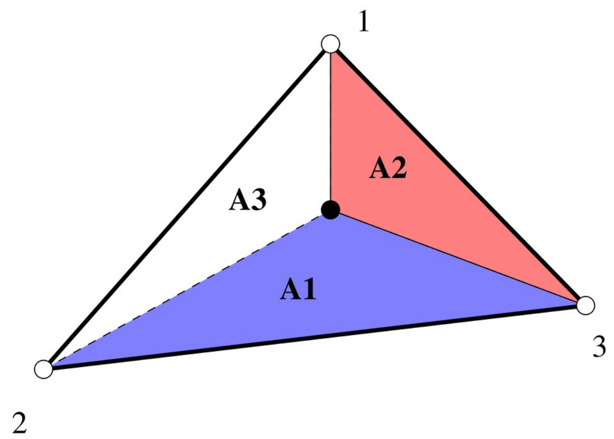
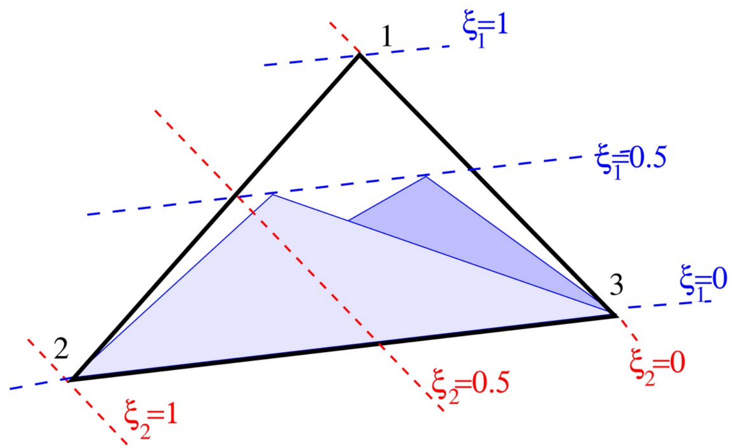
Isoparametric approximation for triangular elements

Triangular coordinates of a given point P are defined as the ratio:

$$\xi_i = \frac{A_i}{A}$$

where A_i is the area of the triangle connecting nodes *j*, *k* and point P , and *A* is the triangle area (1,2,3).





Equations ξ_i = *const* represents the set of lines parallel with the opposite edge of the *i*-th node. Equations corresponds to edges 2-3, 3-1, and 1-2 are $\xi_1 = 0$, $\xi_2 = 0$, and $\xi_3 = 0$. The three nodes have coordinates (1,0,0), (0,1,0), and (0,0,1). Points in centers of edges have coordinates (1/2,1/2,0), (0,1/2,1/2), and (1/2,0,1/2), then the center of gravity (1/3,1/3,1/3).

Triangular coordinates are dependent, and their sum must be equal to 1:

$$\xi_1 + \xi_2 + \xi_3 = 1$$

Kronecker delta property:

$\xi_i(x_j, y_j) = \delta_{ij}$

The dependence of triangular and real coordinates is linear.

- For linear approximation of a trial function, we can write:

$$\phi^e = \sum_{i=1}^{3} \xi_i \phi_i^e = \xi_1 \phi_1^e + \xi_2 \phi_2^e + \xi_3 \phi_3^e$$

Relation between real and triangular coordinates:

$$x = \sum_i x_i \xi_i; \qquad y = \sum_i y_i \xi_i$$

Transformation of coordinates

Quantities, which are connected with the geometry, are expressed in triangular coordinates. Quantities – displacements, strains, and stresses are functions of Kartesian coordinates (*x*, *y*). Therefore, we need the transformation between both (natural and Kartesian) coordinate systems. Kartesian coordinates are connected with the triangular coordinates by:

$$\left\{ \begin{array}{c} 1 \\ x \\ y \end{array} \right\} = \begin{bmatrix} 1 & 1 & 1 \\ x_1^e & x_2^e & x_3^e \\ y_1^e & y_2^e & y_3^e \end{bmatrix} \left\{ \begin{array}{c} \xi_1 \\ \xi_2 \\ \xi_3 \end{array} \right\}$$

