A posteriori error estimate – ZZ method

FEM solution:

$$u^{\text{FEM}}(x) = N(x)r$$

$$\sigma^{\text{FEM}}(x) = DB(x)r,$$

Where B is the matrix of derivatives of approximation functions.
We assume that the exact stress state is close to this

$$\sigma^*(x) = N(x)r_\sigma$$

Coefficients r_σ sigma are determined so that the error of the difference of the approximated stresses σ^{FEM}
And the augmented stresses sigma_star in the least square meaning must be as small as possible:

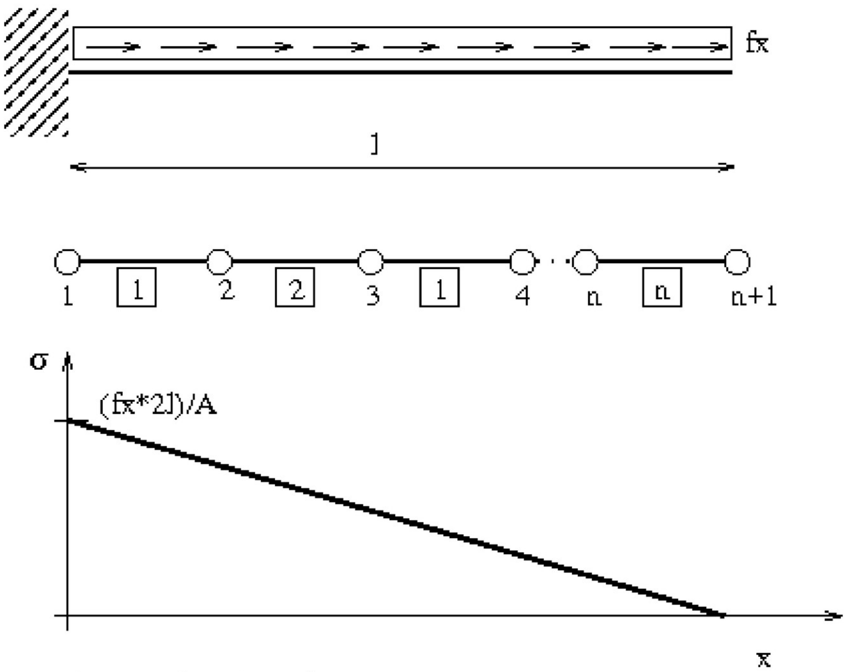
$$\int_\Omega \left(\sigma^* - \sigma^{\text{FEM}}\right)^T \left(\sigma^* - \sigma^{\text{FEM}}\right)\, d\Omega \rightarrow min.$$

After some mathematical operations (see Lecture 11), we obtain:

$$\left(\int_\Omega N^T N\, d\Omega\right) r_\sigma = \left(\int_\Omega N^T \sigma^{\text{FEM}} d\Omega\right)$$

$$Ar_\sigma = b$$

Example 1: Tensile bar



Let's assume: $f_x = 1, l = 3, EA = 1$

Analytical solution:

Stress is determined from the equilibrium equation: $\sigma(x) = f_x(l - x)/A, \;\; x \in (0, l)$

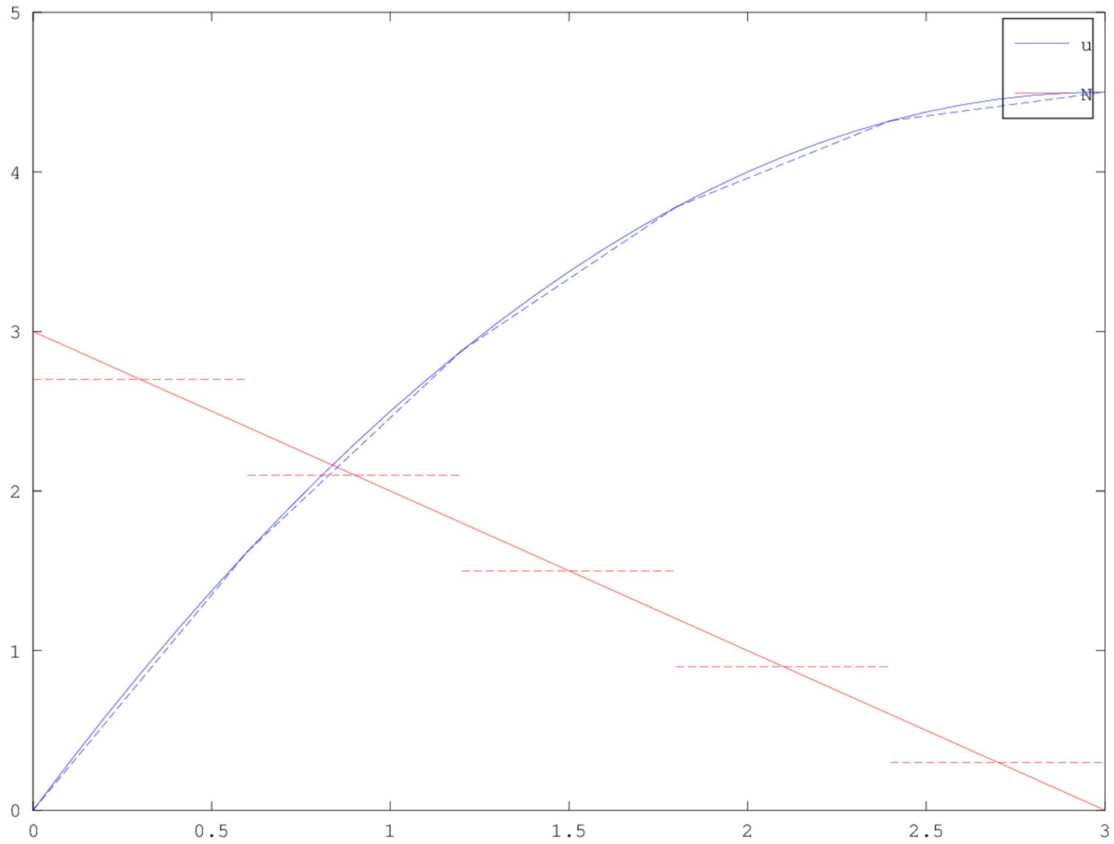
The displacement is derived from the integration:

$$u(x) = \int_0^x \sigma(s)/E ds = \left[(f_x ls - f_x s^2/2)/(EA)\right]_0^x = -(f_x/EA) * (-x^2/2 - lx)$$

