q}(\boldsymbol{x}) = 0
$$

Derivation of heat transfer equation:

- Introducing of Fourier's law , we obtain a partial differential equation of second order for heat transfer:

$$\boldsymbol{\nabla}^{\mathrm{T}}\left(\boldsymbol{\lambda}(\boldsymbol{x})\boldsymbol{\nabla}T(\boldsymbol{x})\right)+\overline{Q}(\boldsymbol{x})=0$$

$\boldsymbol{\lambda}(\boldsymbol{x})$ is the material conductivity matrix:

$$\boldsymbol{\lambda}(\boldsymbol{x})=\begin{bmatrix}\lambda_{xx}(x,y), & \lambda_{xy}(x,y)\\ \lambda_{yx}(x,y), & \lambda_{yy}(x,y)\end{bmatrix}$$

The matrix $(\lambda_{xy}=\lambda_{yx})$ is symmetric and positive definite. For isotropic material reads:

$$\boldsymbol{\lambda}(\boldsymbol{x})=\begin{bmatrix}\lambda(\boldsymbol{x}), & 0\\ 0, & \lambda(\boldsymbol{x})\end{bmatrix}$$

Boundary conditions:

- Dirichlet b.c. - prescribed temperature on the boundary:

$$T(\boldsymbol{x}) = \overline{T}(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Gamma_T$$

- Neumann b.c - prescribed flux on the boundary:

$$q_{\boldsymbol{n}}(\boldsymbol{x}) = \bar{q}_{\boldsymbol{n}}(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Gamma_{qp},$$

  where

$$q_{\boldsymbol{n}}(\boldsymbol{x}) = \left[\ n_x(\boldsymbol{x}), n_y(\boldsymbol{x})\ \right] \left[\begin{array}{c} q_x(\boldsymbol{x}) \\ q_y(\boldsymbol{x}) \end{array} \right] = \boldsymbol{n}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{q}(\boldsymbol{x})$$

- Cauchy - Newton b.c. - heat transfer on the boundary:

$$q_{\boldsymbol{n}}(\boldsymbol{x}) = \alpha(\boldsymbol{x})\left(T(\boldsymbol{x}) - T_\infty(\boldsymbol{x})\right), \qquad \boldsymbol{x} \in \Gamma_{qc}$$

- Non-linear b.c. (Newton) - heat radiation on the boundary:

$$q_{\boldsymbol{n}}(\boldsymbol{x}) = \varepsilon(\boldsymbol{x})\sigma(\boldsymbol{x})\left(T^4(\boldsymbol{x}) - T^4_\infty(\boldsymbol{x})\right), \qquad \boldsymbol{x} \in \Gamma_{qr}$$

Weighted residual method (Galerkin method):

- We are looking for a solution $T(\boldsymbol{x})$ smooth enough, which fulfills $\boldsymbol{x} \in \Omega$:

$$\boldsymbol{\nabla}^{\mathrm{T}} \left(\boldsymbol{\lambda}(\boldsymbol{x}) \boldsymbol{\nabla} T(\boldsymbol{x})\right) + \overline{Q}(\boldsymbol{x}) = 0$$

- for $\boldsymbol{x} \in \Gamma_T$:

$$T(\boldsymbol{x}) = \overline{T}(\boldsymbol{x})$$

- for $x \in \Gamma_q$:

$$-\boldsymbol{n}^{\mathrm{T}}(\boldsymbol{x}) \boldsymbol{\lambda}(\boldsymbol{x}) \boldsymbol{\nabla} T(\boldsymbol{x}) = \bar{q}_{\boldsymbol{n}}(\boldsymbol{x}),$$

  where:

  - for $\boldsymbol{x} \in \Gamma_{qp}$: $\bar{q}_{\boldsymbol{n}}(\boldsymbol{x})$ is prescribed
  - for $\boldsymbol{x} \in \Gamma_{qc}$: $\bar{q}_{\boldsymbol{n}}(\boldsymbol{x}) = \alpha(\boldsymbol{x}) \left(T(\boldsymbol{x}) - T_\infty(\boldsymbol{x})\right)$
  - for $\boldsymbol{x} \in \Gamma_{qr}$: $\bar{q}_{\boldsymbol{n}}(\boldsymbol{x}) = \varepsilon(\boldsymbol{x}) \sigma(\boldsymbol{x}) \left(T^4(\boldsymbol{x}) - T_\infty^4(\boldsymbol{x})\right)$ (it is avoided in the derivation)

Weighted residual method (Galerkin method):

- For an arbitrary weight function δT so that $\delta T(\boldsymbol{x}) = 0$ for $\boldsymbol{x} \in \Gamma_T$, it holds:

$$\int_\Omega \delta T(\boldsymbol{x}) \left(\boldsymbol{\nabla}^{\mathrm{T}} \left(\boldsymbol{\lambda}(\boldsymbol{x}) \boldsymbol{\nabla} T(\boldsymbol{x}) \right) + \overline{Q}(\boldsymbol{x}) \right) \mathrm{d}\Omega = 0, \qquad \boldsymbol{x} \in \Omega$$

- Integration by parts:

$$\begin{aligned}
\int_\Omega g(\boldsymbol{x}) \frac{\partial f_x}{\partial x}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} &= \int_\Gamma g(\boldsymbol{x}) n_x(\boldsymbol{x}) f_x(\boldsymbol{x}) \mathrm{d}s - \int_\Omega \frac{\partial g}{\partial x}(\boldsymbol{x}) f_x(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \\
\int_\Omega g(\boldsymbol{x}) \frac{\partial f_y}{\partial y}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} &= \int_\Gamma g(\boldsymbol{x}) n_y(\boldsymbol{x}) f_y(\boldsymbol{x}) \mathrm{d}s - \int_\Omega \frac{\partial g}{\partial y}(\boldsymbol{x}) f_y(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \\
\hline
\int_\Omega g(\boldsymbol{x}) \boldsymbol{\nabla}^{\mathrm{T}} \boldsymbol{f}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} &= \int_\Gamma g(\boldsymbol{x}) \boldsymbol{n}^{\mathrm{T}}(\boldsymbol{x}) \boldsymbol{f}(\boldsymbol{x}) \mathrm{d}s - \int_\Omega \left(\boldsymbol{\nabla} g(\boldsymbol{x}) \right)^{\mathrm{T}} \boldsymbol{f}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x},
\end{aligned}$$

where

$$\boldsymbol{\nabla}^{\mathrm{T}} \boldsymbol{f}(\boldsymbol{x}) = \left[\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y} \right] \begin{bmatrix} f_x(\boldsymbol{x}) \\ f_y(\boldsymbol{x}) \end{bmatrix} = \frac{\partial f_x}{\partial x}(\boldsymbol{x}) + \frac{\partial f_y}{\partial y}(\boldsymbol{x})$$

Weighted residual method (Galerkin method):

- We apply the divergence theorem for $\delta T = g$ a $\boldsymbol{f} = \boldsymbol{\lambda}(\boldsymbol{x})\boldsymbol{\nabla}T(\boldsymbol{x})$:

$$\int_\Omega \delta T(\boldsymbol{x}) \left(\boldsymbol{\nabla}^{\mathrm{T}} \left(\boldsymbol{\lambda}(\boldsymbol{x})\boldsymbol{\nabla}T(\boldsymbol{x})\right) + \overline{Q}(\boldsymbol{x})\right) \mathrm{d}\Omega =$$

