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# Mixed convection flow in a vertical circular duct with time-periodic boundary conditions: steady-periodic regime

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**Abstract**

Mixed convection flow in a vertical circular duct subjected to a periodic sinusoidal temperature change at the wall is investigated. The analysis is performed by considering fully-developed parallel flow and steady-periodic regime. The local momentum and energy balance equations, together with the constraint equations which arise from the definition of mean velocity and mean temperature, are written in a dimensionless form and mapped into equations in the complex domain. One obtains two independent boundary value problems, which provide the mean value and the oscillating term of the velocity and temperature distributions. These boundary value problems are solved analytically, and the velocity and temperature distributions are obtained as functions of three parameters: the Prandtl number, $Pr$, the dimensionless frequency $\Omega$, the ratio between the Grashof number $Gr$ and the Reynolds number $Re$. The Fanning friction factor and the dimensionless heat flux are evaluated.


*Keywords:* Laminar flow; Mixed convection; Circular duct; Analytical methods; Unsteady solutions

## 1. Introduction

Unsteady boundary conditions compatible with a steady-periodic regime have been widely studied with reference to forced convection flows [1–5]. On the contrary, the effects of buoyancy on steady-periodic flows are not so widely investigated. The interest deserved to this argument is mainly due to its technological applications, such as, for instance, the thermal control of electric resistors in alternating current or the development of heat-exchange enhancement techniques based on flows with time-oscillating mass rates.

In Refs. [6–9], natural convection is studied numerically in a two-dimensional rectangular cavity with two horizontal adiabatic walls, one vertical wall with uniform and constant temperature and the other vertical wall subjected either to a uniform temperature distribution which varies in time according to a sinusoidal law [6,7], or to a uniform heat flux distribution which undergoes a periodic square wave time-change [8,9]. In Refs. [7–9], the existence of a resonance phenomenon is predicted, namely the existence of a frequency of the wall thermal oscillations which corresponds to a maximum of the heat flux amplitude throughout any vertical surface in the cavity. Discrepant results have been found in Refs. [6,7] about the influence of the oscillation frequency on the time-averaged heat flux: this influence is considered as negligible in Ref. [6], while in Ref. [7] it is shown that, for wall temperature oscillations having a sufficiently high amplitude, there exists a critical frequency which corresponds to a maximum of the time-averaged heat flux. In Ref. [10], free convection is studied around a vertical plane subjected to a uniform and time-periodic temperature distribution. In Ref. [11], mixed convection in an inclined parallel-plate channel is investigated, with thermal boundary conditions such

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**Nomenclature**

| | |
|---|---|
| $A(t)$ | function of time, defined in Eq. (7) |
| $A_1$, $A_2$ | integration constants |
| $f$ | Fanning friction factor, defined by Eq. (15) |
| $f_a^*$, $f_b^*$ | dimensionless complex valued functions, defined in Eq. (37) |
| $F(s)$ | function defined in Eq. (32) |
| $g$ | magnitude of the gravitational acceleration |
| $G_1$, $G_2$, $G_3$, $G_4$ | Functions defined in Eq. (34) |
| $Gr$ | Grashof number, defined in Eq. (10) |
| i | imaginary unit |
| $I_n$ | modified Bessel function of first kind and order $n$ |
| $K_n$ | modified Bessel function of second kind and order $n$ |
| $k$ | thermal conductivity |
| $n$ | integer number |
| $p$ | pressure |
| $P$ | difference between the pressure and the hydrostatic pressure |
| $Pr$ | Prandtl number, defined in Eq. (10) |
| $q$ | heat flux per unit area |
| $R$ | radial coordinate |
| $R_0$ | radius of the cylinder |
| $r$ | $= R/(2R_0)$ dimensionless radial coordinate |
| $Re$ | Reynolds number, defined in Eq. (10) |
| $\mathscr{R}e$ | real part of a complex number |
| $s$ | complex number |
| $t$ | time |
| $T$ | temperature |
| $T_0$ | mean temperature in a duct section, defined by Eq. (4) |
| $T_1$ | mean wall temperature |
| $u$ | $= U/U_0$, dimensionless velocity |
| $u^*$ | dimensionless complex-valued function, defined in Eq. (19) |
| $u_a^*$, $u_b^*$ | dimensionless complex-valued functions, defined in Eq. (21) |
| $\mathbf{U}$ | fluid velocity |
| $U$ | $X$-component of the fluid velocity |
| $U_0$ | mean fluid velocity in a duct section, defined by Eq. (3) |
| $W$ | wronskian |
| $X$ | axial coordinate |

*Greek symbols*

| | |
|---|---|
| $\alpha$ | thermal diffusivity |
| $\beta$ | volumetric coefficient of thermal expansion |
| $\Delta T$ | amplitude of the wall temperature oscillations |
| $\lambda$ | dimensionless parameter, defined in Eq. (10) |
| $\lambda^*$ | dimensionless complex-valued function, defined in Eq. (19) |
| $\lambda_a^*$, $\lambda_b^*$ | dimensionless complex-valued functions, defined in Eq. (21) |
| $\eta$ | dimensionless parameter, defined in Eq. (10) |
| $\theta$ | dimensionless temperature, defined in Eq. (10) |
| $\theta^*$ | dimensionless complex-valued function, defined in Eq. (19) |
| $\theta_a^*$, $\theta_b^*$ | dimensionless complex-valued functions, defined in Eq. (21) |
| $\mu$ | dynamic viscosity |
| $\nu$ | kinematic viscosity |
| $\Phi$ | dimensionless heat flux, defined in Eq. (40) |
| $\Phi_a^*$, $\Phi_b^*$ | dimensionless complex-valued functions, defined in Eq. (41) |
| $\xi$ | dimensionless parameter, defined in Eq. (10) |
| $\varrho$ | mass density |
| $\varrho_0$ | mass density for $T = T_0$ |
| $\overline{\tau}_w$ | average wall shear stress, defined by Eq. (29) |
| $\omega$ | frequency of the wall temperature oscillations, defined in Eq. (1) |
| $\Omega$ | dimensionless frequency, defined in Eq. (10) |

