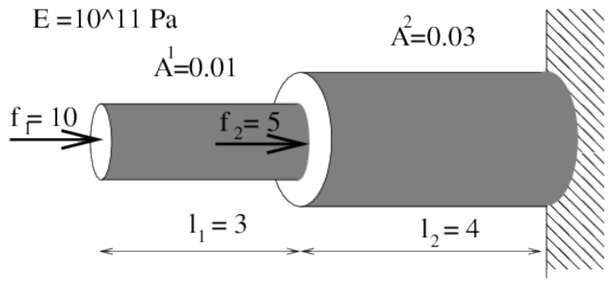
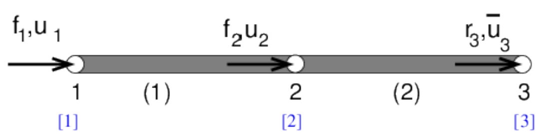
1 Seminar 2 – 1D elastic bar element: Localization, simple benchmarks

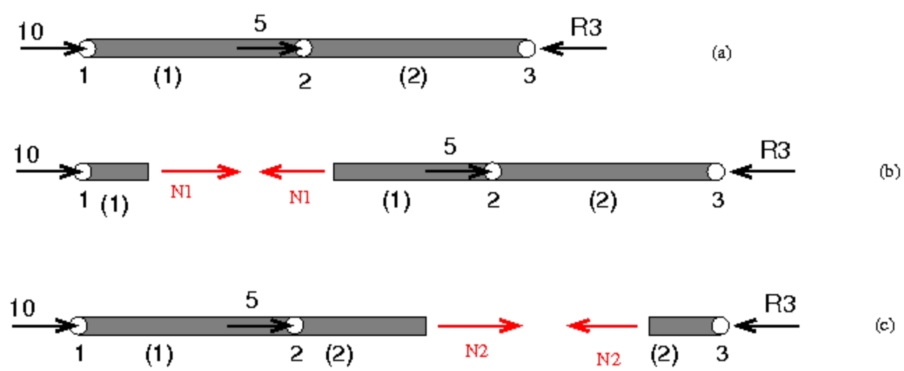
1. 1 Example 1





1.2 "Hand calculation"

The structure is statically determined, it is possible to calculate reactions and internal forces directly



Equilibrium equation for the whole structure (a): $10 + 5 - R_3 = 0 \rightarrow R_3 = 15$
Equilibrium equations for the separated parts (individual elements) from the left side (b) + (c)

- $10 + N_1 = 0 \rightarrow N_1 = -10$
- $10 + 5 + N_2 = 0 \rightarrow N_2 = -15$

Strain calculation:

- $\varepsilon_1 = \frac{\sigma_1}{E} = \frac{N_1}{A_1 E} = -\frac{10}{0.01 * 10^{11}} = -1.10^{-8}$
- $\varepsilon_2 = \frac{\sigma_2}{E} = \frac{N_2}{A_2 E} = -\frac{15}{0.03 * 10^{11}} = -5.10^{-9}$

Displacements are obtained by integration of strains starting from the right side from the known displacement $u_3 = 0$

- $u_2 = (u_3 - \varepsilon_2 * l_2) = (0 + 5.10^{-9} * 4) = 2.10^{-8}$
- $u_1 = (u_2 - \varepsilon_1 * l_1) = (2.10^{-8} + 1.10^{-8} * 3) = 5.10^{-8}$

1.3 Analytical solution of differential equation

Recall the differential equation for bar element (tension-compression)

$EA\,\frac{d^2u}{dx^2} + f_x(x) = 0$

For element 1

(1) $EA_1\,\frac{d^2u_1}{dx^2} = 0,\ x \in (0,\,3)$

For element 2

(2) $EA_2\,\frac{d^2u_2}{dx^2} = 0,\ x \in (3,\,5)$

Subsequent integration of Eq. (1) $EA_1\,\frac{du_1}{dx} + C_1 = 0$* $EA_1\,u_1 + C_1x + C_2 = 0$