FEM solution:

```
# number of elements
n = 5


l = 3/n;

E = 1;
A = 1;

# fem solution
ki = (E*A/l)*[1 -1; -1 1];
fi = [l^2/(2*l) l^2/(2*l)];
K = zeros (n+1);
F = zeros (n+1, 1);
for i=1:n
    loc = [i i+1];
    K(loc, loc) += ki;
    F(loc)+= fi';
endfor
u = K(2:n+1, 2:n+1)\(F(2:n+1,1));
U = [0 ; u]

#plot analytical solution
hold on;
x = 0:0.1:3;
plot (x, -x.^2/2+3*x, "b;u;", x, -x+3, "r;N;")

sig = zeros(n,1);
#evaluate stress and plot obtained solution
for i=1:n
    eps = (U(i+1)-U(i))/l;
    sig(i) = E*eps;
    N = A*E*eps;
    plot ([(i-1)*l i*l], [U(i) U(i+1)], "b--")
    plot ([(i-1)*l i*l], [N N], "r--")
endfor
```

```
n =  5
U =

   0.00000
   1.62000
   2.88000
   3.78000
   4.32000
   4.50000
```



The smoothed stress: σ^*

$$A^e = \int_l \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \begin{bmatrix} N_1 & N_2 \end{bmatrix} dx = \begin{bmatrix} l/3 & l/6 \\ l/6 & l/3 \end{bmatrix} = \begin{bmatrix} l/2 & \\ & l/2 \end{bmatrix}$$

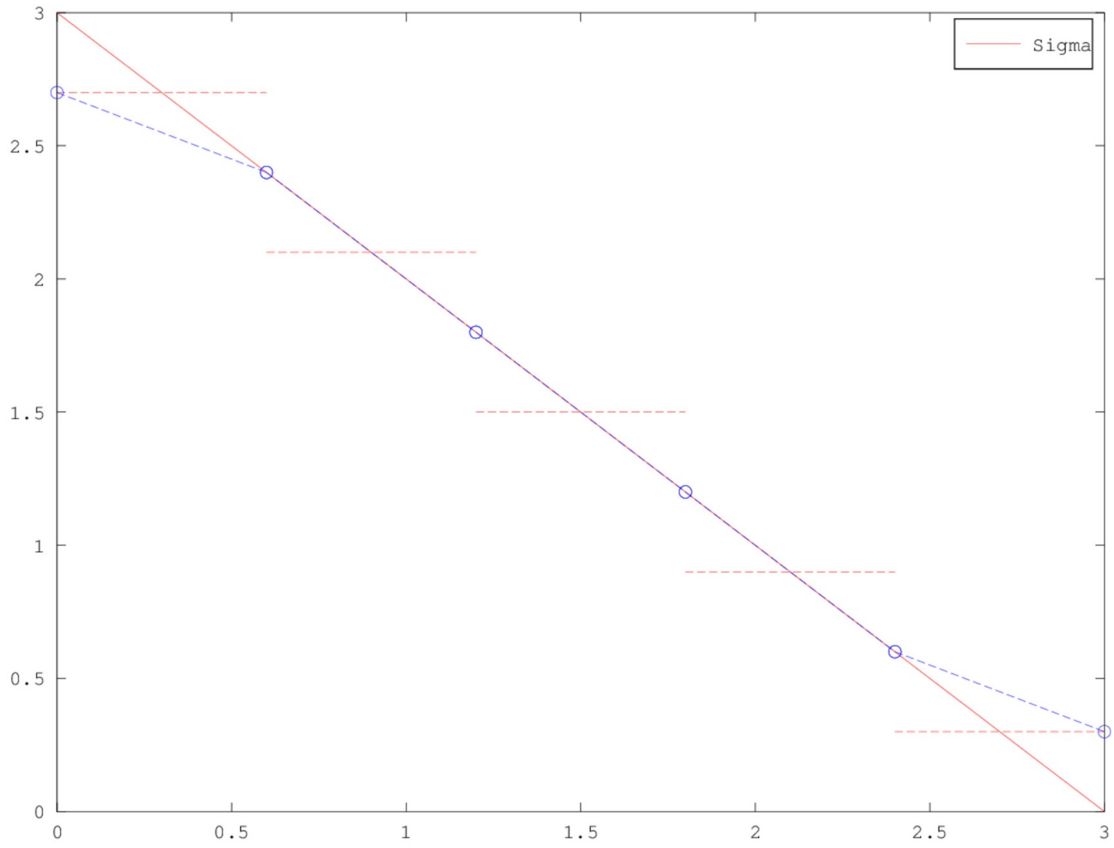
$$b^e = \int N^T \sigma^e \, dx = \begin{Bmatrix} \sigma^e l/2 \\ \sigma^e l/2 \end{Bmatrix}$$

Nodal values of the smoothed stress: $A r_{\sigma} = b$

where the matrix A and vector b is assembled by the localization from A^e and b^e.

