Finite Element Formulation for 2D Problems - Linear Elasticity

Tomáš Krejčí

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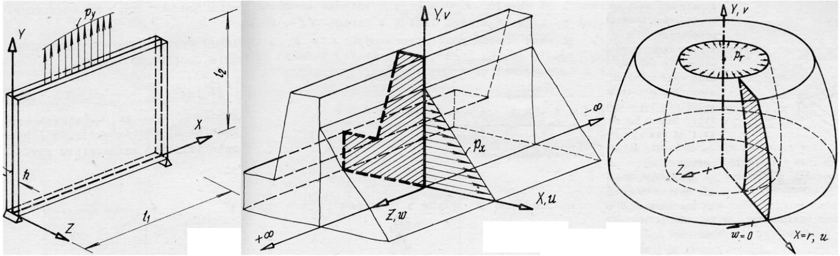
1. Basic equations
   - Kinematic equations
   - Equilibrium equations
   - Constitutive equations
   - Boundary conditions
2. Weak form
3. FEM Discretization

- Basic assumptions for linear elasticity:
  - Deformations are small
  - The behavior of the material is linear
  - Dynamic effects are neglected
  - No gaps or overlaps occur during the deformation of the solid



- Quantities (geometry, material properties, loading) are independent of one coordinate (dimension):
  - Plane stress (historically, the first FEM application [Turner, 1956])
  - Plane strain
  - Axisymmetric problem

- Coordinate vector:

$$\boldsymbol{x} = \{x, y\}^{\mathrm{T}}$$

- Displacements:

$$\boldsymbol{u}(\boldsymbol{x}) = \{u(\boldsymbol{x}), v(\boldsymbol{x})\}^{\mathrm{T}}$$

- Strain vector (independent components):

$$\boldsymbol{\varepsilon}(\boldsymbol{x}) = \{\varepsilon_x(\boldsymbol{x}), \varepsilon_y(\boldsymbol{x}), \gamma_{xy}(\boldsymbol{x})\}^{\mathrm{T}}$$

- Kinematic equations:

$$\left\{ \begin{array}{c} \varepsilon_x(\boldsymbol{x}) \\ \varepsilon_y(\boldsymbol{x}) \\ \gamma_{xy}(\boldsymbol{x}) \end{array} \right\} = \left[\begin{array}{cc} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{array} \right] \left\{ \begin{array}{c} u(\boldsymbol{x}) \\ v(\boldsymbol{x}) \end{array} \right\},$$

$$\boldsymbol{\varepsilon}(\boldsymbol{x}) = \boldsymbol{\partial}^{\mathrm{T}} \boldsymbol{u}(\boldsymbol{x})$$

- In planestrain, $\varepsilon_z = 0$, and in planestress, $\varepsilon_z \neq 0$ is calculated from constitutive equations

Equilibrium equations

- Stress vector (independent components):

$$\boldsymbol{\sigma}(\boldsymbol{x}) = \{\sigma_x(\boldsymbol{x}), \sigma_y(\boldsymbol{x}), \tau_{xy}(\boldsymbol{x})\}^{\mathrm{T}}$$

- Equlibrium equation (static equations):

$$\begin{bmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix} \left\{ \begin{array}{c} \sigma_x(\boldsymbol{x}) \\ \sigma_y(\boldsymbol{x}) \\ \tau_{xy}(\boldsymbol{x}) \end{array} \right\} + \left\{ \begin{array}{c} \overline{X}(\boldsymbol{x}) \\ \overline{Y}(\boldsymbol{x}) \end{array} \right\} = \left\{ \begin{array}{c} 0 \\ 0 \end{array} \right\}$$

$$\boldsymbol{\partial}\boldsymbol{\sigma}(\boldsymbol{x}) + \overline{\boldsymbol{X}}(\boldsymbol{x}) = \mathbf{0}$$

- In planestress, $\sigma_z = 0$, and in planestrain, $\sigma_z \neq 0$ is calculated from constitutive equations

- Plane stress:

$$\left\{ \begin{array}{c} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{array} \right\} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \left\{ \begin{array}{c} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{array} \right\}$$

$$\varepsilon_z = -\frac{\nu}{E}(\varepsilon_x + \varepsilon_y)$$

- Plane strain:

$$\left\{ \begin{array}{c} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{array} \right\} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \left\{ \begin{array}{c} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{array} \right\}$$

$$\sigma_z = \frac{E}{1+\nu} \left[\frac{\nu}{1-2\nu}(\varepsilon_x + \varepsilon_y) \right]$$