Similarly for Eq. (2) $EA_2\,\frac{du_2}{dx} + C_3 = 0$* $EA_2\,u_2 + C_3x + C_4 = 0$

The integration constant must be find for the solution $C_1,\,C_2,\,C_3,\,C_4$ from the boundary conditions:

- (3.1) Static condition on the left side $N_1(0) = EA_1\varepsilon_1 = EA_1\frac{du_1}{dx}(0) = -10 \rightarrow EA_1\,\frac{du_1}{dx}(0) = -10$
- (3.2) Compatibility condition at node 2: $u_1(3) = u_2(3)$
- (3.3) Equilibrium condition at node 2: $-N_1(3)+N_2(3)+5 = 0 \rightarrow -EA_1\frac{du_1}{dx}(3)+EA_2\frac{du_2}{dx}(3)+5 = 0$
- (3.4) Kinematic condition on the right side in $u_2(7) = 0$

Introducing into conditions (3), we obtain the system of linear equations for integration constants

- $-C_1 = -10$
- $-\frac{3C_1+C_2}{EA_1} = -\frac{3C_3+C_4}{EA_2}$
- $C_1 - C_3 + 5 = 0$
- $-\frac{7C_3+C_4}{EA_2} = 0$

```
In [17]: e=1.e11;
         a1=0.01;
         a2=0.03;
         A = [-1, 0,0,0;
              -3/(e*a1),-1/(e*a1), 3/(e*a2), 1/(e*a2);
              1, 0, -1, 0;
              0, 0, -7/(e*a2), -1/(e*a2)];
         b = [-10; 0; -5; 0];
         c=A\b

c =
    10.000
   -50.000
    15.000
  -105.000
```

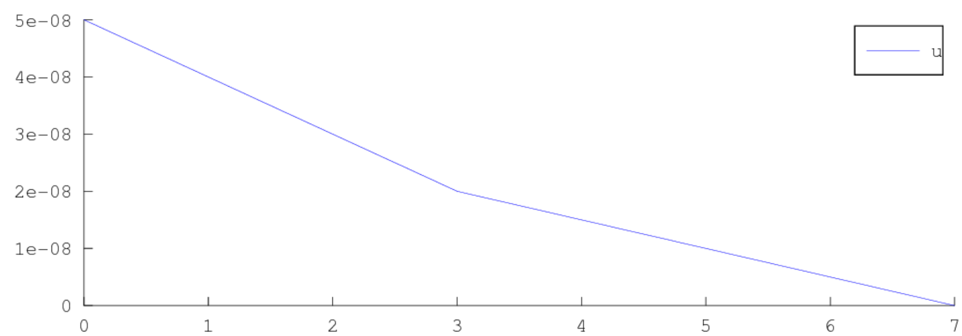
We have: $N_1(x) = EA_1\frac{du_1}{dx}(x) = -C_1 = -10.0$

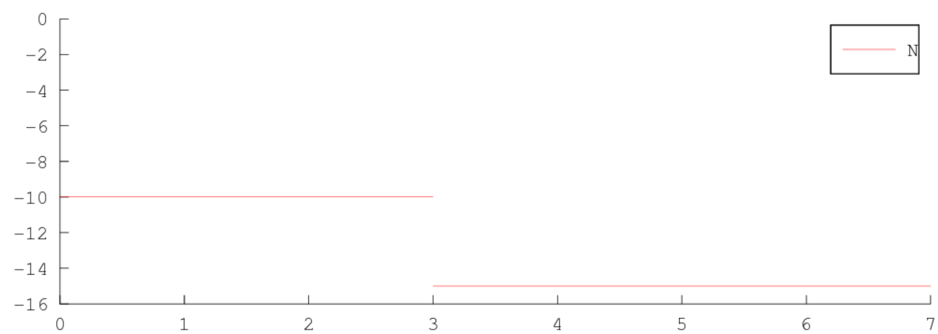
$u_1(x) = -\frac{C_1x+C_2}{EA_1} = \frac{-10x+50}{EA_1}$, then $u_1(0) = -5.10^{-8}$ and $u_1(3) = -2.10^{-8}$