In [4]:

```
# evaluate recovered stresses
Ai = [l/2, 0; 0, l/2];
bi = [l/2;l/2];
A = zeros(n+1);
B = zeros(n+1,1);
for i=1:n
    loc = [i i+1];
    A(loc, loc) += Ai;
    B(loc)+= sig(i)*bi;
endfor
sig2 = A\B;

#plot analytical solution
hold on;
x = 0:0.1:3;
plot (x, -x+3, "r;Sigma;")

#plot recovered stresses
hold on;
for i=1:n
    plot ([(i-1)*l i*l], [sig(i) sig(i)], "r--")
    plot ([(i-1)*l i*l], [sig2(i) sig2(i+1)], "o--")
endfor
```



Error estimate:

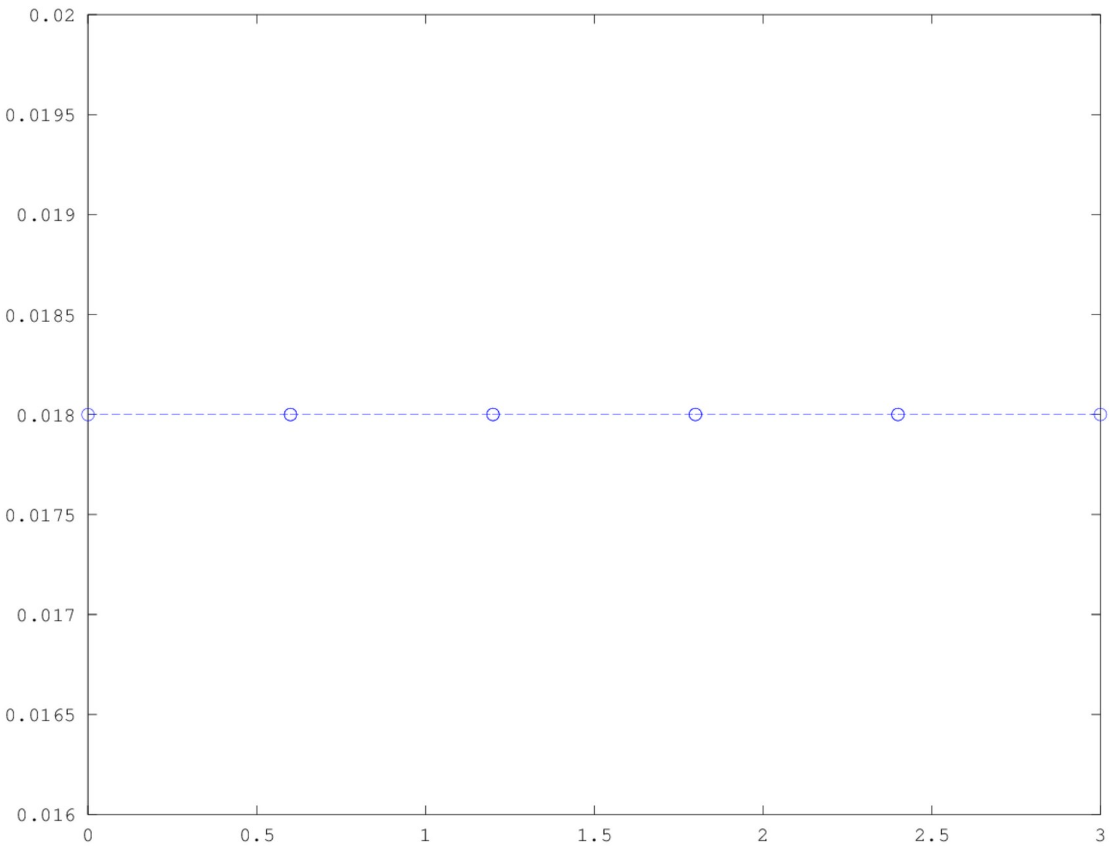
$|e| = \int_\Omega \frac{1}{E(x)} \left(\sigma_x^*(x) - \sigma_x^{\text{FEM}}(x)\right)^2 dx.$

$|e| = \sum_e \int_{l_e} \frac{1}{E(x)} \left(\sigma_x^*(x) - \sigma_x^{\text{FEM}}(x)\right)^2 dx.$

