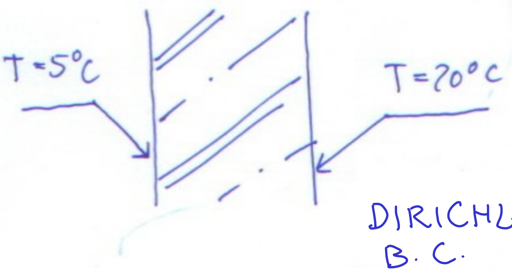
2D HEAT CONDUCTION PROBLEM

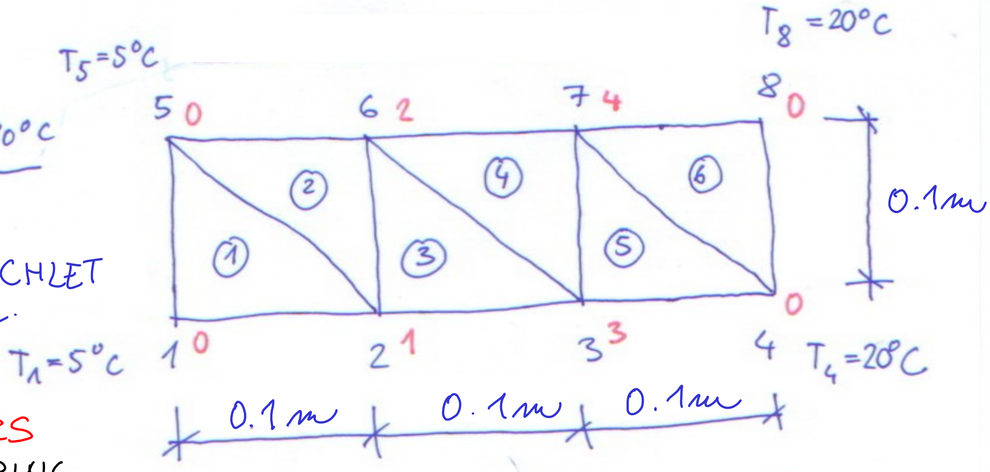
BENCHMARK NO. 1

$\lambda = 2 \ W \cdot m^{-1} \cdot K^{-1}$

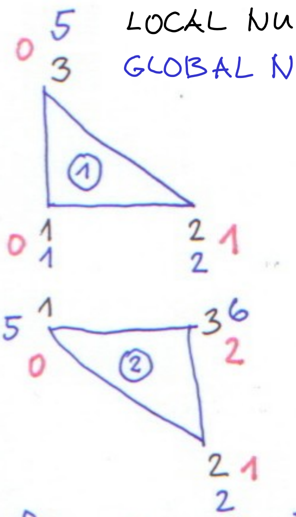
CONCRETE WALL



DIRICHLET B. C.



CODE NUMBERS
LOCAL NUMBERING
GLOBAL NUMBERING



$$B_1^e = \frac{1}{2A_1}\begin{bmatrix} -0{,}1 & 0{,}1 & 0 \\ -0{,}1 & 0 & 0{,}1 \end{bmatrix}$$

$$B_2^e = \frac{1}{2A_2}\begin{bmatrix} -0{,}1 & 0 & 0{,}1 \\ 0 & -0{,}1 & 0{,}1 \end{bmatrix}$$

$A_1 = A_2$

$$B^e = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} \end{bmatrix}$$

$$B^e = \frac{1}{2A}\begin{bmatrix} y_{23} & y_{31} & y_{12} \\ x_{32} & x_{13} & x_{21} \end{bmatrix}$$

$$K_1^e = \frac{\lambda}{4A_1} \begin{array}{c} \begin{bmatrix} 0{,}02 & -0{,}01 & -0{,}01 \\ -0{,}01 & 0{,}01 & 0 \\ -0{,}01 & 0 & 0{,}01 \end{bmatrix} \begin{array}{ccc} 0 & 1 & 3 \\ 1 & 3 & 0 \\ 0 & 2 & 4 \end{array} \\ \begin{array}{ccc} 0 & 1 & 0 \end{array} \end{array}$$

