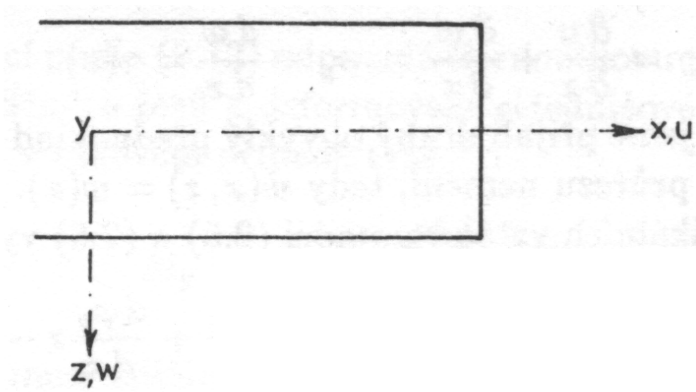
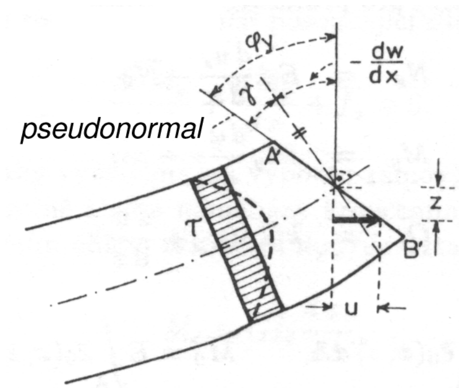
1 Bending of beams – Mindlin theory





Cross-section kinematics assumptions

- Distributed load acts in the xz plane, which is also a plane of symmetry of a body $\Omega \Rightarrow v(\underline{x}) = 0$ m
- Vertical displacement does not vary along the height of the beam (when compared to the value of the displacement) $\Rightarrow w(\underline{x}) = w(x)$.
- The cross sections remain planar but *not necessarily perpendicular* to the deformed beam axis $\Rightarrow u(\underline{x}) = u(x, z) = \varphi_y(x)z$

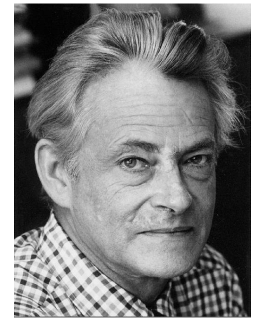
- These hypotheses were independently proposed by Timoshenko [6], Reissner [5] and Mindlin [4].

J. Bernoulli

J.-L. Lagrange

C.-L. Navier

R.-D. Mindlin



B. F. de Veubeke

2 Strain-displacement equations

Cross-section kinematics assumptions imply that only non-zero strain components are

$$
\begin{aligned}
\boxed{\varepsilon_x(\underline{x})} &= \frac{\partial u(\underline{x})}{\partial x} = \frac{\partial}{\partial x}\left(\varphi_y(x) z\right) = \boxed{\frac{\mathrm{d}\varphi_y(x)}{\mathrm{d}x} z = \kappa_y(x) z} \\
\boxed{\gamma_{zx}(\underline{x})} &= \frac{\partial w(\underline{x})}{\partial x} + \frac{\partial u(\underline{x})}{\partial z} = \frac{\mathrm{d}w(x)}{\mathrm{d}x} + \frac{\partial}{\partial z}\left(\varphi_y(x) z\right) = \boxed{\frac{\mathrm{d}w(x)}{\mathrm{d}x} + \varphi_y(x)},
\end{aligned}
$$

when κ_y denotes the *pseudo-curvature* of the deformed beam centerline.

	Bernoulli-Navier [7, kap. II.2]	***Mindlin***
Valid for	$h/L < 1/10$	$h/L < 1/3$
Cross-section	planar, perpendicular	planar
γ_{zx}	0	$\neq 0$ (shear effects)
Unknowns	$w(x)$	$w(x), \varphi_y(x)$
	$\varphi_y(x) = -\dfrac{\mathrm{d}w(x)}{\mathrm{d}x}$	*independent*