$$= \int_\Gamma \delta T(\boldsymbol{x})\boldsymbol{n}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{\lambda}(\boldsymbol{x})\boldsymbol{\nabla}T(\boldsymbol{x})\mathrm{d}\Gamma - \int_\Omega \left(\boldsymbol{\nabla}\delta T(\boldsymbol{x})\right)^{\mathrm{T}} \boldsymbol{\lambda}(\boldsymbol{x})\boldsymbol{\nabla}T(\boldsymbol{x})(\boldsymbol{x})\mathrm{d}\Omega + \int_\Omega \delta T(\boldsymbol{x})\overline{Q}(\boldsymbol{x})\mathrm{d}\Omega,$$

where the integral on the boundary can be split into two parts:

$$\int_\Gamma \delta T(\boldsymbol{x})\boldsymbol{n}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{\lambda}(\boldsymbol{x})\boldsymbol{\nabla}T(\boldsymbol{x})\mathrm{d}\Gamma = \int_{\Gamma_T} \overbrace{\delta T(\boldsymbol{x})}^{=0} \boldsymbol{n}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{\lambda}(\boldsymbol{x})\boldsymbol{\nabla}T(\boldsymbol{x})\mathrm{d}\Gamma + \int_{\Gamma_q} \delta T(x) \underbrace{\boldsymbol{n}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{\lambda}(\boldsymbol{x})\boldsymbol{\nabla}T(\boldsymbol{x})}_{=-q\boldsymbol{n}} \mathrm{d}\Gamma$$

It follows:

$$\int_{\Gamma_q} \delta T(\boldsymbol{x}) q_{\boldsymbol{n}}(\boldsymbol{x}) \mathrm{d}\Gamma = \int_{\Gamma_{qp}} \delta T(\boldsymbol{x}) \overline{q}_{\boldsymbol{n}}(\boldsymbol{x}) \mathrm{d}\Gamma + \int_{\Gamma_{qc}} \delta T(\boldsymbol{x}) \alpha(\boldsymbol{x}) \left(T(\boldsymbol{x}) - T_\infty(\boldsymbol{x}) \right) \mathrm{d}\Gamma$$

Weak form:

$$\int_\Omega \boldsymbol{\nabla}^{\mathrm{T}} \delta T(\boldsymbol{x}) \boldsymbol{\lambda}(\boldsymbol{x}) \boldsymbol{\nabla} T(\boldsymbol{x}) \mathrm{d}\Omega + \int_{\Gamma_{qc}} \delta T(\boldsymbol{x}) \alpha(\boldsymbol{x}) T(\boldsymbol{x}) \mathrm{d}\Gamma = \int_{\Gamma_{qp}} \delta T(\boldsymbol{x}) \overline{q}_{\boldsymbol{n}}(\boldsymbol{x}) \mathrm{d}\Gamma + \\ + \int_{\Gamma_{qc}} \delta T(\boldsymbol{x}) \alpha(\boldsymbol{x}) T_\infty(\boldsymbol{x}) \mathrm{d}\Gamma + \int_\Omega \delta T(\boldsymbol{x}) \overline{Q}(\boldsymbol{x}) \mathrm{d}\Omega$$

We are looking for such an admissible trial solution $T(x)$ which satisfies the above formulation.

Finite Element Method:

- The domain Ω is discretized by finite elements and nodes
- The weak form is expressed by an approximation of the trial solution and the weight function in elements. The approximation solution requires C^0 continuity
- The temperature function T is approximated in each element in the following shape:

$$T^e(\boldsymbol{x}) \approx \boldsymbol{N}^e(\boldsymbol{x})\boldsymbol{r}^e, \quad \boldsymbol{\nabla} T^e(\boldsymbol{x}) \approx \boldsymbol{B}^e(\boldsymbol{x})\boldsymbol{r}^e, \quad \delta T^e(\boldsymbol{x}) \approx \boldsymbol{N}^e(\boldsymbol{x})\boldsymbol{w}^e, \quad \boldsymbol{\nabla}\delta T^e(\boldsymbol{x}) \approx \boldsymbol{B}^e(\boldsymbol{x})\boldsymbol{w}^e$$