$N_2(x) = EA_2\frac{du_2}{dx}(x) = -C_3 = -15.0$

$u_2(x) = -\frac{C_3x+C_4}{EA_2} = \frac{-15x+105}{EA_2}$, then $u_2(3) = -2.10^{-8}$ and $u_1(7) = 0$

```
In [36]: x1 = 0:0.1:3;
         x2 = 3:0.1:7;

         subplot(211)
         hold on;
         plot (x1, (-10*x1+50)/(e*a1), "b;u;")
         plot (x2, (-15*x2+105)/(e*a2), "b")
         subplot(212)
         hold on;
         ylim([-16 0])
         plot (x1, -10+0*x1, "r;N;")
         plot (x2, -15+0*x2, "r")
```





1.4 Direct approach

```
In [1]: e=1.e11;
        a1=0.01;
        a2=0.03;
        l1=3;
        l2=4;
```

```
k1 = (e*a1/l1)*[1 -1; -1 1]
k2 = (e*a2/l2)*[1 -1; -1 1]
```

```
k1 =

   3.3333e+08  -3.3333e+08
  -3.3333e+08   3.3333e+08

k2 =

   750000000  -750000000
  -750000000   750000000
```

1.5 Equilibrium equation assembly - Localization

Nodal forces are expressed from the nodal displacements on each element

$$\left\{ \begin{array}{c} F_1^i \\ F_2^i \end{array} \right\} = \frac{EA^i}{l^i} \left[\begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right] \left\{ \begin{array}{c} u_1^i \\ u_2^i \end{array} \right\}$$

- Equilibrium equation at node 1: $-F_1^1 + f_1 = 0$

Nodal force: $F_1^1 = \frac{EA^1}{l^1}(u_1 - u_2)$

We can write:

$$\frac{EA^1}{l^1}(u_1 - u_2) = f_1$$

- Equilibrium at node 2: $-F_2^1 - F_1^2 + f_2 = 0$

$$\frac{EA^1}{l^1}(u_2 - u_1) + \frac{EA^2}{l^2}(u_2 - u_3) = f_2$$

```
In [2]: loc1 = [1 2];
        loc2 = [2 3];

        K = zeros(3);
        K(loc1, loc1) += k1
        K(loc2, loc2) += k2
```

```
K =

   3.3333e+08  -3.3333e+08   0.0000e+00
  -3.3333e+08   3.3333e+08   0.0000e+00
   0.0000e+00   0.0000e+00   0.0000e+00
K =

   3.3333e+08  -3.3333e+08   0.0000e+00
  -3.3333e+08   1.0833e+09  -7.5000e+08
   0.0000e+00  -7.5000e+08   7.5000e+08
```

1.6 Solution

```
In [3]: f = [10; 5]
        u = K(1:2, 1:2)\f
        # displacement vector for the who structure
        U = [u(1), u(2), 0]
```

```
f =

   10
    5

u =

   5.0000e-08
   2.0000e-08

U =

   5.0000e-08   2.0000e-08   0.0000e+00
```

1.7 Calculation of strains and internal forces

Strains are computed from the nodal displacements for each element $\varepsilon^i = (u_2^i - u_1^i)/l_i$ and then axial forces $N^\imath = A^\imath \sigma^\imath = A^\imath E^\imath \varepsilon^\imath$

