Multi-objective optimization

- Optimization of several objective function at once
- More math needed
- Generally:

$$\text{minimize } \mathbf{y} = \mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_k(\mathbf{x}))$$

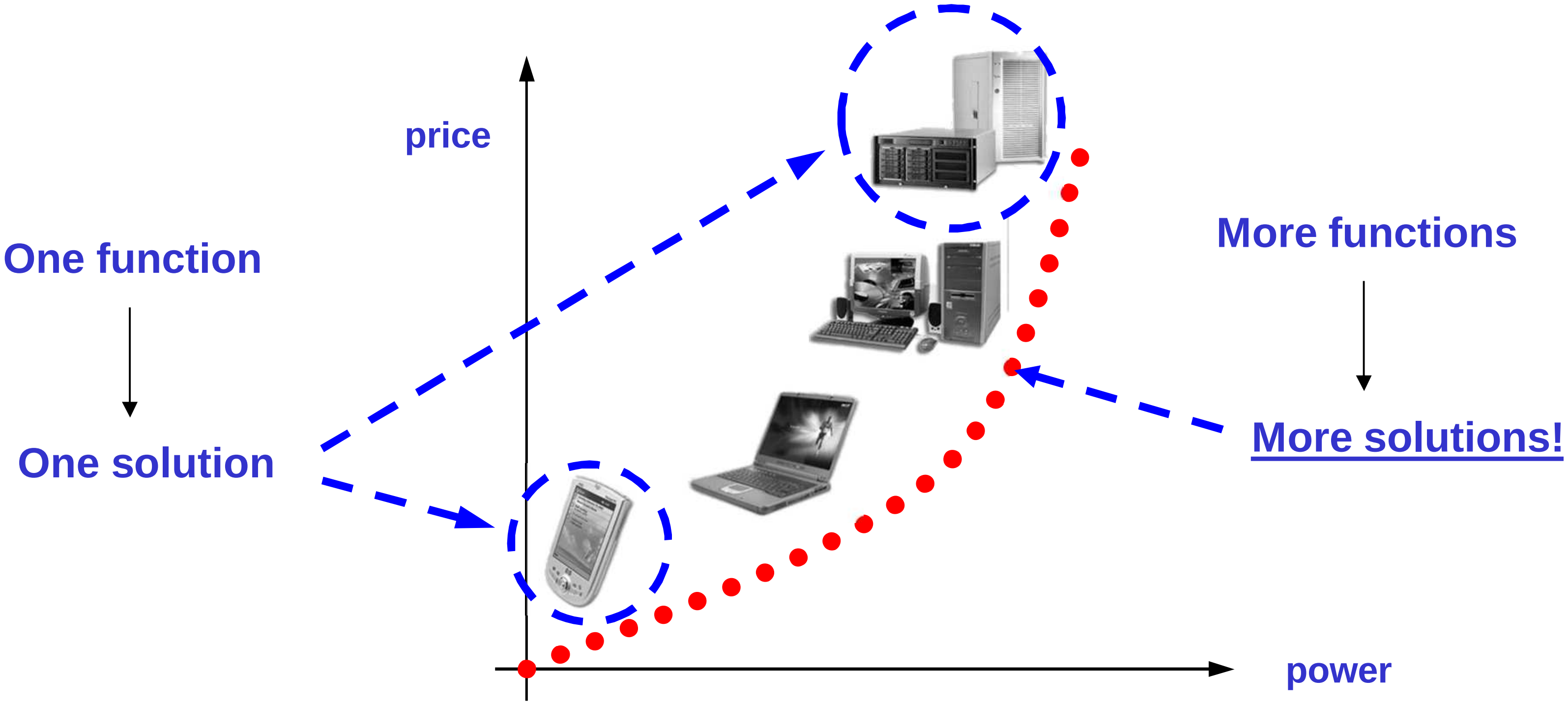
$$\text{subjected to } g_j(\mathbf{x}) = 0, \quad j = 1, \ldots, ne,$$

$$g_j(\mathbf{x}) \le 0, \quad j = ne+1, \ldots, m = ne + ni,$$

$$\text{where } \mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbf{X}, \quad \mathbf{X} \subset \{N, R\}^n,$$

$$\mathbf{y} = (y_1, y_2, \ldots, y_k) \in \mathbf{Y}, \quad \mathbf{Y} \subseteq R^k,$$

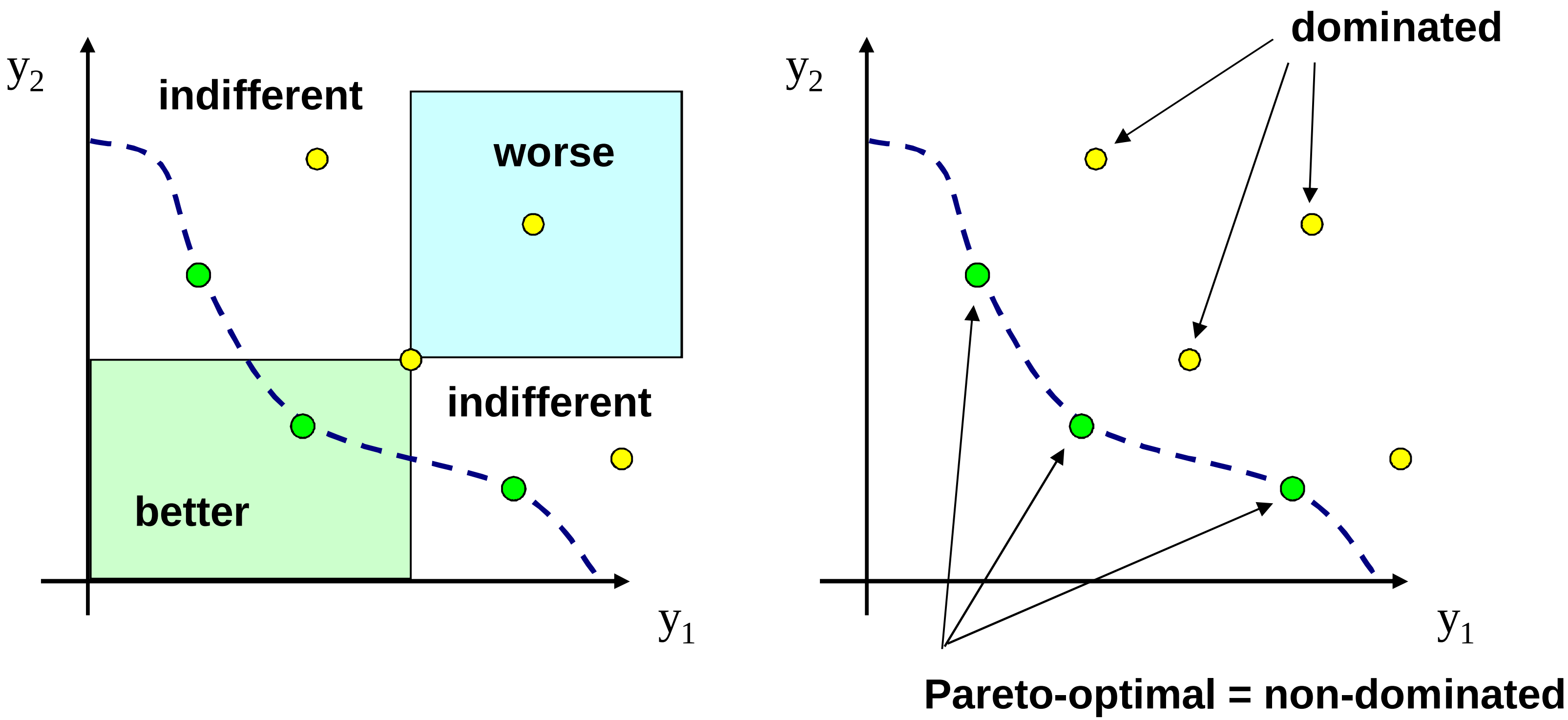
Motivation



Single-objective optimization is a special case of multi-objective optimization (but not vice-versa!).

Basic notation

$$\text{Min } (y_1, y_2, \ldots, y_k) = \mathbf{f}\,(x_1, x_2, \ldots, x_n)$$

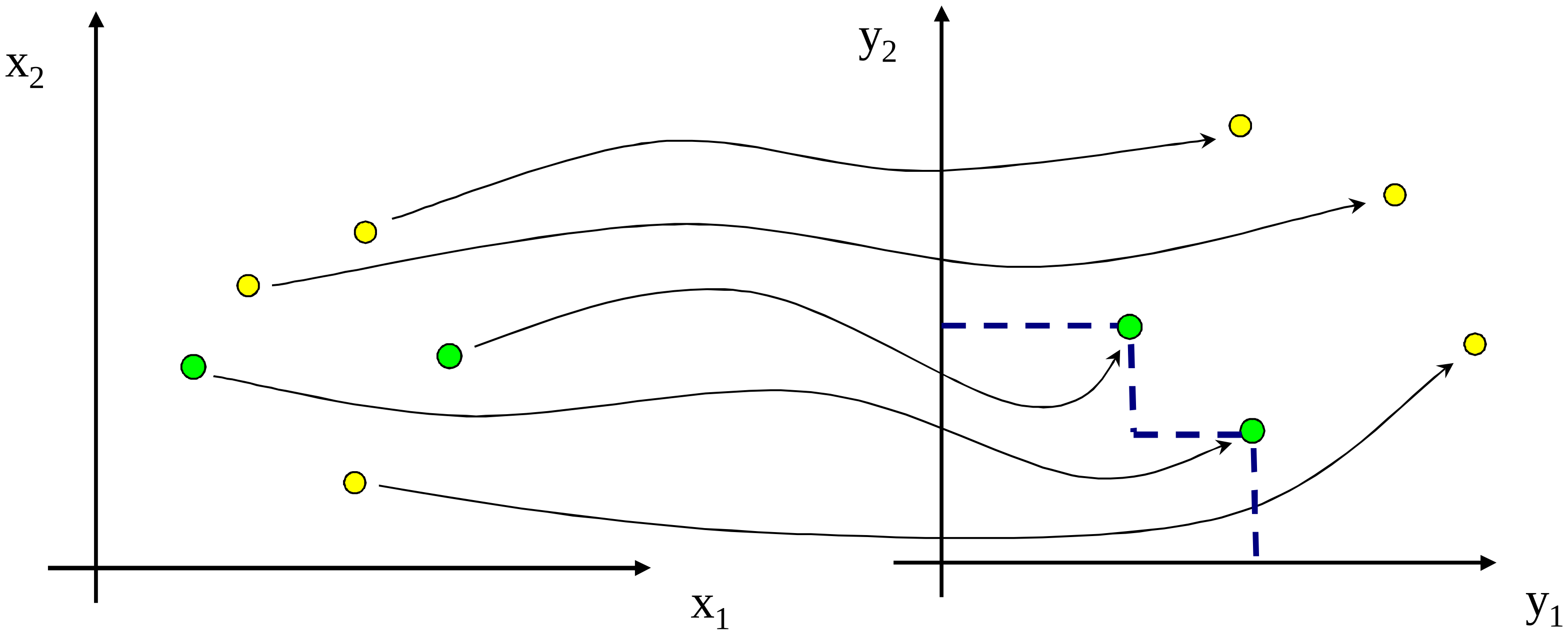


$$\mathbf{a} \succ \mathbf{b} \text{ } (\mathbf{a} \text{ dominates } \mathbf{b}) \text{ } \textit{iff} \quad \forall i : f_i(\mathbf{a}) \leq f_i(\mathbf{b}) \wedge \exists i : f_i(\mathbf{a}) < f_i(\mathbf{b}),$$

