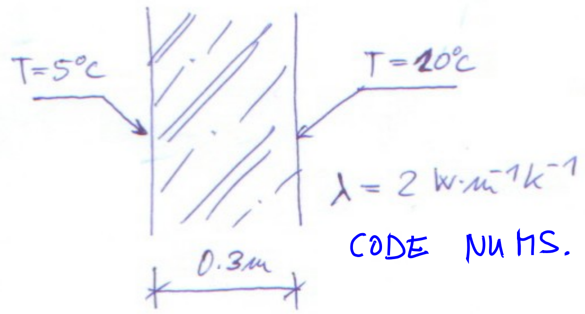
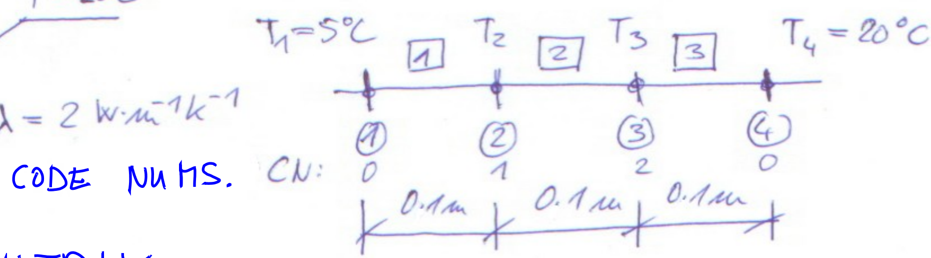
1/ BENCHMARK No. 1 – STEADY STATE

DETERMINE THE TEMPERATURE DISTRIBUTION IN CONCRETE WALL BY FEM.



DISCRETIZATION:



CONDUCTIVITY MATRIX OF 1D ELEMENT:

$$\underline{N}^e = \left[\frac{x_2 - x}{\ell} ; \frac{x - x_1}{\ell} \right]$$

$$\underline{k}^e = \int_\Omega \underline{B}^{eT} \lambda \, \underline{B}^e \, d\Omega$$

$$\frac{d\underline{N}^e}{dx} = \underline{B}^e = \left[\frac{-1}{\ell} ; \frac{1}{\ell} \right]$$

AFTER INTEGRATION

$$\underline{k}^e = \frac{\lambda}{\ell} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\underline{k}^{1,2,3} = \frac{2}{0{,}1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{array}{c|c} \begin{matrix} 2 & 0 \\ 1 & 2 \\ 0 & 1 \end{matrix} & \\ \begin{bmatrix} 20 & -20 \\ -20 & 20 \end{bmatrix} & \begin{matrix} 0\,1\,2 \\ 1\,2\,0 \end{matrix} \end{array}$$

CONDUCTIVITY MATRIX OF THE STRUCTURE:

$$\underline{K} = \begin{array}{cc} \begin{matrix} 1 & \quad 2 \end{matrix} & \\ \begin{bmatrix} 20+20 & -20 \\ -20 & 20+20 \end{bmatrix} & \begin{matrix} 1 \\ 2 \end{matrix} \end{array} = \begin{bmatrix} 40 & -20 \\ -20 & 40 \end{bmatrix}$$

SYSTEM OF EQUATION

$$\underline{k} \cdot \underline{r} = \underline{f}$$

$$\begin{bmatrix} 40 & -20 \\ -20 & 40 \end{bmatrix} \begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad ?!!

CONTRIBUTION FROM DIRICHLET B.C. INTO THE RHS VECTOR (SEE LECTURE No. 5)

$$\underline{K}_{TT}\,\underline{r}_T = \underline{f} - \underline{K}_{Td}\,\underline{r}_d$$