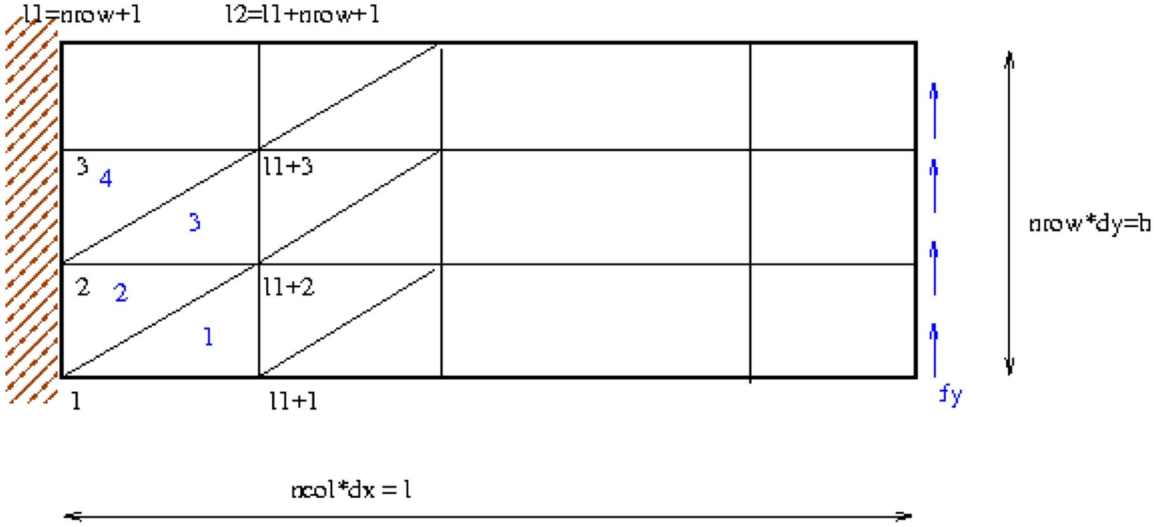
In [5]:

```
# numerical integration
w = [1.0; 1.0];
xi=[-1./sqrt(3); 1/sqrt(3)];

# error contributions from elements
e = zeros(n,1);
err = 0.0;
hold on;
for i=1:n
    for ip=1:2
        sig2xi = (1-xi(ip))/2*sig2(i)+(1+xi(ip))/2*sig2(i+1);
        dsig = sig2xi-sig(i);
        e(i) += (dsig^2/E)*w(ip)*l/2;
    endfor
    err = err+e(i);
endfor
e
err
for i=1:n
    plot ([(i-1)*l i*l], [e(i) e(i)], "o--")
endfor
```

```
e =

   0.018000
   0.018000
   0.018000
   0.018000
   0.018000

err =  0.090000
```



Example 2: Plane stress problem – cantilever beam loaded by a force

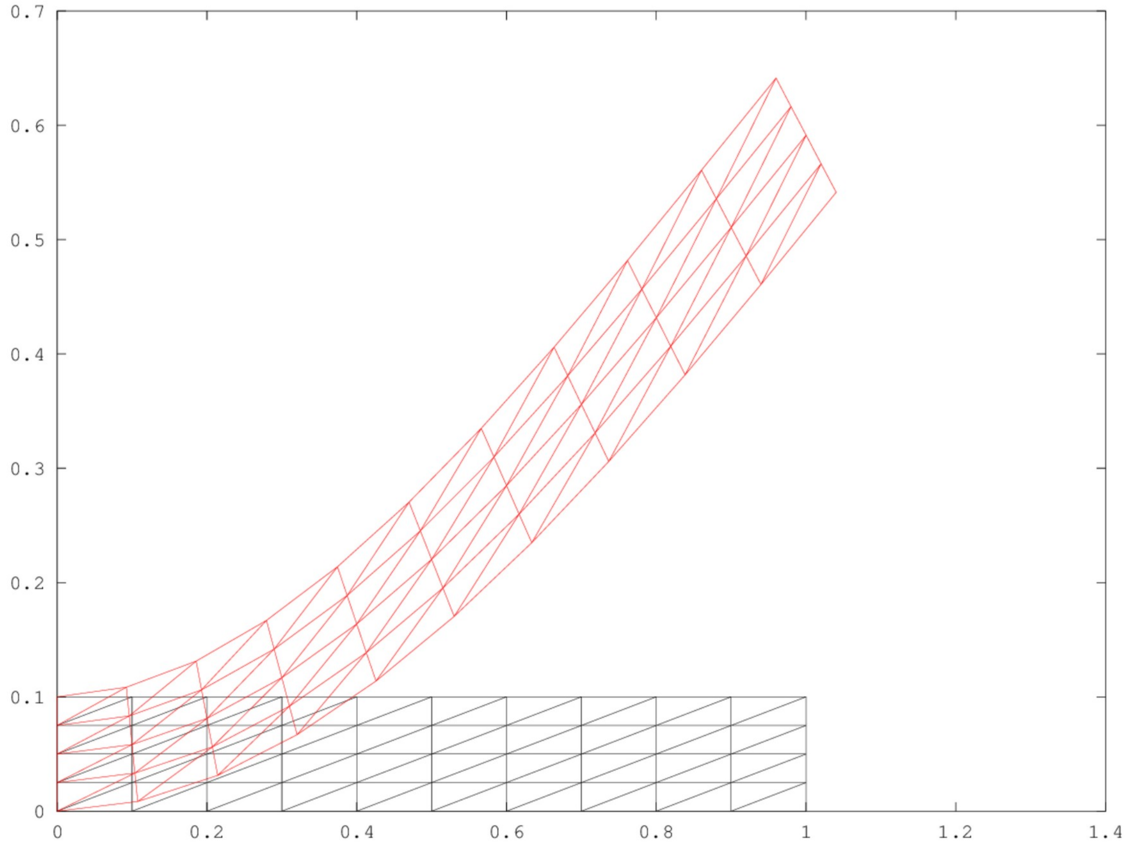


E = 3000 l=1.0, h=0.1

Note: The FEM solution is not presented.

In [7]:

```
# mesh draw
hold on;
sig;
triplot(enodes,x,y,'k')
triplot(enodes,xnew,ynew,'r')
#for ie=1:nelem
#    trimesh(enodes(ie,:),xnew(enodes(ie,:)),ynew(enodes(ie,:)),[sig(ie,1),sig(i
e,1),sig(ie,1)])
#endfor
```



Error estimate:

Nodal values of the smoothed stress:

$$A^e = \int_{A_e} N^T N dA = \int_{A_e} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} [\, N_1 \quad N_2 \quad N_3 \,]\, dA$$

$$b_i = \int_{A_e} N^T \sigma dA = \int_{A_e} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} \sigma(x) dx$$