Basic notation

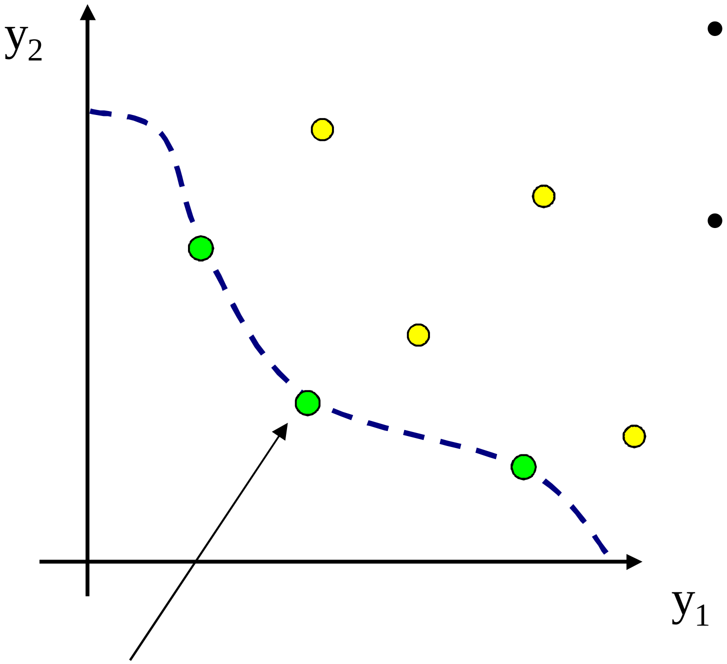
Decision/design space X   Objective space Y

Pareto set   Pareto front/surface



decision vector $(x_1, x_2, \ldots, x_n) \longrightarrow$ **f** $\longrightarrow (y_1, y_2, \ldots, y_k)$ **objective vector**

Optimization & Decision making



- Decision making before search (defining single objective) => EAs
- Decision making after search (find Pareto-optimal set first) => MOEAs
- Decision making during search (interactively) => interactive algorithms, e.g. NIMBUS

Pareto-optimality: set of optimal trade-offs
(all objectives equally important)

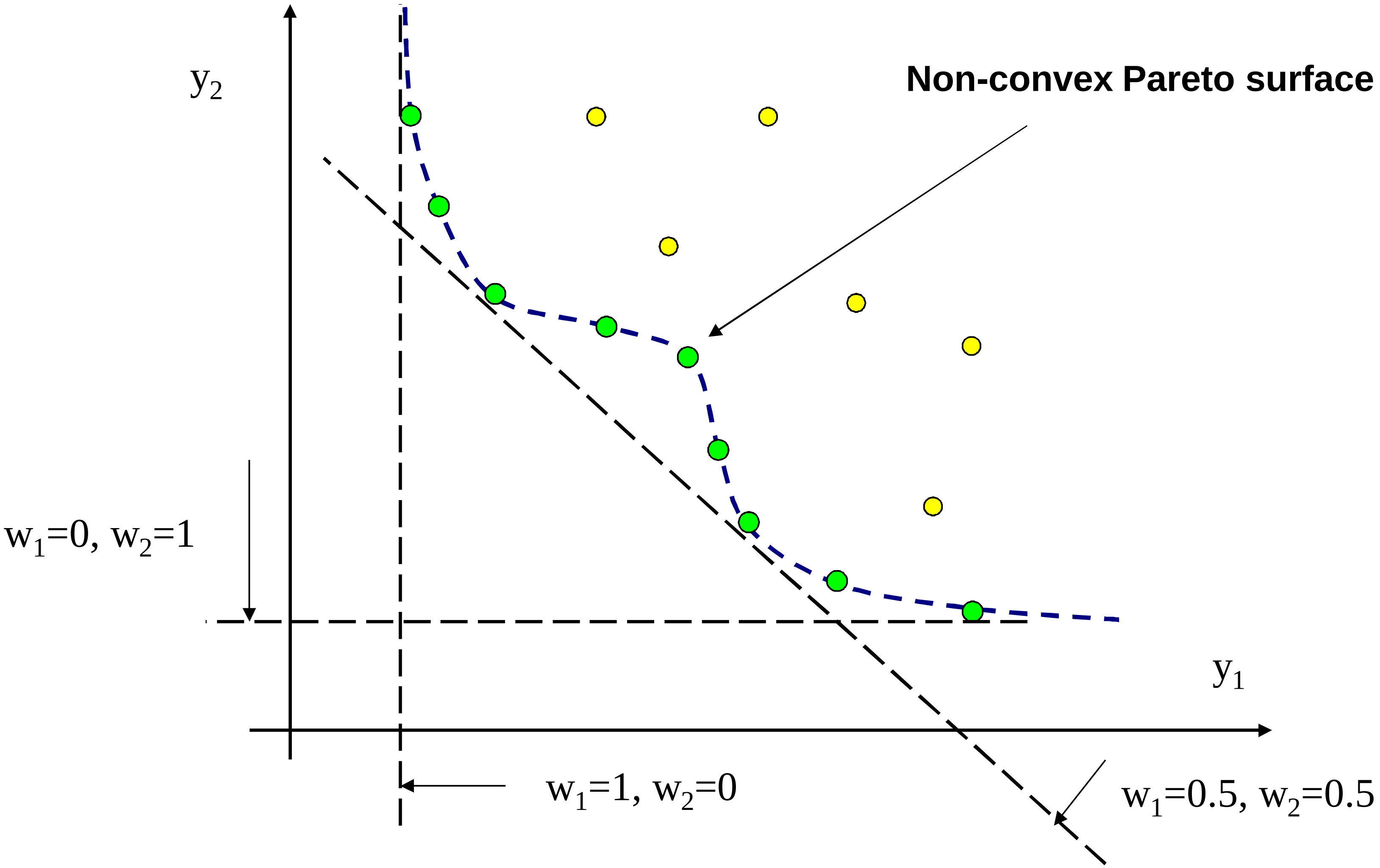
Single-objectivization

- Commonly using weights:

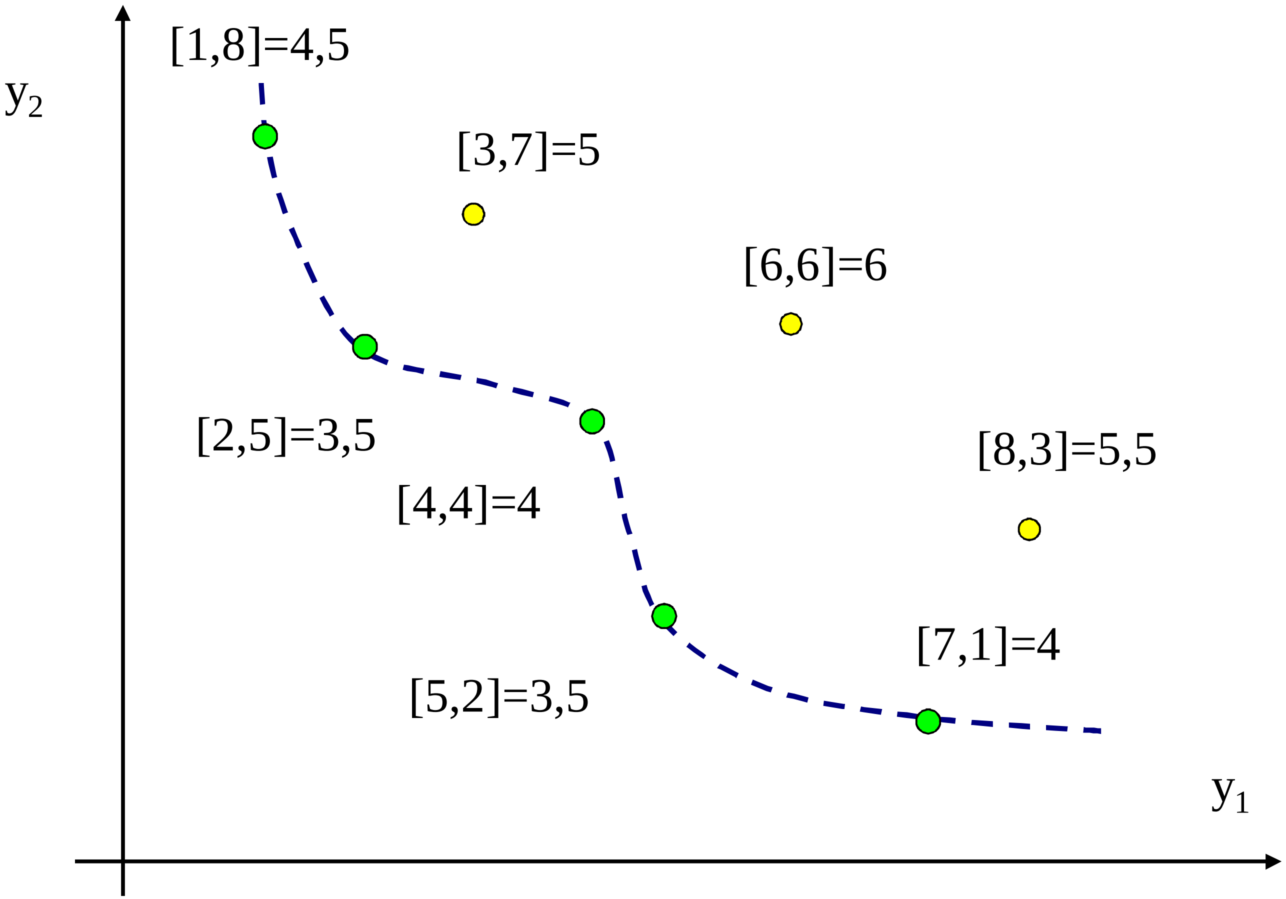
$$F'(\mathbf{x}) = \sum_{i}^{k} w_i f_i(\mathbf{x}), \qquad \text{where } \sum_{i}^{k} w_i = 1$$