that one wall is kept at a constant temperature and the other is subjected to a sinusoidal temperature oscillation. The authors show that, for any value of the Prandtl number greater that 0.277, there exists a resonance frequency of the friction factor evaluated at the wall with an unsteady temperature. Moreover, for every plane which lies between the midplane of the channel and the wall subjected to the oscillating boundary condition, any value of the Prandtl number corresponds to a resonance frequency of the dimensionless heat flux.

The aim of the present paper is to investigate laminar mixed convection in the steady-periodic regime for a Newtonian fluid in a vertical circular duct. Reference is made to the fully developed region, and the thermal boundary condition is given by a uniform wall temperature distribution which undergoes a sinusoidal time-change. The study is performed with analytical methods. In particular, the local momentum and energy balance equations, together with the boundary conditions and the constraint equations which arise from the definition of mean velocity and mean temperature, are written in a dimensionless form and mapped onto differential equations and constraint equations in the complex domain. A pair of independent boundary value problems is obtained, which provides both the mean value and the oscillating part of the temperature and velocity distributions. The pair of boundary value problems is solved analytically and the dimensionless distributions are ob-

tained as functions of three parameters: the Prandtl number $Pr$, the dimensionless angular frequency $\Omega$, the ratio between the Grashof number $Gr$ and the Reynolds number $Re$.

## 2. Governing equations

Let us consider a Newtonian fluid which steadily flows in a vertical duct with an infinite length and a circular cross section. The thermal conductivity $k$, the thermal diffusivity $\alpha$ and the dynamic viscosity $\mu$ of the fluid are treated as constants. Let $X$ be the axial coordinate, chosen parallel to the gravitational acceleration $\mathbf{g}$ but with opposite direction. Let us assume that the flow is laminar and parallel, so that only the $X$-component $U$ of the velocity vector $\mathbf{U}$ is nonzero. A parallel-velocity regime always exists in the case of fully developed flow inside a vertical duct, provided that the buoyancy forces are not too intense. In other words, a parallel-velocity regime for vertical mixed convection flows is stable for sufficiently small values of the Grashof number. For higher values of the Grashof number, multi-cell patterns arise, so that even in the fully developed region no parallel flow exists anymore.

The effect of viscous dissipation in the fluid is neglected and the Boussinesq approximation is employed. Since this approximation implies that the velocity field is solenoidal, one has $\partial U/\partial X = 0$, i.e. $U = U(R,t)$. Let us assume that the wall at $R = R_0$ is kept at an oscillating temperature, namely

$$T(X,R_0,t) = T_1 + \Delta T\cos(\omega t). \tag{1}$$

Moreover, since the thermal boundary condition (1) does not yield any net fluid heating or cooling, heat transfer occurs only in the radial direction, so that

$$\frac{\partial T}{\partial X} = 0, \tag{2}$$

i.e. $T = T(R,t)$. The prescribed mass flow rate is assumed to be stationary, so that the mean velocity in a duct cross section, defined as

$$U_0 = \frac{2}{R_0^2}\int_0^{R_0} \mathrm{d}R\, U(R,t)R \tag{3}$$

is time-independent. Moreover, with reference to the Boussinesq approximation, let us choose the reference temperature as the average value of the fluid temperature with respect both to a duct cross section and to a period of time, namely

$$T_0 = \frac{\omega}{\pi R_0^2}\int_0^{2\pi/\omega}\mathrm{d}t\int_0^{R_0}\mathrm{d}R\, T(R,t)R. \tag{4}$$

Obviously, since $\partial T/\partial X = 0$, $T_0$ is a constant. On account of the above assumptions, the $X$-component and the $R$-component of the momentum balance equation are

$$\varrho_0\frac{\partial U}{\partial t} = -\frac{\partial P}{\partial X} + \varrho_0 g\beta(T - T_0) + \mu\nabla^2 U, \tag{5}$$

$$\frac{\partial P}{\partial R} = 0, \tag{6}$$

where $P = p + \varrho_0 g\beta$ is the difference between the pressure and the hydrostatic pressure. By differentiating both sides of Eq. (5) with respect to $X$, one obtains $\partial^2 P/\partial X^2 = 0$. This result implies the existence of a function $A(t)$ such that

$$\frac{\partial P}{\partial X} = -A(t). \tag{7}$$

Then, Eq. (5) can be rewritten as

$$\varrho_0\frac{\partial U}{\partial t} = A(t) + \varrho_0 g\beta(T - T_0) + \frac{\mu}{R}\frac{\partial}{\partial R}\left(R\frac{\partial U}{\partial R}\right). \tag{8}$$

The energy balance equation is given by

$$\frac{\partial T}{\partial t} = \frac{\alpha}{R}\frac{\partial}{\partial R}\left(R\frac{\partial T}{\partial R}\right). \tag{9}$$

