Modeling of Localized Inelastic Deformation

Milan Jirásek

General outline:

A. Introduction
B. Elastoplasticity
C. Damage mechanics
D. Strain localization
E. Regularized continuum models
F. **Strong discontinuity models**

F. Strong discontinuity models

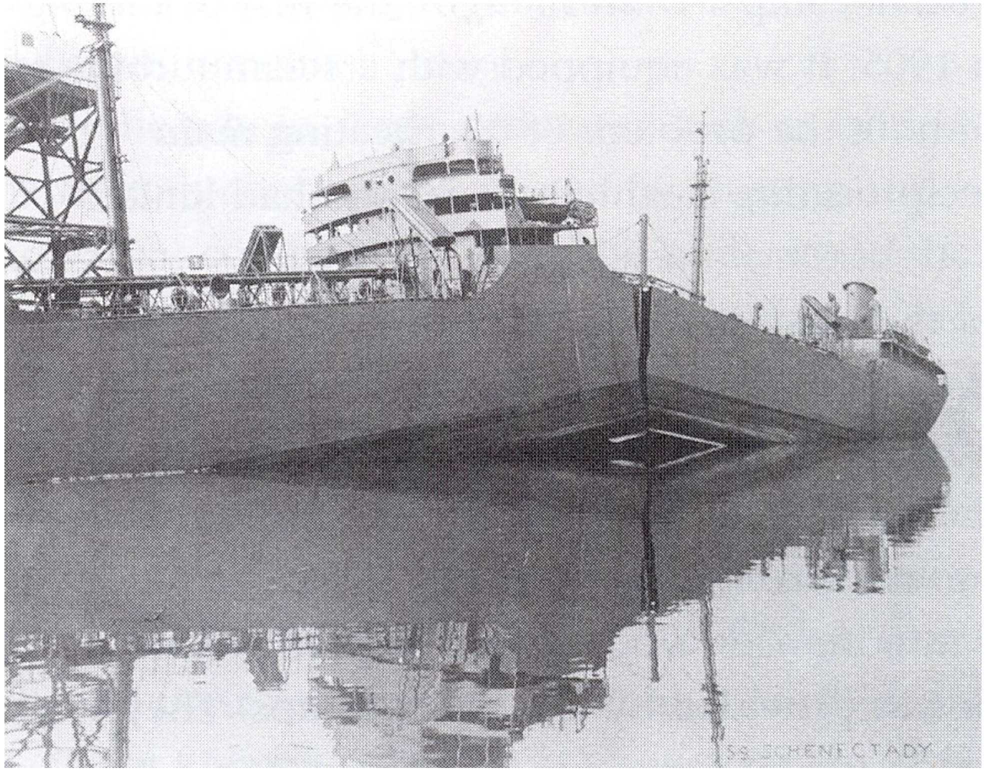
F.1 Fundamentals of fracture mechanics

F.2 Finite elements with discontinuities - introduction

F.3 Embedded discontinuities (EED-EAS)

F.4 Extended finite elements (XFEM-PUM)