$$K_2^e = \frac{\lambda}{4A_2} \begin{array}{c} \begin{bmatrix} 0{,}01 & 0 & -0{,}01 \\ 0 & 0{,}01 & -0{,}01 \\ -0{,}01 & -0{,}01 & 0{,}02 \end{bmatrix} \begin{array}{ccc} 0 & 2 & 4 \\ 1 & 3 & 0 \\ 2 & 4 & 0 \end{array} \\ \begin{array}{ccc} 0 & 1 & 2 \end{array} \end{array}$$

$$K^e = A \, B^{eT} \lambda B^e$$

ELEMENT 1

$$K_1^e = \frac{\lambda}{4A_1}\begin{bmatrix} -0{,}1 & -0{,}1 \\ 0{,}1 & 0 \\ 0 & 0{,}1 \end{bmatrix}\begin{bmatrix} -0{,}1 & 0{,}1 & 0 \\ -0{,}1 & 0 & 0{,}1 \end{bmatrix}$$

LOCALIZATION $\Rightarrow$

ELEMENTS: 1 3 5
2 4 6

GLOBAL CONDUCTIVITY MATRIX:
- SYMMETRIC

$$K = \frac{\lambda}{4A} \begin{array}{c c} \left[\begin{array}{c|c|c|c} 0{,}04 & -0{,}02 & -0{,}01 & 0 \\ \hline & 0{,}04 & 0 & -0{,}01 \\ \hline & & 0{,}04 & -0{,}02 \\ \hline & & & 0{,}04 \end{array}\right] & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \\ \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} & \end{array}$$

RIGHT-HAND SIDE VECTOR:

$$f = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\underline{k}\,\underline{r} = \underline{f} - \underline{d}$$

$$\underline{d} = \underline{k_d}\,\underline{r_d}$$ = CONTRIBUTION FROM DIRICHLET B.C.
- PRESCRIBED TEMPERATURES

$$d = \frac{\lambda}{4A} \begin{array}{cc} & 0 \\ \begin{Bmatrix} -0{,}05 \\ -0{,}05 \\ -0{,}2 \\ -0{,}2 \end{Bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \end{array}$$

ELEMENT ①

$$\underbrace{k_1 \cdot r_{d1}}\ = \ d_1$$

$$\frac{\lambda}{4A} \begin{array}{c} \left[\begin{array}{c|c|c} 0{,}02 & -0{,}01 & -0{,}01 \\ \hline -0{,}01 & 0{,}01 & 0 \\ \hline -0{,}01 & 0 & 0{,}01 \end{array}\right] \\ \begin{array}{ccc} 0 & 1 & 0 \end{array} \end{array} \begin{array}{cc} \begin{Bmatrix} 5 \\ 0 \\ 5 \end{Bmatrix} & \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \end{array} = \begin{array}{cc} \begin{Bmatrix} 0{,}05 \\ -0{,}05 \\ 0 \end{Bmatrix} & \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \end{array}$$

ELEMENT ②

$$\frac{\lambda}{4A} \begin{array}{c} \left[\begin{array}{c|c|c} 0{,}01 & 0 & -0{,}01 \\ \hline 0 & 0{,}01 & -0{,}01 \\ \hline -0{,}01 & -0{,}01 & 0{,}02 \end{array}\right] \\ \begin{array}{ccc} 0 & 1 & 2 \end{array} \end{array} \begin{array}{cc} \begin{Bmatrix} 5 \\ 0 \\ 0 \end{Bmatrix} & \begin{array}{c} 0 \\ 1 \\ 2 \end{array} \end{array} = \begin{array}{cc} \begin{Bmatrix} 0{,}05 \\ 0 \\ -0{,}05 \end{Bmatrix} & \begin{array}{c} 0 \\ 1 \\ 2 \end{array} \end{array} \quad d_2$$