Let us define the following dimensionless variables:

$$\begin{aligned} &\theta = \frac{T - T_0}{\Delta T}, \quad r = \frac{R}{2R_0}, \quad u = \frac{U}{U_0}, \quad \eta = \omega t,\\ &\Omega = \frac{4R_0^2\omega}{\nu}, \quad \lambda = \frac{4R_0^2 A(t)}{\mu U_0}, \quad Re = \frac{2R_0U_0}{\nu},\\ &Gr = \frac{8g\beta\Delta T R_0^3}{\nu^2}, \quad \xi = \frac{T_1 - T_0}{\Delta T}, \quad Pr = \frac{\nu}{\alpha}. \end{aligned} \tag{10}$$

Eqs. (8) and (9) can be rewritten in the following dimensionless form:

$$\Omega\frac{\partial u}{\partial \eta} = \lambda + \frac{Gr}{Re}\theta + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right), \tag{11}$$

$$\Omega Pr\frac{\partial \theta}{\partial \eta} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \theta}{\partial r}\right). \tag{12}$$

The boundary conditions for the dimensionless velocity distribution $u(r,\eta)$ and for the dimensionless temperature distribution $\theta(r,\eta)$ are as follows:

$$u(1/2,\eta) = 0, \quad \theta(1/2,\eta) = \xi + \cos\eta. \tag{13}$$

Moreover, on account of Eqs. (3) and (4), one obtains two constraint equations for the dimensionless velocity and for the dimensionless temperature respectively, namely

$$\begin{aligned} &\int_0^{1/2}\mathrm{d}r\, u(r,\eta)r = \frac{1}{8},\\ &\int_0^{2\pi}\mathrm{d}\eta\int_0^{1/2}\mathrm{d}r\,\theta(r,\eta)r = 0. \end{aligned} \tag{14}$$

One can define the Fanning friction factor as

$$f = \frac{2\bar{\tau}_w}{\varrho_0 U_0^2} = -\frac{2}{Re}\frac{\partial u}{\partial r}\bigg|_{r=1/2}. \tag{15}$$

By differentiating with respect to $\eta$ both sides of the integral constraint on $u(r,\eta)$ expressed in Eq. (14), one obtains

$$\int_0^{1/2} \mathrm{d}r \frac{\partial u(r,\eta)}{\partial \eta} r = 0. \tag{16}$$

If one multiplies both sides of Eq. (11) by $r$ and integrates with respect to $r$ in the interval $[0,1/2]$, one is led to the integral balance equation

$$\frac{\partial u}{\partial r}\bigg|_{r=1/2} + \frac{\lambda}{4} + 2\frac{Gr}{Re}\int_0^{1/2} \mathrm{d}r\,\theta(r,\eta) r = 0. \tag{17}$$

It can be easily shown that, as a consequence of Eq. (17), the parameters $f$ and $\lambda$ are related through the expression

$$fRe = \frac{\lambda}{2} + 4\frac{Gr}{Re}\int_0^{1/2} \mathrm{d}r\,\theta(r,\eta) r. \tag{18}$$

## 3. Analytical solution: velocity and temperature distributions

In the steady periodic regime, one can solve analytically the momentum and energy balance equations (11), (12), together with the boundary conditions (13) and the constraints (14), by considering the functions $u(r,\eta)$, $\theta(r,\eta)$ and $\lambda(\eta)$ as the real parts of three complex valued functions, namely:

$$\begin{aligned} u(r,\eta) &= \mathscr{R}e[u^*(r,\eta)],\\ \theta(r,\eta) &= \mathscr{R}e[\theta^*(r,\eta)],\\ \lambda(\eta) &= \mathscr{R}e[\lambda^*(\eta)]. \end{aligned} \tag{19}$$

On account of Eqs. (11)–(14), the complex valued functions $u^*(r,\eta)$, $\theta^*(r,\eta)$ and $\lambda^*(\eta)$ must be the solution of the following boundary value problem:

$$\Omega\frac{\partial u^*}{\partial \eta} = \lambda^* + \frac{Gr}{Re}\theta^* + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u^*}{\partial r}\right),$$

$$\Omega Pr\frac{\partial \theta^*}{\partial \eta} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \theta^*}{\partial r}\right),$$

$$u^*(1/2,\eta) = 0, \quad \theta^*(1/2,\eta) = \xi + \mathrm{e}^{\mathrm{i}\eta},$$

$$\int_0^{1/2} \mathrm{d}r\,u^*(r,\eta) r = \frac{1}{8}, \quad \int_0^{2\pi}\mathrm{d}\eta\int_0^{1/2}\mathrm{d}r\,\theta^*(r,\eta) r = 0. \tag{20}$$

Therefore, one has

$$\begin{aligned} u^*(r,\eta) &= u_a^*(r) + \frac{Gr}{Re}u_b^*(r)\mathrm{e}^{\mathrm{i}\eta},\\ \theta^*(r,\eta) &= \theta_a^*(r) + \theta_b^*(r)\mathrm{e}^{\mathrm{i}\eta},\\ \lambda^*(\eta) &= \lambda_a^* + \frac{Gr}{Re}\lambda_b^*\mathrm{e}^{\mathrm{i}\eta}. \end{aligned} \tag{21}$$

By substituting Eq. (21) into Eq. (20), one obtains two independent boundary value problems. The first boundary value problem is expressed as

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u_a^*}{\mathrm{d}r}\right) + \lambda_a^* + \frac{Gr}{Re}\theta_a^* = 0,$$