Failure of Liberty (and other) ships during WW II

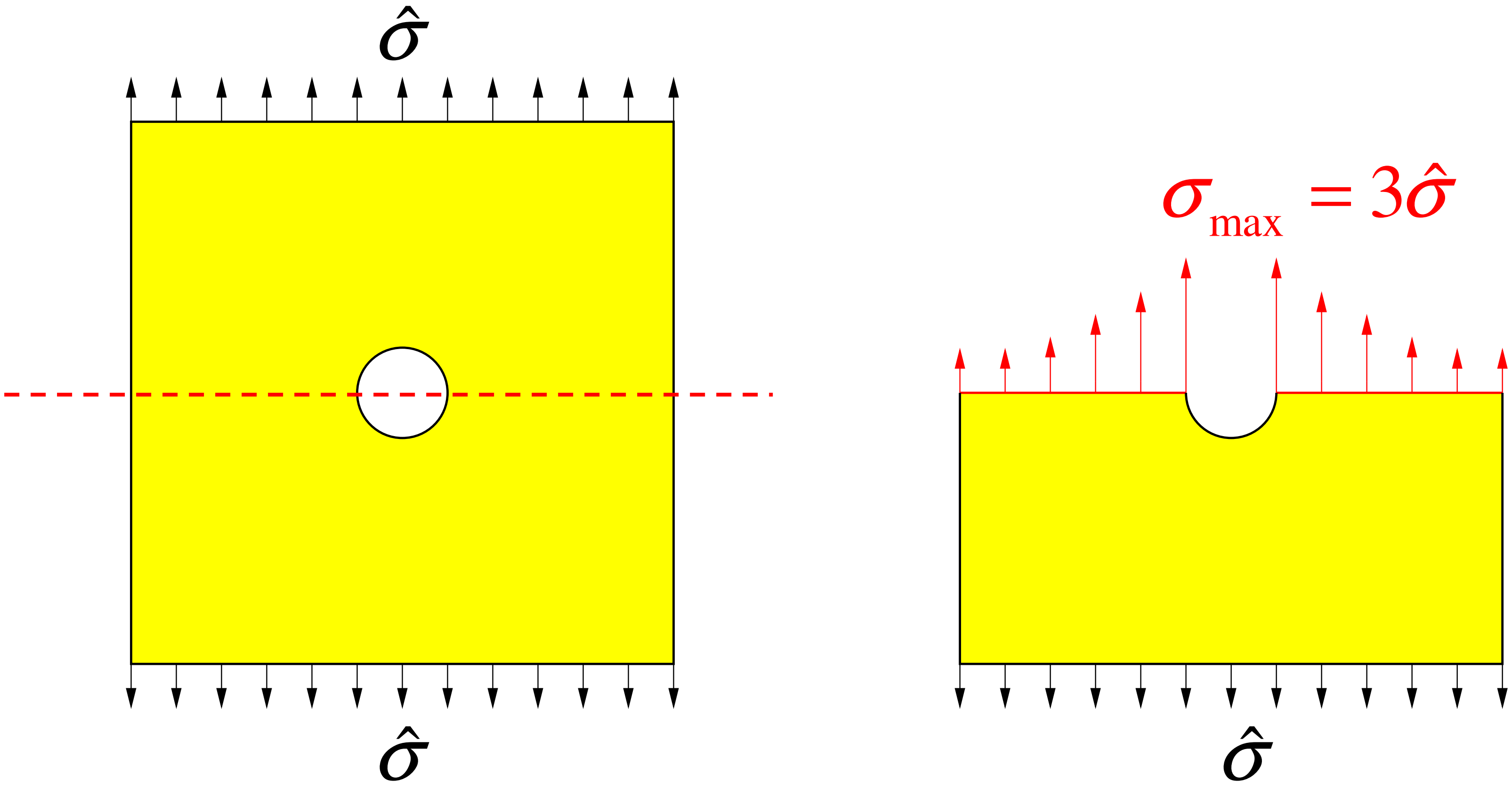


reason:
brittle fracture

19 ships broke in half without warning

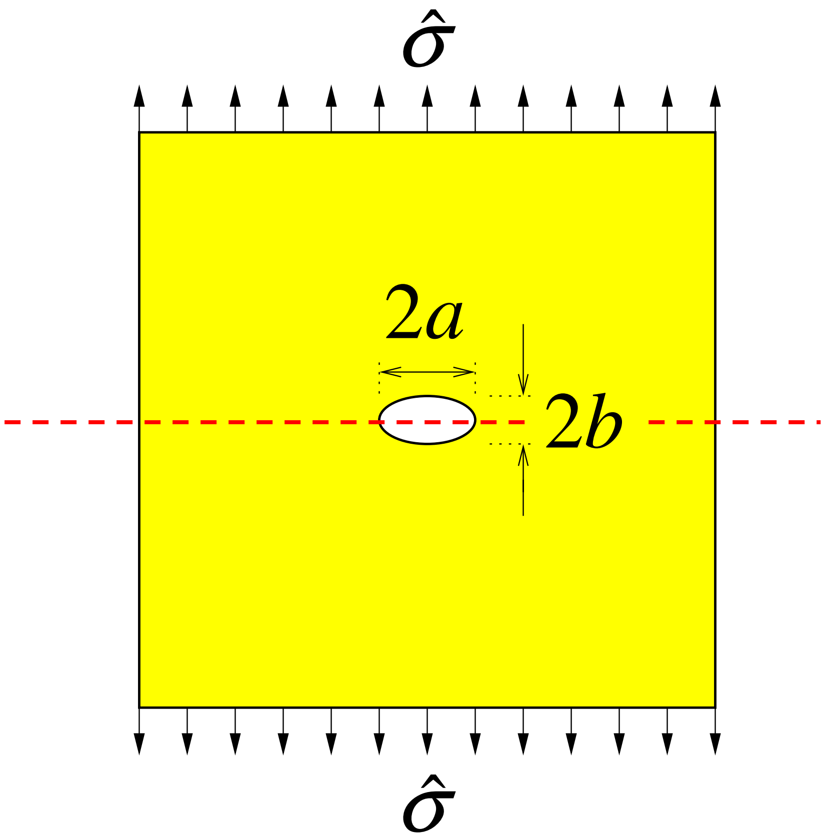
Stress concentration near defects

panel weakened by a spherical hole

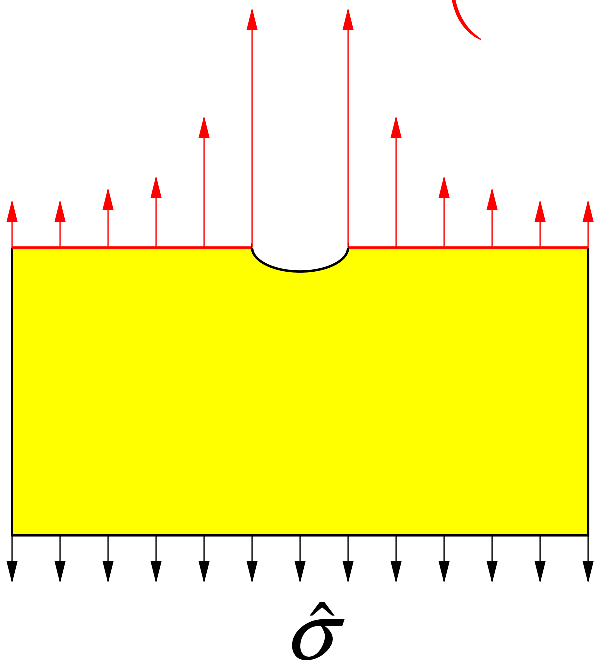


Stress concentration near defects

panel weakened by an eliptical hole

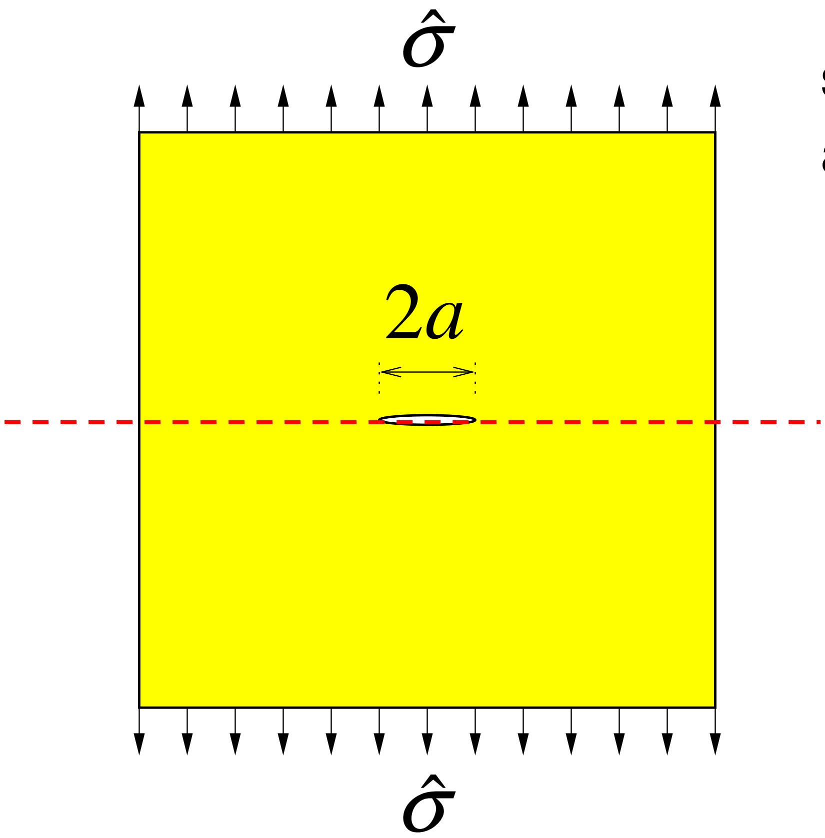


$$\sigma_{\max} = \left(1 + \frac{2a}{b}\right)\hat{\sigma}$$



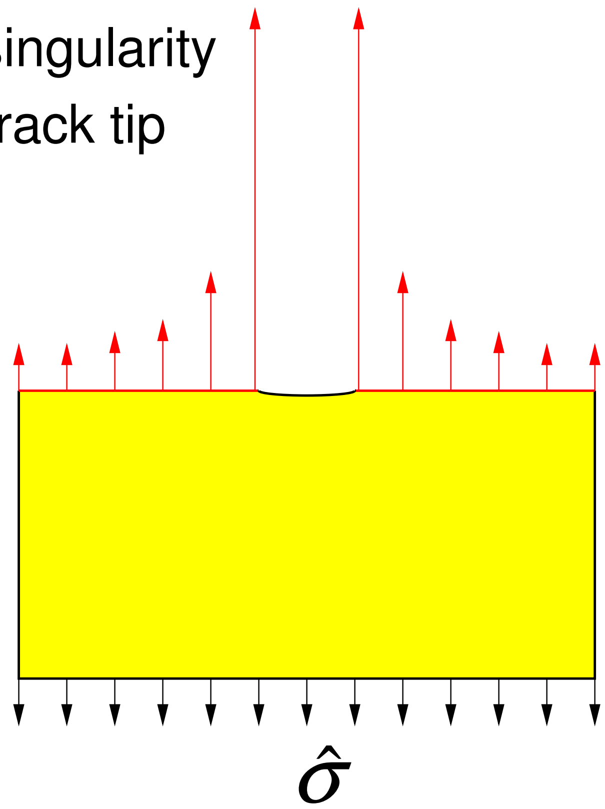
Stress concentration near defects

panel weakened by a crack

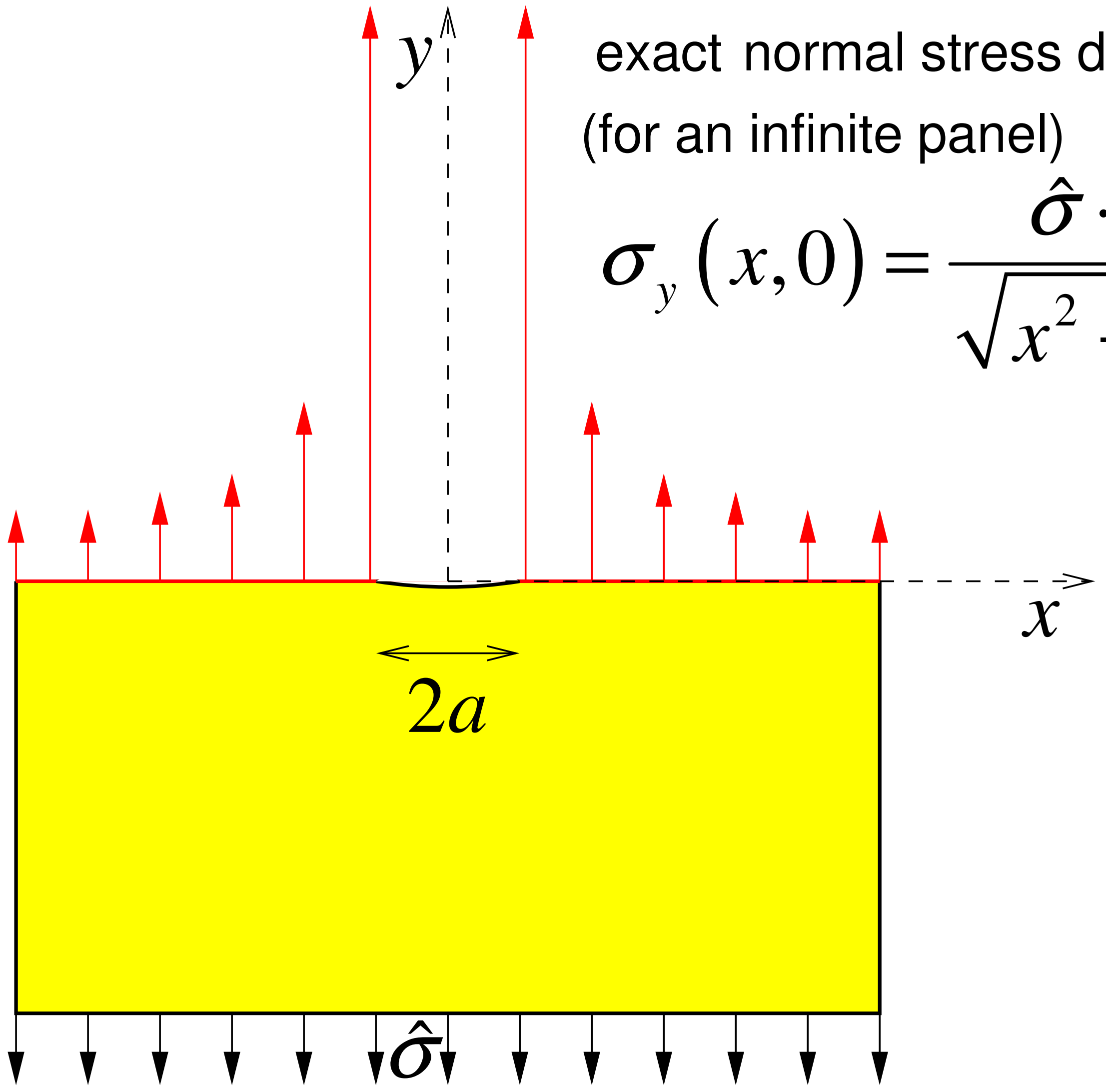


stress singularity
at the crack tip