3 Stress-strain relations

- For simplicity, we will assume $\underline{\varepsilon}^0 = \underline{0}$

$$\begin{aligned} \sigma_x(x,z) &= E(x)\varepsilon_x(x,z) = E(x)\kappa_y(x)z \\ \tau_{zx}(x) &= G(x)\gamma_{zx}(x) = G(x)\left(\frac{\mathrm{d}w(x)}{\mathrm{d}x} + \varphi_y(x)\right) \end{aligned}$$

- Non-zero internal forces:

$$
\begin{aligned}
\boxed{M_y(x)} &= \int_{A(x)} \sigma_x(x,z) z \, \mathrm{d}y \, \mathrm{d}z = E(x)\kappa_y(x) \int_{A(x)} z^2 \, \mathrm{d}y \, \mathrm{d}z \\
&= E(x) I_y(x) \kappa_y(x) = \boxed{E(x) I_y(x) \frac{\mathrm{d}\varphi_y(x)}{\mathrm{d}x}} \qquad (1) \\
Q_z^c(x) &= \int_{A(x)} \tau_{zx}(x) \, \mathrm{d}y \, \mathrm{d}z = G(x) \left(\frac{\mathrm{d}w(x)}{\mathrm{d}x} + \varphi_y(x) \right) \int_{A(x)} \mathrm{d}y \, \mathrm{d}z \\
&= G(x) A(x) \left(\frac{\mathrm{d}w(x)}{\mathrm{d}x} + \varphi_y(x) \right)
\end{aligned}
$$

- Distribution of shear stresses τ_{zx} for a rectangular cross-section

	Bernoulli-Navier	***Mindlin***
Constitutive eqs: $\tau = G\gamma$	0	constant
Equilibrium eqs	quadratic	?
	[7, kap. II.2.5]	

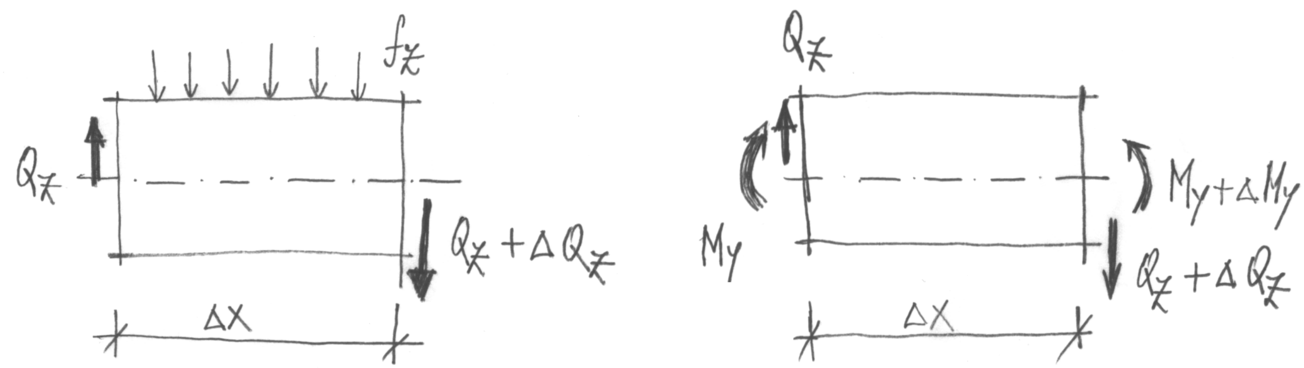
- Therefore, we modify the shear force relation in order to take into account equilibrium equations, at least in the sense of average work of shear components

$$Q_z(x) = k(x)Q_z^c(x) = k(x)G(x)A(x)\left(\frac{\mathrm{d}w(x)}{\mathrm{d}x} + \varphi_y(x)\right) \tag{2}$$

- The multiplier $k(x)$ depends on a shape of a cross-section, for a rectangular cross-section, $k = 5/6$.

Homework 1. Derive the relation for the constant k for a general cross-section: $k = I_y^2/(A\int_A \frac{S_y^2(z)}{b^2(z)}\,\mathrm{d}A)$.

4 Equilibrium equations



(a) (b)

- Equilibrium equation of vertical forces (a)

$$\frac{\mathrm{d}Q_z(x)}{\mathrm{d}x} + \overline{f}_z(x) = 0 \tag{3}$$

- Equilibrium equation of moments (b)

$$\frac{\mathrm{d}M_y(x)}{\mathrm{d}x} - Q_z(x) = 0 \tag{4}$$

- For a detailed derivation see Lecture 1, Homework 1.