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\theta_a^*}{\mathrm{d}r}\right) = 0,$$

$$u_a^*(1/2) = 0, \quad \theta_a^*(1/2) = \xi,$$

$$\int_0^{1/2}\mathrm{d}r\,u_a^*(r) r = \frac{1}{8}, \quad \int_0^{1/2}\mathrm{d}r\,\theta_a^*(r) r = 0, \tag{22}$$

while the second is given by

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u_b^*}{\mathrm{d}r}\right) + \lambda_b^* + \theta_b^* = \mathrm{i}\Omega u_b^*,$$

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\theta_b^*}{\mathrm{d}r}\right) = \mathrm{i}\Omega Pr\theta_b^*,$$

$$u_b^*(1/2) = 0, \quad \theta_b^*(1/2) = 1,$$

$$\int_0^{1/2}\mathrm{d}r\,u_b^*(r) r = 0. \tag{23}$$

The boundary value problem (22) provides the average values of the complex fields $u^*$ and $\theta^*$, as well as the average value of $\lambda^*$. On the contrary, the solution of the boundary value problem (23) yields the oscillating components of these quantities. Indeed, the values of $\lambda_a^*$, $\lambda_b^*$ and $\xi$ can be determined by employing the constraint equations. For instance, the constraint on $\theta_a^*(r)$ yields $\xi = 0$, i.e. $T_1 = T_0$. The solution of Eq. (22) is

$$\theta_a^*(r) = 0, \quad u_a^*(r) = 2(1-4r^2), \quad \lambda_a^* = 32. \tag{24}$$

As expected, the velocity distribution $u_a^*(r)$ is the well known Poiseuille velocity profile. On the other hand, the dimensionless temperature distribution $\theta_b^*(r)$ which fulfils Eq. (23) is given by

$$\theta_b^*(r) = \frac{I_0\left(r\sqrt{\mathrm{i}\Omega Pr}\right)}{I_0\left(\sqrt{\mathrm{i}\Omega Pr}/2\right)}. \tag{25}$$

By substituting Eq. (25) into Eq. (23), one obtains the following boundary value problem

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u_b^*}{\mathrm{d}r}\right) - \mathrm{i}\Omega u_b^* + \lambda_b^* + \frac{I_0\left(r\sqrt{\mathrm{i}\Omega Pr}\right)}{I_0\left(\sqrt{\mathrm{i}\Omega Pr}/2\right)} = 0,$$

$$u_b^*(1/2)=0,\quad \int_0^{1/2}\mathrm{d}r\,u_b^*(r)r=0. \tag{26}$$

The homogeneous equation associated with the boundary value problem (26) is

$$\frac{\mathrm{d}^2u_b^*}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}u_b^*}{\mathrm{d}r}-\mathrm{i}\Omega u_b^*=0. \tag{27}$$

A pair of linearly independent solutions of Eq. (27) is $I_0(r\sqrt{\mathrm{i}\Omega})$ and $K_0(r\sqrt{\mathrm{i}\Omega})$ [12]. The Wronskian of these solutions is given by

$$W(r)=I_0\left(r\sqrt{\mathrm{i}\Omega}\right)\frac{\mathrm{d}K_0\left(r\sqrt{\mathrm{i}\Omega}\right)}{\mathrm{d}r}-K_0\left(r\sqrt{\mathrm{i}\Omega}\right)\frac{\mathrm{d}I_0\left(r\sqrt{\mathrm{i}\Omega}\right)}{\mathrm{d}r}. \tag{28}$$

On account of the identity [12]

$$K_0(s)\frac{\mathrm{d}I_0(s)}{\mathrm{d}s}-I_0(s)\frac{\mathrm{d}K_0(s)}{\mathrm{d}s}=\frac{1}{s}, \tag{29}$$

one obtains

$$W(r)=-\frac{1}{r}. \tag{30}$$

As a consequence of a theorem on linear differential equations [13], the general solution of the inhomogeneous differential equation in the boundary value problem (26) has the form

$$\begin{aligned}u_b^*(r)&=A_1I_0\left(r\sqrt{\mathrm{i}\Omega}\right)+A_2K_0\left(r\sqrt{\mathrm{i}\Omega}\right)\\&\quad+I_0\left(r\sqrt{\mathrm{i}\Omega}\right)\int_0^r\mathrm{d}s\,K_0\left(s\sqrt{\mathrm{i}\Omega}\right)sF(s)\\&\quad-K_0\left(r\sqrt{\mathrm{i}\Omega}\right)\int_0^r\mathrm{d}s\,I_0\left(s\sqrt{\mathrm{i}\Omega}\right)sF(s),\end{aligned} \tag{31}$$

where $A_1$ and $A_2$ are two integration constants and the function $F(s)$ is given by the inhomogeneous term of the differential equation which appears in Eq. (26), namely

$$F(s)=-\lambda_b^*-\frac{I_0\left(s\sqrt{\mathrm{i}\Omega Pr}\right)}{I_0\left(\sqrt{\mathrm{i}\Omega Pr}/2\right)}. \tag{32}$$