ELEMENT No. 1

$$\underline{K}^1_{Td}\,\underline{r}^1_d = \begin{array}{cc} 0 & 1 \end{array} \qquad \begin{array}{c} 0 \end{array} \qquad \begin{array}{c} 0 \end{array}$$

$$\underline{K}^1_{Td}\,\underline{r}^1_d = \begin{array}{c} \begin{bmatrix} 20 & -20 \\ -20 & 20 \end{bmatrix} \end{array} \begin{array}{c} 0 \\ 1 \end{array} \begin{Bmatrix} 5 \\ 0 \end{Bmatrix} \begin{array}{c} 0 \\ 1 \end{array} = \begin{Bmatrix} 100 \\ -100 \end{Bmatrix} \begin{array}{c} 0 \\ 1 \end{array}$$

ELEMENT No. 2

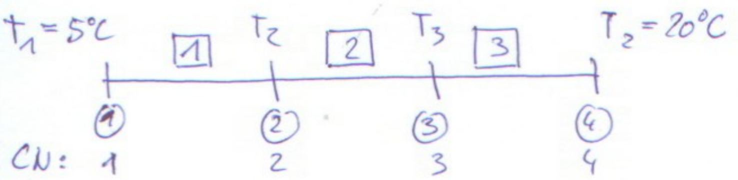
$$\underline{K}^3_{Td}\,\underline{r}^3_d = \begin{array}{cc} 2 & 0 \end{array} \qquad \begin{array}{c} 0 \end{array} \qquad \begin{array}{c} 0 \end{array}$$

$$\underline{K}^3_{Td}\,\underline{r}^3_d = \begin{bmatrix} 20 & -20 \\ -20 & 20 \end{bmatrix} \begin{array}{c} 2 \\ 0 \end{array} \begin{Bmatrix} 0 \\ 20 \end{Bmatrix} \begin{array}{c} 2 \\ 0 \end{array} = \begin{Bmatrix} -400 \\ 400 \end{Bmatrix} \begin{array}{c} 2 \\ 0 \end{array}$$

RESULTING SYSTEM OF EQS. :

$$\begin{bmatrix} 40 & -20 \\ -20 & 40 \end{bmatrix} \begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} -100 \\ -400 \end{Bmatrix}$$

CODE NUMBERS FOR ALL NODAL VALUES :



$$K^1 = \begin{bmatrix} 20 & -20 \\ -20 & 20 \end{bmatrix}$$

$$\underline{K} = \begin{bmatrix} 20 & -20 & & \\ -20 & 40 & -20 & \\ & -20 & 40 & -20 \\ & & -20 & 20 \end{bmatrix} \underbrace{\begin{Bmatrix} 5 \\ T_2 \\ T_3 \\ 20 \end{Bmatrix}}_{\underline{r}} = \begin{Bmatrix} R_1 \\ 0 \\ 0 \\ R_4 \end{Bmatrix}$$

RESIDUA = NODAL FLUXES ON BOUNDARY

3/

2^{ND} ROW:

$$-20\cdot 5 + 40\cdot T_2 - 20\cdot T_3 = 0$$

3^{RD} ROW:

$$-20\cdot T_2 + 40 T_3 - 20\cdot 20 = 0$$

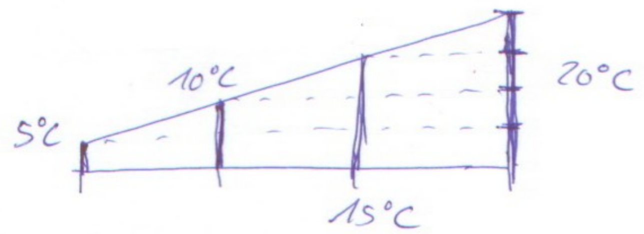
SOLUTION $T_2 = 10$

$T_3 = 15$

SOLUTINON

$$r = \begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 10 \\ 15 \end{Bmatrix}$$

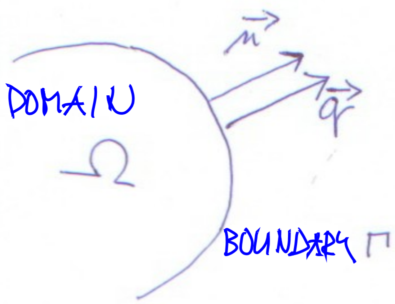
DISTRIBUTION:



DISCUSSION ABOUT THE HEAT FLUX ON THE BOUNDARY

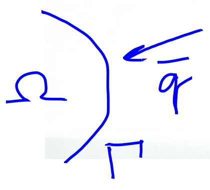
- POSITIVE HEAT FL. $q\,[W\cdot m^{-2}]$ IS DEFINED IN THE DIRECTION OF OUTER NORMAL VECTOR

$$q = -\lambda \frac{\partial T}{\partial x} \vec{n}$$



- FROM FEM, INTEGRAL $f_{\bar{q}} = -\int_{\Gamma_{qn}} \underline{N}^{e^T} \bar{q}\, d\Gamma$

- IF WE ASSUME THE POSITIVE FLUX HEATING A BODY

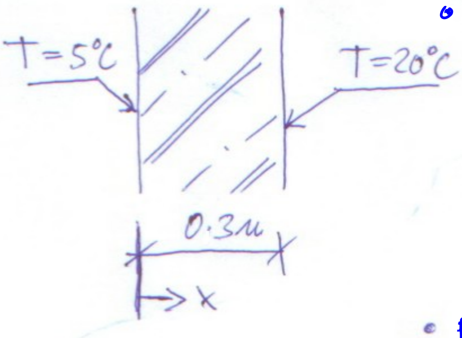


$$f_{\bar{q}} = + \int \underline{N}^{e^T} \bar{q}\, d\Gamma$$

THE SIGN MUST BE +

4/

ANALYTICAL SOLUTION



- WE ARE LOOKING FOR SMOOTH ENOUGH SOLUTION SATISFYING :
  - FOR $x \in \Omega$:

$$\frac{d}{dx}\left(\lambda(x)\frac{dT(x)}{dx}\right) = 0$$

- FOR $x \in \Gamma_T$: $T(x) = \bar{T}(x)$ Dirichlet.

STRONG FORM :

$$\lambda \frac{d^2T(x)}{dx^2} = 0$$

INTEGRATION :

1ST: $\lambda \frac{dT(x)}{dx} = C_1$

2ND: $\lambda T(x) = C_1 x + C_2$

BOUNDARY CONDITIONS :

$T(x=0) = 5°C \Rightarrow C_2 = +10$

$T(x=0{,}3) = 20°C \Rightarrow C_1 = 100$

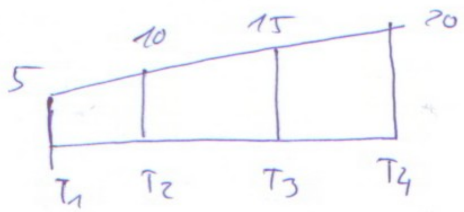
THE SOLUTION :

$$T(x) = \frac{100}{2}x + \frac{10}{2} = 50x + 50$$

VALUES :

$T(x=0{,}1) = T_2 = 10°C$

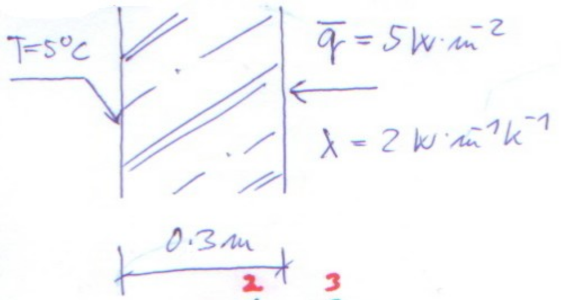
$T(x=0{,}2) = T_3 = 15°C$

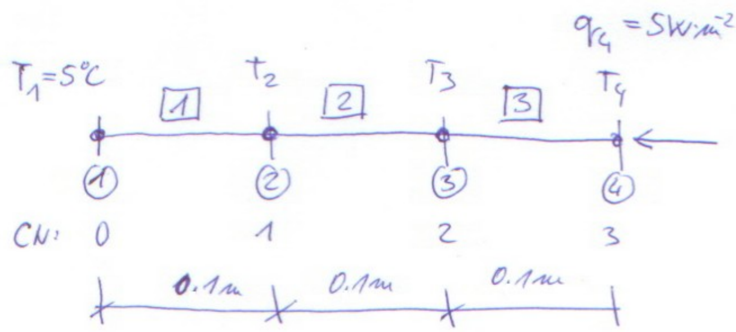


5/ BENCHMARK No. 2

PRESCRIBED FLUX ON BOUNDARY

DISCRETIZATION:





$$\underline{k}^{1,2,3} = \begin{array}{c|c} \begin{matrix} 2 & 3 \\ 1 & 2 \\ 0 & 1 \end{matrix} & \\ \begin{bmatrix} 20 & -20 \\ -20 & 20 \end{bmatrix} & \begin{matrix} 0 & 1 & 2 \\ 1 & 2 & 3 \end{matrix} \end{array}$$

$$\underline{r} = \begin{Bmatrix} T_2 \\ T_3 \\ T_4 \end{Bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \end{matrix}$$

$$\underline{K} = \begin{array}{c c} \begin{matrix} 1 & 2 & 3 \end{matrix} & \\ \begin{bmatrix} 40 & -20 & 0 \\ -20 & 40 & -20 \\ 0 & -20 & 20 \end{bmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \end{array}$$

CONTRIBUTION FROM THE FLUX:

NEUMANN B.C.

$$\underline{f}^{e}_{\Gamma_p} = \int_{\Gamma_{qp}} \underline{N}^{eT} \underline{N}^{e} \bar{q}_e \, d\Gamma = \int_{\Gamma_{qp}} \underline{N}^{eT} \bar{q} \, d\Gamma$$

$$\underline{f}^{e}_{\Gamma_p} = \int_{\Gamma_{qp}} \begin{bmatrix} \frac{x_2 - x}{l} \\ \frac{x - x_1}{l} \end{bmatrix} 5 \Big|_{x_2} d\Gamma = \begin{bmatrix} \frac{x_2 - x}{l} \\ \frac{x - x_1}{l} \end{bmatrix} 5 \Big|_{x_2} = \begin{bmatrix} 0 \\ 5 \end{bmatrix} \begin{matrix} 2 \\ 3 \end{matrix}$$

INTEGRATION AT NODE !

$$\underline{f} = \begin{Bmatrix} 0 \\ 0 \\ 5 \end{Bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \end{matrix}$$

CONTRIBUTION FROM DIRICHLET B.C.

$$\underline{k}^{1}_{td} \underline{r}^{1}_{d} = \begin{array}{c c c c} \begin{matrix} 0 & 1 \end{matrix} & & \begin{matrix} 0 \end{matrix} & \\ \begin{bmatrix} 20 & -20 \\ -20 & 20 \end{bmatrix} & \begin{matrix} 0 \\ 1 \end{matrix} & \begin{Bmatrix} 5 \\ 0 \end{Bmatrix} & \begin{matrix} 0 \\ 1 \end{matrix} \end{array} = \begin{array}{c c} 0 & \\ \begin{Bmatrix} 100 \\ -100 \end{Bmatrix} & \begin{matrix} 0 \\ 1 \end{matrix} \end{array}$$

$$\underline{K}_{td} \underline{r}_{d} = \begin{Bmatrix} -100 \\ 0 \\ 0 \end{Bmatrix}$$

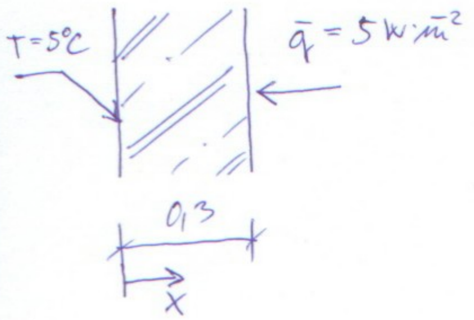
6/ SYSTEM OF EQS.:

$$\underline{k}\,\underline{r} = \underline{f} - \underline{K}_{rd}\,\underline{r}_d$$

$$\begin{bmatrix} 40 & -20 & 0 \\ -20 & 40 & -20 \\ 0 & -20 & 20 \end{bmatrix} \begin{Bmatrix} T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 5 \end{Bmatrix} - \begin{Bmatrix} -100 \\ 0 \\ 0 \end{Bmatrix}$$

$$\underline{r} = \begin{Bmatrix} T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 5{,}25 \\ 5{,}5 \\ 5{,}75 \end{Bmatrix} \; [^\circ C]$$