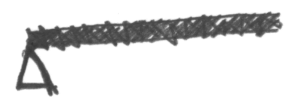
5 Governing equations

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(k(x)G(x)A(x)\left(\frac{\mathrm{d}w(x)}{\mathrm{d}x}+\varphi_y(x)\right)\right)+\overline{f}_z(x) = 0 \quad (5)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(E(x)I_y(x)\frac{\mathrm{d}\varphi_y(x)}{\mathrm{d}x}\right)-k(x)G(x)A(x)\left(\frac{\mathrm{d}w(x)}{\mathrm{d}x}+\varphi_y(x)\right) = 0 \quad (6)$$

5.1 Kinematic boundary conditions: $x \in I_u$

Pinned end: $w = 0$



Clamped end: $w = 0$, $\varphi_y = 0$

5.2 Static boundary conditions: $x \in I_p$

$$Q_z(x) = \overline{Q_z}(x), \qquad M_y(x) = \overline{M_y}(x).$$

6 Weak solution

- For notational simplicity, we will use relations (3)–(4) instead of (5)–(6).
- We will "weight" Eq. (3) by term $\delta w(x)$, Eq. (4) by $\delta\varphi_y(x)$ and integrate them on I. This leads to conditions

$$\begin{aligned} 0 &= \int_I \delta w(x) \left(\frac{\mathrm{d}Q_z(x)}{\mathrm{d}x} + \overline{f}_z(x) \right) \mathrm{d}x, \\ 0 &= \int_I \delta\varphi_y(x) \left(\frac{\mathrm{d}M_y(x)}{\mathrm{d}x} - Q_z(x) \right) \mathrm{d}x, \end{aligned}$$

which are to be satisfied for all $\delta w(x)$ and $\delta\varphi_y(x)$ compatible with the kinematic boundary conditions.

- By parts integration

$$0 = [\delta w(x) Q_z(x)]_a^b - \int_I \frac{\mathrm{d}(\delta w(x))}{\mathrm{d}x} Q_z(x)\,\mathrm{d}x + \int_I \delta w(x) \overline{f}_z(x)\,\mathrm{d}x$$

$$0 = [\delta\varphi_y(x) M_y(x)]_a^b - \int_I \frac{\mathrm{d}(\delta\varphi_y(x))}{\mathrm{d}x} M_y(x)\,\mathrm{d}x - \int_I \delta\varphi_y(x) Q_z(x)\,\mathrm{d}x$$

- Enforcement of boundary conditions

$$0 = \left[\delta w(x) \overline{Q_z}(x)\right]_{I_p} - \int_I \frac{\mathrm{d}(\delta w(x))}{\mathrm{d}x} Q_z(x)\,\mathrm{d}x + \int_I \delta w(x) \overline{f}_z(x)\,\mathrm{d}x$$

$$0 = \left[\delta\varphi_y(x) \overline{M_y}(x)\right]_{I_p} - \int_I \frac{\mathrm{d}(\delta\varphi_y(x))}{\mathrm{d}x} M_y(x)\,\mathrm{d}x - \int_I \delta\varphi_y(x) Q_z(x)\,\mathrm{d}x$$

- The weak of equilibrium equations (we insert (1) for M_y and (2) for Q_z)

$$\int_I \frac{\mathrm{d}(\delta w(x))}{\mathrm{d}x} k(x)G(x)A(x)\left(\frac{\mathrm{d}w(x)}{\mathrm{d}x} + \varphi_y(x)\right)\mathrm{d}x = \left[\delta w(x)\overline{Q_z}(x)\right]_{I_p} + \int_I \delta w(x)\overline{f}_z(x)\,\mathrm{d}x \tag{7}$$

$$\int_I \frac{\mathrm{d}(\delta\varphi_y(x))}{\mathrm{d}x} E(x)I_y(x)\frac{\mathrm{d}\varphi_y(x)}{\mathrm{d}x}\,\mathrm{d}x + \int_I \delta\varphi_y(x)k(x)G(x)A(x)\left(\frac{\mathrm{d}w(x)}{\mathrm{d}x} + \varphi_y(x)\right)\mathrm{d}x = \left[\delta\varphi_y(x)\overline{M_y}(x)\right]_{I_p} \tag{8}$$