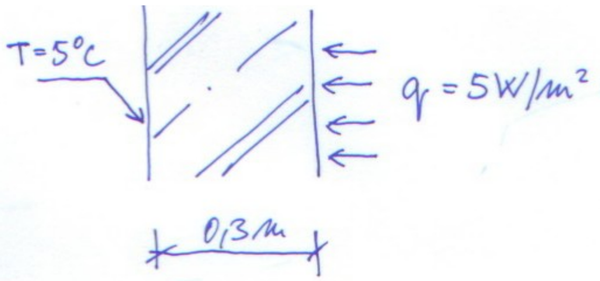
THE SAME PRINCIPLE OF LOCALIZATION FOR ELEMENTS ⑤ AND ⑥

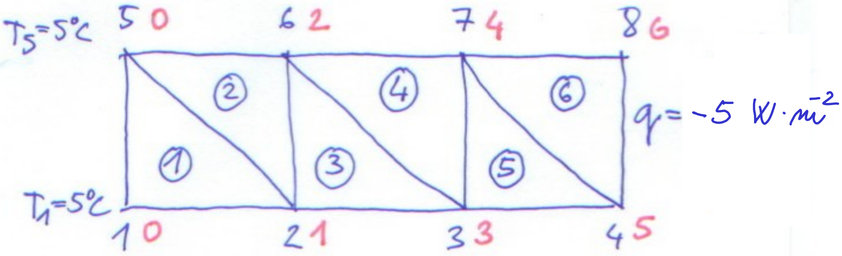
$$\frac{\lambda}{4 \cdot A} = \frac{2}{4 \cdot 0.1 \cdot 0{,}1/2} = 100$$

PROBLEM SOLUTION:

$$r = \begin{array}{cc} \begin{Bmatrix} T_2 \\ T_6 \\ T_3 \\ T_7 \end{Bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \end{array} = \begin{Bmatrix} 10 \\ 10 \\ 15 \\ 15 \end{Bmatrix}$$

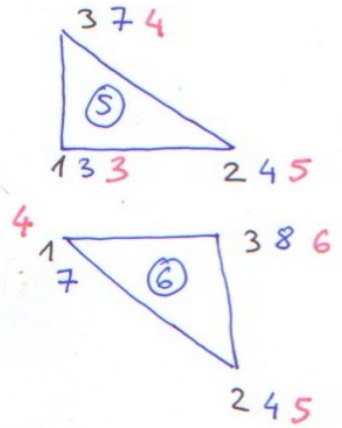
BENCHMARK NO. 2





CONDUCTIVITY MATRIX OF THE PROBLEM

- WE USE COND. MATRIX FROM PREVIOUS EXAMPLE, AND EXPAND IT BY COMPONENTS IN POSITIONS 3 TO 6



$$K = \frac{\lambda}{4A}\begin{array}{cc}\begin{bmatrix} 0{,}04 & -0{,}02 & -0{,}01 & 0 & 0 & 0 \\ & 0{,}04 & 0 & -0{,}01 & 0 & 0 \\ & & 0{,}04 & -0{,}02 & -0{,}01 & 0 \\ & & & 0{,}04 & 0 & -0{,}01 \\ & & & & 0{,}02 & -0{,}01 \\ & & & & & 0{,}02 \end{bmatrix} & \begin{matrix}1\\2\\3\\4\\5\\6\end{matrix} \\ \begin{matrix}1 & \quad 2 & \quad 3 & \quad 4 & \quad 5 & \quad 6\end{matrix} & \end{array}$$

$$K_5^e = \frac{\lambda}{4A}\begin{array}{cc}\begin{bmatrix} 0{,}02 & -0{,}01 & -0{,}01 \\ -0{,}01 & 0{,}01 & 0 \\ -0{,}01 & 0 & 0{,}01 \end{bmatrix} & \begin{matrix}3\\4\\5\end{matrix} \\ \begin{matrix}3 & \quad 4 & \quad 5\end{matrix} & \end{array}$$

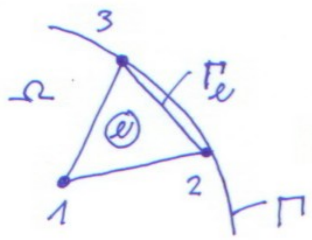
$$K_6^e = \frac{\lambda}{4A}\begin{array}{cc}\begin{bmatrix} 0{,}01 & 0 & -0{,}01 \\ 0 & 0{,}01 & -0{,}01 \\ -0{,}01 & -0{,}01 & 0{,}02 \end{bmatrix} & \begin{matrix}4\\5\\6\end{matrix} \\ \begin{matrix}4 & \quad 5 & \quad 6\end{matrix} & \end{array}$$

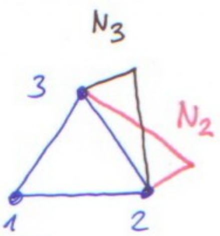
RIGHT-HAND SIDE:

$\underline{K_d} \cdot \underline{r_d} = \underline{d}$ = CONTRIBUTION FROM DIRICHLET B.C. IN ELEMENTS NO. 1 AND 2

$$d = \begin{Bmatrix} -0{,}05 \\ -0{,}05 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

LOADING VECTOR FROM PRESCRIBED FLUX:





$$\underline{N}^T = \begin{bmatrix} 0 \\ N_2 \\ N_3 \end{bmatrix}$$

LINEAR APPROXIMATION ON THE EDGE !!