- Introducing approximations of trial solution and weight function into the weak form (for all $\boldsymbol{w}^e$ that $\boldsymbol{w}^e = 0$ on Γ_T), we obtain the following equation, where the integrals of the weak form are transferred into the sum of integrals in elements:

$$\sum_{e=1}^{n} \boldsymbol{w}^{e\mathrm{T}} \Bigg\{ \overbrace{\int_{\Omega^e} \boldsymbol{B}^{e\mathrm{T}}(x)\lambda^e(x)\boldsymbol{B}^e(x)\mathrm{d}\Omega}^{\boldsymbol{K}^e_\Omega}\, \boldsymbol{r}^e + \overbrace{\int_{\Gamma^e} \boldsymbol{N}^{e\mathrm{T}}(x)\alpha^e(x)\boldsymbol{N}^e(x)\mathrm{d}\Gamma}^{\boldsymbol{K}^e_\Gamma}\, \boldsymbol{r}^e +$$

$$- \overbrace{\int_{\Gamma^e} \boldsymbol{N}^{e\mathrm{T}}(x)\alpha^e(x)\boldsymbol{N}^e(x)\mathrm{d}\Gamma \boldsymbol{T}^e_0}^{\boldsymbol{f}^e_{\Gamma_c}} + \overbrace{\int_{\Gamma^e} \boldsymbol{N}^{e\mathrm{T}}(x)\boldsymbol{N}^e(x)\mathrm{d}\Gamma \overline{\boldsymbol{q}}^e}^{-\boldsymbol{f}^e_{\Gamma_p}} - \overbrace{\int_{\Omega^e} \boldsymbol{N}^{e\mathrm{T}}(x)\boldsymbol{N}^e(x)\mathrm{d}\Omega \overline{\boldsymbol{Q}}^e}^{\boldsymbol{f}^e_\Omega} \Bigg\} = 0$$

Finite Element Method:

- Global quantities - localization, for which is introduced the distribution function for an each element: $\boldsymbol{L}^e$ such that it is valid $\boldsymbol{r}^e = \boldsymbol{L}^e\boldsymbol{r}$:

$$\boldsymbol{w}^{\mathrm{T}}\sum_{e=1}^{n}\left((\boldsymbol{L}^{e\mathrm{T}}\boldsymbol{K}^e_{\Omega}\boldsymbol{L}^e + \boldsymbol{L}^{e\mathrm{T}}\boldsymbol{K}^e_{\Gamma}\boldsymbol{L}^e)\boldsymbol{r} - \boldsymbol{L}^{e\mathrm{T}}\boldsymbol{f}^e_{\Gamma_c} - \boldsymbol{L}^{e\mathrm{T}}\boldsymbol{f}^e_{\Gamma_p} - \boldsymbol{L}^{e\mathrm{T}}\boldsymbol{f}^e_{\Omega}\right) = 0,$$

  We can write:

$$\boldsymbol{w}^{\mathrm{T}}\left(\sum_{e=1}^{n}\hat{\boldsymbol{K}}^e\boldsymbol{r} - \sum_{e=1}^{n}\hat{\boldsymbol{f}}^e\right) = 0$$

- Finaly, we have:

$$\boldsymbol{K}\boldsymbol{r} = \boldsymbol{f}$$

- Decomposition of the conductivity matrix according to Dirichlet b.c. ($\boldsymbol{r}_d$):

$$\begin{bmatrix} \boldsymbol{K}_{TT} & \boldsymbol{K}_{Td} \\ \boldsymbol{K}_{dT} & \boldsymbol{K}_{dd} \end{bmatrix}\begin{bmatrix} \boldsymbol{r}_T \\ \boldsymbol{r}_d \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_T \\ \boldsymbol{f}_d \end{bmatrix} + \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{R} \end{bmatrix},$$

where

$$\boldsymbol{K}_{TT}\boldsymbol{r}_T = \boldsymbol{f}_T - \boldsymbol{K}_{Td}\boldsymbol{r}_d$$

References

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