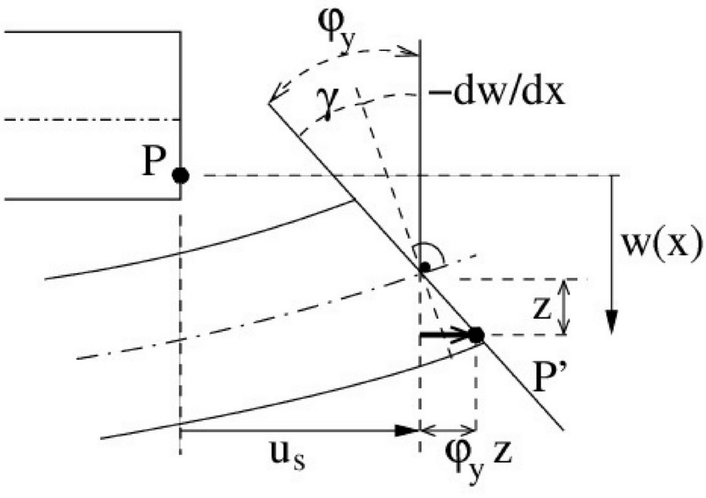
Mindlin theory



Assumptions:

- Constant deflection (vertical displacement) does not vary along the height of the beam
- The cross sections remain planar but not necessarily perpendicular to the deformed beam axis
- Kinematics of the cross-section (plane xz):

$$u(x,z) = u_s(x) + \varphi_y(x) z$$

$$v = 0$$

$$w(x,z) = w(x)$$

- Non-zero strain components:

$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{\partial u_s}{\partial x} + \frac{\partial \varphi_y}{\partial x} z = \varepsilon_s + \kappa_y z$$

$$\gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x} + \varphi_y$$

- Governing differential equations:

$$\frac{d}{dx}\left(EA\frac{du_s}{dx}\right) + f_x = 0$$

$$\frac{d}{dx}\left(kGA\left(\frac{dw}{dx} + \varphi_y\right)\right) + f_z = 0$$

$$\frac{d}{dx}\left(EI_y\frac{d\varphi_y}{dx}\right) - kGA\left(\frac{dw}{dx} + \varphi_y\right) = 0$$

- Boundary conditions:
  - Kinematic b.c. (prescribed: u_s, w, φ_y)
  - Static b.c. $\left(N(x) = \bar{N}(x), V(x) = \bar{V}(x), M(x) = \bar{M}(x)\right)$

Discretized problem

$$\begin{bmatrix} \boldsymbol{K^e_{uu}} & 0 & 0 \\ 0 & \boldsymbol{K^e_{ww}} & \boldsymbol{K^e_{w\varphi}} \\ 0 & \boldsymbol{K^e_{\varphi w}} & \boldsymbol{K^e_{\varphi\varphi}} \end{bmatrix} \begin{Bmatrix} r^e_u \\ r^e_w \\ r^e_\varphi \end{Bmatrix} = \begin{Bmatrix} R^e_u \\ R^e_w \\ R^e_\varphi \end{Bmatrix}$$

$$\boldsymbol{K^e_{uu}} = \int \boldsymbol{B}^{eT}_u EA \boldsymbol{B}^e_u \, dx$$

$$\boldsymbol{K^e_{ww}} = \int \boldsymbol{B}^{eT}_w kGA \boldsymbol{B}^e_w \, dx$$

$$\boldsymbol{K^e_{w\varphi}} = \int \boldsymbol{B}^{eT}_w kGA \boldsymbol{N}^e_\varphi \, dx$$

$$\boldsymbol{K^e_{\varphi w}} = \int \boldsymbol{N}^{eT}_\varphi kGA \boldsymbol{B}^e_w$$

$$\boldsymbol{K^e_{\varphi\varphi}} = \int \boldsymbol{B}^{eT}_\varphi EI \boldsymbol{B}^e_\varphi \, dx + \int \boldsymbol{N}^e_\varphi kGA \boldsymbol{N}^e_\varphi \, dx$$

Linear approximation function

- Matrix of approximation functions:

$$\boldsymbol{N}^e = \left[\frac{l^e - x}{l^e}, \frac{x}{l^e} \right]$$

- Kinematic matrix (derivatives of approximation functions):

$$\boldsymbol{B}^e = \left[\frac{-1}{l^e}, \frac{1}{l^e} \right]$$

- Sub-matrices of the stiffness matrix:

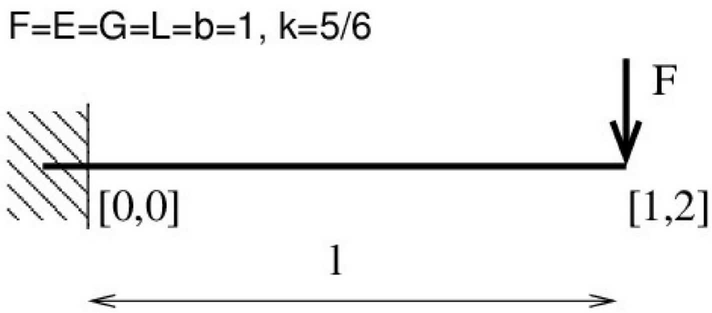
$$\boldsymbol{K^e_{uu}} = \frac{EA}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\boldsymbol{K^e_{ww}} = \frac{kGA}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\boldsymbol{K^e_{w\varphi}} = \boldsymbol{K^e_{\varphi w}} = kGA \begin{bmatrix} -1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$\boldsymbol{K^e_{\varphi\varphi}} = \frac{EI}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + kGA \begin{bmatrix} l/3 & l/6 \\ l/6 & l/3 \end{bmatrix}$$

Example 1



In [3]:

```
function Ke = beam2d_stiffness(xz, EA, kGA, EI)

    l = sqrt((xz(3) - xz(1))^2 + (xz(4) - xz(2))^2)

    Kuu = EA/l*[1 -1; -1 1];
    Kww = kGA/l*[1 -1; -1 1];
    Kwf = kGA*[-1/2 -1/2; 1/2 1/2];
    Kff = EI/l*[1 -1; -1 1] + kGA*[l/3 l/6; l/6 l/3];
    % Kff = EI/l*[1 -1; -1 1] + kGA*[l/4 l/4; l/4 l/4];

    O = zeros(2);

    Ke = [Kuu O O; O Kww Kwf; O Kwf' Kff]
end
```

Shear locking

In the same type of approximation of deflection and rotation, the approximation of shear force is of one degree higher than the approximation of bending moment. It doesn't satisfy the Schwedler relation – resulting response in too "stiff" → excessive influence of shear terms, sc. shear locking.

- Shear force: $V(x) = kGA\left(\frac{dw}{dx} + \varphi_y\right)$ = linear
- Bending moment: $M(x) = EI\frac{d\varphi_y}{dx}$ = constant
- Schwedler relation: $\frac{dM(x)}{dx} - V(x) = 0$

The shear locking problem can be solved by the use of a selective integration or a hierarchical function.

Selective (reduced) integration

The conflict is reduced by a selective integration of the $\boldsymbol{K}_{\varphi\varphi}$, the part of the stiffness matrix, which corresponds to the contribution of the shear force.

Shear strain $\gamma = \frac{dw}{dx} + \varphi$ is assumed constant, then $\bar{\gamma} = \frac{w_2 - w_1}{l} + \frac{\varphi_1 + \varphi_2}{2}$, which corresponds to the reduced integration.

Assuming the constant shear strain influences the parts of the stiffness matrix with N_φ. In the case of constant approximation, only matrix $\boldsymbol{K}_{\varphi\varphi}$, is changed, because parts of the integral have parabolic shape. If we prefer linear functions, the one-point integration is equal to the full one.

$$\boldsymbol{K}^e_{\varphi\varphi} = \frac{EI}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + kGA\begin{bmatrix} \boldsymbol{l/4} & \boldsymbol{l/4} \\ \boldsymbol{l/4} & \boldsymbol{l/4} \end{bmatrix}$$

Bubble (hierarchical) function

In case of linear approximation for the shear strain, it is valid:

$$\gamma = \frac{dw}{dx} + \varphi = \frac{w_2 - w_1}{l} + \varphi_1 + \frac{x}{l}(\varphi_2 - \varphi_1) \quad = \text{const} + \frac{x}{l}(\varphi_2 - \varphi_1)$$

For the constant shear strain, we add a quadratic term to the deflection approximation. Then the linear part is vanished after in the derivative:

$$w(x) = w^{lin} + \frac{1}{2l}(\varphi_2 - \varphi_1)x(x - l)$$

Resulting approximation:

$$u_s(x) = N_1 u_{s1} + N_2 u_{s2}$$

$$w(x) = N_1 w_1 + N_2 w_2 - \frac{N_3}{2l}\varphi_1 + \frac{N_3}{2l}\varphi_2$$

$$\varphi_y(x) = N_1\varphi_1 + N_2\varphi_2$$

Reordering of nodal values of deflection and rotation is suitable for the calculation:
$r_{w,\varphi} = \{w_1, \varphi_1, w_2, \varphi_2\}$.

Interpolation of deflection and rotation:

$$w(x) = N_w r_{w,\varphi} = [1 - \frac{x}{l}, \frac{x(x-l)}{2l}, \frac{x}{l}. \frac{-x(x-l)}{2l}]$$