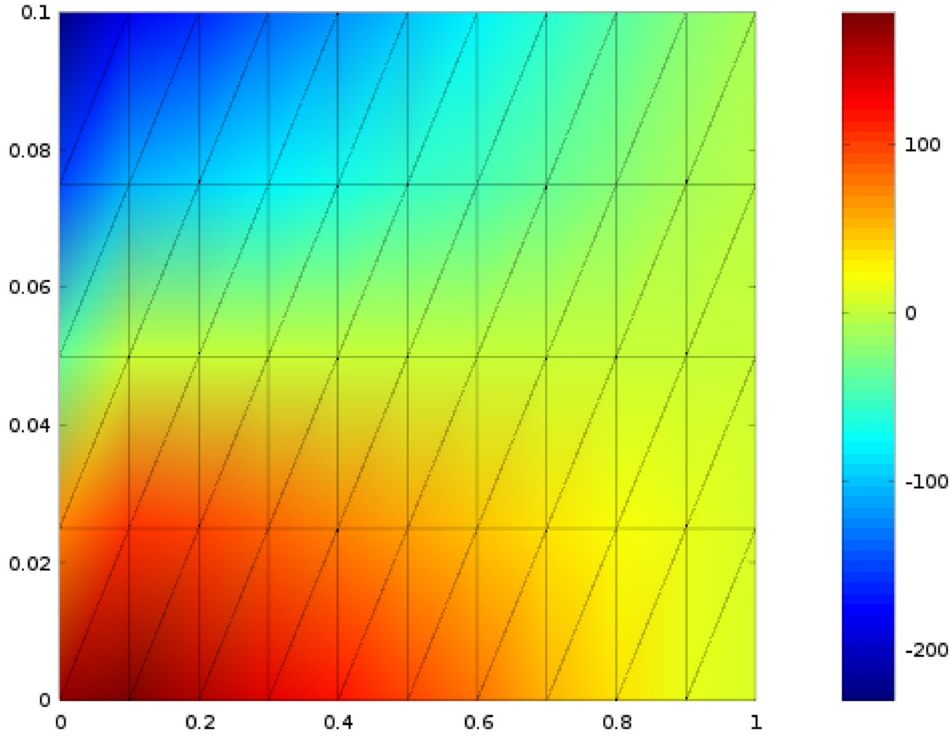
$$A r_{\sigma^*} = b$$

In [8]:

```
function [Ai, bi] = plane_stress_reco (xe,ye,sig)

    # numerical integration
    w = [0.26014666666; 0.26014666666;0.26014666666; -0.28125];
    xi=[0.2, 0.2; 0.6, 0.2; 0.2, 0.6; 0.3333333333, 0.3333333333];

    Ae=(1/2)*((xe(2)*ye(3)-xe(3)*ye(2))-(xe(1)*ye(3)-xe(3)*ye(1))+(xe(1)*ye(2)-x
e(2)*ye(1))) ;
    AAi=zeros(3);
    Ai=zeros(3);
    bi=zeros(3,3); # right hand side for 3 components
    for ip = 1:4
        Ni=[xi(ip,1), xi(ip,2), 1-xi(ip,1)-xi(ip,2)]; # bazove funkce
        AAi = AAi + Ni'*Ni*w(ip)*2*Ae;
        bi = bi+Ni'*sig*w(ip)*2*Ae;
    endfor
    for i=1:3
        Ai(i,i)=sum(AAi(i,:));
    endfor
end

# evaluate recovered stresses

A = zeros(nnodes);
B = zeros(nnodes,3);
for en=1:nelem
    xe = [x(enodes(en,:))] ;
    ye = [y(enodes(en,:))] ;
    [Ai,bi] = plane_stress_reco (xe,ye,sig(en,:));
    lm = [enodes(en,1), enodes(en,2), enodes(en,3)];# code numbers = node numbers

                 Ai;
    B(lm,[1,2,3])+= bi;
endfor
sig2 = A\B

if (0)
    hold on;
    caxis ("auto")
    colorbar;
    for en=1:nelem
        xe = [x(enodes(en,:));  x(enodes(en,1))];
        ye = [y(enodes(en,:));  y(enodes(en,1))];
        sig2e = [sig2(enodes(en,:),1); sig2(enodes(en,1),1)];
        patch (xe, ye, sig2e);
    endfor
endif
```

```
sig2 =

   1.7218e+02   8.3082e+00   9.6599e+00
   7.6158e+01   5.3983e+00   4.4783e+01
  -3.8137e+01  -2.5533e+00   4.4558e+01
  -1.5283e+02  -1.0527e+01   4.4393e+01
  -2.3074e+02  -2.3074e+01   1.1516e+02
   1.7892e+02   6.4295e-01  -2.6171e+01
   1.1051e+02   1.7933e+00   8.4175e+00
   1.9143e+00  -2.5754e-01   9.2766e+00
  -1.0632e+02  -2.4918e+00   7.9001e+00
  -1.8312e+02  -8.8015e+00   4.0493e+01
   1.5869e+02   3.5503e-01  -2.2594e+01
   9.8389e+01  -1.6859e+00   8.0161e+00
   2.1021e+00  -1.6623e+00   9.7197e+00
  -9.4360e+01  -1.7128e+00   8.2875e+00
  -1.6286e+02  -3.0333e+00   3.5981e+01
   1.3870e+02   1.5161e-01  -1.8919e+01
   8.6325e+01  -1.6739e+00   8.1711e+00
   2.0146e+00  -1.6998e+00   9.6890e+00
  -8.2309e+01  -1.6958e+00   8.1974e+00
  -1.4265e+02  -2.7036e+00   3.2204e+01
   1.1861e+02  -5.0728e-02  -1.5266e+01
   7.4282e+01  -1.6730e+00   8.1861e+00
   2.0074e+00  -1.6751e+00   9.6924e+00
  -7.0266e+01  -1.6754e+00   8.1866e+00
  -1.2254e+02  -2.4861e+00   2.8535e+01
   9.8515e+01  -2.5346e-01  -1.1616e+01
   6.2236e+01  -1.6730e+00   8.1864e+00
   2.0075e+00  -1.6730e+00   9.6921e+00
  -5.8221e+01  -1.6729e+00   8.1863e+00
  -1.0245e+02  -2.2812e+00   2.4886e+01
   7.8418e+01  -4.5632e-01  -7.9659e+00
   5.0190e+01  -1.6731e+00   8.1863e+00
   2.0076e+00  -1.6731e+00   9.6921e+00
  -4.6175e+01  -1.6730e+00   8.1863e+00
  -8.2352e+01  -2.0785e+00   2.1236e+01
   5.8322e+01  -6.5940e-01  -4.3150e+00
   3.8145e+01  -1.6746e+00   8.1871e+00
   2.0080e+00  -1.6750e+00   9.6920e+00
  -3.4129e+01  -1.6731e+00   8.1853e+00
  -6.2257e+01  -1.8758e+00   1.7585e+01
   3.8217e+01  -8.6687e-01  -6.6091e-01
   2.6097e+01  -1.6769e+00   8.1966e+00
   2.0150e+00  -1.6772e+00   9.6951e+00
  -2.2082e+01  -1.6739e+00   8.1733e+00
  -4.2168e+01  -1.6730e+00   1.3933e+01
   1.8098e+01  -1.5020e+00   3.0253e+00
   1.4038e+01  -1.7895e+00   8.1952e+00
   2.0173e+00  -1.5587e+00   9.6894e+00
  -1.0016e+01  -1.4535e+00   8.1699e+00
  -2.2076e+01  -1.4645e+00   1.0284e+01
   1.2078e+01  -1.7451e+00   3.0195e+00
   8.0066e+00  -2.1230e+00   7.7018e+00
   2.0061e+00  -1.6627e+00   9.6658e+00
  -3.9772e+00  -1.2099e+00   8.6548e+00
  -8.9892e+00  -1.0215e+00   8.4720e+00
```