$\sigma_{max} \rightarrow \infty$



Stress concentration near defects



exact normal stress distribution

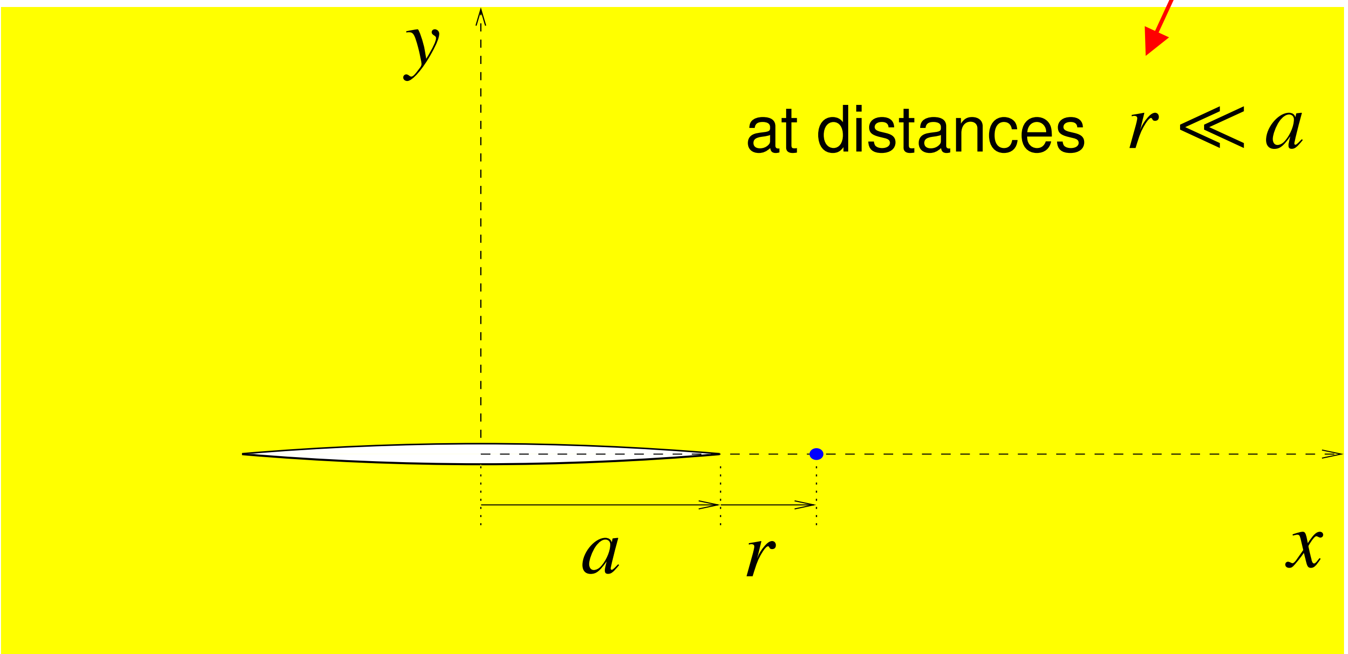
(for an infinite panel)

$$\sigma_y(x,0) = \frac{\hat{\sigma} \cdot x}{\sqrt{x^2 - a^2}} \quad \text{for } x > a$$

Singular stress field near the crack tip

exact   approximation near the tip

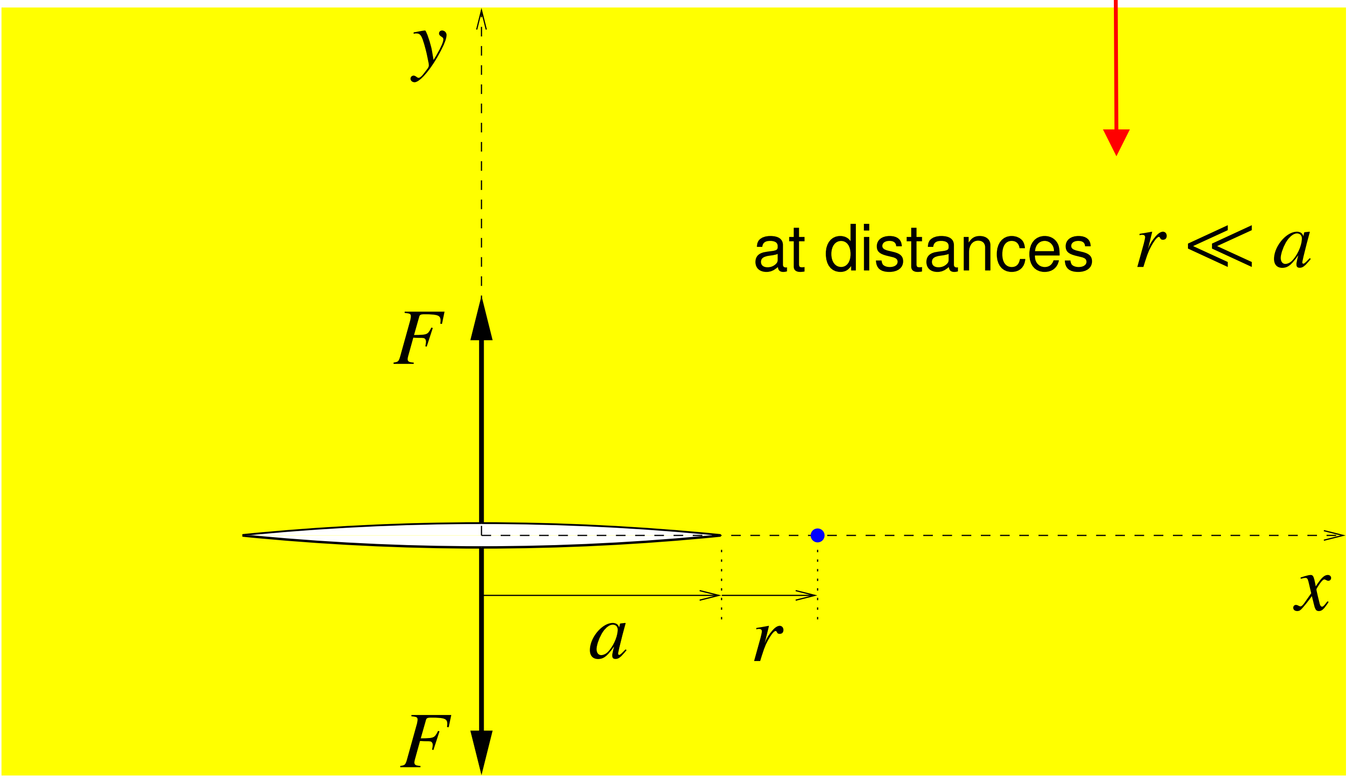
$$\sigma_y(x,0) = \frac{\hat{\sigma}\cdot x}{\sqrt{x^2 - a^2}} = \frac{\hat{\sigma}\cdot(a+r)}{\sqrt{(a+r)^2 - a^2}} \approx \frac{\hat{\sigma}\cdot a}{\sqrt{2ar}} = \hat{\sigma}\sqrt{\frac{a}{2}}\cdot\frac{1}{\sqrt{r}}$$



stress is inversely proportional to the square root of distance from the crack tip