The integration constant $A_2$ must be zero, since the the velocity distribution must be finite at $r=0$. On the contrary, the integration constant $A_1$ can be determined on account of the boundary condition expressed in Eq. (26). One can rewrite Eq. (31) as follows:

$$\begin{aligned}u_b^*(r)&=A_1I_0\left(r\sqrt{\mathrm{i}\Omega}\right)+\lambda_b^*[G_3(r,\Omega)-G_1(r,\Omega)]\\&\quad+G_4(r,\Omega,Pr)-G_2(r,\Omega,Pr),\end{aligned} \tag{33}$$

where the functions $G_1$, $G_2$, $G_3$ and $G_4$ are defined as:

$$G_1(r,\Omega)=\frac{I_0\left(r\sqrt{\mathrm{i}\Omega}\right)}{\mathrm{i}\Omega}\left[1-r\sqrt{\mathrm{i}\Omega}K_1\left(r\sqrt{\mathrm{i}\Omega}\right)\right],$$

$$\begin{aligned}G_2(r,\Omega,Pr)&=\frac{I_0\left(r\sqrt{\mathrm{i}\Omega}\right)}{\mathrm{i}\Omega(1-Pr)I_0\left(\sqrt{\mathrm{i}\Omega Pr}/2\right)}\\&\quad\times\left[1-r\sqrt{\mathrm{i}\Omega Pr}\times K_0\left(r\sqrt{\mathrm{i}\Omega}\right)I_1\left(r\sqrt{\mathrm{i}\Omega Pr}\right)\right.\\&\quad\left.-r\sqrt{\mathrm{i}\Omega}K_1\left(r\sqrt{\mathrm{i}\Omega}\right)I_0\left(r\sqrt{\mathrm{i}\Omega Pr}\right)\right],\end{aligned}$$

$$G_3(r,\Omega)=\frac{r}{\sqrt{\mathrm{i}\Omega}}K_0\left(r\sqrt{\mathrm{i}\Omega}\right)I_1\left(r\sqrt{\mathrm{i}\Omega}\right),$$

$$\begin{aligned}G_4(r,\Omega,Pr)&=\frac{rK_0\left(r\sqrt{\mathrm{i}\Omega}\right)}{\mathrm{i}\Omega(1-Pr)I_0\left(\sqrt{\mathrm{i}\Omega Pr}/2\right)}\\&\quad\times\left[\sqrt{\mathrm{i}\Omega}I_1\left(r\sqrt{i\Omega}\right)I_0\left(r\sqrt{\mathrm{i}\Omega Pr}\right)\right.\\&\quad\left.-\sqrt{\mathrm{i}\Omega Pr}I_0\left(r\sqrt{\mathrm{i}\Omega}\right)I_1\left(r\sqrt{\mathrm{i}\Omega Pr}\right)\right].\end{aligned} \tag{34}$$

As a consequence of the boundary condition in Eq. (26), the integration constant $A_1$ is given by

$$\begin{aligned}A_1&=\lambda_b^*\frac{G_1(1/2,\Omega)-G_3(1/2,\Omega)}{I_0\left(\sqrt{\mathrm{i}\Omega}/2\right)}\\&\quad+\frac{G_2(1/2,\Omega,Pr)-G_4(1/2,\Omega,Pr)}{I_0\left(\sqrt{\mathrm{i}\Omega}/2\right)}.\end{aligned} \tag{35}$$

Moreover, the parameter $\lambda_b^*$ can be evaluated by utilizing the constraint equation which appears in Eq. (26), namely

$$\begin{aligned}\lambda_b^*&=-\left[G_3(1/2,\Omega)\frac{G_2(1/2,\Omega,Pr)-G_4(1/2,\Omega,Pr)}{I_0\left(\sqrt{\mathrm{i}\Omega}/2\right)K_0\left(\sqrt{\mathrm{i}\Omega}/2\right)}\right.\\&\quad\left.-\int_0^{1/2}\mathrm{d}r\,G_2(r,\Omega,Pr)r+\int_0^{1/2}\mathrm{d}r\,G_4(r,\Omega,Pr)r\right]\\&\quad\times\left[G_3(1/2,\Omega)\frac{G_1(1/2,\Omega)-G_3(1/2,\Omega)}{I_0\left(\sqrt{\mathrm{i}\Omega}/2\right)K_0\left(\sqrt{\mathrm{i}\Omega}/2\right)}\right.\\&\quad\left.-\int_0^{1/2}\mathrm{d}r\,G_1(r,\Omega)r+\int_0^{1/2}\mathrm{d}r\,G_3(r,\Omega)r\right]^{-1}.\end{aligned} \tag{36}$$

On account of Eqs. (15), (19) and (21), also the Fanning friction factor can be rewritten as follows:

$$fRe=\mathscr{R}e(f_a^*Re+f_b^*Gr\mathrm{e}^{\mathrm{i}\eta}), \tag{37}$$

where $f_a^*$ and $f_b^*$ are respectively given by

$$f_a^*Re=-2\left.\frac{\mathrm{d}u_a^*}{\mathrm{d}r}\right|_{r=1/2}=16 \tag{38}$$

and

$$f_b^* Re = -2 \left. \frac{\mathrm{d}u_b^*}{\mathrm{d}r} \right|_{r=1/2}. \tag{39}$$

Finally, a dimensionless heat flux can be defined as

$$\Phi = \frac{2R_0 q}{k\Delta T} = \frac{\partial \theta}{\partial r} = \mathscr{R}e\left(\frac{\partial \theta^*}{\partial r}\right) = \mathscr{R}e(\Phi^*). \tag{40}$$

As a consequence of Eqs. (21), (24) and (25), the complex valued function $\Phi^* = \partial\theta^*/\partial r$ can be expressed as

$$\Phi^* = \Phi_a^* + \Phi_b^* \mathrm{e}^{\mathrm{i}\eta}, \tag{41}$$

where $\Phi_a^* = 0$ and

$$\Phi_b^* = \frac{\partial \theta_b^*}{\partial r} = \frac{\sqrt{\mathrm{i}\Omega Pr} I_1\left(r\sqrt{\mathrm{i}\Omega Pr}\right)}{I_0\left(\sqrt{\mathrm{i}\Omega Pr}/2\right)}. \tag{42}$$

## 4. Discussion of the results

In this section, a survey of the solution obtained above is performed by representing the radial distributions of the oscillation amplitudes of the dimensionless temperature as well as of the dimensionless velocity, for fixed values of the parameters $Gr/Re$, $\Omega$, and $Pr$. Moreover, the influence of the dimensionless parameters $\Omega$ and $Pr$ on the oscillation amplitudes of $\lambda$ and of $fRe$ is analyzed.