7 FEM discretization

- We replace a continuous structure with n nodal points and $(n-1)$ (finite) elements.
- In every nodal point we introduce two *independent* quantities – a deflection w_i and a rotation φ_{y_i} of the i-th nodal point.
- On the level of whole structure, we collect the unknowns into vectors of deflections $\underline{r}_w$ and rotations $\underline{r}_\varphi$.
- Discretization of unknown quantities and their derivatives

$$w(x) \approx \underline{N_w}(x)\underline{r_w}\,, \qquad \frac{\mathrm{d}w(x)}{\mathrm{d}x} \approx \underline{B_w}(x)\underline{r_w}\,,$$

$$\varphi_y(x) \approx \underline{N_\varphi}(x)\underline{r_\varphi}\,, \qquad \frac{\mathrm{d}\varphi_y(x)}{\mathrm{d}x} \approx \underline{B_\varphi}(x)\underline{r_\varphi}\,.$$

- Discretization of weight functions

$$\delta w(x) \approx \underline{N_w}(x)\underline{\delta r_w} \qquad \frac{\mathrm{d}(\delta w(x))}{\mathrm{d}x} \approx \underline{B_w}(x)\underline{\delta r_w}$$

$$\delta \varphi_y(x) \approx \underline{N_\varphi}(x)\underline{\delta r_\varphi} \qquad \frac{\mathrm{d}(\delta \varphi_y(x))}{\mathrm{d}x} \approx \underline{B_\varphi}(x)\underline{\delta r_\varphi}$$

- The linear system of discretized equilibrium equations

$$\begin{aligned} \underline{\underline{K_{ww}}}\,\underline{r_w} + \underline{\underline{K_{w\varphi}}}\,\underline{r_\varphi} &= \underline{R_w} \\ \underline{\underline{K_{\varphi w}}}\,\underline{r_w} + \underline{\underline{K_{\varphi\varphi}}}\,\underline{r_\varphi} &= \underline{R_\varphi} \end{aligned}$$

- Compact notation

$$\begin{bmatrix} \underline{\underline{K_{ww}}} & \underline{\underline{K_{w\varphi}}} \\ \underline{\underline{K_{\varphi w}}} & \underline{\underline{K_{\varphi\varphi}}} \end{bmatrix} \begin{Bmatrix} \underline{r_w} \\ \underline{r_\varphi} \end{Bmatrix} = \begin{Bmatrix} \underline{R_w} \\ \underline{R_\varphi} \end{Bmatrix}$$

$$\underline{\underline{K}}_{(2n\times 2n)}\underline{r}_{(2n\times 1)} = \underline{R}_{(2n\times 1)}$$

- $\underline{\underline{K_{\varphi w}}} = \underline{\underline{K_{w\varphi}}}^{\mathsf{T}}$ $\Rightarrow$ the stiffness matrix $\underline{\underline{K}}$ is symmetric thanks to appearance of the terms $\int_I (\delta w(x))' kGA(x)\varphi_y(x)\,\mathrm{d}x$ in (7) and $\int \delta\varphi_y(x) kGA(x) w'(x)\,\mathrm{d}x$ in (8).

Homework 2. Derive explicit relations for matrices $\underline{\underline{K_{ww}}}, \underline{\underline{K_{w\varphi}}}, \underline{\underline{K_{\varphi w}}}, \underline{\underline{K_{\varphi\varphi}}}$ and vectors $\underline{R_w}, \underline{R_\varphi}$.