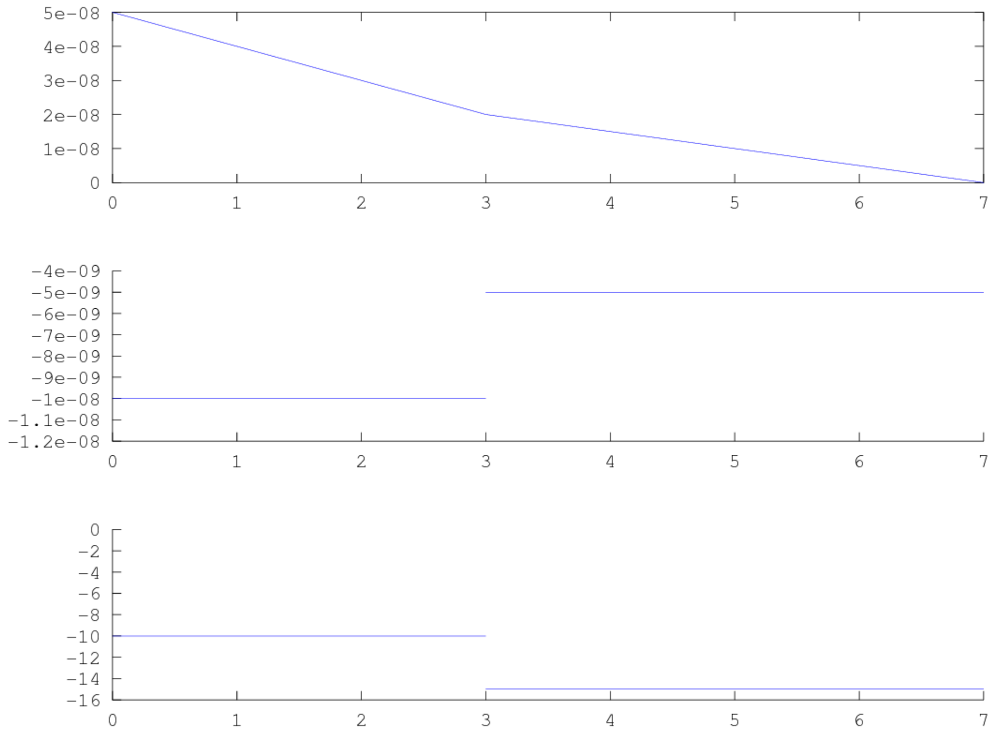
```
In [4]: eps1=(U(2)-U(1))/l1
        eps2=(U(3)-U(2))/l2

        N1 = e*a1*eps1
        N2 = e*a2*eps2
```

```
eps1 =   -1.0000e-08
eps2 =   -5.0000e-09
N1 = -10.000
N2 = -15
```

1.8 Plotting

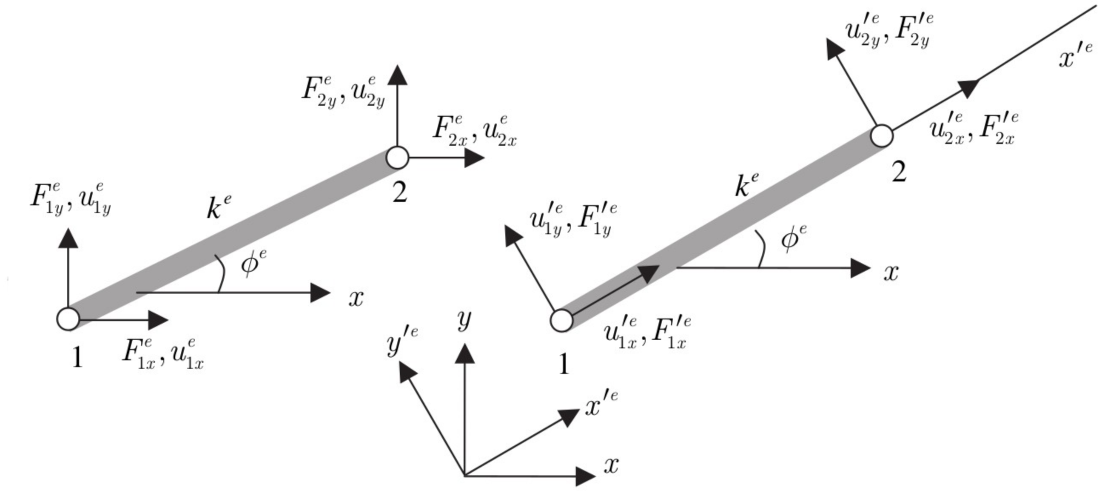
```
In [5]: subplot(311)
        plot ([0 l1 l1+l2], U)
        #title("Displacements")
```

```
subplot(312)
hold on
plot ([0 l1], [eps1 eps1])
plot ([l1 l1+l2], [eps2 eps2])
#title("strains")
subplot(313)
hold on
ylim ([-16 0])
plot ([0 l1], [N1 N1])
plot ([l1 l1+l2], [N2 N2])
#title("axial forces")
```




```
In [ ]:
```