LOADING VECTOR FROM PRESCRIBED FLUX:

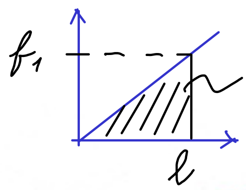
$\bar{q}$ IS NEGATIVE AND CONSTANT

$$\underline{f} = -\int_{\Gamma_{qp}} \underline{N}^{e^T} \underbrace{\underline{N}\,\bar{\underline{q}}}_{\bar{q}(x)}\, d\Gamma = \int_{\Gamma_{qp}} -\underline{N}^{e^T} \bar{q}\, d\Gamma$$

$$\underline{f}^e_6 = -\int_{\ell} \begin{bmatrix} 0 \\ N_2 \\ N_3 \end{bmatrix} [-5]\, d\Gamma = \frac{1}{2}\ell \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix} = \begin{array}{c} 0 \\ \begin{bmatrix} 0 \\ 0{,}25 \\ 0{,}25 \end{bmatrix} \end{array} \begin{array}{c} \\ 4 \\ 5 \\ 6 \end{array}$$

NOTE: INTEGRAL OF MULTIPLICATION OF LINEAR AND CONSTANT FUNCTION:

$$\int_{\ell} \overset{f_1}{\triangle}\ \overset{f_2}{\square}\, dx = \frac{1}{2} f_1 f_2 \ell$$



$\text{AREA} = \frac{1}{2} f_1 \ell \cdot \underset{\text{CONST.}}{f_2}$

LOCALIZATION

$$\underline{f} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0{,}25 \\ 0{,}25 \end{Bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

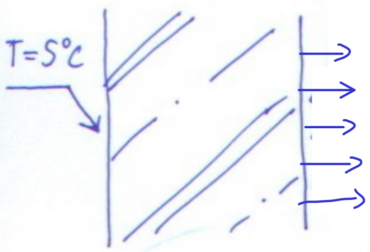
PROBLEM SOLUTION:

$$\underline{K}\,\underline{r} = \underline{f} - \underline{d}$$

$$\underline{r} = \begin{Bmatrix} T_2 \\ T_6 \\ T_3 \\ T_7 \\ T_4 \\ T_8 \end{Bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} = \begin{Bmatrix} 5{,}25 \\ 5{,}25 \\ 5{,}5 \\ 5{,}5 \\ 5{,}75 \\ 5{,}75 \end{Bmatrix}$$

BENCHMARK NO. 3

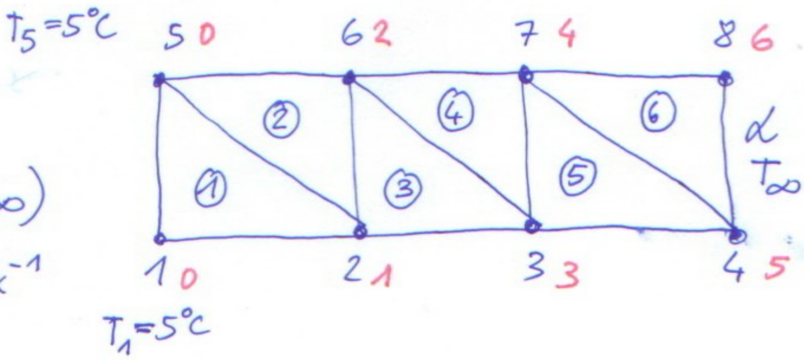
DISCRETIZATION:



$$q = \alpha (T - T_\infty)$$

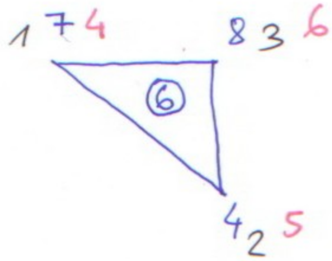
$$\alpha = 4\,W \cdot m^{-2} \cdot K^{-1}$$