Singular stress field near the crack tip

$$\sigma_y(x,0) = \frac{Fa}{t\pi x\sqrt{x^2-a^2}} \approx \frac{F}{t\pi\sqrt{2ar}} = \frac{F}{t\pi\sqrt{2a}} \cdot \frac{1}{\sqrt{r}}$$

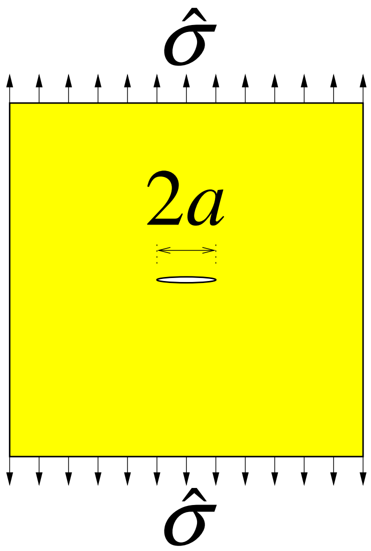


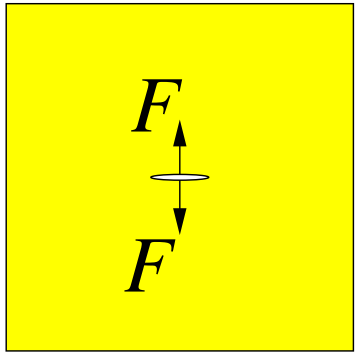
stress is inversely proportional to the square root of distance from the crack tip

Singular stress field near the crack tip

$2a = 20\ \text{mm}$

$\hat{\sigma} = 10\ \text{MPa}$





$2a = 30\ \text{mm}$

$t = 5\ \text{mm}$

$F = 1.9238\ \text{kN}$

near-tip stress approximation

$$\hat{\sigma}\sqrt{\frac{a}{2}} \cdot \frac{1}{\sqrt{r}} \qquad \frac{F}{t\pi\sqrt{2a}} \cdot \frac{1}{\sqrt{r}}$$

same value of this factor

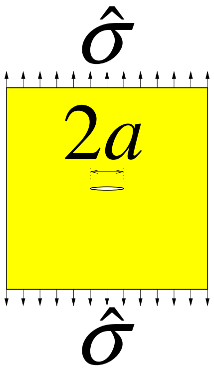
⇒ same stress concentration near the tip

Singular stress field near the crack tip

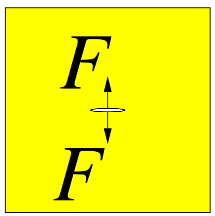
general expression for the singular part of stress field
that dominates near the crack tip

$$\sigma_y(x,0) \approx \frac{K_I}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{r}}$$

K_I … stress intensity factor



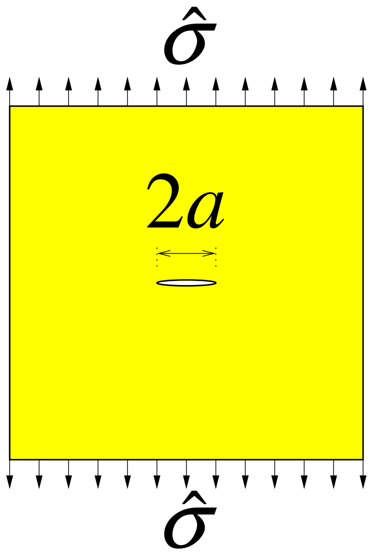
$$\sigma_y(x,0) \approx \hat{\sigma}\sqrt{\frac{a}{2}} \cdot \frac{1}{\sqrt{r}} \qquad \ldots \qquad K_I = \hat{\sigma}\sqrt{\pi a}$$

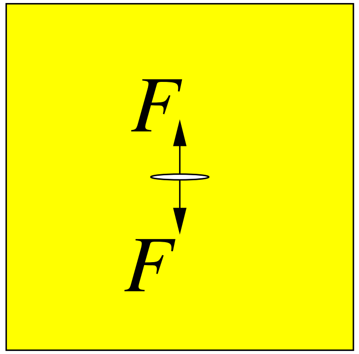


$$\sigma_y(x,0) \approx \frac{F}{t\pi\sqrt{2a}} \cdot \frac{1}{\sqrt{r}} \qquad \ldots \qquad K_I = \frac{F}{t\sqrt{\pi a}}$$

Singular stress field near the crack tip

$$2a = 20 \text{ mm}$$
$$\hat{\sigma} = 10 \text{ MPa}$$





$$2a = 30 \text{ mm}$$
$$t = 5 \text{ mm}$$
$$F = 1.9238 \text{ kN}$$

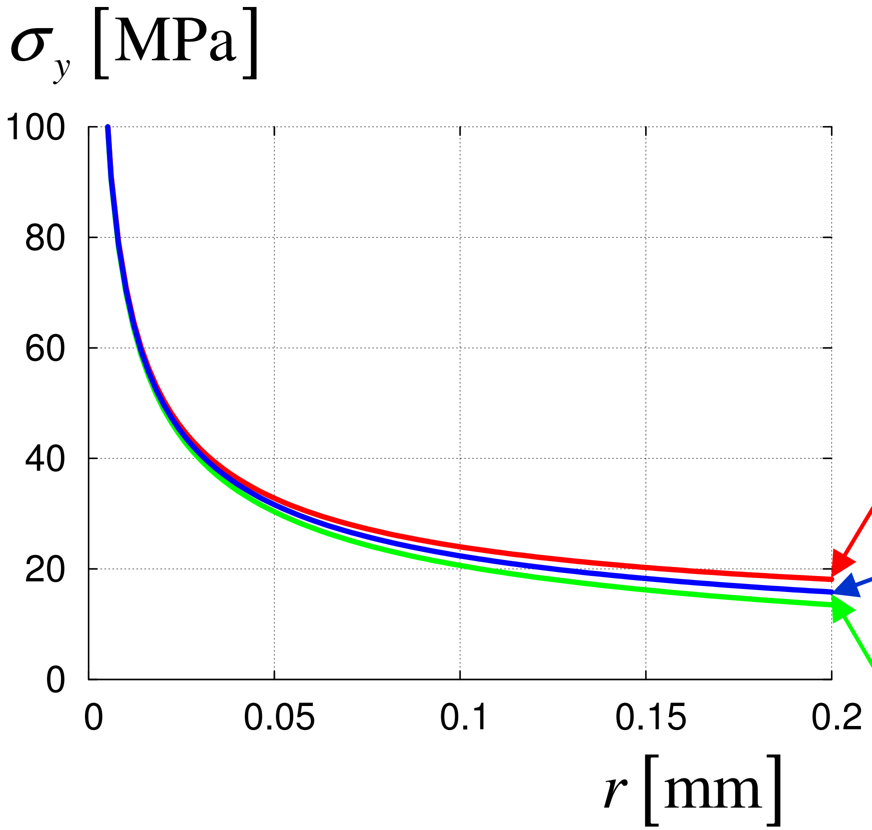
$$K_I = \hat{\sigma}\sqrt{\pi a} = 1{,}772 \cdot 10^6 \, \text{Nm}^{-3/2}$$