- Problem:
  - Search for combination of weights that leads to compact Pareto set (i.e. optimization of optimization)
  - Unable to cover non-convex Pareto surface

Single-objectivization



Average ranking



Multi-objective EAs (MOEAs)

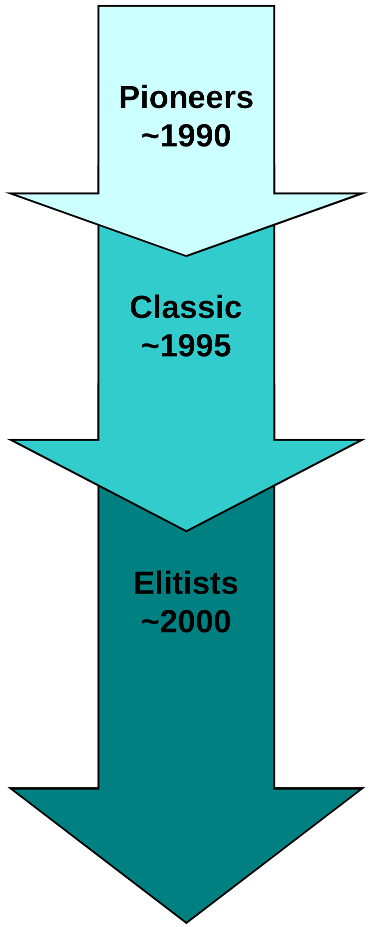
- There is need to store a set of Pareto solutions => analogy with population within EAs
- Easiest implementation – Pareto-dominance included in selection mechanism (e.g. Generalized Differential Evolution)

Brief history of MOEAs

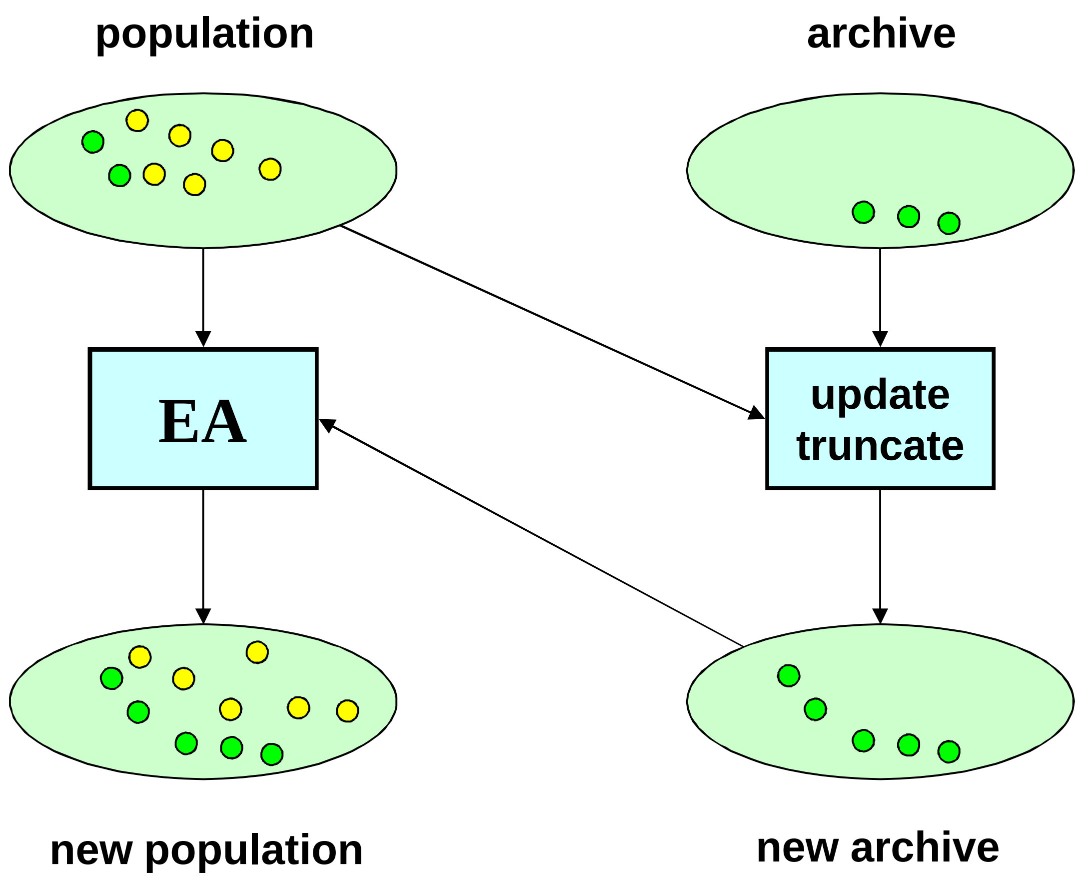
VEGA [Schaffer, 1985]

MOGA [Fonesca & Fleming, 1993]
NPGA [Horn & Nafpliotis, 1993]
NSGA [Srinivas & Deb, 1994]

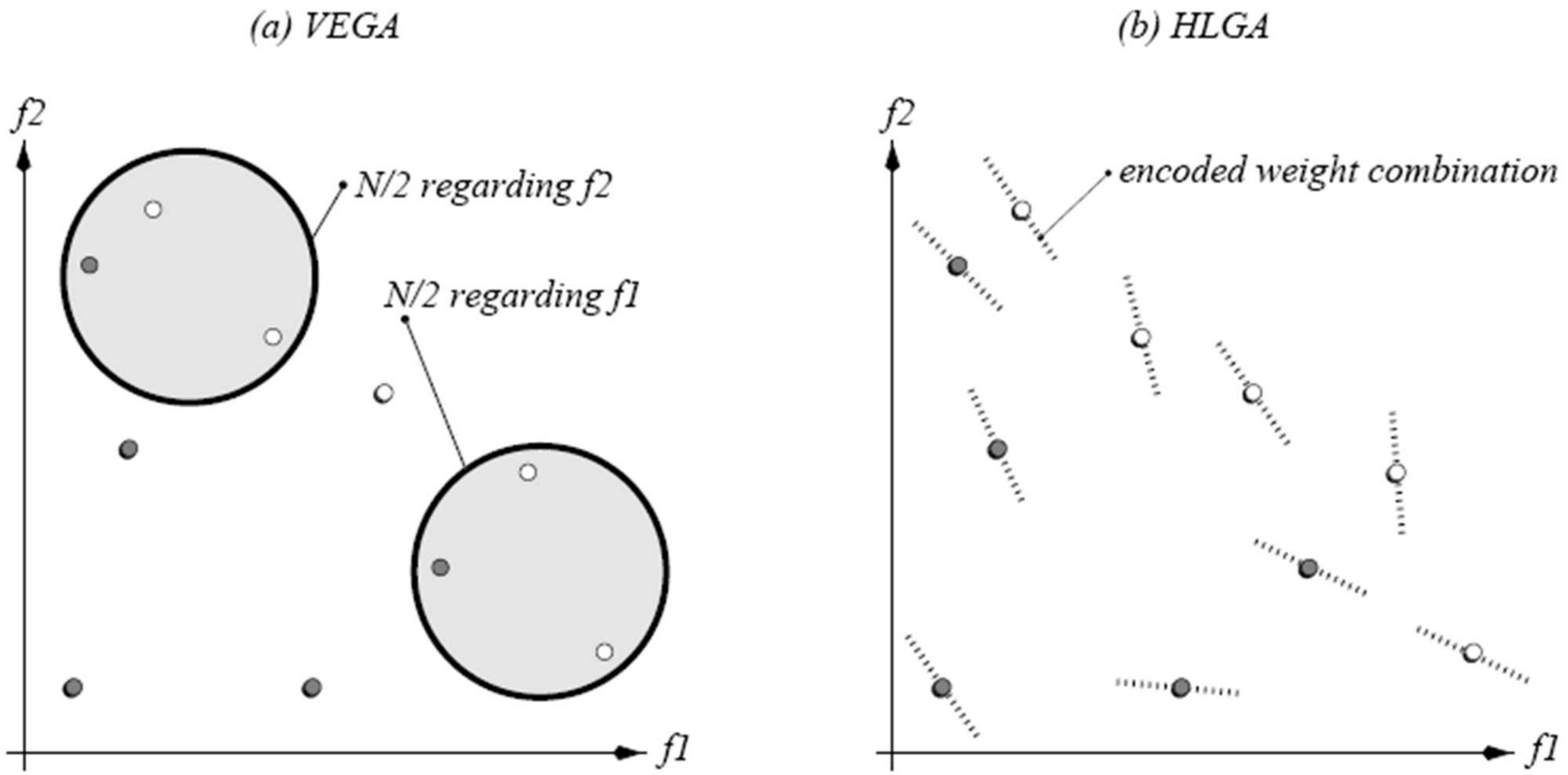
SPEA [Zitzler & Thiele, 1999]
PAES, PESA [Knowles & Corne, 1999]
NSGA-II [Deb et al., 2000]
SPEA 2 [Zitzler & Thiele, 2001]