$$\varphi(x) = N_\varphi r_{w,\varphi} = [0, \frac{1-x}{l}, 0, \frac{x}{l}]$$

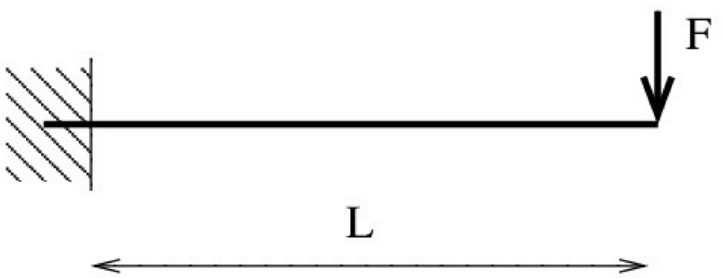
Final form of sub-matrices: $K_{ww} = \int B_w^T KGAB_w dx = \begin{bmatrix} & 0 & -\frac{kGA}{l} & 0 \\ 0 & \frac{kGAl}{12} & 0 & -\frac{kGAl}{12} \\ -\frac{kga}{l} & 0 & \frac{kGA}{l} & 0 \\ 0 & -\frac{kGAl}{12} & 0 & \frac{kGAl}{12} \end{bmatrix}$

$$K_{w\varphi} = \int B_W^T kGAN_\varphi dx = \begin{bmatrix} & -\frac{kGA}{2} & 0 & -\frac{kGA}{2} \\ 0 & -\frac{kGAl}{12} & 0 & \frac{kGAl}{12} \\ 0 & \frac{kGA}{2} & 0 & \frac{kGA}{2} \\ 0 & \frac{kGAl}{12} & 0 & -\frac{kGAl}{12} \end{bmatrix}$$

$$K_{\varphi,\varphi} = \int B_\varphi^T EIB_\varphi dx + \int N_\varphi^T kGAN\varphi dx = \begin{bmatrix} & 0 & 0 & 0 \\ 0 & \frac{EI}{l} + \frac{kGAl}{3} & 0 & -\frac{EI}{l} + \frac{kGAl}{6} \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{EI}{l} + \frac{kGAl}{6} & 0 & \frac{EI}{l} + \frac{kGAl}{3} \end{bmatrix}$$

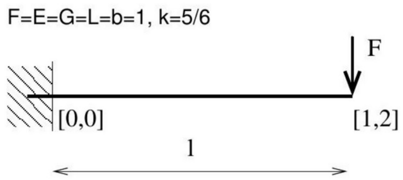
This gives the same results as for the reduced integration

Comparison for a cantilever beam:



Full integration

- **Cantilever beam:**



$$\begin{bmatrix} kGA/l & kGA/2 \\ kGA/2 & EI/l + kGAl/3 \end{bmatrix} \begin{Bmatrix} w \\ \varphi \end{Bmatrix} = \begin{Bmatrix} F \\ 0 \end{Bmatrix}$$

$h = 0.1 \rightarrow w_{(0.1)} = 47.6$

$h = 0.01 \rightarrow w_{(0.01)} = 479.9$ Ratio

$w_{(0.01)}/w_{(0.1)} = 10$

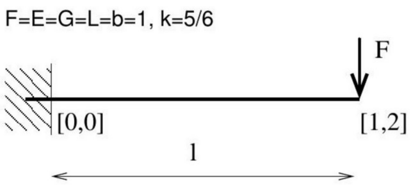
Deflection – analytical solution: $w = FL^3/3EI + FL/kGA.$

Our case: $w_{(0.1)} = 4x10^3 + 12, w_{(0.01)} = 4x10^6 + 120,$

$w_{(0.01)}/w_{(0.1)} = 1000$!!!!

Reduced integration

- **Cantilever beam:**



$$\begin{bmatrix} kGA/l & kGA/2 \\ kGA/2 & EI/l + kGAl/4 \end{bmatrix} \begin{Bmatrix} w \\ \varphi \end{Bmatrix} = \begin{Bmatrix} F \\ 0 \end{Bmatrix}$$

$h = 0.1 \rightarrow w_{(0.1)} = 3012$

$h = 0.01 \rightarrow w_{(0.01)} = 3x10^6$ Ratio

$w_{(0.01)}/w_{(0.1)} = 1000$

Deflection – analytical solution: $w = FL^3/3EI + FL/kGA.$

Our case: $w_{(0.1)} = 4x10^3 + 12, w_{(0.01)} = 4x10^6 + 120,$

$w_{(0.01)}/w_{(0.1)} = 1000$ ok.

Comparison of full integration and reduced integration scheme for one and more elements

- Full integration (2 points):

h/L > 1/3:

nelem	w/w_e
1	0.0416
2	0.445
4	0.762
8	0.927

h/L < 1/10:

nelem	w/w_e
1	0.0002
2	0.0008
4	0.0003
8	0.0013

- Reduced integration (1 point):

h/L > 1/3:

nelem	w/w_e
1	0.762
2	0.940
4	0.985
8	0.996

h/L < 1/10:

nelem	w/w_e
1	0.750
2	0.938
4	0.984
8	0.996

References

- English course of “Numerical analysis of structures” by J. Zeman (jan.zeman@fsv.cvut.cz)

- Czech course of “Numerická analýza konstrukcí” (Numerical analysis of structures) by B. Patzák (borek.patzak@fsv.cvut.cz)

- J. Fish and T. Belytschko: A First Course in Finite Elements, John Wiley & Sons, 2007