8 Shear locking

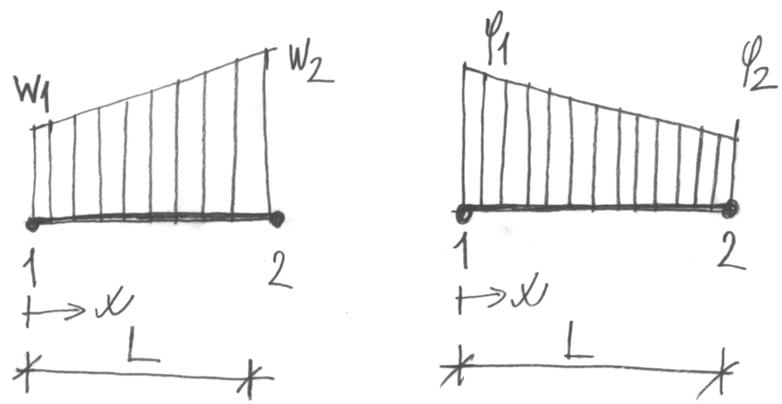
- For $h/L \to 0$, the response of a Mindlin theory-based element should approach the classical slender beam (negligible shear effects).
- If the basis functions $\underline{N_w}$ a $\underline{N_\varphi}$ are chosen as piecewise *linear*, resulting response in too “stiff” $\to$ excessive influence of shear terms, sc. *shear locking*.

8.1 Statics-based analysis

- Shear force: $Q_z(x) = k(x)G(x)A(x)\left(\frac{\mathrm{d}w(x)}{\mathrm{d}x} + \varphi_y(x)\right)$ – linear
- Bending moment: $M_y(x) = E(x)I_y(x)\frac{\mathrm{d}\varphi_y(x)}{\mathrm{d}x}$ – constant
- Severe violation of the Schwedler relation

$$\frac{\mathrm{d}M_y(x)}{\mathrm{d}x} - Q_z(x) = 0$$

8.2 Kinematics-based explanation



- The approximate solution must be able to correctly reproduce the *pure bending* mode, see [3, Section 3.1]):

$$\kappa_y(x) = \frac{\mathrm{d}\varphi_y(x)}{\mathrm{d}x} = \kappa = \text{const} \qquad \gamma_{zx}(x) = \frac{\mathrm{d}w(x)}{\mathrm{d}x} + \varphi_y(x) = 0$$

- For the given discretization

$$w(x) \approx w_1\left(1 - \frac{x}{L}\right) + w_2\frac{x}{L} \qquad \frac{\mathrm{d}w(x)}{\mathrm{d}x} \approx \frac{1}{L}(w_2 - w_1)$$

$$\varphi_y(x) \approx \varphi_1\left(1 - \frac{x}{L}\right) + \varphi_2\frac{x}{L} \qquad \frac{\mathrm{d}\varphi_y(x)}{\mathrm{d}x} \approx \frac{1}{L}(\varphi_2 - \varphi_1)$$

- The requirement of zero shear strain leads to

$$\gamma_{zx}(x) \approx \frac{1}{L}(w_2 - w_1) + \varphi_1 + \frac{x}{L}(\varphi_2 - \varphi_1) = 0.$$

- Therefore, the previous relation must be *independent* of the x coordinate ⇒

$$\varphi_2 - \varphi_1 = 0 \Rightarrow \kappa_y \approx \frac{1}{L}(\varphi_2 - \varphi_1) = 0 \neq \kappa$$

9 Selective integration

- The shear strain is assumed to be constant on a given interval, its value is derived from the value in the center of an interval

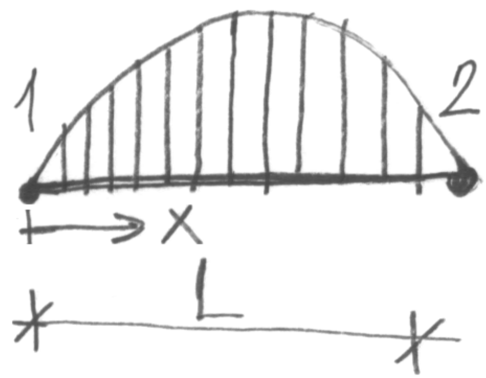
$$\gamma_{zx}(x) \approx \frac{1}{L}(w_2 - w_1) + \varphi_1 + \frac{1}{2}(\varphi_2 - \varphi_1) = \frac{1}{L}(w_2 - w_1) + \frac{1}{2}(\varphi_1 + \varphi_2)$$

- Kinematics: the element behaves correctly, it enables to describe the pure bending mode.

- Statics: $Q_z(x) = k(x)G(x)A(x)\gamma_{xz}(x)$ – constant, M_y – constant $\leftarrow$ the Schwedler condition is not "severely violated".

10 Bubble (hierarchical) function

- It follows from analysis of the kinematics that the shear locking is caused by insufficient degree of polynomial approximation of the displacement $w(x)$.



