Modern Methods of Optimization

Lecture 1: General overview

Faculty of Civil Engineering / CTU in Prague

Outline

Motivation [Why?]

How to eat as an economist?

Objective choose the *most economic* menu

Constraints must meet *daily nutrition requirements*

Item	Price [CZK/1kg]	Nutrition [Cal/100g]
Roast pork	120,10	273
Chicken	51,49	331
Potatoes	8,39	93
Bread	15,56	260
...		

- Optimal diet problem
- First studied in 40's in military applications

Motivation [Why?]

How to invest your money?

Objective invest your money with the *minimal risk*

Constraint achieve at least the given *profit*

Share	Price [CZK]				
	2001	2002	2003	2004	...
Commercial Bank (KB ČR)	1000	2000	2250	3100	...
ČEZ group	850	990	140	330	...
Unipetrol group	45	40	60	80	...
Philip Morris ČR	8,000	11,000	15,000	17,500	...
...					

- Basic version of the portfolio management problem
- Applications in various problems of risk management

Motivation [Why?]

How to travel efficiently?

Objective find the *shortest possible* route

Constraint each city must be visited *at least once*

	Praha	Brno	Ostrava	Cheb
Praha	0 km	205	381	176
Brno	207	0 km	180	381
Ostrava	380	181	0 km	550
Cheb	176	384	556	0 km

Distances between cities

- Traveling salesman problem, formulated in 1930
- Applications in circuit design, production planning ...

Motivation [Why?]

Which computer should I buy?

Objective select the *best* notebook

Criteria prize, performance, memory and disk sizes, screen size ...

Type	Price [CZK]	Screen Size ["]	Disk size [GB]	Weight [kg]
AMILO Pro V2035	**14 999**	15,4	60	2,9
Aspire 9814WKMi	65 996	**20**	240	7,5
Qosmio G30-194	93 210	17	**320**	4,5
Lifebook Q2010 U1400	105 743	12	80	**0,9**

Parameters of selected notebooks (as of December 2006)

- Typical multi-objective problem
- Used in decision-making theory

Formalization of optimization problems

General statement Find $\mathbf{x}_{opt} \in \mathscr{O}$, such that

$$f(\mathbf{x}_{opt}) \overset{\mathscr{H}}{\preceq} f(\mathbf{x}) \quad \text{for all} \quad \mathbf{x} \in \mathscr{O}$$

- $\mathbf{x}$ – unknown *variables* parametrizing the underlying model
- $\mathscr{O}$ – set of admissible solution (reflects *constraints*)
- f – *objective* function $[f : \mathscr{O} \to \mathscr{H}]$
- $\mathscr{H}$ – range of the objective function
- $\overset{\mathscr{H}}{\preceq}$ – [partial] ordering in the range of objective function – allows to compare two candidate solutions
- $\mathbf{x}_{opt}$ – *optimal solution*

- N items on the menu
- Unknowns – quantities of individual items (in kg): $x_i \in \mathscr{R}, x_i \geq 0$
- Objective function

$$f(\mathbf{x}) = x_1 c_1 + x_2 c_2 + \ldots x_N c_N,$$

  where c_i is the price of the i-th item (per 1 kg)
- Range of the function $\mathscr{H} = \mathscr{R}_0^+$
- $\overset{\mathscr{H}}{\preceq}$ corresponds to $\leq$
- Constraints

$$x_1 k_1 + x_2 k_2 + \ldots x_N k_N \geq K,$$

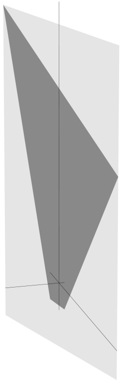
  where k_i denotes the nutrition amount of the i-th item and K is the required nutrition value
- $\mathscr{O}$ corresponds to a subspace of $\mathscr{R}^N$

Optimal 2-diet problem

Objective minimize prize

Constraints nutrition at least $1,000$ cal

Item	Price [CZK/1kg]	Nutrition [Cal/100g]
Chicken	51,49	331
Potatoes	8,39	93



Formalization of optimization problems

Purchase of a computer

- Unknown – index of a notebook $x \in \mathscr{N}$
- Objective function: M different criteria

$$\mathbf{f}(x) = [f_1(x), f_2(x), \ldots, f_M(x)], \qquad \mathscr{H} \in \mathscr{R}^M$$