$$K_I = \frac{F}{t\sqrt{\pi a}} = 1{,}772 \cdot 10^6 \, \text{Nm}^{-3/2}$$

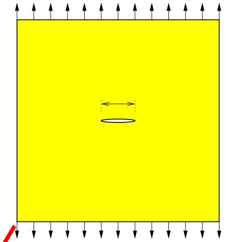
same stress intensity factor

⇨ same stress concentration near the tip

Singular stress field near the crack tip



$$a_1 = 10 \text{ mm}, \ \hat{\sigma} = 10 \text{ MPa}$$

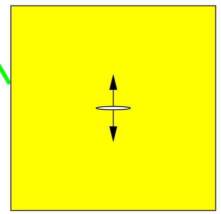


$$\frac{\hat{\sigma} \cdot (a_1 + r)}{\sqrt{(a_1 + r)^2 - a_1^2}}$$

asymptotic stress field

$$\frac{K_I}{\sqrt{2\pi r}}$$

$$K_I = 1.772 \text{ MNm}^{-3/2}$$



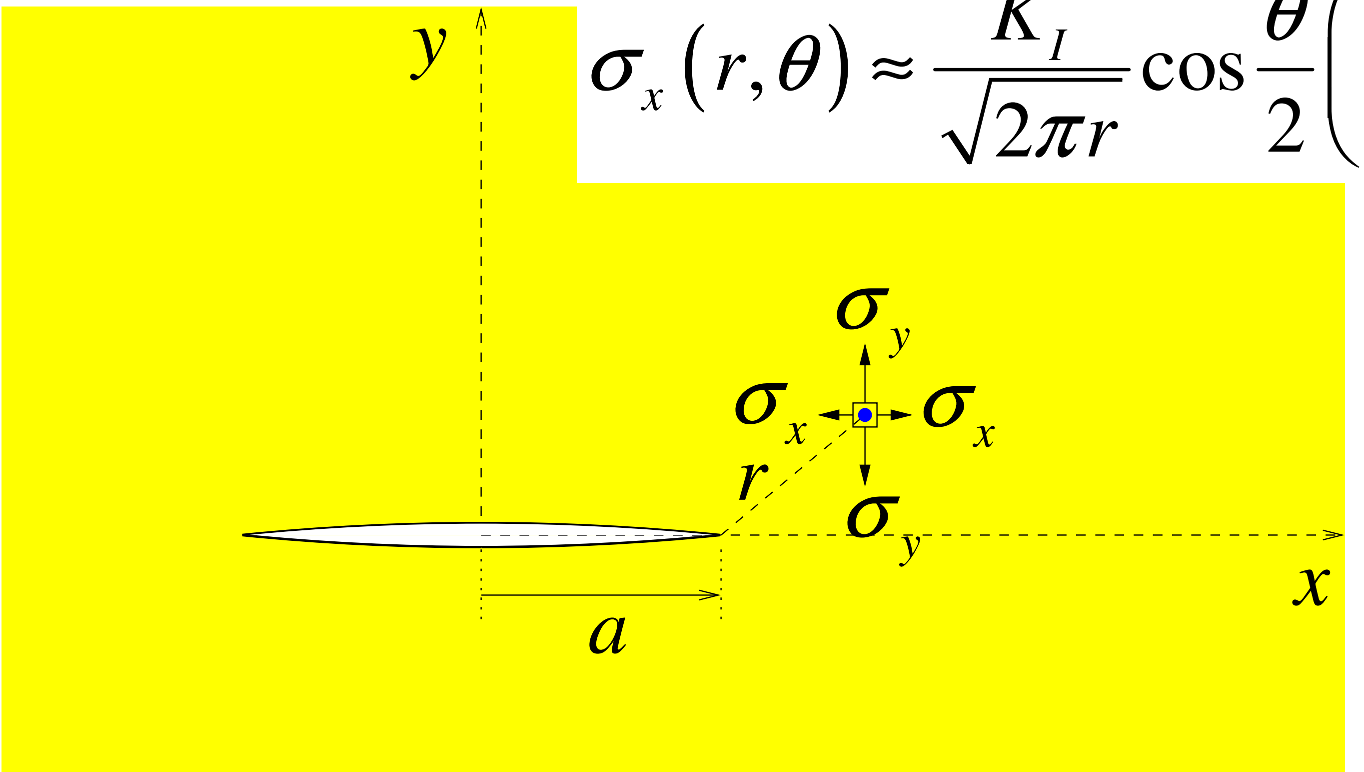
$$\frac{F a_2}{t\pi (a_2 + r)\sqrt{(a_2 + r)^2 - a_2^2}}$$

$$a_2 = 15 \text{ mm}, \ F/t = 385 \text{ kN/m}$$

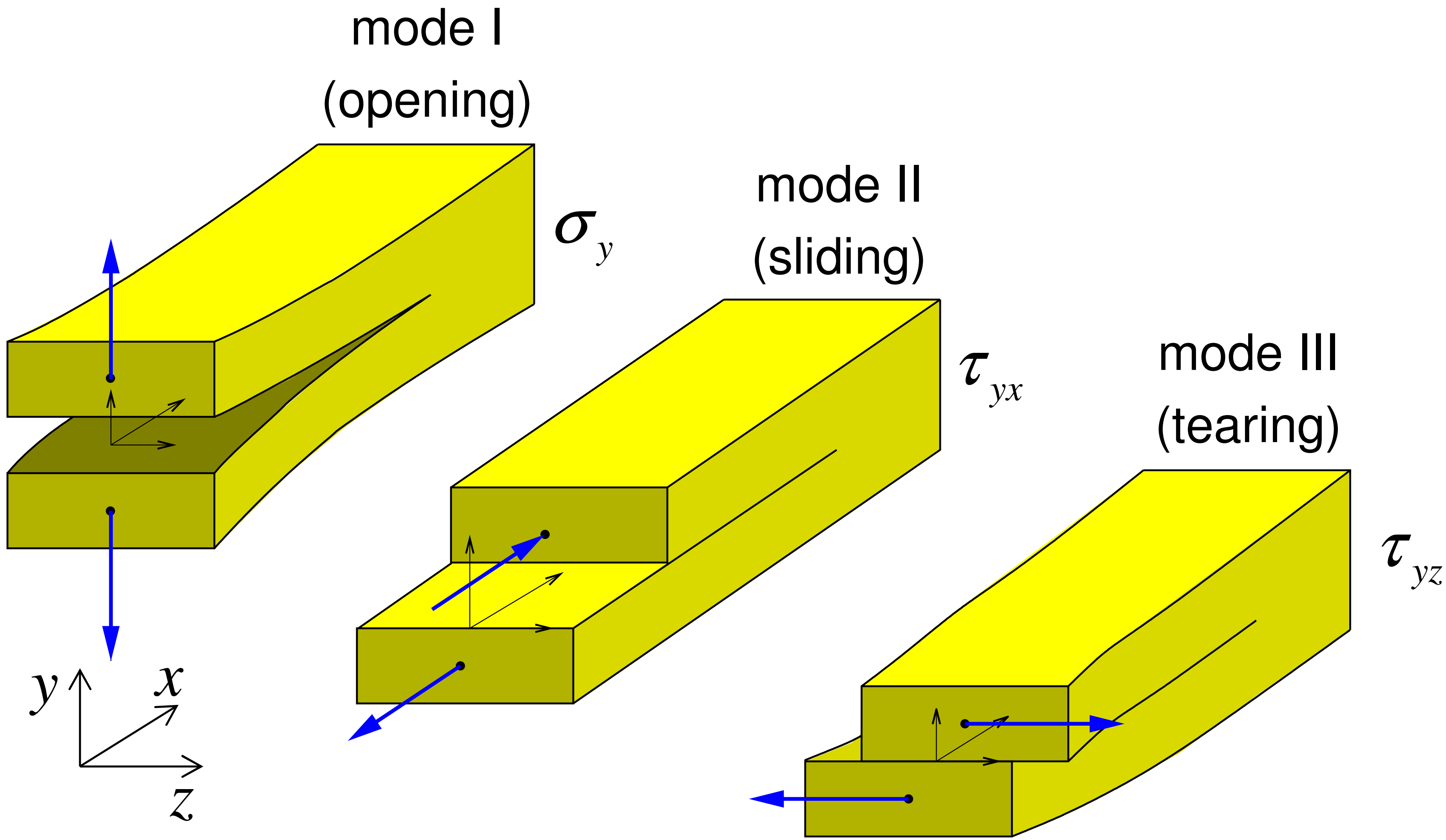
Singular stress field near the crack tip

$$\sigma_y(r,\theta) \approx \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1+\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)$$

$$\sigma_x(r,\theta) \approx \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)$$



Basic fracture modes



Near-tip asymptotic fields

crack loaded in a mixed mode (combination of modes I and II):

$$\sigma_y(r,\theta) \approx \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1+\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) - \frac{K_{II}}{\sqrt{2\pi r}} \sin\frac{\theta}{2}\left(2+\cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right)$$

$$\sigma_x(r,\theta) \approx \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) + \frac{K_{II}}{\sqrt{2\pi r}} \sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2}$$

$$\tau_{xy}(r,\theta) \approx \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{3\theta}{2} + \frac{K_{II}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)$$