- Therefore, we add a quadratic term to approximation of $w(x)$:

$$w(x) \approx w_1 \left(1 - \frac{x}{L}\right) + w_2 \frac{x}{L} + \alpha x(x - L)$$

- Pure bending mode requirement

$$\begin{aligned}
\gamma_{zx}(x) &= \frac{\mathrm{d}w(x)}{\mathrm{d}x} + \varphi_y(x) \\
&\approx \frac{1}{L}(w_2 - w_1) + \alpha(2x - L) + \varphi_1 + \frac{x}{L}(\varphi_2 - \varphi_1) \\
&= \frac{1}{L}(w_2 - w_1) - \alpha L + \varphi_1 + \frac{x}{L}(\varphi_2 - \varphi_1 + 2\alpha L) = 0
\end{aligned}$$

- Requirement of independence of coordinate $x \Rightarrow$

$$\alpha = \frac{1}{2L}\left(\varphi_1 - \varphi_2\right)$$

- Final approximations

$$\begin{aligned} w(x) &\approx w_1\left(1 - \frac{x}{L}\right) + w_2\frac{x}{L} + \frac{1}{2L}\left(\varphi_1 - \varphi_2\right)x(x - L) \\ \varphi_y(x) &\approx \varphi_1\left(1 - \frac{x}{L}\right) + \varphi_2\frac{x}{L} \end{aligned}$$

- From the "static" point of view the element behaves similarly to previous formulation – Q_z is constant, M_y is constant.
- Approximation of the w displacement not based not only on the values of deflections nodal, but also on the values of nodal rotations [2] – sc. *linked interpolation.*

11 Method of Lagrange multipliers

- Recall the weak form of the bending moment equilibrium equations (8) for a beam with with $\overline{M_y} = 0$, constant values of E, G and a rectangular cross-section.

$$\begin{aligned} 0 &= EI_y \int_I \frac{\mathrm{d}(\delta\varphi_y(x))}{\mathrm{d}x} \frac{\mathrm{d}\varphi_y(x)}{\mathrm{d}x}\,\mathrm{d}x + kGA \int_I \delta\varphi_y(x) \left(\frac{\mathrm{d}w(x)}{\mathrm{d}x} + \varphi_y(x) \right) \mathrm{d}x \\ &= E\frac{bh^3}{12} \int_I \frac{\mathrm{d}(\delta\varphi_y(x))}{\mathrm{d}x} \frac{\mathrm{d}\varphi_y(x)}{\mathrm{d}x}\,\mathrm{d}x \\ &+ \frac{5}{6}\frac{E}{2(1+\nu)} bh \int_I \delta\varphi_y(x) \left(\frac{\mathrm{d}w(x)}{\mathrm{d}x} + \varphi_y(x) \right) \mathrm{d}x \Bigg/ \frac{12}{Ebh^3} \end{aligned}$$

$$\int_I \frac{\mathrm{d}(\delta\varphi_y(x))}{\mathrm{d}x} \frac{\mathrm{d}\varphi_y(x)}{\mathrm{d}x}\,\mathrm{d}x + \frac{5}{1+\nu} \boxed{\frac{1}{h^2} \int_I \delta\varphi_y(x) \left(\frac{\mathrm{d}w(x)}{\mathrm{d}x} + \varphi_y(x) \right) \mathrm{d}x} = 0$$

- The condition of zero shear strain for $h \to 0$ is imposed via the sc. *penalty term.*

- For slender beams and linear-linear approximation this leads to the shear locking as

$$\overbrace{\frac{1}{h^2}}^{\to\infty} \int_I \overbrace{\delta\varphi_y(x)}^{\text{arbitrary}} \overbrace{\left(\frac{\mathrm{d}w(x)}{\mathrm{d}x} + \varphi_y(x)\right)}^{\to 0 \text{ for all } x \in I} \mathrm{d}x = 0.$$

- If we introduce a *new independent* variable for imposing the condition $\gamma_{xz} = 0$ for $h \to 0$, we suppress influence of the choice of approximation of unknowns $w(x)$ a $\varphi_y(x)$.