ANALYTICAL SOLUTION



DIF. EQUATION

$$\lambda \frac{d^2 T(x)}{dx^2} = 0$$

INTEGRATION:

1^{ST} $$\lambda \frac{d\,T(x)}{dx} = C_1$$

2^{ND} $$\lambda\, T(x) = C_1 x + C_2$$

BOUNDARY CONDITIONS:

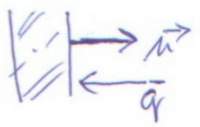
$$T(x=0) = 5^\circ C$$

$$C_2 = 10$$

DIRECTION OF OUTER NORMAL VECTOR

$$T(x=0{,}3) = -\lambda \frac{\partial T(x)}{\partial x}\vec{n} = -5$$

$$C_1 = +5$$



SOLUTION = LINEAR FUNCT.

$$\lambda T(x) = +5x + 10$$

$$T(x) = +2.5\,x + 5$$

$T(x=0{,}1) = 5.25\,^\circ C$

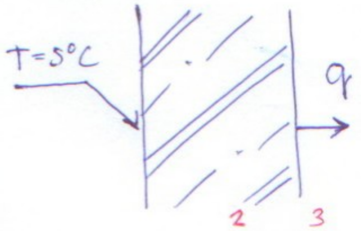
$T(x=0{,}2) = 5.5\,^\circ C$

$T(x=0{,}3) = 5.75\,^\circ C$

7/

BENCHMARK No. 3

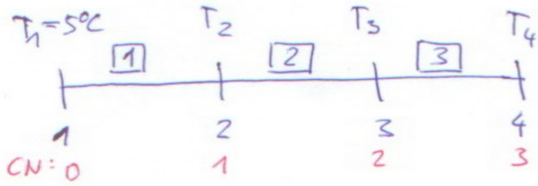
· HEAT TRANSFER ON BOUNDARY:



$q = \alpha (T - T_\infty)$

$\alpha = 4\ W \cdot m^{-2} K^{-1}$

$T_\infty = 20°C$



$$\begin{array}{cccccc} & 2 & 3 & & & \\ & 1 & 2 & & & \\ & 0 & 1 & & & \\ \underline{k}^1 = & \left[\begin{matrix} 20 & -20 \\ -20 & 20 \end{matrix}\right. & \left.\right] & \begin{matrix} 0 \\ 1 \end{matrix} & \begin{matrix} 1 \\ 2 \end{matrix} & \begin{matrix} 2 \\ 3 \end{matrix} \end{array}$$

$$\underline{r} = \begin{Bmatrix} T_2 \\ T_3 \\ T_4 \end{Bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \end{matrix}$$

$$\underline{k} = \begin{bmatrix} 40 & -20 & \cancel{0} \\ -20 & 40 & -20 \\ \cancel{0} & -20 & 20+4 \end{bmatrix}$$

CONTRIB. FROM HEAT TRANSFER INTO THE CONDUCTIVITY MATRIX:

$$\underline{k}^3_\Gamma = \int_{\Gamma_{qc}} \underline{N}^{e^T} \alpha^e \underline{N}^e \, d\Gamma =$$

$$\underline{k}^3_\Gamma = \begin{bmatrix} \frac{(x_2 - x)}{\ell} \\ \frac{(x - x_1)}{\ell} \end{bmatrix} \alpha \left[\frac{x_2 - x}{\ell} ; \frac{x - x_1}{\ell} \right] \Bigg|_{x_2} =$$

INTEGRATION AT NODE

$$\begin{array}{ccc} & 2 & 3 & \\ \underline{k}^3_\Gamma = & \left[\begin{matrix} 0 \\ 0 \end{matrix}\right. & \left.\begin{matrix} 0 \\ 4 \end{matrix}\right] & \begin{matrix} 2 \\ 3 \end{matrix} \end{array}$$