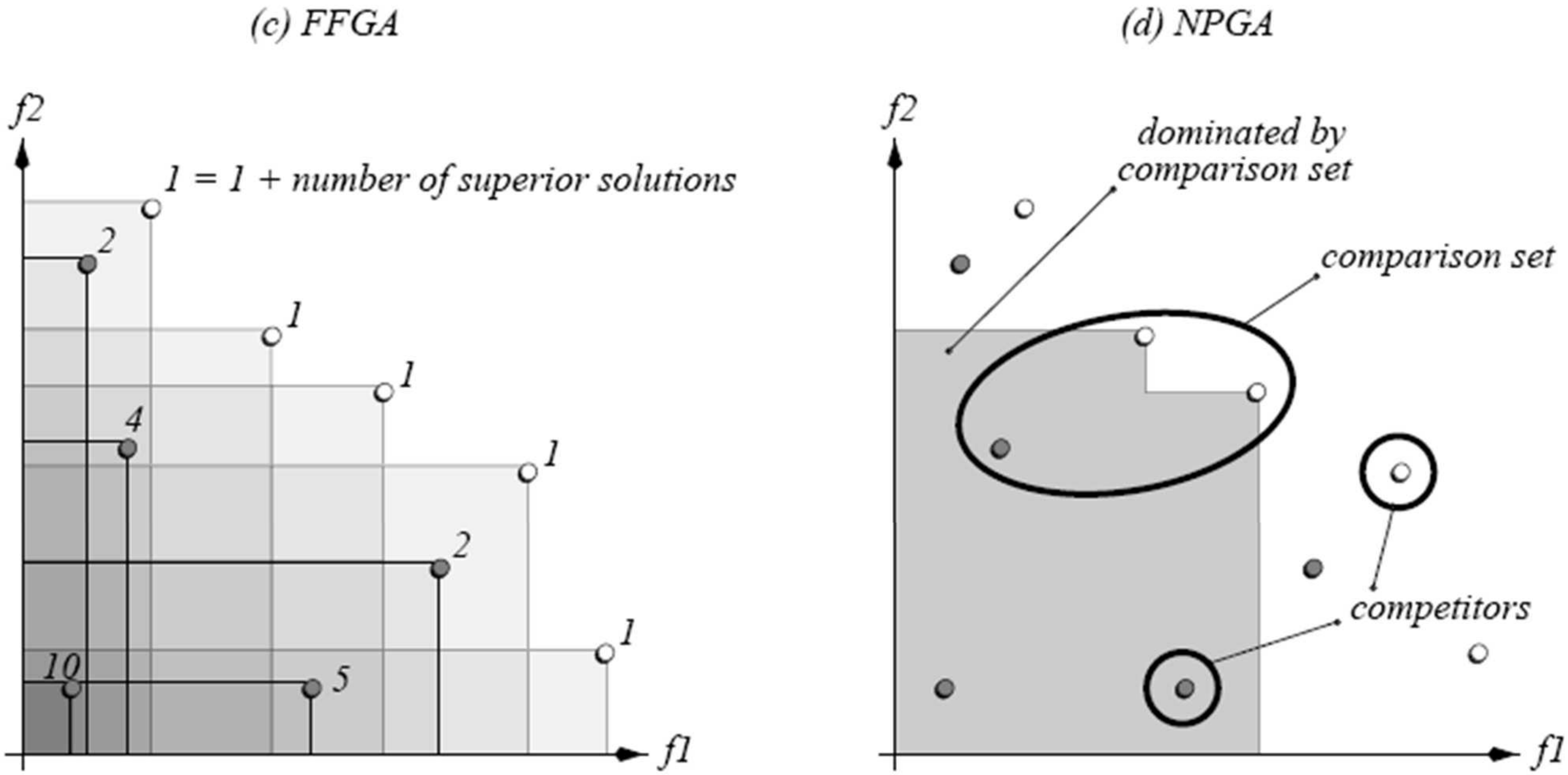
General „elitist“ MOEA



How to evaluate solutions?

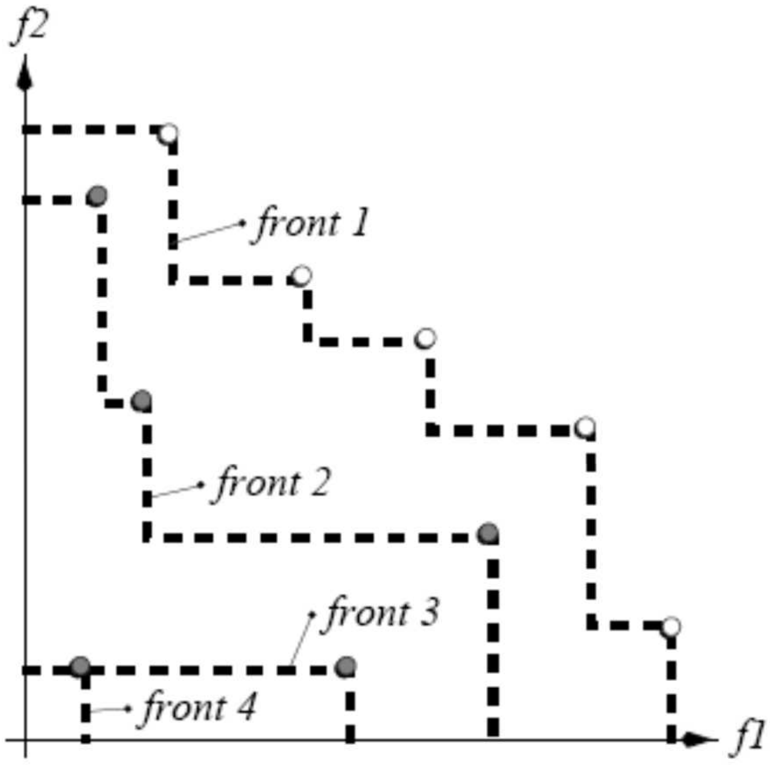


How to evaluate solutions?

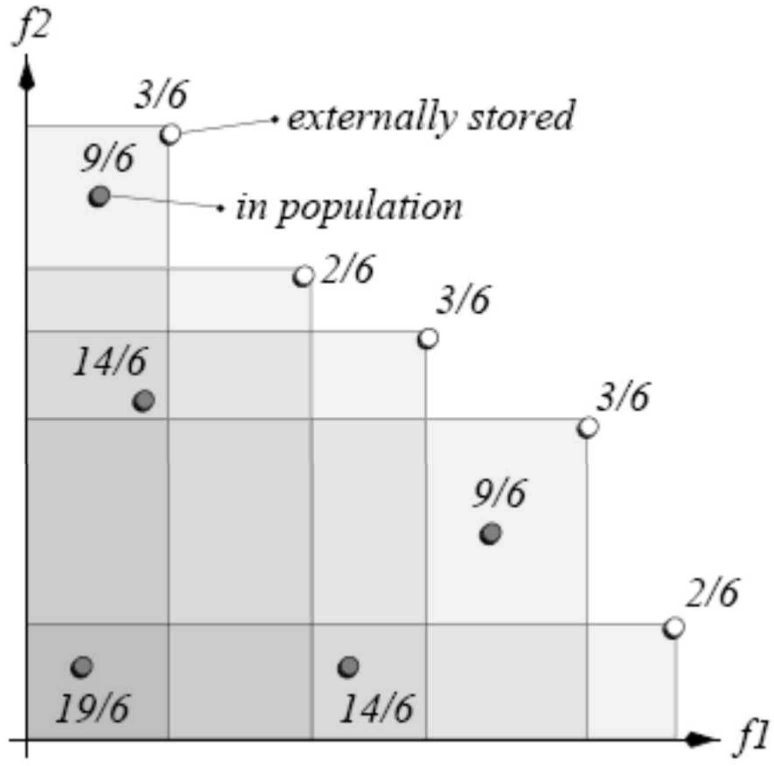


How to evaluate solutions?

(e) NSGA



(f) SPEA



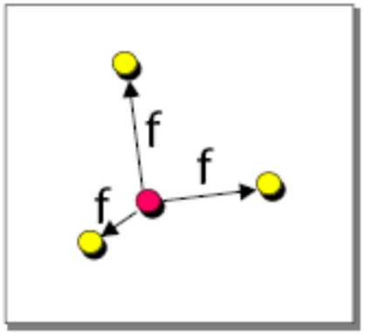
How to ensure diversity?

Density estimation techniques: [Silverman 86]

Kernel

MOGA, NPGA

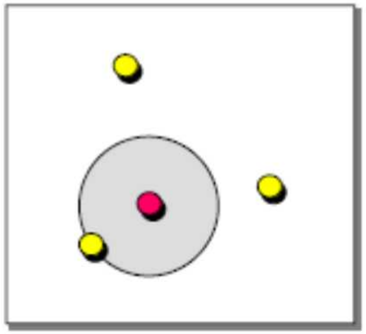
density estimate = sum of f values where f is a function of the distance



Nearest neighbor

NSGA-II, SPEA2

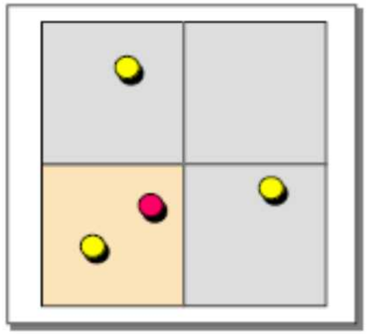
density estimate = volume of the sphere defined by the nearest neighbor