2 2D trusses - transformation



First, we augment relation of nodal forces vector by transversal forces

$$F_{1y}^{le} = F_{2y}^{le} = 0:$$

$$\left\{ \begin{array}{c} F_{1x}^{le} \\ F_{2x}^{le} \end{array} \right\} = \left[\begin{array}{cc} k^e & -k^e \\ -k^e & k^e \end{array} \right] \left\{ \begin{array}{c} u_{1x}^{le} \\ u_{2x}^{le} \end{array} \right\} \quad \left\{ \begin{array}{c} F_{1x}^{le} \\ F_{1y}^{le} \\ F_{2x}^{le} \\ F_{2y}^{le} \end{array} \right\} = \left[\begin{array}{cccc} k^e & 0 & -k^e & 0 \\ 0 & 0 & 0 & 0 \\ -k^e & 0 & k^e & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \left\{ \begin{array}{c} u_{1x}^{le} \\ u_{1y}^{le} \\ u_{2x}^{le} \\ u_{2y}^{le} \end{array} \right\}$$

Transformation of nodal displacements from local to global coordinate system system gives:

$$\begin{aligned} u_{1x}^{le} &= u_{1x}^{ge}\cos(\phi^e) + u_{1y}^{ge}\sin(\phi^e) \\ u_{1y}^{le} &= -u_{1x}^{ge}\sin(\phi^e) + u_{1y}^{ge}\cos(\phi^e) \end{aligned}$$

Matrix form of nodal displacement and nodal forces:

$$\boldsymbol{r}^{le} = \boldsymbol{T}^e \boldsymbol{r}^{ge}; \ \boldsymbol{F}^{le} = \boldsymbol{T}^e \boldsymbol{F}^{ge}$$

$$\boldsymbol{r}^{ge} = \{u_{1x}^{ge}, u_{1y}^{ge}, u_{2x}^{ge}, u_{2y}^{ge}\}^T; \quad \boldsymbol{T}^e = \left[\begin{array}{cccc} \cos(\phi) & \sin(\phi) & 0 & 0 \\ -\sin(\phi) & \cos(\phi) & 0 & 0 \\ 0 & 0 & \cos(\phi) & \sin(\phi) \\ 0 & 0 & -\sin(\phi) & \cos(\phi) \end{array} \right]$$

Note: Transformation matrix is orthogonal, so the inverse matrix is transposed:

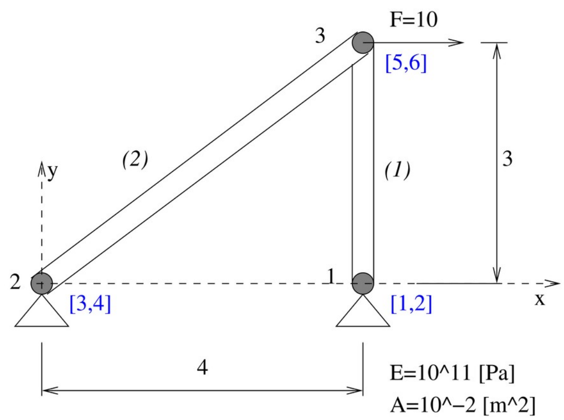
$$(\boldsymbol{T}^e)^T\boldsymbol{T}^e = \boldsymbol{T}^e(\boldsymbol{T}^e)^T = \boldsymbol{I}; \quad (\boldsymbol{T}^e)^T = (\boldsymbol{T}^e)^{-1}$$

$$\begin{aligned}\boldsymbol{F}^{ge} &= (\boldsymbol{T}^e)^T\boldsymbol{F}^{le} \\ &= (\boldsymbol{T}^e)^T\boldsymbol{K}^{le}\boldsymbol{r}^{le} \\ &= \underbrace{(\boldsymbol{T}^e)^T\boldsymbol{K}^{le}\boldsymbol{T}^e}_{\boldsymbol{K}^{ge}}\boldsymbol{r}^{ge}\end{aligned}$$

Stiffness matrix of one bar element in global coordinate system:

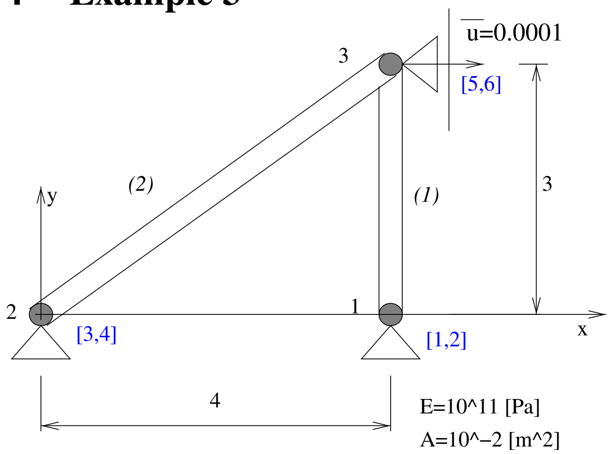
$$\boldsymbol{K}^{ge} = \frac{EA}{l}\begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}; \quad \begin{array}{c} c = \cos(\phi^e) \\ s = \sin(\phi^e) \end{array}$$

3 Example 2



```
ke1 =
   1.0e+08  *

          0          0          0          0
          0     3.3333          0    -3.3333
          0          0          0          0
          0    -3.3333          0     3.3333


ke2 =
   1.0e+08  *

     1.2800     0.9600    -1.2800    -0.9600
     0.9600     0.7200    -0.9600    -0.7200
    -1.2800    -0.9600     1.2800     0.9600
    -0.9600    -0.7200     0.9600     0.7200


k =
   1.0e+08  *

          0          0          0          0          0          0
          0     3.3333          0          0          0    -3.3333
          0          0     1.2800     0.9600    -1.2800    -0.9600
          0          0     0.9600     0.7200    -0.9600    -0.7200
          0          0    -1.2800    -0.9600     1.2800     0.9600
          0    -3.3333    -0.9600    -0.7200     0.9600     4.0533


r  = 1.0e-07  * [0; 0; 0; 0; 0.9500; -0.2250]
fr = [0; 7.5000; -10.0000; -7.5000]


Element 1   eps=   -0.000000, s=    -7.500000
Element 2   eps=    0.000000, s=    12.500000
```

4 Example 3




```
r =

   0.0000e+00
   0.0000e+00
   0.0000e+00
   0.0000e+00
   1.0000e-04
  -2.3684e-05

fr =

   0.0000e+00
   7.8947e+03
  -1.0526e+04
  -7.8947e+03
   1.0526e+04

Element 1    eps=    -0.000008, s=-7894.736842
Element 2    eps=     0.000013, s=13157.894737
```

Reference: Czech course of “Numerická analýza konstrukcí” (Numerical analysis of structures) by B.
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