The first equation expresses the sum of all coordinates (it is equal to 1). The second and the third express coordinates x and y as a linear combination of ξ_i. Inverting, we obtain:

$$\left\{ \begin{array}{c} \xi_1 \\ \xi_2 \\ \xi_3 \end{array} \right\} = \frac{1}{2A} \begin{bmatrix} x_2^e y_3^e - x_3^e y_2^e & y_2^e - y_3^e & x_3^e - x_2^e \\ x_3^e y_1^e - x_1^e y_3^e & y_3^e - y_1^e & x_1^e - x_3^e \\ x_1^e y_2^e - x_2^e y_1^e & y_1^e - y_2^e & x_2^e - x_1^e \end{bmatrix} \left\{ \begin{array}{c} 1 \\ x \\ y \end{array} \right\}$$

For partial derivatives, it applies:

$$\frac{\partial x}{\partial \xi_i} = x_i \qquad \frac{\partial y}{\partial \xi_i} = y_i$$
$$2A\frac{\partial \xi_i}{\partial x} = y_{jk} \quad 2A\frac{\partial \xi_i}{\partial y} = x_{kj}$$

where $x_{ij} = x_i^e - x_j^e,\ y_{ij} = y_i^e - y_j^e$ and indices in the last row are connected by a cyclic permutation, e. g., for: $i = 2$ is $j = 3$ and $k = 1$.

The derivative of a given function $f(\xi_1, \xi_2, \xi_3)$ with respect to coordinates x, y comes out from the derivative of a composite function

$$\frac{\partial f}{\partial x} = \frac{1}{2A}\left(\frac{\partial f}{\partial \xi_1}y_{23} + \frac{\partial f}{\partial \xi_2}y_{31} + \frac{\partial f}{\partial \xi_3}y_{12}\right)$$
$$\frac{\partial f}{\partial y} = \frac{1}{2A}\left(\frac{\partial f}{\partial \xi_1}x_{32} + \frac{\partial f}{\partial \xi_2}x_{13} + \frac{\partial f}{\partial \xi_3}x_{21}\right)$$

Matrix form:

$$\left\{ \begin{array}{c} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{array} \right\} = \frac{1}{2A}\left[\begin{array}{ccc} y_{23} & y_{31} & y_{12} \\ x_{32} & x_{13} & x_{21} \end{array} \right] \left\{ \frac{\partial f}{\partial \xi_1}, \frac{\partial f}{\partial \xi_2}, \frac{\partial f}{\partial \xi_3} \right\}^T$$

Stiffness matrix of linear triangular element

- Interpolation functions are equal to triangular coordinates $N_i = \xi_i$
- Displacements approximation

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$\boldsymbol{u}^e = \boldsymbol{N}^e \boldsymbol{r}^e$$

- Kinematic matrix $\boldsymbol{B}^e$ calculation:

$$\frac{\partial N_i}{\partial x} = \frac{\partial N_i}{\partial \xi_1}\frac{\partial \xi_1}{\partial x} + \frac{\partial N_i}{\partial \xi_2}\frac{\partial \xi_2}{\partial x} + \frac{\partial N_i}{\partial \xi_3}\frac{\partial \xi_3}{\partial x} = \frac{y_{jk}}{2A}$$

$$\frac{\partial N_i}{\partial y} = \frac{\partial N_i}{\partial \xi_1}\frac{\partial \xi_1}{\partial y} + \frac{\partial \xi_i}{\partial L_2}\frac{\partial \xi_2}{\partial y} + \frac{\partial \xi_i}{\partial L_3}\frac{\partial \xi_3}{\partial y} = \frac{x_{kj}}{2A}$$

- $$\boldsymbol{B}^e(\boldsymbol{x}) = \partial^{\mathsf{T}} \boldsymbol{N}^e(\boldsymbol{x})$$

$$\boldsymbol{B}^e = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} \end{bmatrix}$$

$$= \frac{1}{2A} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

- $\boldsymbol{B}^e$ is constant in the element.
- The stiffness matrix $\boldsymbol{K}^e$ is constant (matrix $\boldsymbol{D}^e$ is also constant)

$$(\boldsymbol{K}^e)_{6\times 6} = \int_{\Omega^e} \boldsymbol{B}^{eT} \boldsymbol{D}^e \boldsymbol{B}^e d\Omega = \boldsymbol{B}^{eT} \boldsymbol{D}^e \boldsymbol{B}^e \int_{\Omega^e} d\Omega = A \boldsymbol{B}^{eT} \boldsymbol{D}^e \boldsymbol{B}^e$$