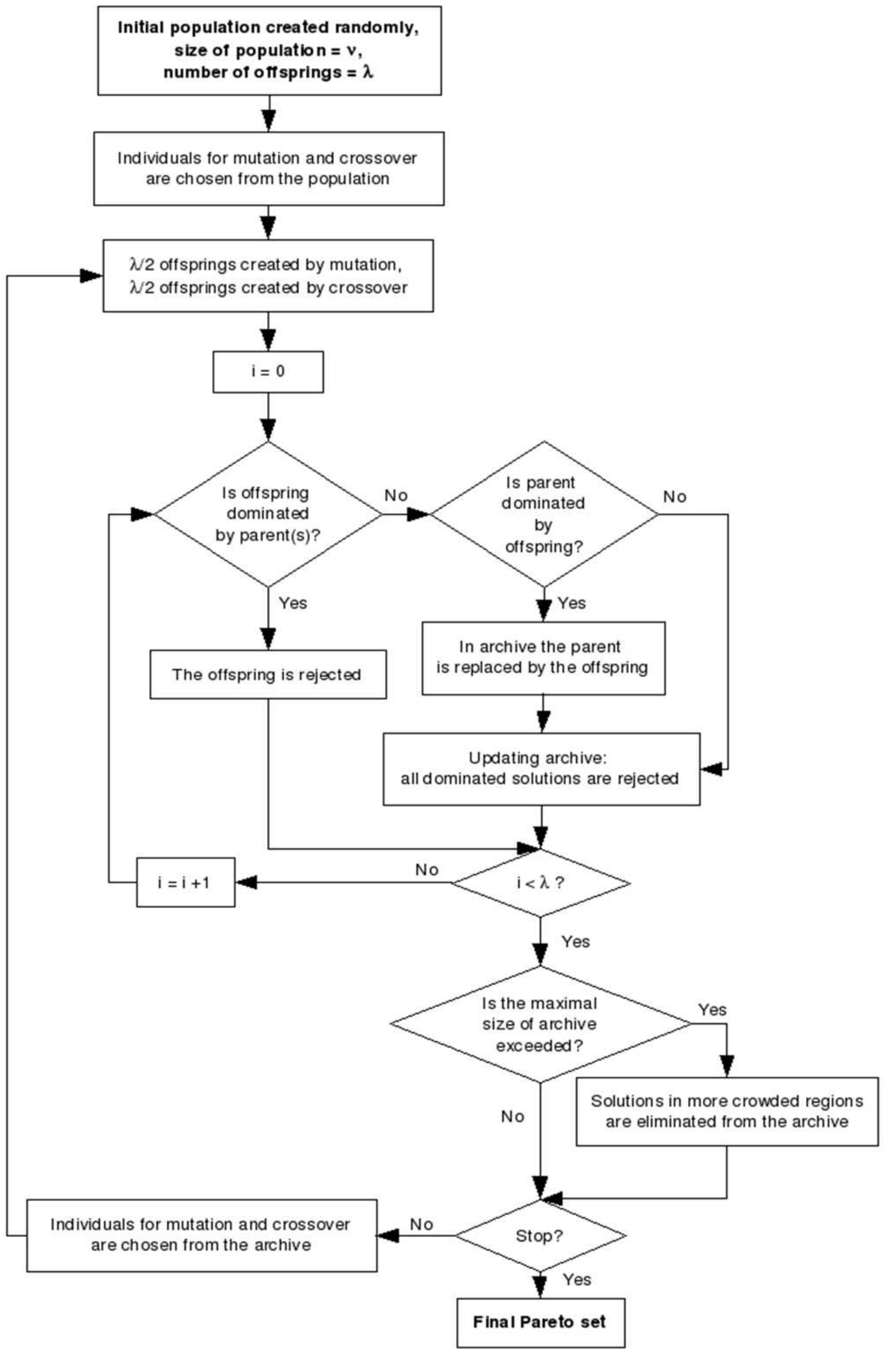
Histogram

PAES, PESA

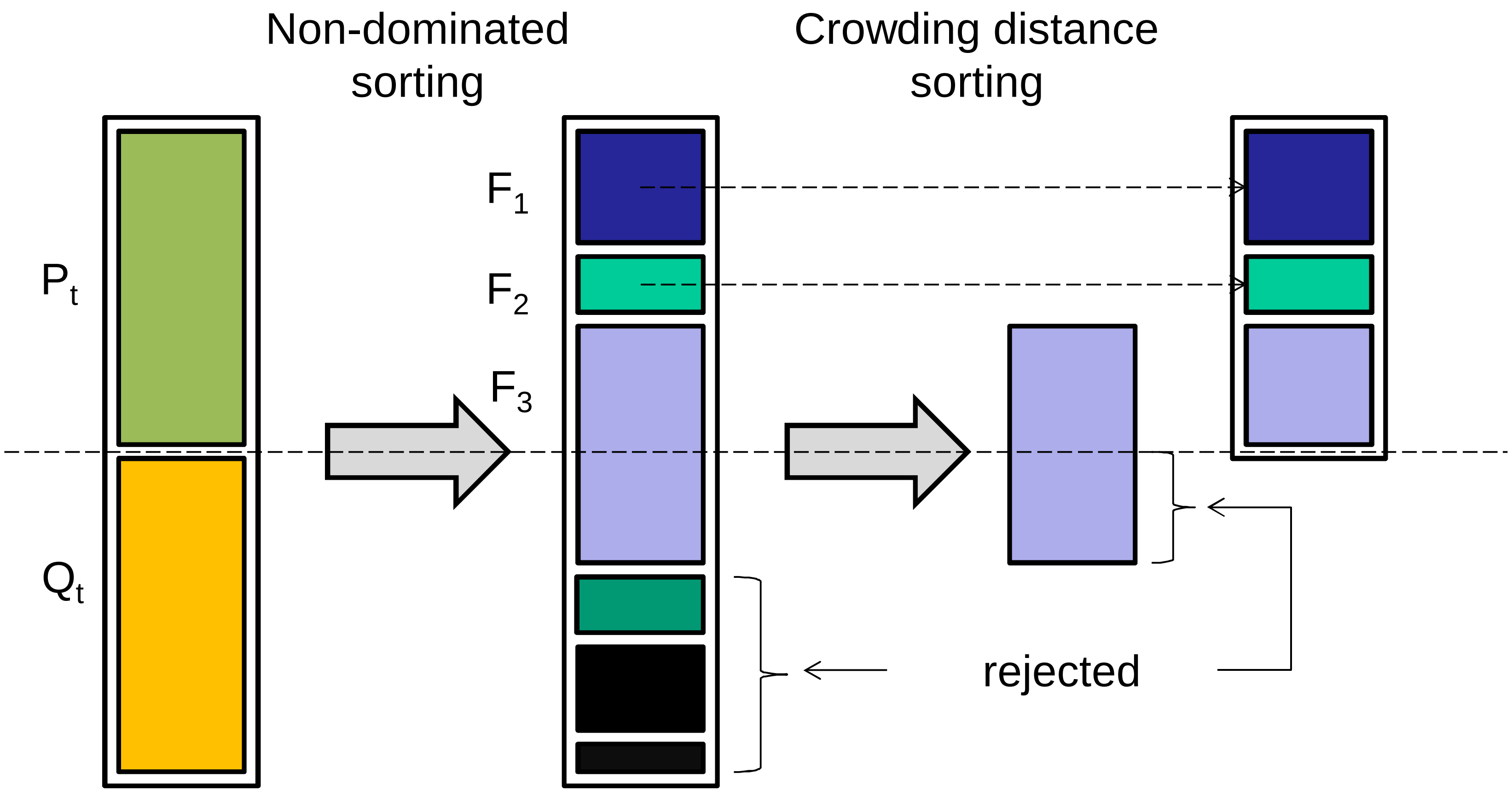
density estimate = number of solutions in the same box



PAES

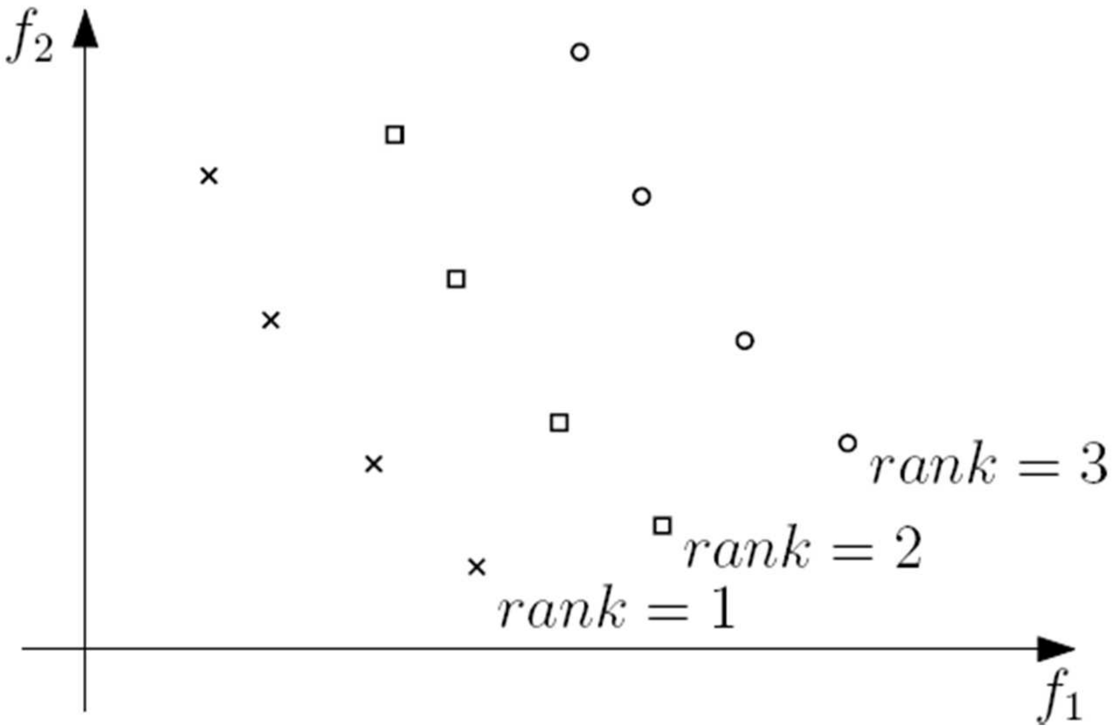

Initial population created randomly, size of population = ν, number of offsprings = λ
Individuals for mutation and crossover are chosen from the population
λ/2 offsprings created by mutation, λ/2 offsprings created by crossover
i = 0
Is offspring dominated by parent(s)?
No
Is parent dominated by offspring?
No
Yes
Yes
The offspring is rejected
In archive the parent is replaced by the offspring
Updating archive: all dominated solutions are rejected
No
i = i +1
i < λ ?
Yes
Is the maximal size of archive exceeded?
Yes
Solutions in more crowded regions are eliminated from the archive
No
Individuals for mutation and crossover are chosen from the archive
No
Stop?
Yes
Final Pareto set