In Fig. 1, the dimensionless temperature distribution $|\theta_b^*|$ is reported versus the dimensionless radius $r$, for different values assumed by the product $\Omega Pr$. This figure shows that $|\theta_b^*|$ is a monotonic increasing function of $r$ and that, for increasing values of $\Omega Pr$, the oscillations of the dimensionless temperature tend to be confined in a narrow region close to the duct wall.

Figs. 2–5 represent the radial distribution of $|u_b^*|$, for fixed values of the parameter $Pr$ and for different values assumed by the parameter $\Omega$. In particular, Figs. 2 and 4 refer to $Pr = 7$ while Figs. 3 and 5 refer to $Pr = 0.7$. These figures show that $|u_b^*|$ has two local maxima, one placed on the duct axis and the other close to the the duct wall. Moreover, in Figs. 2–5 the presence of a local minimum arises for $r \approx 0.3$. This local minimum is more

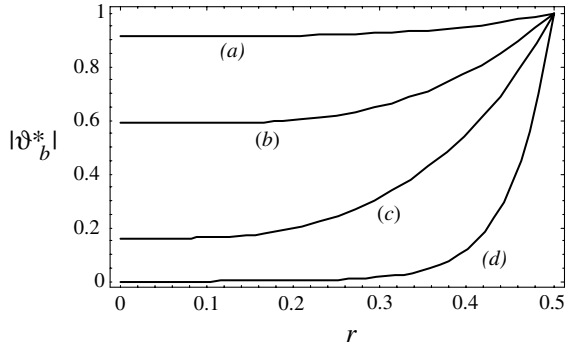


Fig. 1. Radial distribution of $|\theta_b^*|$ for $\Omega Pr = 10$ (a), $\Omega Pr = 30$ (b), $\Omega Pr = 100$ (c) and $\Omega Pr = 1000$ (d).

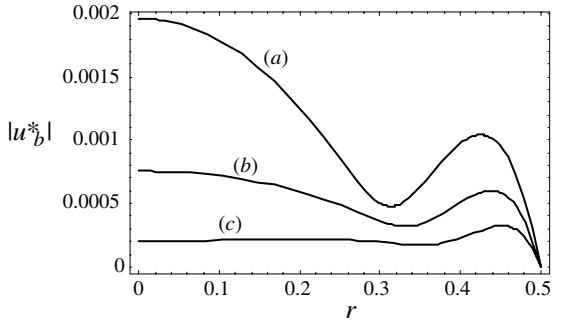


Fig. 2. Radial distribution of $|u_b^*|$ for $Pr = 7$ and for $\Omega = 50$ (a), $\Omega = 100$ (b) and $\Omega = 200$ (c).

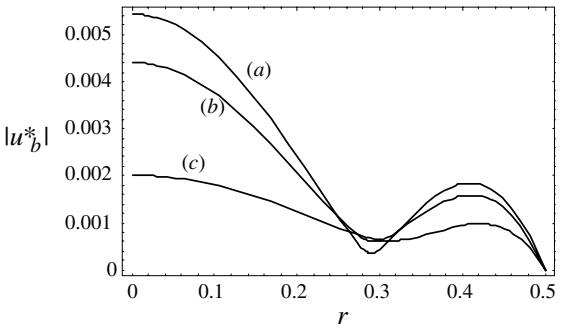


Fig. 3. Radial distribution of $|u_b^*|$ for $Pr = 0.7$ and for $\Omega = 50$ (a), $\Omega = 100$ (b) and $\Omega = 200$ (c).

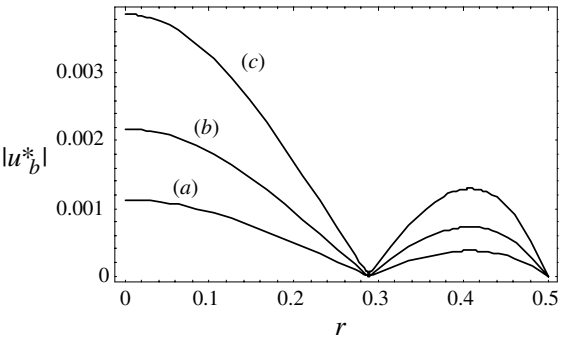


Fig. 4. Radial distribution of $|u_b^*|$ for $Pr = 7$ and for $\Omega = 0.5$ (a), $\Omega = 1$ (b) and $\Omega = 2$ (c).

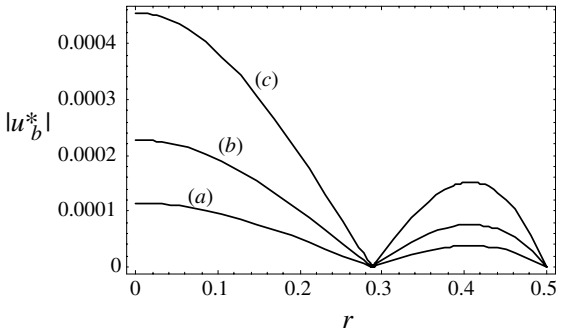


Fig. 5. Radial distribution of $|u^*_b|$ for $Pr = 0.7$ and for $\Omega = 0.5$ (a), $\Omega = 1$ (b) and $\Omega = 2$ (c).