Crack propagation – Irwin (local) criterion

A crack loaded in mode I propagates
if the stress intensity factor at its tip attains a critical value:

$$K_I = K_c$$

stress intensity factor
(depends on loading,
shape and dimensions
of the body
and on the crack size)

fracture toughness
(material property)

$$\left[\mathrm{Nm}^{-3/2} \right]$$

Crack propagation – Griffith (global) criterion

A crack loaded in mode I propagates
if its propagation releases a critical amount of energy:

$$\mathcal{G} = G_\mathrm{f}$$

energy release rate
(depends on loading,
shape and dimensions
of the body
and on the crack size)

fracture energy
(material property)

$$\left[\mathrm{J/m^2} \equiv \mathrm{N/m} \right]$$

Crack propagation criteria

crack propagates if

$$K_I = K_c \qquad\qquad \mathcal{G} = G_f$$

local (Irwin) criterion

global (Griffith) criterion

for plane stress and mode I loading it can be shown that

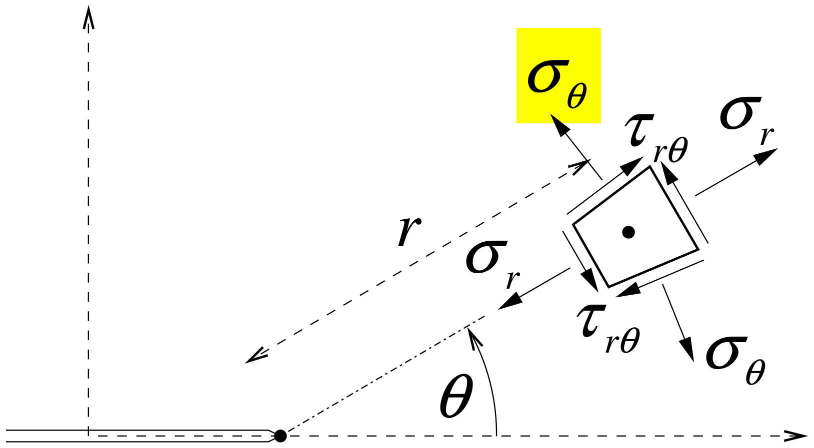
$$\mathcal{G} = \frac{K_I^2}{E}$$

the above criteria are then equivalent and the fracture tougness and fracture energy are linked by

$$G_f = \frac{K_c^2}{E} \qquad\qquad K_c = \sqrt{EG_f}$$

Direction of crack propagation

for mode I loading, the crack can be expected to propagate straight ahead,
but for general mixed-mode loading we need a criterion for the crack direction



maximum circumferential stress criterion
(maximum hoop stress criterion):

crack propagates in the direction perpendicular to the maximum circumferential stress

(evaluated on a circle of a small diameter centered at the tip)

the direction of propagation
is given by the angle θ_c for which

$$\sigma_\theta(r, \theta_c) = \max_{-\pi<\theta<\pi} \sigma_\theta(r, \theta)$$

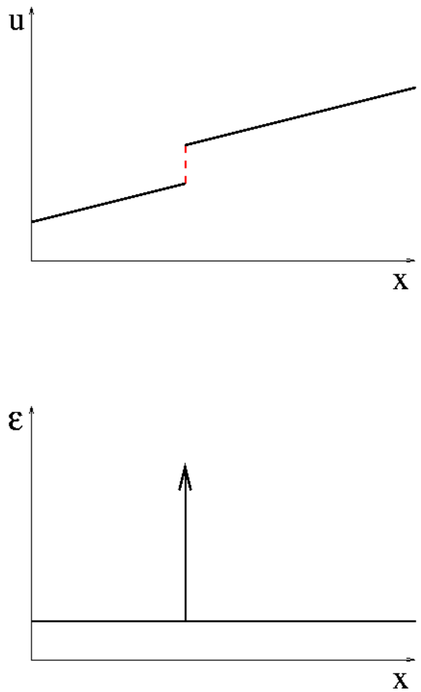
F.2

Finite elements with discontinuities:

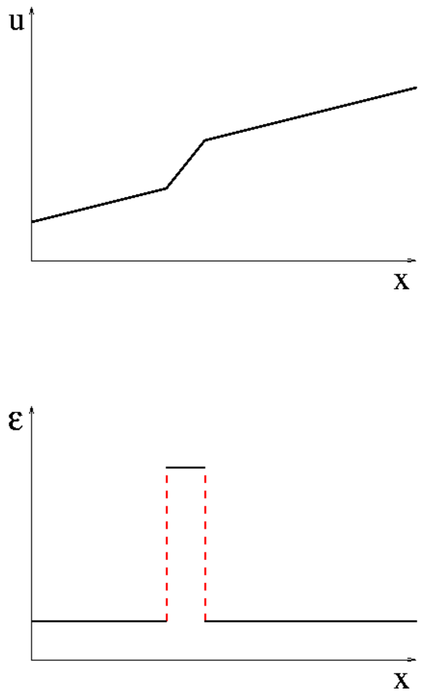
Introduction

Classification of models: kinematic aspects

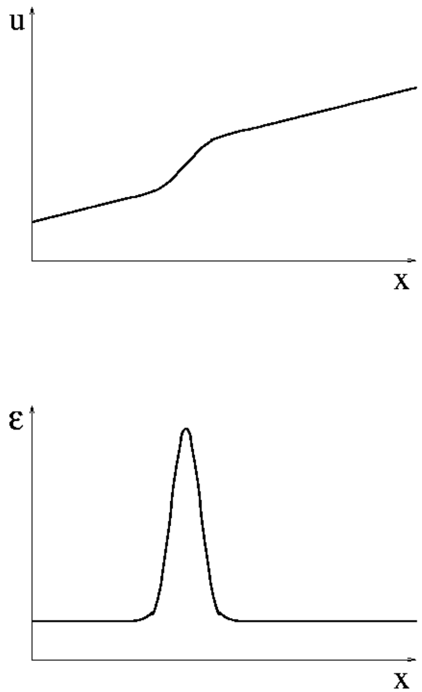
Strong
discontinuity



Weak
discontinuity



Regularized
localization zone

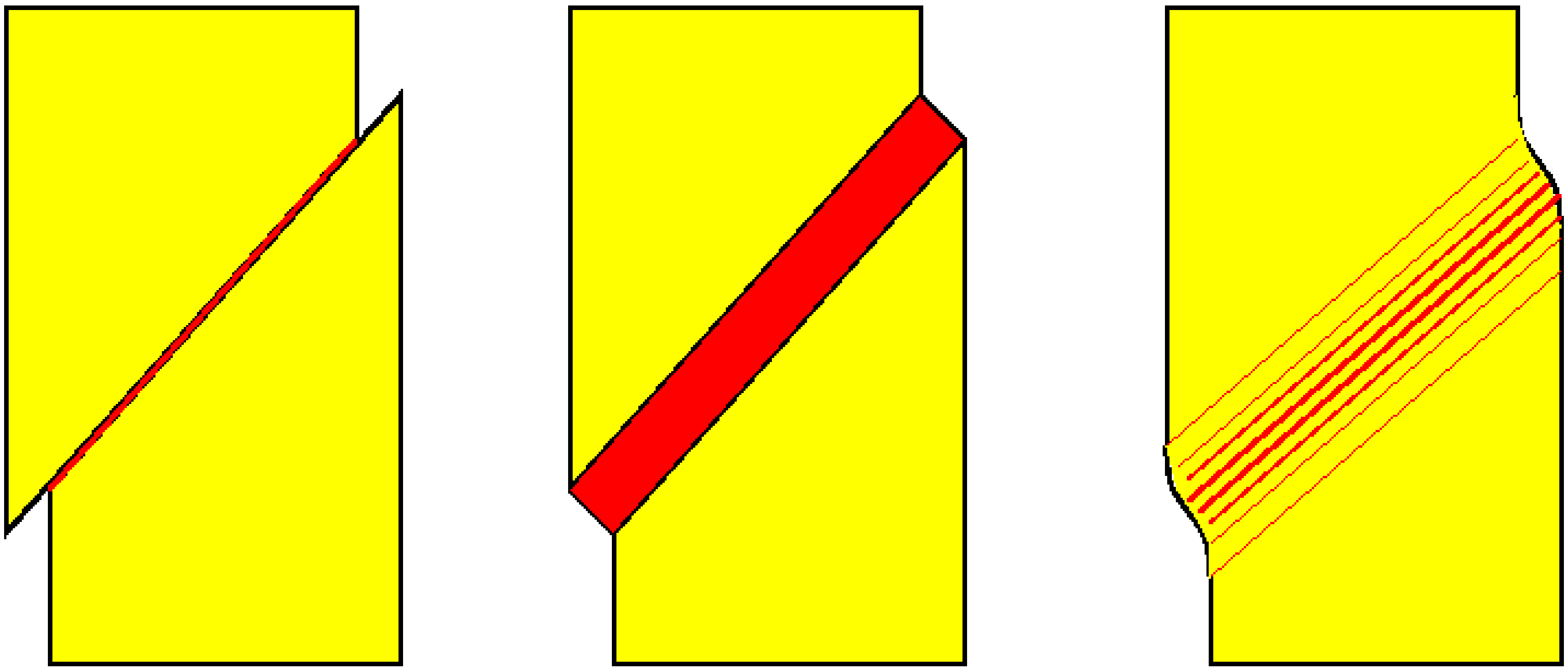


Classification of models: kinematic aspects

Strong discontinuity

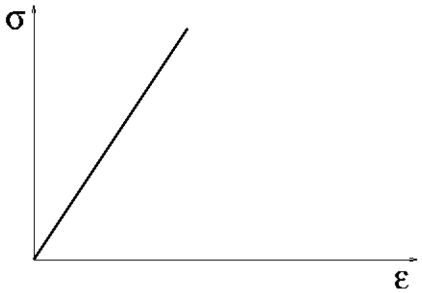
Weak discontinuity

Regularized localization zone

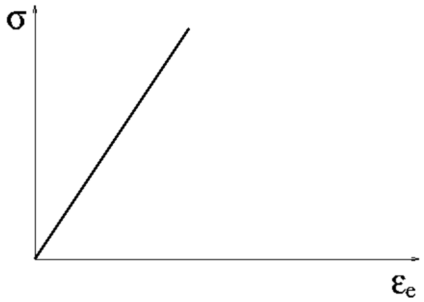


Classification of models: material laws

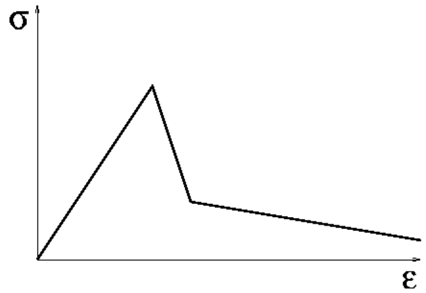
Stress-strain law



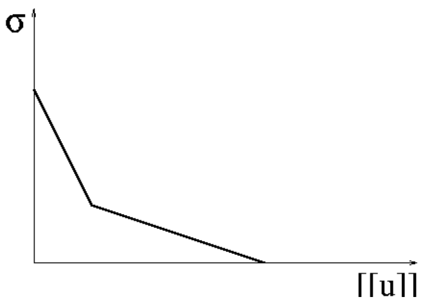
Stress-strain law
(pre-localization part)



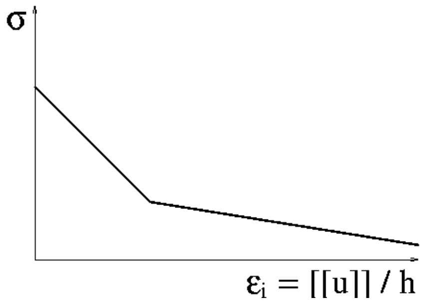
Stress-strain law



Traction-separation law



Stress-strain law
(post-localization part)

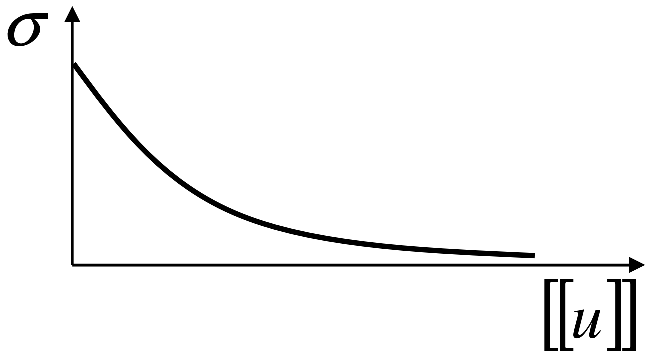


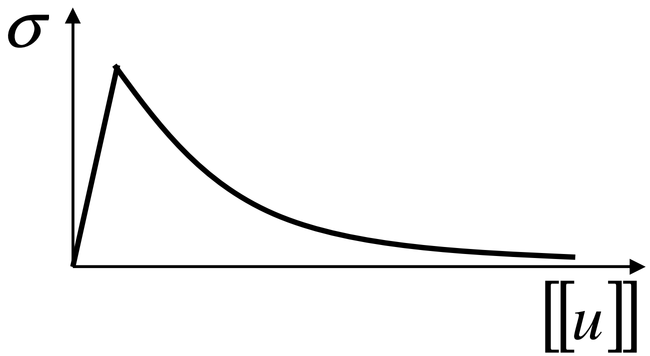
Enrichment acting
as localization limiter:

- nonlocal
- gradient
- Cosserat
- viscosity

Traction-separation laws

1) Formulated directly in the traction-separation space
   a) with nonzero elastic compliance (elasto-plastic, …)
   b) with zero elastic compliance (rigid-plastic, …)





For general applications, we need a link between the separation **vector** (displacement jump vector) and the traction **vector**:

$$[\![\mathbf{u}]\!] \longrightarrow \mathbf{t}$$

Traction-separation laws

2) “Derived“ from a stress-strain law (softening continuum) using the strong discontinuity approach

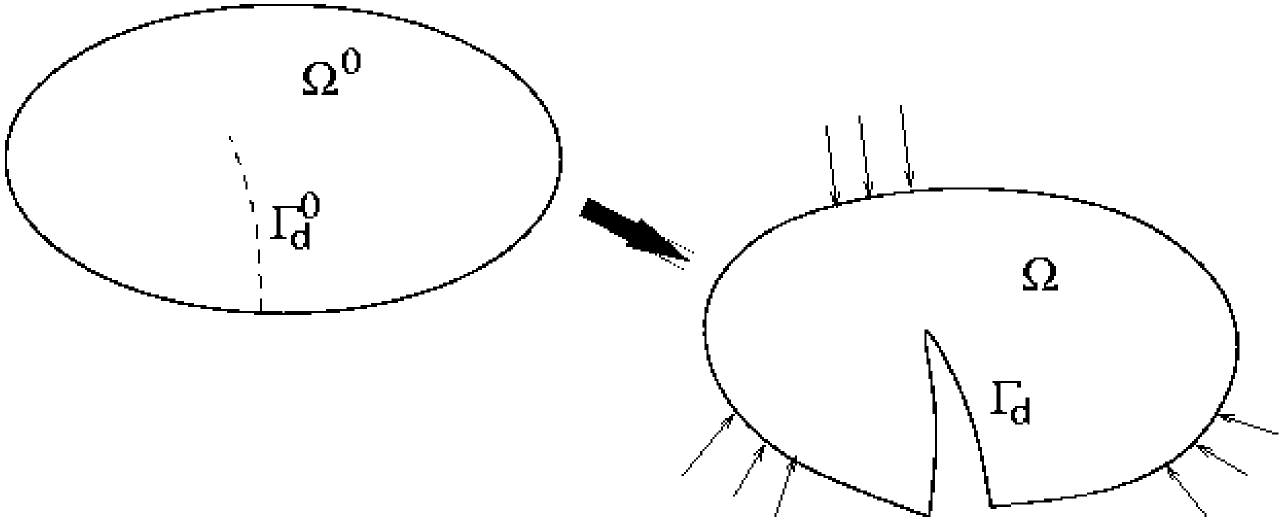
$$[[\mathbf{u}]]$$

$$\mathbf{t} = \mathbf{n} \cdot \boldsymbol{\sigma}$$

$$\boldsymbol{\varepsilon} = \frac{1}{h}\left(\left[[\mathbf{u}]\right] \otimes \mathbf{n}\right)_{sym}$$