NSGA II

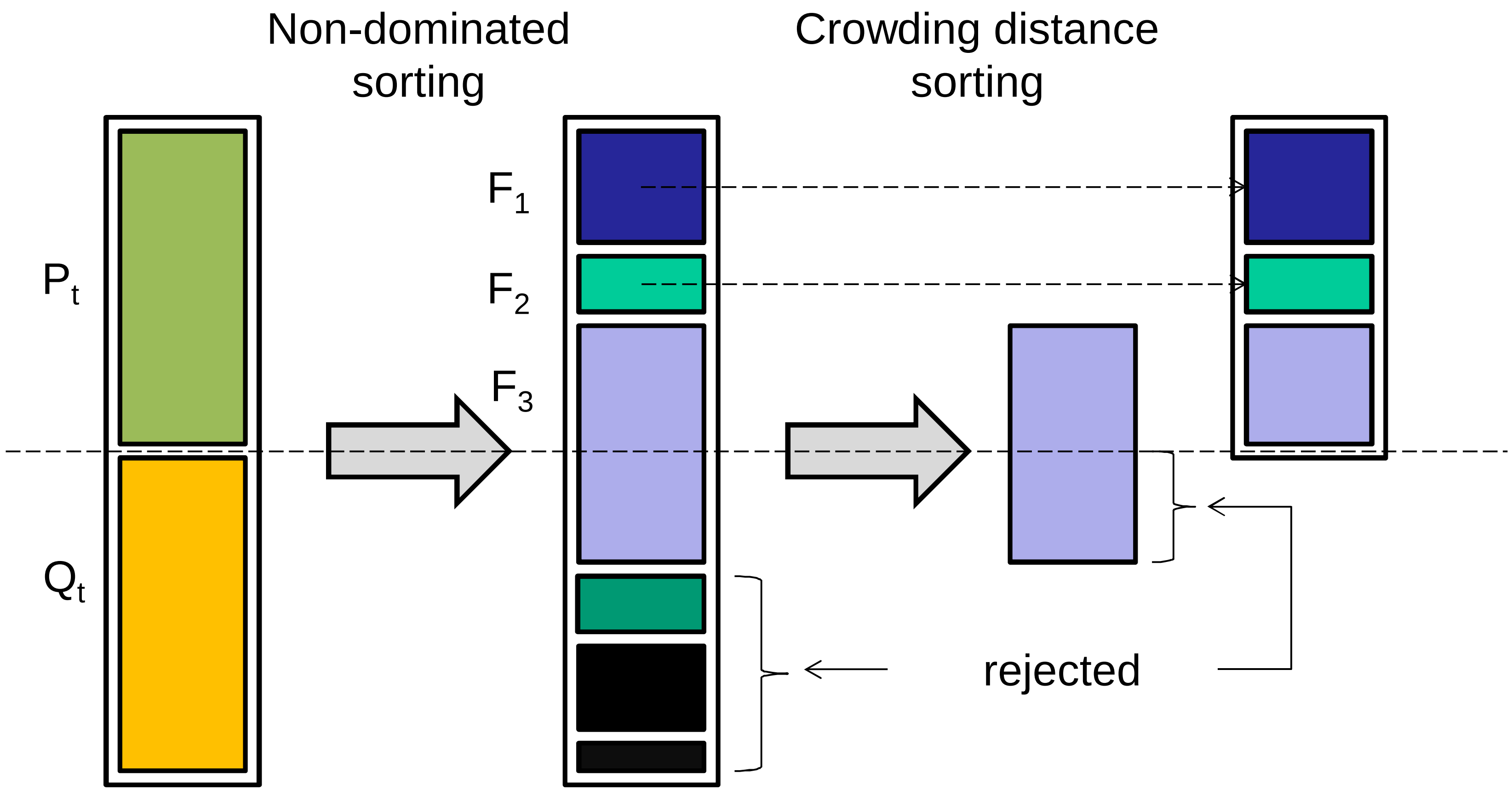

Non-dominated sorting
Crowding distance sorting
P_t
Q_t
F_1
F_2
F_3
rejected

NSGA II

Non-dominated ranking



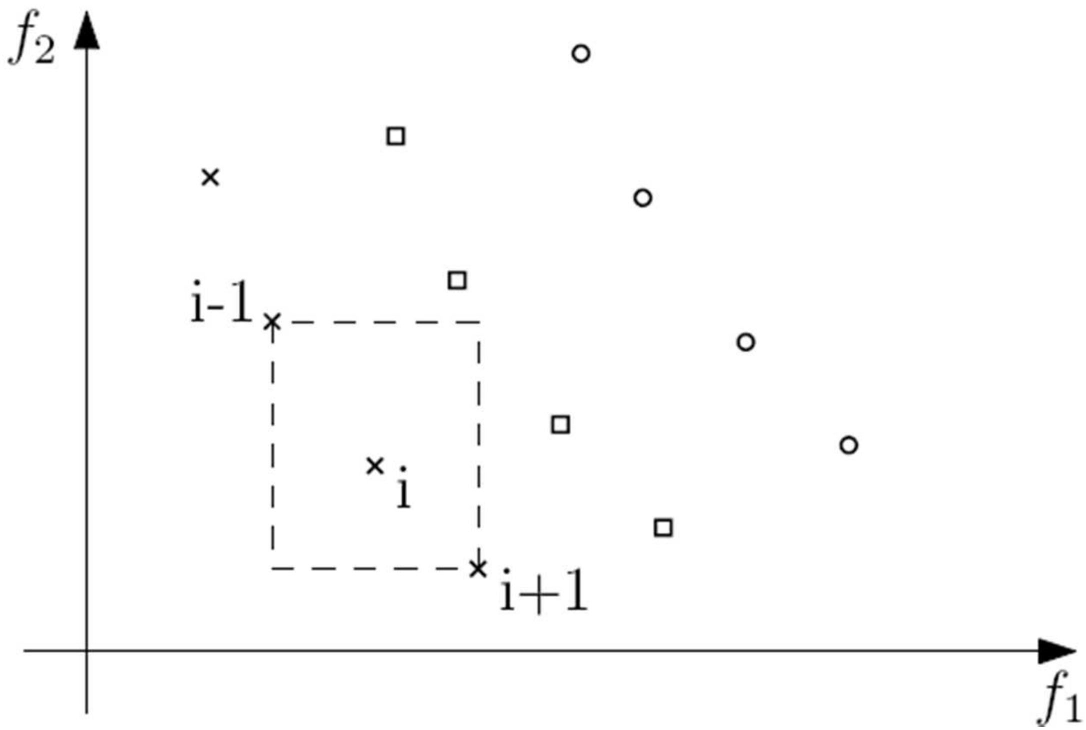
NSGA II



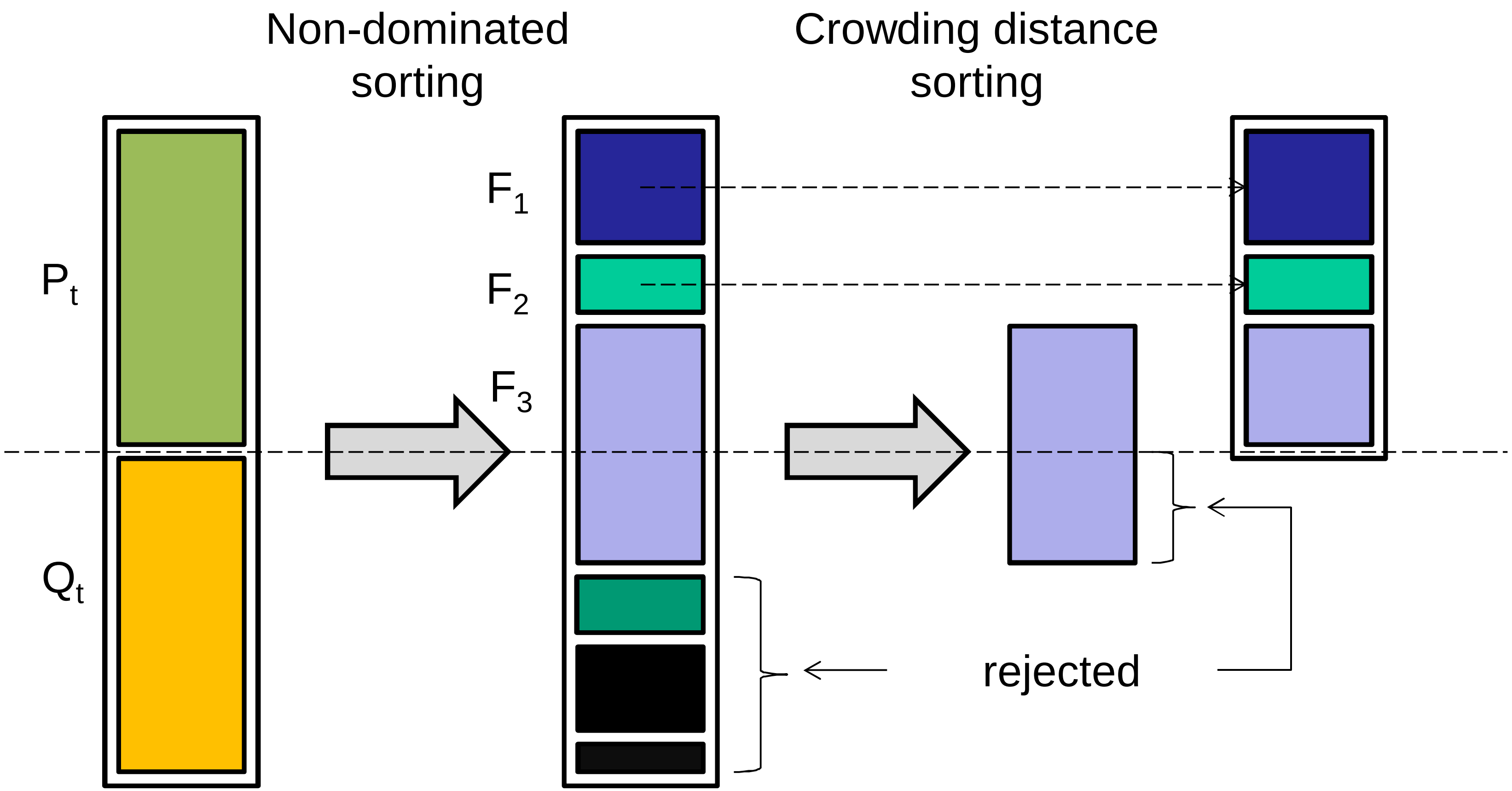
Non-dominated sorting
Crowding distance sorting
P_t
Q_t
F_1
F_2
F_3
rejected