Stress profile σ_x^*



Error estimate:

$$||e|| = \int \left(\sigma^*(x) - \sigma^{\text{FEM}}(x)\right)^T D^{-1} \left(\sigma^*(x) - \sigma^{\text{FEM}}(x)\right) d\Omega$$

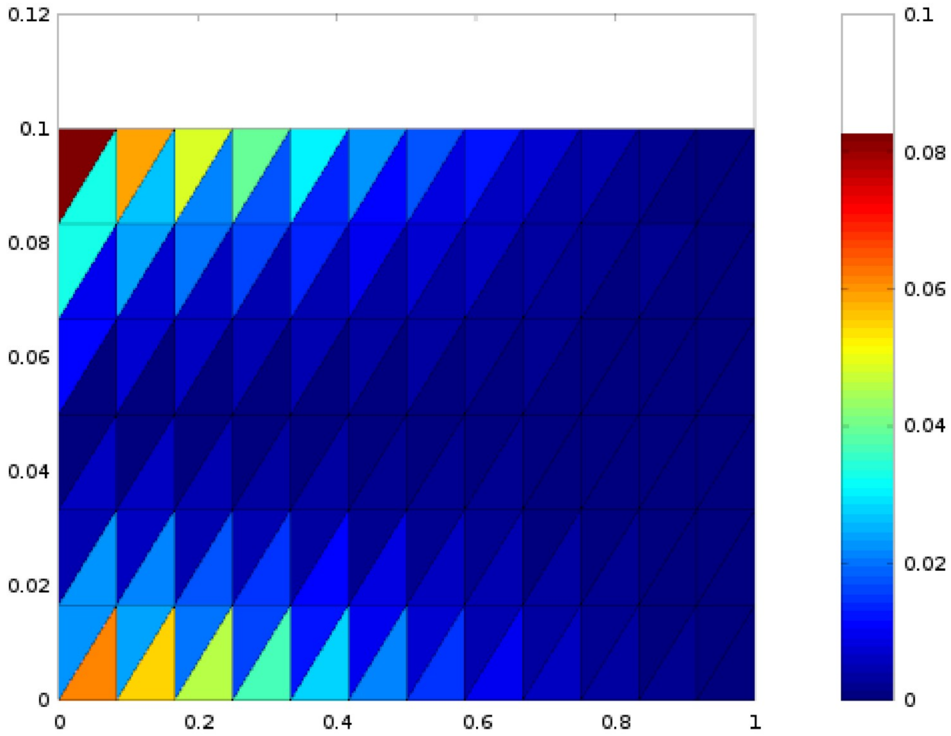
In [9]:

```
# numerical integration
w = [0.26014666666; 0.26014666666;0.26014666666; -0.28125];
xi=[0.2, 0.2; 0.6, 0.2; 0.2, 0.6; 0.3333333333, 0.3333333333];

# error contributions from elements
e = zeros(nelem,1);
err = 0.0;
sig
for en=1:nelem
    sig2e = [sig2(enodes(en,1),:); sig2(enodes(en,2),:);sig2(enodes(en,3),:)];
    xe = [x(enodes(en,:))] ;
    ye = [y(enodes(en,:))] ;
    Ae=(1/2)*((xe(2)*ye(3)-xe(3)*ye(2))-(xe(1)*ye(3)-xe(3)*ye(1))+(xe(1)*ye(2)-x
e(2)*ye(1))) ;
    [ke,dbe,de,be] = plane_stress(xe,ye,E,nu) ; # de

    for ip=1:4
        xiip = [xi(ip,1), xi(ip,2), 1-xi(ip,1)-xi(ip,2)];
        sig2xi = xiip*sig2e;  # stress at integration point
        dsig = sig2xi-sig(en,:); # diff. sigma - sigma_fem
        e(en) += (dsig)*inv(de)*dsig'*w(ip)*Ae*2;

    endfor
    err = err+e(en);
endfor
e
err
```

```
sig =

   2.2973e+02   5.1542e+00  -9.4548e+01
   1.1462e+02   1.1462e+01   1.1387e+02
   1.1395e+02   4.7427e+00  -9.2630e+01
  -9.8909e-02  -9.8909e-03   1.1311e+02
   2.8302e-01   3.8094e+00  -9.2980e+01
  -1.1459e+02  -1.1459e+01   1.1354e+02
  -1.1315e+02   2.9511e+00  -9.5525e+01
  -2.3074e+02  -2.3074e+01   1.1516e+02
   2.0505e+02   4.3949e+00  -8.4683e+01
   1.0199e+02  -7.6203e+00   1.0072e+02
   1.0312e+02   3.7080e+00  -8.1568e+01
  -3.4434e-01  -6.6868e+00   1.0467e+02
   6.3556e-01   3.1122e+00  -8.1496e+01
  -1.0294e+02  -6.5129e+00   1.0499e+02
  -1.0203e+02   2.5426e+00  -8.4465e+01
  -2.0548e+02  -6.2815e+00   1.0184e+02
   1.8120e+02   3.8000e+00  -7.3744e+01
   8.9806e+01  -7.1298e+00   9.0646e+01
   9.0837e+01   3.1833e+00  -7.0663e+01
  -4.7322e-01  -6.6514e+00   9.3648e+01
   4.4925e-01   2.5733e+00  -7.0585e+01
  -9.0775e+01  -6.0289e+00   9.3655e+01
  -8.9975e+01   1.9711e+00  -7.3524e+01
  -1.8107e+02  -5.3610e+00   9.0567e+01
   1.5708e+02   3.1939e+00  -6.2788e+01
   7.7811e+01  -6.5391e+00   7.9774e+01
   7.8723e+01   2.5852e+00  -5.9708e+01
  -4.2705e-01  -5.9431e+00   8.2721e+01
   3.6496e-01   1.9770e+00  -5.9639e+01
  -7.8663e+01  -5.3380e+00   8.2652e+01
  -7.7993e+01   1.3690e+00  -6.2581e+01
  -1.5690e+02  -4.7210e+00   7.9568e+01
   1.3293e+02   2.5856e+00  -5.1836e+01
   6.5825e+01  -5.9316e+00   6.8826e+01
   6.6616e+01   1.9773e+00  -4.8757e+01
  -3.6530e-01  -5.3236e+00   7.1769e+01
   3.0396e-01   1.3689e+00  -4.8688e+01
  -6.6555e+01  -4.7150e+00   7.1700e+01
  -6.6008e+01   7.6057e-01  -5.1631e+01
  -1.3274e+02  -4.1062e+00   6.8619e+01
   1.0878e+02   1.9772e+00  -4.0886e+01