- Therefore, we have to add an additional condition to weak equilibrium equations (7)–(8)

$$\int_I \delta\lambda(x) \left(\gamma_{zx}(x) - \frac{\mathrm{d}w(x)}{\mathrm{d}x} - \varphi_y(x) \right) \mathrm{d}x = 0, \tag{9}$$

where $\gamma(x)$ is now a new variable *independent* of w and φ_y and $\delta\lambda(x)$ is the corresponding weight function.

- Constitutive equations for the shear force Q_z now simplify as

$$Q_z(x) = k(x)G(x)A(x)\gamma_{xz}(x)\ .$$

- Weak form of equilibrium of equations can now be rewritten as

$$\begin{aligned} 0 &= \int_I \frac{\mathrm{d}(\delta w(x))}{\mathrm{d}x} k(x)G(x)A(x)\gamma_{zx}(x)\,\mathrm{d}x - \left[\delta w(x)\overline{Q_z}(x)\right]_{I_p} \\ &- \int_I \delta w(x)\overline{f}_z(x)\,\mathrm{d}x \\ 0 &= \int_I \frac{\mathrm{d}(\delta\varphi_y(x))}{\mathrm{d}x} E(x)I_y(x)\frac{\mathrm{d}\varphi_y(x)}{\mathrm{d}x}\,\mathrm{d}x \\ &+ \int_I \delta\varphi_y(x)k(x)G(x)A(x)\gamma_{zx}(x)\,\mathrm{d}x - \left[\delta\varphi_y(x)\overline{M_y}(x)\right]_{I_p} \\ 0 &= \int_I \delta\lambda(x)\left(\gamma_{zx}(x) - \frac{\mathrm{d}w(x)}{\mathrm{d}x} - \varphi_y(x)\right)\mathrm{d}x \end{aligned}$$

- Observe that is we choose the weight function in the specific form

$$\delta\lambda(x) = k(x)G(x)A(x)\delta\gamma_{xz}(x)\ ,$$

we will finally obtain a symmetric stiffness matrix $\underline{\underline{K}}$.

- The last equation now can be modified as

$$0 = \int_I \delta\gamma_{xz}(x)k(x)G(x)A(x)\left(\gamma_{zx}(x) - \frac{\mathrm{d}w(x)}{\mathrm{d}x} - \varphi_y(x)\right)\mathrm{d}x.$$

- The additional variable γ_{xz} needs to be discretized

$$\gamma_{xz}(x) \approx \underline{N_\gamma}(x)\underline{r_\gamma}$$

and inserted into the weak form of equilibrium equations. This yields, after standard manipulations, the following system of linear equations

$$\begin{bmatrix} \underline{\underline{K_{ww}}} & \underline{\underline{K_{w\varphi}}} & \underline{\underline{K_{w\gamma}}} \\ \underline{\underline{K_{\varphi w}}} & \underline{\underline{K_{\varphi\varphi}}} & \underline{\underline{K_{\varphi\gamma}}} \\ \underline{\underline{K_{\gamma w}}} & \underline{\underline{K_{\gamma\varphi}}} & \underline{\underline{K_{\gamma\gamma}}} \end{bmatrix} \begin{Bmatrix} \underline{r_w} \\ \underline{r_\varphi} \\ \underline{r_\gamma} \end{Bmatrix} = \begin{Bmatrix} \underline{R_w} \\ \underline{R_\varphi} \\ \underline{0} \end{Bmatrix}$$

- The stiffness matrix, resulting from this discretization, is larger only virtually. It can be observed that parameters $\underline{r_\gamma}$ only internal and can be eliminated (expressed via variables $\underline{r_w}$ and $\underline{r_\varphi}$); see, e.g. [1, pp. 234–235] for more details.

- This formulation works even for piecewise linear approximation of w and φ_y; it suffices to approximate γ as a piecewise constant on an element.
- Kinematics: shear locking avoided due to (9).
- Statics: the shear force Q_z is again (piecewise) constant, so is the bending moment M_y.

Homework 3*. Derive the element stiffness matrix based on Lagrange multipliers. Assume the linear approximation of deflections $w(x)$, linear approximation of rotations $\varphi_y(x)$ and constant values of γ_{xz} on a given elements. Show that this procedure yields results identical to the reduced integration and linked interpolation.

□

A humble plea. Please feel free to e-mail any suggestions, errors and typos to `zemanj@cml.fsv.cvut.cz`.

Version 000

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