CONTRIB. INTO RHS VECTOR:

$$\underline{f}^e_{\Gamma_{qc}} = \int_{\Gamma_{qc}} \underline{N}^{e^T} \alpha \, \underline{T_\infty} \, d\Gamma$$

$$= \begin{bmatrix} \frac{(x_2 - x)}{\ell} \\ \frac{(x - x_1)}{\ell} \end{bmatrix} \alpha T_\infty \Bigg|_{x_2} = \begin{Bmatrix} \cancel{0} \\ 80 \end{Bmatrix}$$

FROM DIRICHLET B.C.:

$$\begin{array}{cccccccc} & 0 & 1 & & 0 & & 0 & \\ \underline{k}^1_{Td} \underline{r}^1_d = & \left[\begin{matrix} 20 \\ -20 \end{matrix}\right. & \left.\begin{matrix} -20 \\ 20 \end{matrix}\right] & \begin{matrix} 0 \\ 1 \end{matrix} & \begin{Bmatrix} 5 \\ 0 \end{Bmatrix} & \begin{matrix} 0 \\ 1 \end{matrix} = & \begin{Bmatrix} 100 \\ -100 \end{Bmatrix} & \begin{matrix} 0 \\ 1 \end{matrix} \end{array}$$

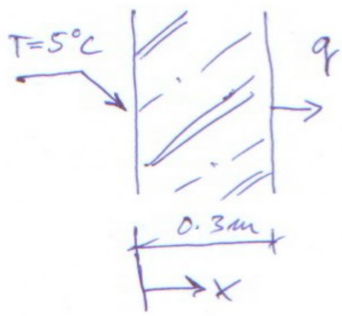
SYSTEM OF EQUATION:

$$\begin{bmatrix} 40 & -20 & 0 \\ -20 & 40 & -20 \\ 0 & -20 & 24 \end{bmatrix} \begin{Bmatrix} T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 80 \end{Bmatrix} - \begin{Bmatrix} -100 \\ 0 \\ 0 \end{Bmatrix}$$

$$\underline{r} = \begin{Bmatrix} T_2 = 55/8 = 6.875 \\ T_3 = 35/4 = 8.75 \\ T_4 = 85/8 = 10.625 \end{Bmatrix}$$

ANALYTICAL SOLUTION:

$$\lambda \frac{d^2 T(x)}{dx^2} = 0$$



$\lambda = 2 \ W \cdot m^{-1} K^{-1}$

$q = \alpha (T - T_\infty)$

$\alpha = 4 \ W \cdot m^{-2} K^{-1}$

$T_\infty = 20°C$

INTEGRATION:

1^{ST}: $\lambda \frac{dT(x)}{dx} = C_1$

2^{ND}: $\lambda T(x) = C_1 x + C_2$

B.Cs:

$T(x=0) = 5°C \Rightarrow C_2 = 10$

FLUX $\begin{cases} q(x=0.3) = -\lambda \left.\frac{dT}{dx}\right|_{x=0,3} \\ q(x=0.3) = \alpha (T(0.3) - T_\infty) + C_1 = 0 \end{cases} \Rightarrow C_1 = -\alpha (T(0.3) - T_\infty)$

$\underline{C_1 = -4 T(0.3) + 80}$

$$\boxed{\lambda T(x) = (-4 T(0.3) + 80) x + 10}$$

$\underline{x=0.3}$ $2 \cdot T(0.3) = -4 T(0.3) \cdot 0.3 + 80 \cdot 0.3 + 10$

$T(0.3)(2 + 4 \cdot 0.3) = 80 \cdot 0.3 + 10$

$\underline{T(0.3)} = (80 \cdot 0.3 + 10) / (2 + 4 \cdot 0.3) = \underline{\underline{10.625 °C}}$

$\underline{\underline{T(0.1) = 6.875 °C}}$

$\underline{\underline{T(0.2) = 8.75 °C}}$