   5.3840e+01  -5.3232e+00   5.7875e+01
   5.4509e+01   1.3688e+00  -3.7806e+01
  -3.0422e-01  -4.7148e+00   6.0818e+01
   2.4330e-01   7.6045e-01  -3.7738e+01
  -5.4449e+01  -4.1063e+00   6.0749e+01
  -5.4023e+01   1.5210e-01  -4.0681e+01
  -1.0859e+02  -3.4980e+00   5.7669e+01
   8.4624e+01   1.3686e+00  -2.9936e+01
   4.1855e+01  -4.7148e+00   4.6924e+01
   4.2403e+01   7.5960e-01  -2.6855e+01
  -2.4340e-01  -4.1064e+00   4.9867e+01
   1.8238e-01   1.5130e-01  -2.6787e+01
  -4.2342e+01  -3.4981e+00   4.9798e+01
  -4.2038e+01  -4.5623e-01  -2.9730e+01
  -8.4441e+01  -2.8897e+00   4.6719e+01
   6.0471e+01   7.5991e-01  -1.8983e+01
   2.9871e+01  -4.1067e+00   3.5974e+01
   3.0296e+01   1.4457e-01  -1.5901e+01
```

```
  -1.8118e-01  -3.4988e+00   3.8917e+01
   1.2217e-01  -4.6533e-01  -1.5837e+01
  -3.0235e+01  -2.8904e+00   3.8848e+01
  -3.0052e+01  -1.0634e+00  -1.8785e+01
  -6.0291e+01  -2.2815e+00   3.5768e+01
   3.6304e+01   1.3911e-01  -8.0244e+00
   1.7875e+01  -3.4996e+00   2.5025e+01
   1.8179e+01  -4.6381e-01  -4.9058e+00
  -1.0865e-01  -2.8959e+00   2.7971e+01
   7.5071e-02  -1.0587e+00  -4.8791e+00
  -1.8113e+01  -2.2889e+00   2.7900e+01
  -1.8051e+01  -1.6614e+00  -7.9038e+00
  -3.6161e+01  -1.6742e+00   2.4817e+01
   1.2078e+01  -1.7451e+00   3.0195e+00
   5.9122e+00  -2.9001e+00   1.4081e+01
   6.0298e+00  -1.7237e+00   6.0050e+00
  -7.1441e-02  -2.2888e+00   1.6990e+01
   5.9889e-02  -9.7554e-01   6.0021e+00
  -6.0300e+00  -1.6692e+00   1.6958e+01
  -5.9616e+00  -9.8495e-01   3.0042e+00
  -1.2017e+01  -1.0580e+00   1.3940e+01

e =

   1.0268e-02
   8.1590e-03
   1.3137e-02
   6.2613e-03
   1.3016e-02
   6.4484e-03
   1.5302e-02
   3.2585e-03
   6.2711e-03
   1.0400e-02
   8.1474e-03
   9.2370e-03
   8.1153e-03
   9.3210e-03
   9.7814e-03
   6.3693e-03
   4.8997e-03
   8.2860e-03
   6.2832e-03
   7.2454e-03
   6.2813e-03
   7.2292e-03
   7.5861e-03
   4.9375e-03
   3.6814e-03
   6.2708e-03
   4.6739e-03
   5.4983e-03
   4.6647e-03
   5.4849e-03
   5.6564e-03
   3.7171e-03
   2.6369e-03
   4.5332e-03
   3.2973e-03
   3.9937e-03
   3.2894e-03
```

```
   3.9830e-03
   4.0134e-03
   2.6681e-03
   1.7669e-03
   3.0776e-03
   2.1601e-03
   2.7300e-03
   2.1538e-03
   2.7211e-03
   2.6524e-03
   1.7925e-03
   1.0715e-03
   1.9038e-03
   1.2627e-03
   1.7063e-03
   1.2581e-03
   1.6991e-03
   1.5730e-03
   1.0914e-03
   5.5086e-04
   1.0114e-03
   6.0532e-04
   9.2250e-04
   6.0229e-04
   9.1727e-04
   7.7534e-04
   5.6490e-04
   2.0461e-04
   4.0079e-04
   1.8672e-04
   3.7867e-04
   1.8629e-04
   3.7522e-04
   2.6127e-04
   2.1316e-04
   5.3013e-06
   8.1341e-05
   1.0145e-05
   7.4529e-05
   1.1178e-05
   6.4665e-05
   2.8891e-05
   2.7579e-05

err =  0.30338
```