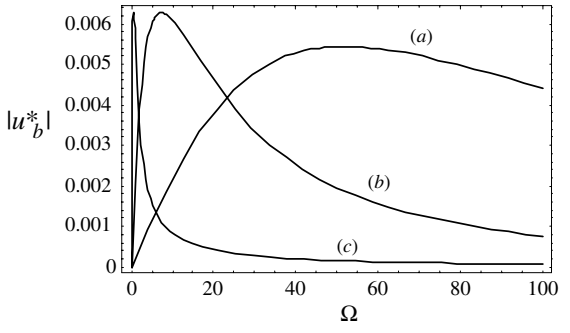


Fig. 6. Distribution of $|u^*_b|$ versus $\Omega$, for $r = 0$ and $Pr = 0.7$ (a), $Pr = 7$ (b) and $Pr = 100$ (c).

evident in Figs. 4 and 5, which refer to lower values of the dimensionless frequency $\Omega$ ($\Omega \leqslant 2$).

Moreover, for any radial position there exists a dimensionless resonance frequency such that the oscillation amplitude of the dimensionless velocity distribution reaches a maximum. This resonance frequency is a function of the radial position and of the Prandtl number. The phenomenon is evident in Figs. 6 and 7, where the dimensionless velocity distribution is reported versus the dimensionless frequency, for fixed values of the radial coordinate and of the parameter $Pr$. In Table 1, the values of the resonance frequency are reported for different values of the parameter $Pr$ and of the dimensionless radial position. This Table shows that, for any value of $Pr$, the resonance frequency is not a monotonic function of $r$. In particular, the resonance frequency reaches a relative maximum for $r \approx 0.3$, which approximately correspond to the radial position in which $|u^*_b|$ has a local minimum.

In Fig. 8, the oscillation amplitude of the dimensionless pressure drop is reported versus the dimensionless frequency $\Omega$, for some values assumed by the parameter $Pr$. This Figure shows that $|\lambda^*_b|$ is a monotonic decreasing function of $\Omega$ and that it decreases more rapidly when the Prandtl number assumes higher values.

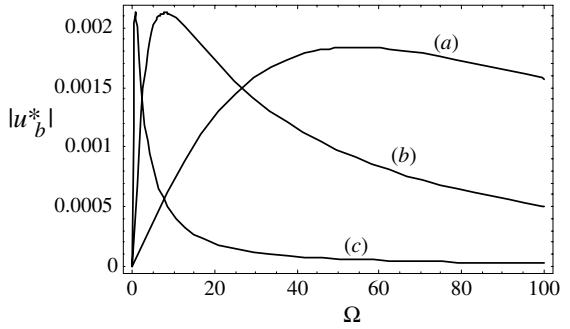


Fig. 7. Distribution of $|u^*_b|$ versus $\Omega$, for $r = 0.4$ and $Pr = 0.7$ (a), $Pr = 7$ (b) and $Pr = 100$ (c).

In Fig. 9 the oscillation amplitude of the friction factor is reported versus the dimensionless frequency $\Omega$, for different values assumed by $Pr$. This Fig. 9 shows that, for any value assumed by the Prandtl number, there exists a resonance frequency which maximizes the oscillation amplitude of the friction factor. In Table 2, those resonance frequencies are reported for $0.01 \leqslant Pr \leqslant 1000$. The table shows that the resonance frequency is a monotonic decreasing function of the Prandtl number.

Table 1
Dimensionless resonance frequencies of $u$ for different values of the parameter $Pr$

| $r$ | $Pr = 0.01$ | $Pr = 0.1$ | $Pr = 0.7$ | $Pr = 7$ | $Pr = 20$ | $Pr = 100$ |
|---|---|---|---|---|---|---|
| 0 | 255.5 | 138.8 | 51.99 | 7.349 | 2.582 | 0.5160 |
| 0.05 | 274.0 | 140.2 | 52.06 | 7.372 | 2.592 | 0.5180 |
| 0.1 | 352.3 | 145.0 | 52.31 | 7.449 | 2.623 | 0.5246 |
| 0.15 | 458.2 | 155.2 | 52.92 | 7.609 | 2.687 | 0.5382 |
| 0.2 | 565.2 | 178.0 | 54.70 | 7.991 | 2.837 | 0.5695 |
| 0.25 | 721.5 | 240.5 | 70.48 | 10.72 | 3.851 | 0.7765 |
| 0.3 | 935.0 | 334.0 | 147.8 | 28.20 | 8.959 | 1.668 |
| 0.35 | 294.9 | 159.0 | 55.58 | 7.684 | 2.682 | 0.5343 |
| 0.4 | 612.1 | 180.1 | 54.52 | 7.949 | 2.820 | 0.5659 |
| 0.45 | 1226 | 214.2 | 55.73 | 8.360 | 2.999 | 0.6047 |

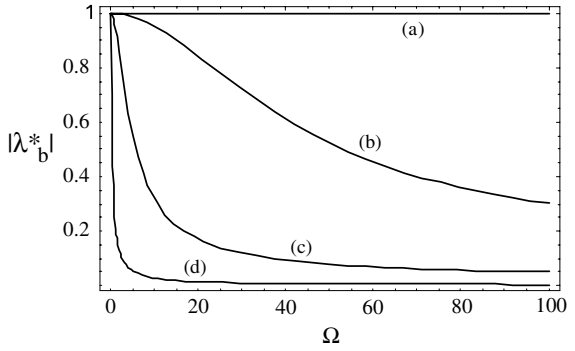


Fig. 8. Distribution of $|\lambda_b^*|$ versus $\Omega$, for $Pr = 0.01$ (a), $Pr = 0.7$ (b), $Pr = 7$ (c) and $Pr = 100$ (d).