NSGA II

Crowding distance

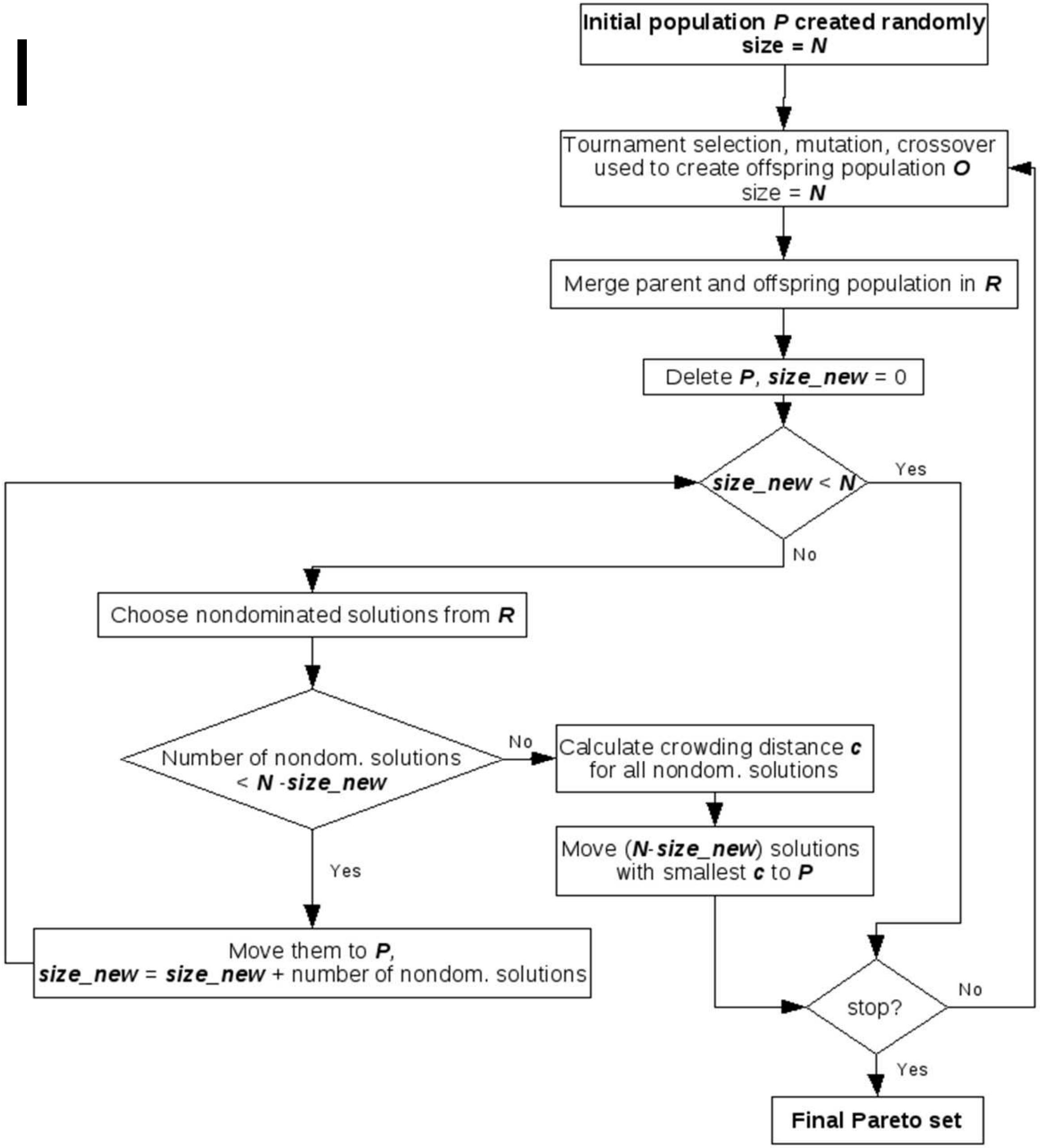


NSGA II



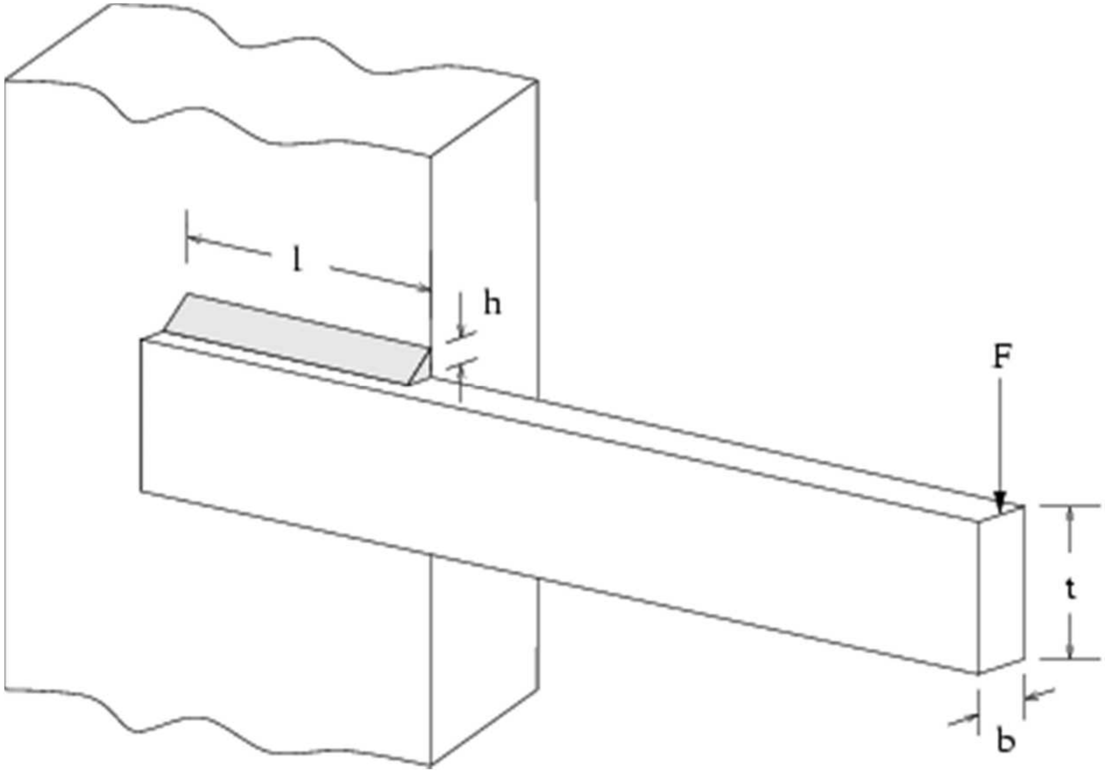
Non-dominated sorting
Crowding distance sorting
P_t
Q_t
F_1
F_2
F_3
rejected

NSGA II



Initial population P created randomly
size = N
Tournament selection, mutation, crossover
used to create offspring population O
size = N
Merge parent and offspring population in R
Delete P, size_new = 0
size_new < N
Yes
No
Choose nondominated solutions from R
Number of nondom. solutions
< N -size_new
No
Calculate crowding distance c
for all nondom. solutions
Move (N-size_new) solutions
with smallest c to P
Yes
Move them to P,
size_new = size_new + number of nondom. solutions
stop?
No
Yes
Final Pareto set

Example



F = 6 000 lb

h,b ∈ (0.125, 5.0) [in]

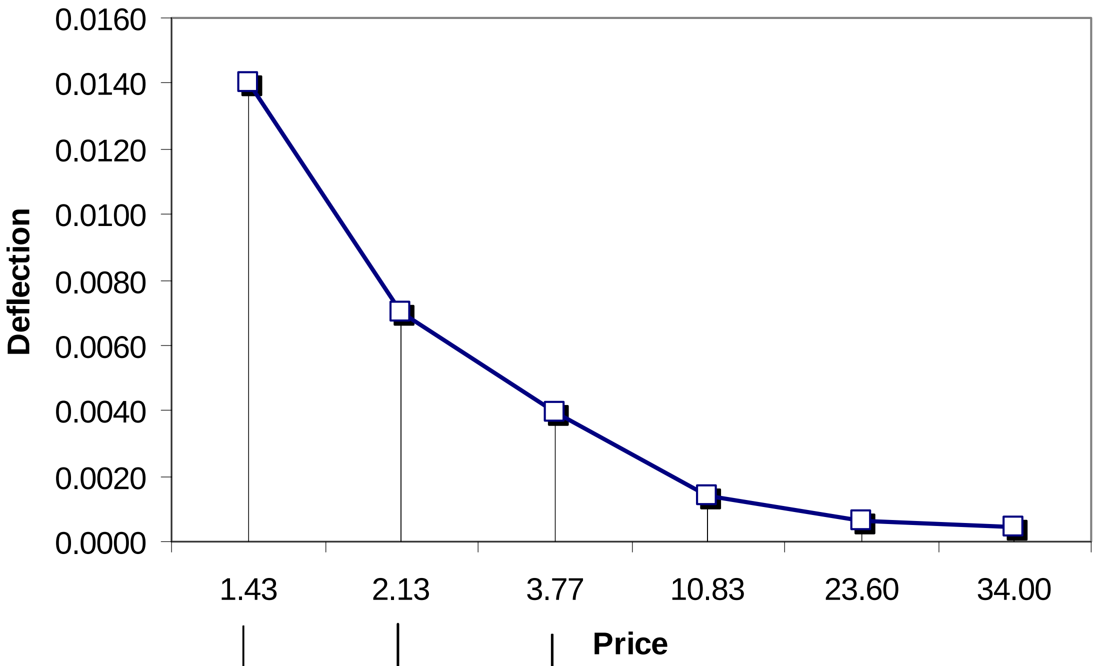
l,t ∈ (0.1, 10.0) [in]

Minimize:

f_1 = price

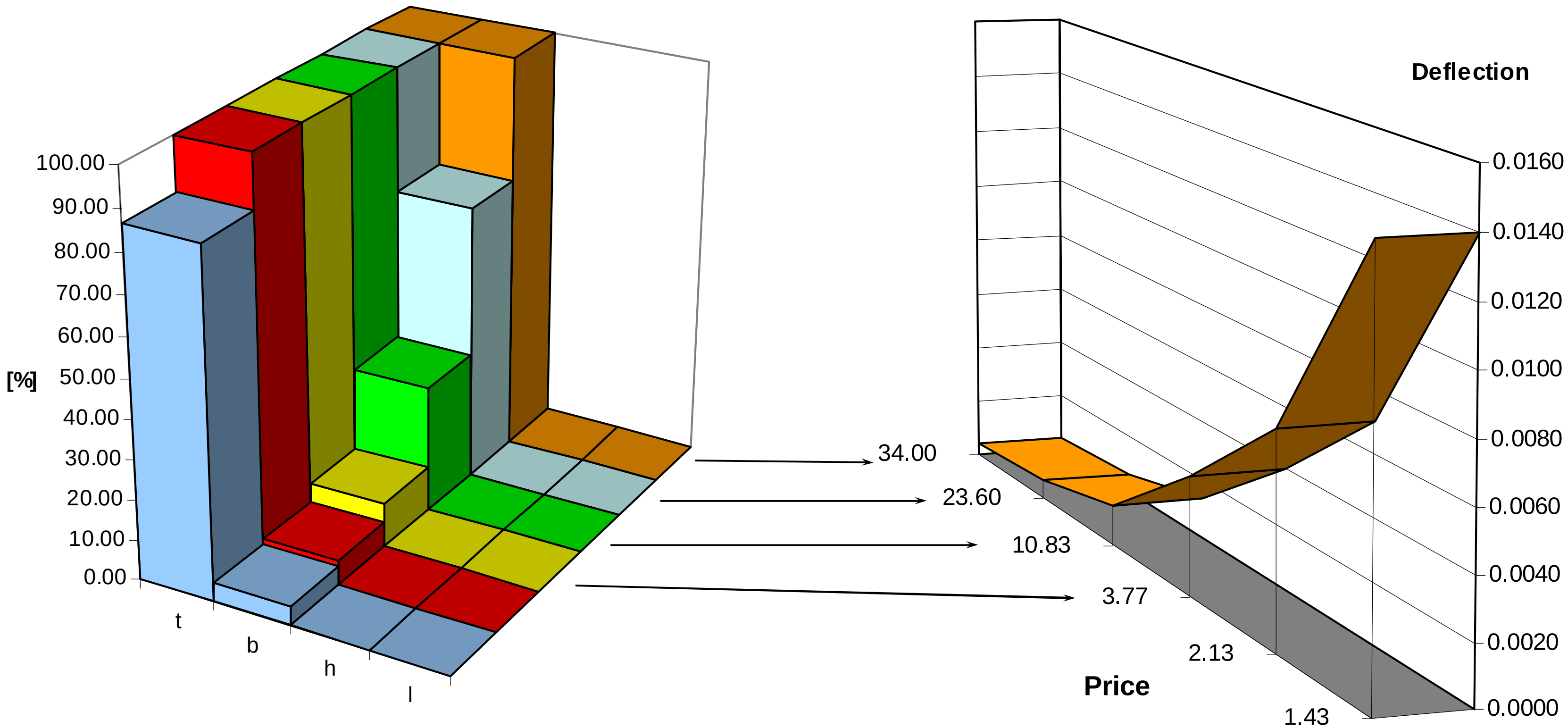
f_2 = deflection

Pareto surface:

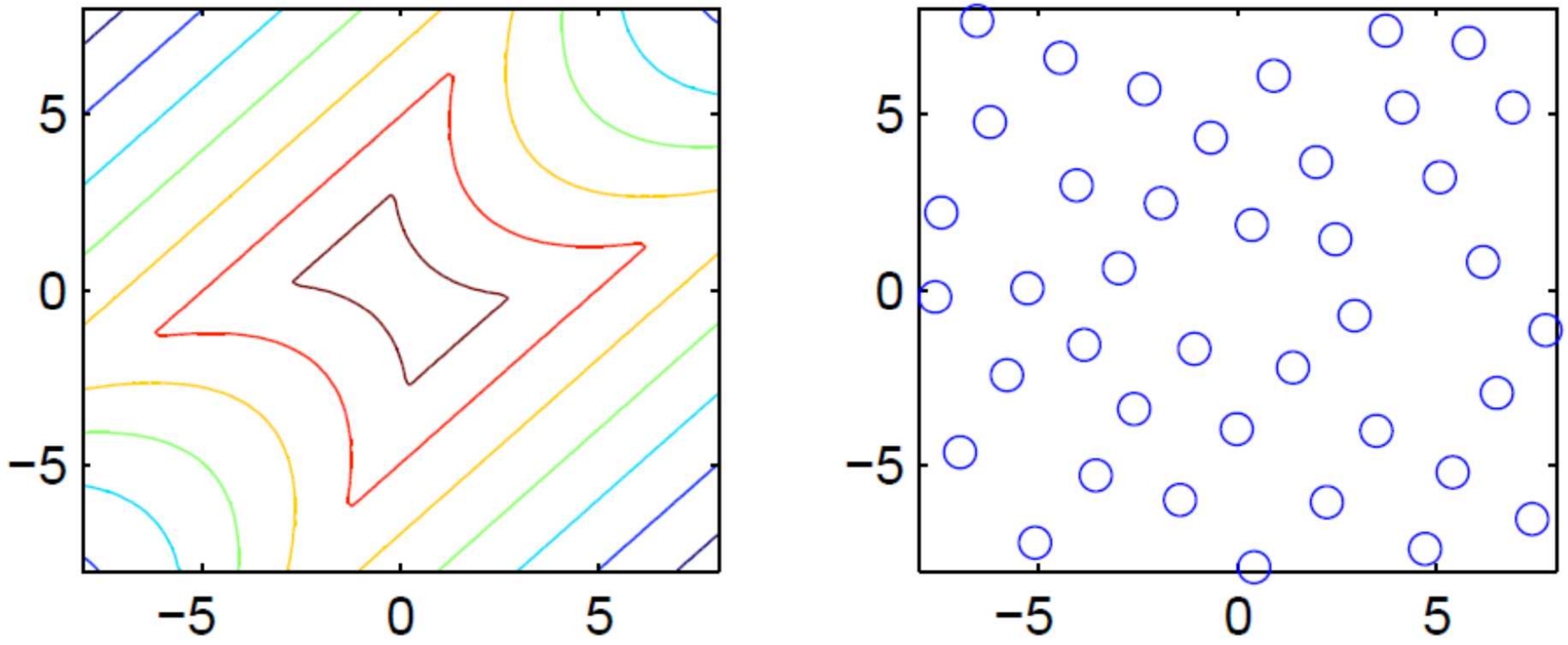


Pareto set:

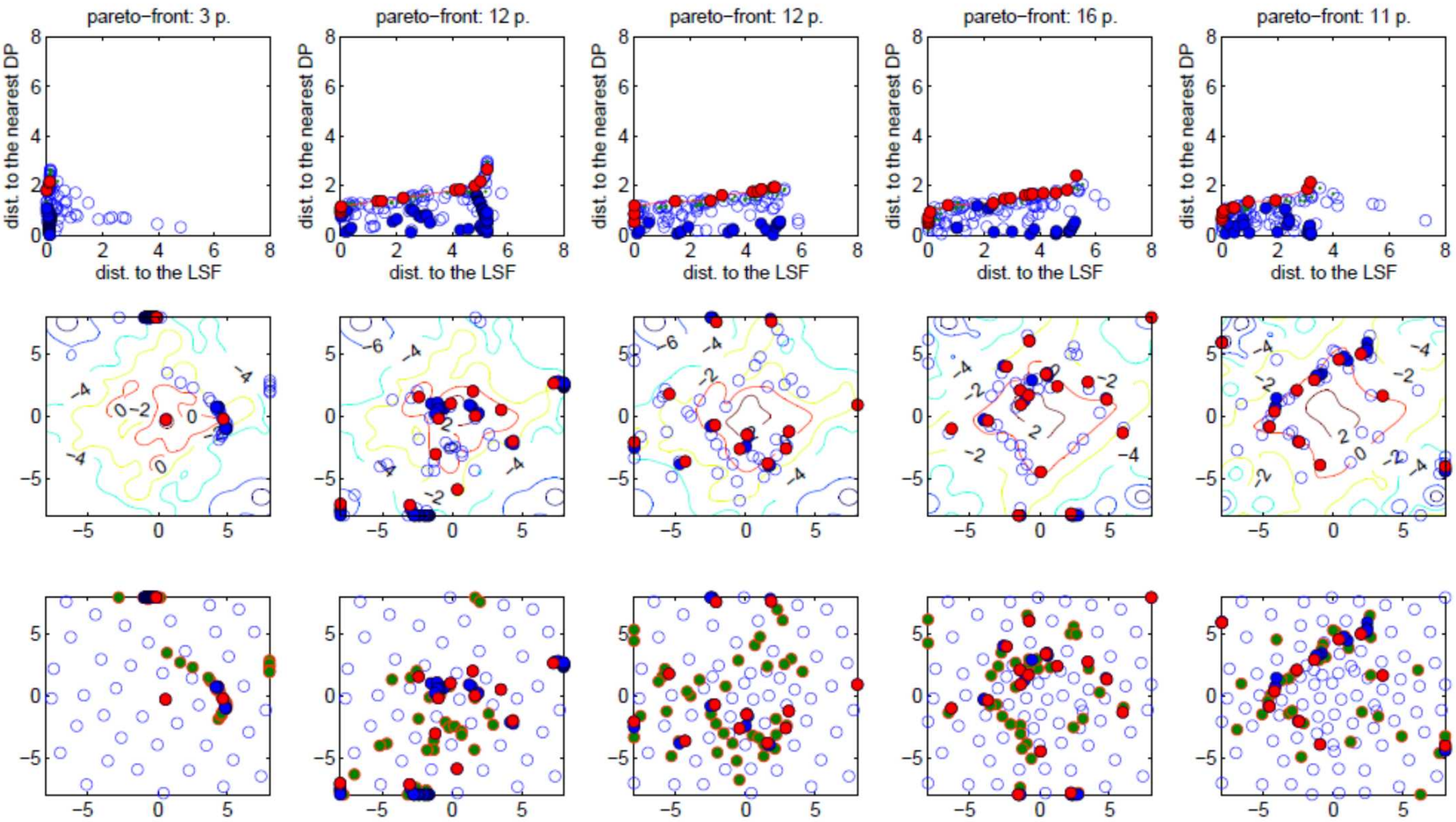
Results


100.00
90.00
80.00
70.00
60.00
[%]
50.00
40.00
30.00
20.00
10.00
0.00
t
b
h
l
34.00
23.60
10.83
3.77
2.13
1.43
Price
Deflection
0.0160
0.0140
0.0120
0.0100
0.0080
0.0060
0.0040
0.0020
0.0000

Example 2:
Adaptive update of meta-model



Contours of the example (left) and starting DoE (right). Note that the red contour is for F(x) = 0.



Pareto front (top), contours of the problem with DoEs (middle) and DoEs' points (bottom).
Key: Red – added and computed solutions, Blue – points that were too close to other Pareto front points, Green – the remaining points of population and Blue empty points – the original DoE.

References

[1] Zitzler, E. (1999). Evolutionary Algorithms for Multiobjective Optimization: Methods and Applications. PhD thesis, Swiss Federal Institute of Technology (ETH), Zurich, Switzerland.

[2] Zitzler, E., Laumanns, M., and Thiele, L. (2001). SPEA2: Improving the Strength Pareto Evolutionary Algorithm. In Giannakoglou, K., Tsahalis, D., Periaux, J., Papailou, P., and Fogarty, T., editors, EUROGEN 2001. Evolutionary Methods for Design, Optimization and Control with Applications to Industrial Problems, Athens, Greece.

[3] Miettinen, K. (1999). Nonlinear Multiobjective Optimization. Kluwer Academic Publishers, Dordrecht.

References

[4] Knowles, J. D. and Corne, D.W. (2000). Approximating the Nondominated Front Using the Pareto Archived Evolution Strategy. Evolutionary Computation, 8(2):149–172.

[5] Kukkonen, S. and Lampinen, J. (2004). Comparison of generalized differential evolution to other multi-objective evolutionary algorithms. In European Congress on Computational Methods in Applied Sciences and Engineering (ECCOMAS 2004).

[6] Coello, C. A. C. (2004). List of references on evolutionary multiobjective optimization. http://www.lania.mx/~ccoello/EMOO/EMOObib.html.

[7] Lepš, M. (2005). Single and Multi-Objective Optimization in Civil Engineering with Applications, PhD thesis, CTU in Prague.

[8] NIMBUS: https://wwwnimbus.it.jyu.fi/

A humble plea. Please feel free to e-mail any suggestions, errors and typos to **`matej.leps@fsv.cvut.cz`**.

Change 26.11.2014: NSGA II and PAES slides added.

Date of the last version: 26.11.2014
Version: 001