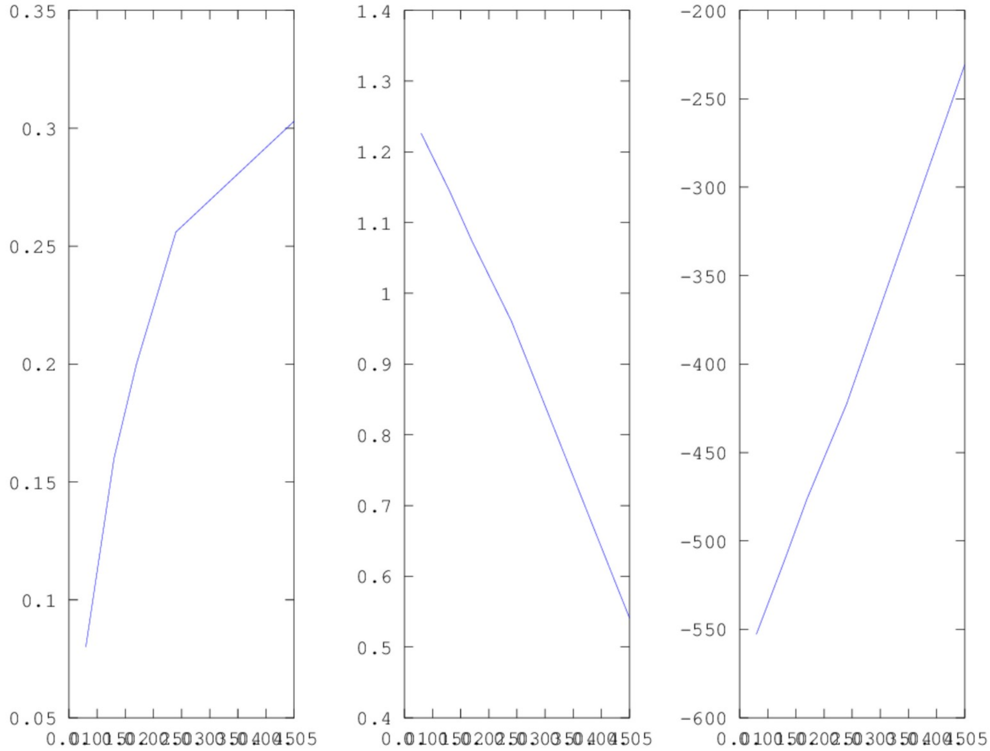


Error estimate for different mesh sizes

sizes	h (char. elem. size $= \sqrt{A_e}$)	w(l)	$\sigma_x(0)$	$\|\|e\|\|$
4x10	0.05	0.541	-230.7	0.303
6x20	0.029	0.961	-422.7	0.265
8x25	0.022	1.074	-476.1	0.20
10x30	0.018	1.1453	-510.8	0.16
15x40	0.013	1.226	-552.7	0.08

In [10]:

```
h=[0.05;0.029;0.022; 0.018; 0.013];
e=[0.303;0.256; 0.20; 0.16; 0.08];
w=[0.541; 0.961; 1.074; 1.145; 1.226];
s=[-230.7; -422.7; -476.1; -510.8; -552.7];
subplot(1,3,1)
plot (h,e)
subplot(1,3,2)
plot (h,w)
subplot(1,3,3)
plot (h,s)
```



In [ ]:

```
```

In [ ]:

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References

- English course of “Numerical analysis of structures” by J. Zeman (jan.zeman@fsv.cvut.cz)

- Czech course of “Numerická analýza konstrukcí” (Numerical analysis of structures) by B. Patzák (borek.patzak@fsv.cvut.cz)

- J. Fish and T. Belytschko: A First Course in Finite Elements, John Wiley & Sons, 2007