- Kinematic boundary conditions:

$$\boldsymbol{u}(\boldsymbol{x}) - \overline{\boldsymbol{u}}(\boldsymbol{x}) = \boldsymbol{0}, \qquad \boldsymbol{x} \in \Gamma_u$$

- Static boundary conditions (prescribed tractions) $\boldsymbol{x} \in \Gamma_p$:

$$\begin{bmatrix} n_x(\boldsymbol{x}) & 0 & n_y(\boldsymbol{x}) \\ 0 & n_y(\boldsymbol{x}) & n_x(\boldsymbol{x}) \end{bmatrix} \left\{ \begin{array}{c} \sigma_x(\boldsymbol{x}) \\ \sigma_y(\boldsymbol{x}) \\ \tau_{xy}(\boldsymbol{x}) \end{array} \right\} - \left\{ \begin{array}{c} \overline{p}_x(\boldsymbol{x}) \\ \overline{p}_y(\boldsymbol{x}) \end{array} \right\} = \left\{ \begin{array}{c} 0 \\ 0 \end{array} \right\}$$

$$\boldsymbol{n}(\boldsymbol{x})\boldsymbol{\sigma}(\boldsymbol{x}) - \overline{\boldsymbol{p}}(\boldsymbol{x}) = \boldsymbol{0}$$

- The divergence (Clapeyron) theorem:

$$\int_\Omega \boldsymbol{\sigma}^{\mathrm{T}} \boldsymbol{\partial}^{\mathrm{T}} \boldsymbol{u} \mathrm{d}\Omega = \int_\Gamma \boldsymbol{u}^{\mathrm{T}} \boldsymbol{n} \boldsymbol{\sigma} \mathrm{d}\Gamma - \int_\Omega \boldsymbol{u}^{\mathrm{T}} \boldsymbol{\partial} \boldsymbol{\sigma} \mathrm{d}\Omega$$

Weighted residual method (Galerkin method):

- For an arbitrary weight function $\delta\boldsymbol{u}$ so that $\delta\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{0}$ for $\boldsymbol{x} \in \Gamma_u$, it holds:

$$\int_\Omega \delta\boldsymbol{u}(\boldsymbol{x})^{\mathrm{T}} \left(\boldsymbol{\partial\sigma}(\boldsymbol{x}) + \overline{\boldsymbol{X}}\right) \mathrm{d}\Omega = 0$$

Clapeyron theorem:

$$\int_{\Gamma_u} \overbrace{\delta\boldsymbol{u}(\boldsymbol{x})^{\mathrm{T}}}^{=\boldsymbol{0}} \boldsymbol{n}(\boldsymbol{x})\boldsymbol{\sigma}(\boldsymbol{x})\mathrm{d}\Gamma + \int_{\Gamma_p} \delta\boldsymbol{u}(\boldsymbol{x})^{\mathrm{T}} \overbrace{\boldsymbol{n}(\boldsymbol{x})\boldsymbol{\sigma}(\boldsymbol{x})}^{=\overline{\boldsymbol{p}}}\mathrm{d}\Gamma$$

$$- \int_\Omega \left(\boldsymbol{\partial}^{\mathrm{T}}\delta\boldsymbol{u}(\boldsymbol{x})\right)^{\mathrm{T}} \boldsymbol{\sigma}(\boldsymbol{x})\mathrm{d}\Omega + \int_\Omega \delta\boldsymbol{u}(\boldsymbol{x})^{\mathrm{T}}\overline{\boldsymbol{X}}\mathrm{d}\Omega = 0$$

- If the weight function $\delta\boldsymbol{u}$ has the physical meaning of the virtual displacement, the expression $\boldsymbol{\partial}^{\mathrm{T}}\delta\boldsymbol{u}(\boldsymbol{x})$ can be identified as the virtual strain $\delta\boldsymbol{\varepsilon}(\boldsymbol{x})$

$$\begin{aligned} \int_\Omega \delta\boldsymbol{\varepsilon}(\boldsymbol{x})^{\mathrm{T}}\boldsymbol{\sigma}(\boldsymbol{x})\mathrm{d}\Omega &= \int_{\Gamma_p} \delta\boldsymbol{u}(\boldsymbol{x})^{\mathrm{T}}\overline{\boldsymbol{p}}(\boldsymbol{x})\mathrm{d}\Gamma + \int_\Omega \delta\boldsymbol{u}(\boldsymbol{x})^{\mathrm{T}}\overline{\boldsymbol{X}}\mathrm{d}\Omega, \\ \delta W_{\mathrm{int}} &= \delta W_{\mathrm{ext}} \end{aligned}$$

- The weighted residual method can be explained as the generalisation of the virtual displacement principle

Finite Element Method:

- The domain Ω is discretized by finite elements and nodes
- The weak form is expressed by an approximation of the trial solution and the weight function in elements
- Displacement approximation:

$$\boldsymbol{u}^e(\boldsymbol{x}) \approx \boldsymbol{N}^e(\boldsymbol{x})\boldsymbol{d}^e$$

- Approximations of strains and stresses:

$$\begin{aligned}\boldsymbol{\varepsilon}^e(\boldsymbol{x}) &\approx \boldsymbol{B}^e(\boldsymbol{x})\boldsymbol{d}^e,\\ \boldsymbol{\sigma}^e(\boldsymbol{x}) &\approx \boldsymbol{D}^e(\boldsymbol{x})\left(\boldsymbol{B}^e(\boldsymbol{x})\boldsymbol{d}^e\right)\end{aligned}$$

- Weight functions approximation:

$$\begin{aligned}\delta\boldsymbol{u}^e(\boldsymbol{x}) &\approx \boldsymbol{N}^e(\boldsymbol{x})\boldsymbol{w}^e,\\ \delta\boldsymbol{\varepsilon}^e(\boldsymbol{x}) &\approx \boldsymbol{B}^e(\boldsymbol{x})\boldsymbol{w}^e\end{aligned}$$

Finite Element Method:

- Introducing approximations of trial solution and weight function into the weak form (for all $\boldsymbol{w}^e$ that $\boldsymbol{w}^e = 0$ on Γ_T), we obtain the following equation, where the integrals of the weak form are transferred into the sum of integrals in elements:

$$\sum_{e=1}^{n} \boldsymbol{w}^{e\mathrm{T}} \left\{ \overbrace{\int_{\Omega^e} \boldsymbol{B}^{e\mathrm{T}}(x)\boldsymbol{D}^e(\boldsymbol{x})\boldsymbol{B}^e(x)\mathrm{d}\Omega}^{\boldsymbol{K}^e}\,\boldsymbol{d}^e - \overbrace{\int_{\Gamma^e} \boldsymbol{N}^{e\mathrm{T}}(x)\boldsymbol{N}^e(x)\mathrm{d}\Gamma\overline{\boldsymbol{p}}^e}^{\boldsymbol{f}_\Gamma^e} - \overbrace{\int_{\Omega^e} \boldsymbol{N}^{e\mathrm{T}}(x)\boldsymbol{N}^e(x)\mathrm{d}\Omega\overline{\boldsymbol{X}}^e}^{\boldsymbol{f}_\Omega^e} \right\} = 0$$

- Global quantities - localization, for which is introduced the distribution function for an each element: $\boldsymbol{L}^e$ such that it is valid $\boldsymbol{d}^e = \boldsymbol{L}^e\boldsymbol{d}$:

$$\boldsymbol{w}^{\mathrm{T}} \sum_{e=1}^{n} \left(\boldsymbol{L}^{e\mathrm{T}}\boldsymbol{K}^e\boldsymbol{L}^e\boldsymbol{d} - \boldsymbol{L}^{e\mathrm{T}}\boldsymbol{f}^e_{\Gamma} - \boldsymbol{L}^{e\mathrm{T}}\boldsymbol{f}^e_{\Omega}\right) = 0,$$

  We can write:

$$\boldsymbol{w}^{\mathrm{T}} \left(\sum_{e=1}^{n} \hat{\boldsymbol{K}}^e\boldsymbol{d} - \sum_{e=1}^{n} \hat{\boldsymbol{f}}^e\right) = 0$$

- Finaly, we have:

$$\boldsymbol{K}\boldsymbol{d} = \boldsymbol{f}$$

- Decomposition of the stiffness matrix according to constrained and prescribed DOF's ($\boldsymbol{d}_u$):

$$\begin{bmatrix} \boldsymbol{K}_{pp} & \boldsymbol{K}_{pu} \\ \boldsymbol{K}_{up} & \boldsymbol{K}_{uu} \end{bmatrix} \begin{bmatrix} \boldsymbol{d}_p \\ \boldsymbol{d}_u \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_p \\ \boldsymbol{f}_u \end{bmatrix} + \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{R} \end{bmatrix},$$

where

$$\boldsymbol{K}_{pp}\boldsymbol{d}_p = \boldsymbol{f}_p - \boldsymbol{K}_{pu}\boldsymbol{d}_u$$

and residuum vector - vector of nodal heat fluxes on the boundary:

$$\boldsymbol{R} = \boldsymbol{K}_{up}\boldsymbol{d}_p + \boldsymbol{K}_{uu}\boldsymbol{d}_u - \boldsymbol{f}_u$$

References

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- J. Fish and T. Belytschko: A First Course in Finite Elements, John Wiley & Sons, 2007