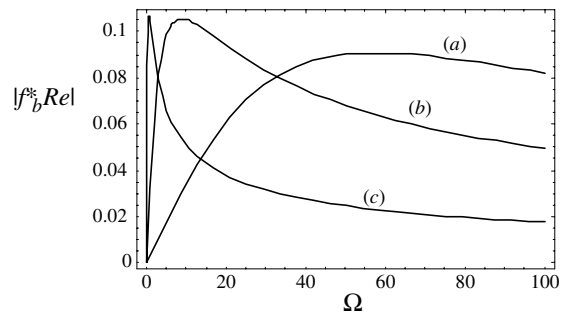


Fig. 9. Distribution of $|f_b^* Re|$ versus $\Omega$, for $Pr = 0.7$ (a), $Pr = 7$ (b) and $Pr = 100$ (c).

Table 2
Dimensionless resonance frequencies of *fRe* for different values of *Pr*

| *Pr* | Ω | *Pr* | Ω |
|---|---|---|---|
| 0.01 | 2462 | 5 | 12.20 |
| 0.02 | 1249 | 10 | 6.354 |
| 0.05 | 514.3 | 20 | 3.237 |
| 0.1 | 266.2 | 50 | 1.309 |
| 0.2 | 142.9 | 100 | 0.6565 |
| 0.5 | 72.99 | 200 | 0.3288 |
| 1 | 45.28 | 500 | 0.1316 |
| 2 | 26.87 | 1000 | 0.06584 |

For any radial position, also the oscillation amplitude of the dimensionless heat flux displays a maximum in correspondence of a particular value of the product $\Omega Pr$. These resonance values of $\Omega Pr$ are reported in Table 3. Finally, in Fig. 10 the oscillation amplitude of the dimensionless heat flux is reported versus the dimensionless radial coordinate for different values assumed by the product $\Omega Pr$. The figure shows that $|\Phi_b^*|$ is a monotonic increasing function of $r$.

Table 3
Values of the product $\Omega Pr$ which correspond to resonances of $\Phi$

| *r* | $\Omega Pr$ | *r* | $\Omega Pr$ |
|---|---|---|---|
| 0 | 49.23 | 0.275 | 56.73 |
| 0.05 | 49.23 | 0.3 | 61.50 |
| 0.075 | 49.26 | 0.325 | 70.60 |
| 0.1 | 49.33 | 0.35 | 92.48 |
| 0.125 | 49.48 | 0.375 | 143.7 |
| 0.15 | 49.75 | 0.4 | 223.0 |
| 0.175 | 50.21 | 0.425 | 384.4 |
| 0.2 | 50.95 | 0.45 | 842.4 |
| 0.225 | 52.10 | 0.475 | 3282 |
| 0.25 | 53.89 | 0.495 | 80 400 |

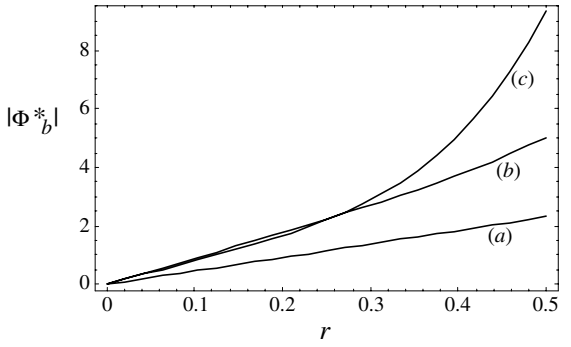


Fig. 10. Distribution of $|\Phi_b^*|$ versus $r$, for $\Omega Pr = 10$ (a), $\Omega Pr = 30$ (b) and $\Omega Pr = 100$ (c).

## 5. Conclusions

An analysis of the fully developed laminar mixed convection of a Newtonian fluid in a vertical cylindrical duct with circular cross section has been performed. Reference has been made to the steady periodic regime, by assuming as thermal boundary condition that the wall temperature distribution is a sinusoidal function of time. The velocity and temperature fields, as well as the pressure drop, have been written in a dimensionless form and expressed as real parts of complex valued functions. Thus, two independent boundary value problems have been obtained and solved analytically. The first one accounts for the mean component of each physical quantity, while the second one provides the oscillating component. The dimensionless velocity and temperature have been obtained as functions of three different parameters: the dimensionless frequency $\Omega$, the Prandtl number $Pr$ and the ratio between the Grashof number and the Reynolds number $Gr/Re$. The oscillating components of the velocity and temperature distributions obtained by the second boundary value problem have been reported versus the dimensionless radial coordinate, for some values of $\Omega$ and $Pr$. Moreover, the

oscillating component of the pressure drop has been reported versus the dimensionless angular frequency $\Omega$. It has been also shown that there exists a resonance frequency such that the velocity oscillations reach a maximum. This resonance frequency depends on the value assumed by $Pr$ and on the radial position.

Finally, the Fanning friction factor and the dimensionless heat flux have been expressed as the sum of a mean component and an oscillating component and evaluated analytically. It has been shown that there exists a resonance frequency which maximizes the amplitude of the friction factor and that this frequency is a monotonic decreasing function of $Pr$. Finally, it has been shown that there also exists a resonance frequency for the dimensionless heat flux, which is a function of the dimensionless radial position and of the product $\Omega Pr$.

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