- Comparison of two solutions

$$x_1 \overset{\mathscr{H}}{\preceq} x_2 \quad \equiv \quad \left\{ \begin{array}{lcl} f_1(x_1) & \leq & f_1(x_2) \\ f_2(x_1) & \geq & f_2(x_2) \\ f_3(x_1) & \geq & f_3(x_2) \\ f_4(x_1) & \leq & f_4(x_2) \end{array} \right.$$

- Optimal portfolio $\rightarrow$ Next lecture
- Traveling salesman $\rightarrow$ Lecture #6

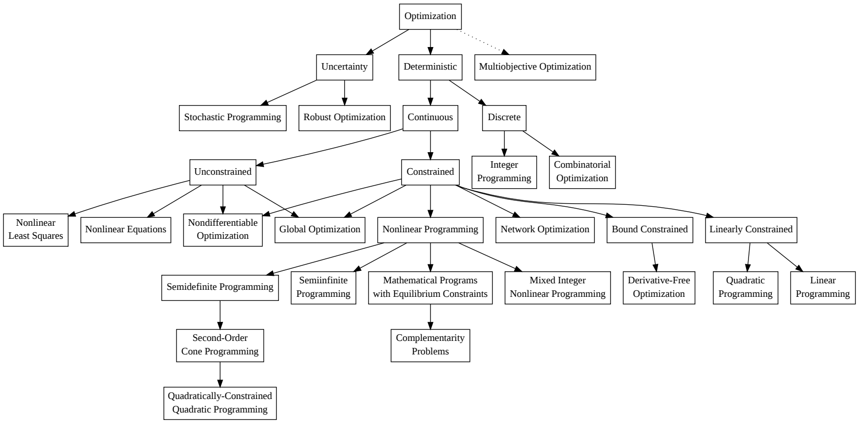
Rough characterization of optimization problems

- *Dimension* of the problem N – number of [independent] components of $\mathbf{x}$
- How the number of steps needed to find $\mathbf{x}_{opt}$ growths with N?
  - $\approx CN^k$ – *Polynomial* problem (P)
    [diet, portfolio, computer purchase]
  - $\approx 2^N$ – *Exponential* problem (E)
  - *Non-deterministic polynomial* problem (NP) – problem where decision can be made in a polynomial time by a non-deterministic oracle [traveling salesman]

$$P \stackrel{?}{=} NP \prec E$$

- "?" amounts to 1 mil. USD

Practical characterization of optimization problems

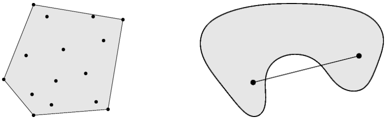


Example of “optimization tree”

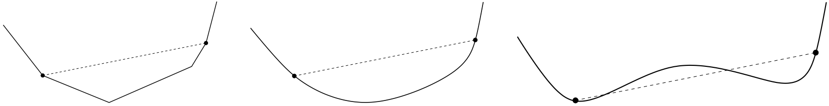
`https://neos-guide.org/content/optimization-taxonomy`

What can be solved efficiently?

- Convex set



- [Strictly] convex function



$$\mathscr{O} \text{ convex} + f \text{ convex} \rightarrow P$$

- Strictly convex function on a convex set possesses the *unique minimum*
- Efficient methods of *mathematical programming* [1]

Summary

- Optimization problem $\equiv$
  - [Mathematical] model of the problem
  - Optimization variables $\mathbf{x}$
  - Objective function f
  - Set of admissible solutions $\mathscr{O}$
  - Range of objective function $\mathscr{H}$ + partial order
- Solution methods
  - Convex problem $\to$ P $\to$ mathematical programming $\to$ Lectures #2 – #3
  - Non-convex problems $\to$ NP/E $\to$ heuristic approaches $\to$ Lectures 4+

S. Boyd and L. Vandenberghe, *Convex optimization*, Cambridge University Press, Cambridge, United Kingdom, 2004, `http://www.stanford.edu/~boyd/cvxbook`.