$$T_\infty = 20°C$$



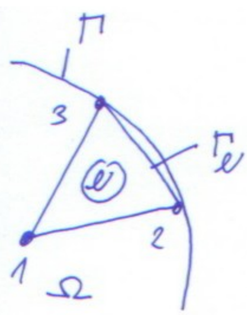
WE USE THE COND. MATRIX FROM PREVIOUS BENCHMARK:

$$K = \frac{\lambda}{4A} \begin{bmatrix} 0{,}04 & -0{,}02 & -0{,}01 & 0 & 0 & 0 \\ & 0{,}04 & 0 & -0{,}01 & 0 & 0 \\ & & 0{,}04 & -0{,}02 & -0{,}01 & 0 \\ & & & 0{,}04 & 0 & -0{,}01 \\ & & & & 0{,}02 & -0{,}01 \\ & & & & & 0{,}02 \end{bmatrix}$$



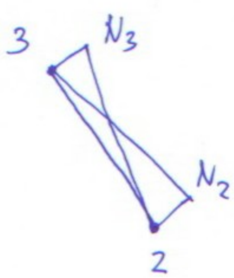
THE CONTRIBUTION FROM THE HEAT TRANSMISSION B.C. IS LOCALIZED IN THIS PART. BEFORE THAT, THE MATRIX IS MULTIPLIED BY $\frac{\lambda}{4 \cdot A} = 100$

BOUNDARY CONDITION:



$$\underline{N} = [0, N_2, N_3]$$

$$K^e_{\Gamma,6} = \int_{\Gamma^e_{qc}} \underline{N}^{eT} \alpha \underline{N}^e \, d\Gamma = \int_l \begin{bmatrix} 0 \\ N_2 \\ N_3 \end{bmatrix} \alpha \, [0, N_2\ N_3] \, d\Gamma$$



$$= \alpha \int_l \begin{bmatrix} 0 & 0 & 0 \\ 0 & N_2N_2 & N_2N_3 \\ 0 & N_3N_2 & N_3N_3 \end{bmatrix} d\Gamma = \alpha l \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1/3 & 1/6 \\ 0 & 1/6 & 1/3 \end{bmatrix}$$

INTEGRAL OF TWO LINEAR FUNCTION MULTIPLICATION:

$$\int_l f_1 \, f_2 \, dx = \frac{1}{3} f_1 f_2 l \quad ; \quad \int_l f_1 \, f_2 \, dx = \frac{1}{6} f_1 f_2 l$$

$$\underline{K}^{e}_{\Gamma,6} = 4 \cdot 0{,}1 \begin{array}{c|c} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1/3 & 1/6 \\ 0 & 1/6 & 1/3 \end{bmatrix} & \begin{matrix} 4 \\ 5 \\ 6 \end{matrix} \\ \begin{matrix} 4 & 5 & 6 \end{matrix} & \end{array} \Rightarrow \text{LOCALIZATION}$$

CONTRIBUTION TO THE RIGHT-HAND SIDE:

CONSTANT PARAMETERS

$$\underline{f}^{e}_{\Gamma_{qc}} = \int_{\Gamma^{e}_{qc}} \underline{N}^{e^T} \alpha(x)\, T_\infty(x)\, d\Gamma = \int_{e} \underline{N}^{e^T} \alpha\, T_\infty\, d\Gamma =$$

$$= \int_{l} \begin{bmatrix} 0 \\ N_2 \\ N_3 \end{bmatrix} 4 \cdot 20\, d\Gamma = 4 \cdot 20 \cdot \frac{1}{2} l \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix} \begin{matrix} 4 \\ 5 \\ 6 \end{matrix} \Rightarrow$$

LOCALIZATION

RESULTING SYSTEM OF EQUATION:

FROM THE PREV. BENCHMARK

$$\underline{K} \cdot \underline{r} = \underline{f} - \underline{d}$$

$$\begin{bmatrix} & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{bmatrix} \begin{Bmatrix} T_2 \\ T_6 \\ T_3 \\ T_7 \\ T_4 \\ T_8 \end{Bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 4 \\ 4 \end{Bmatrix} - \begin{Bmatrix} -0{,}05 \\ -0{,}05 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\underline{r} = \begin{Bmatrix} 6{,}875 \\ 6{,}875 \\ 8{,}75 \\ 8{,}75 \\ 10{,}625 \\ 10{,}625 \end{Bmatrix}$$

SOLUTION