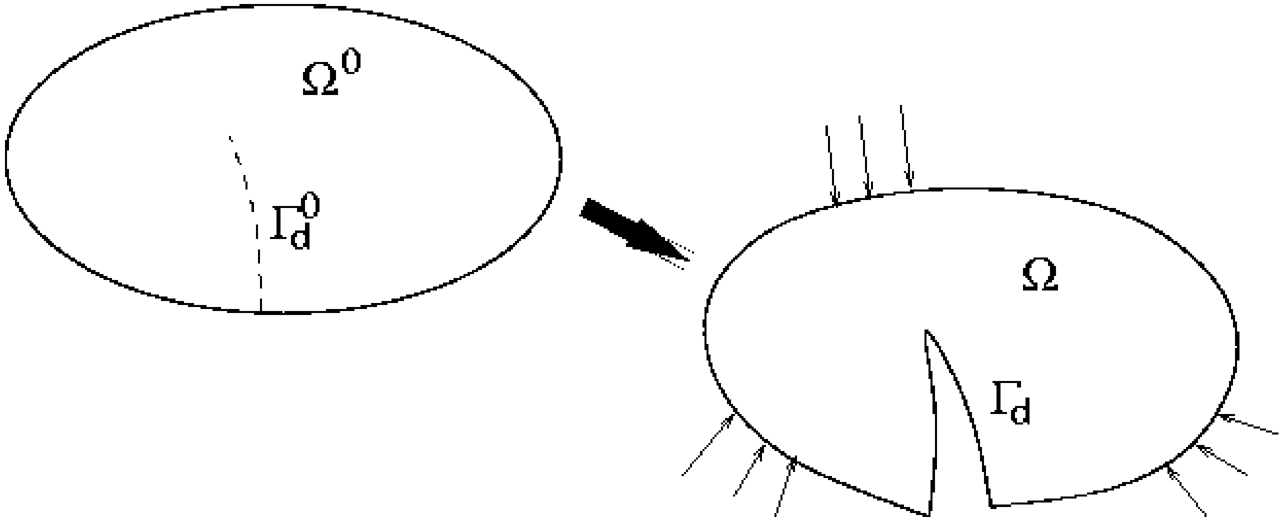
$$\boldsymbol{\sigma} = \tilde{\boldsymbol{\sigma}}\left(\boldsymbol{\varepsilon}, \ldots ; h\right)$$

Finite element representation of strong discontinuities



1) Discontinuities at element interfaces:
   a) Remeshing
   b) Interspersed potential discontinuities

Finite element representation of strong discontinuities



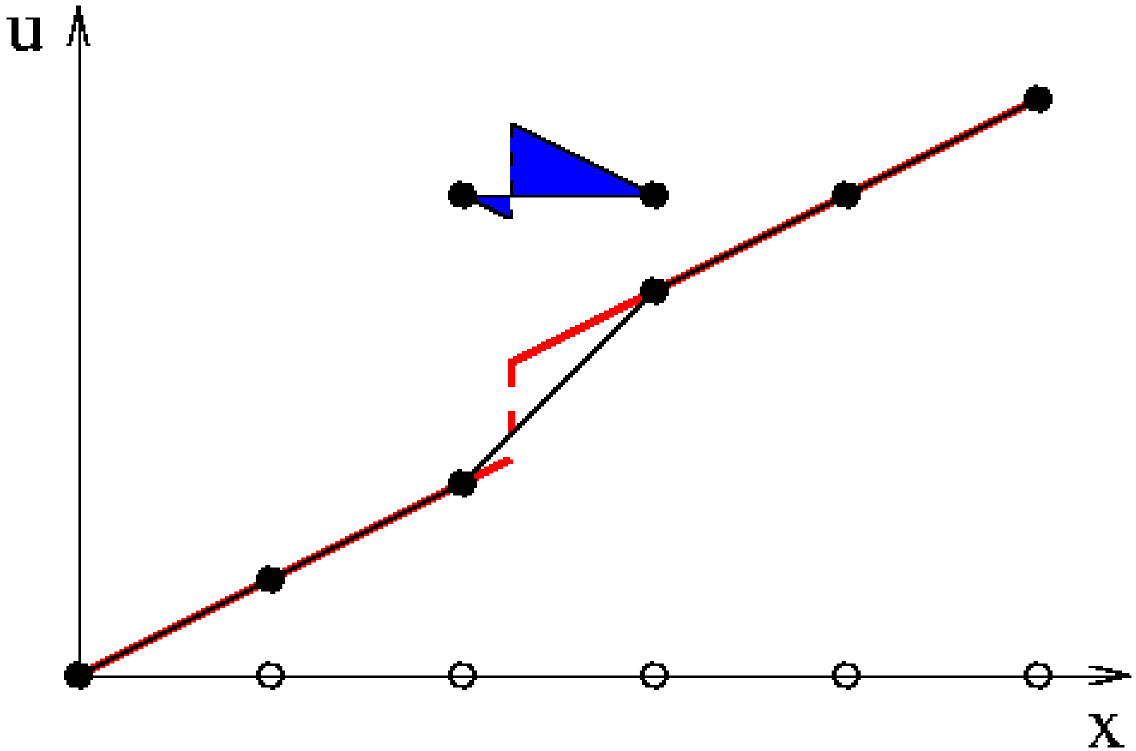
2) Arbitrary discontinuities across elements:

a) Elements with embedded discontinuities using the enhanced assumed strain formulation (EED-EAS)

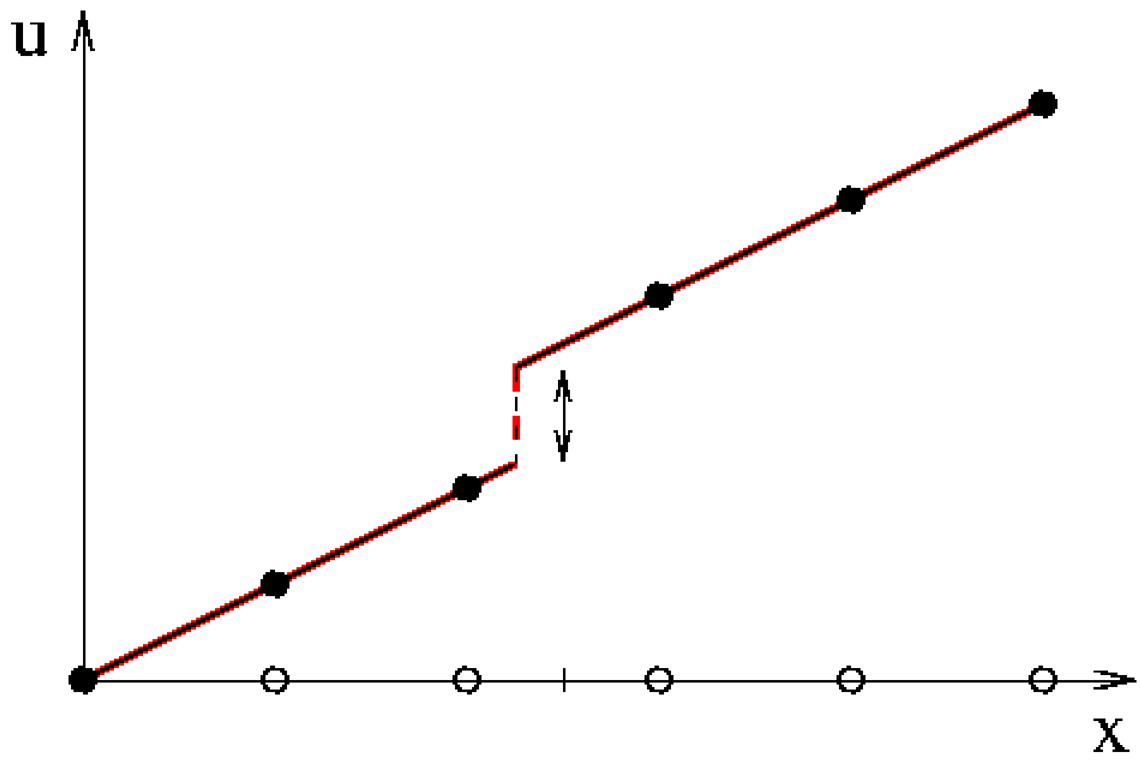
aka EFEM, SDA, GSDA, …

b) Extended finite elements based on the partition-of-unity concept (XFEM-PUM) aka GFEM, …

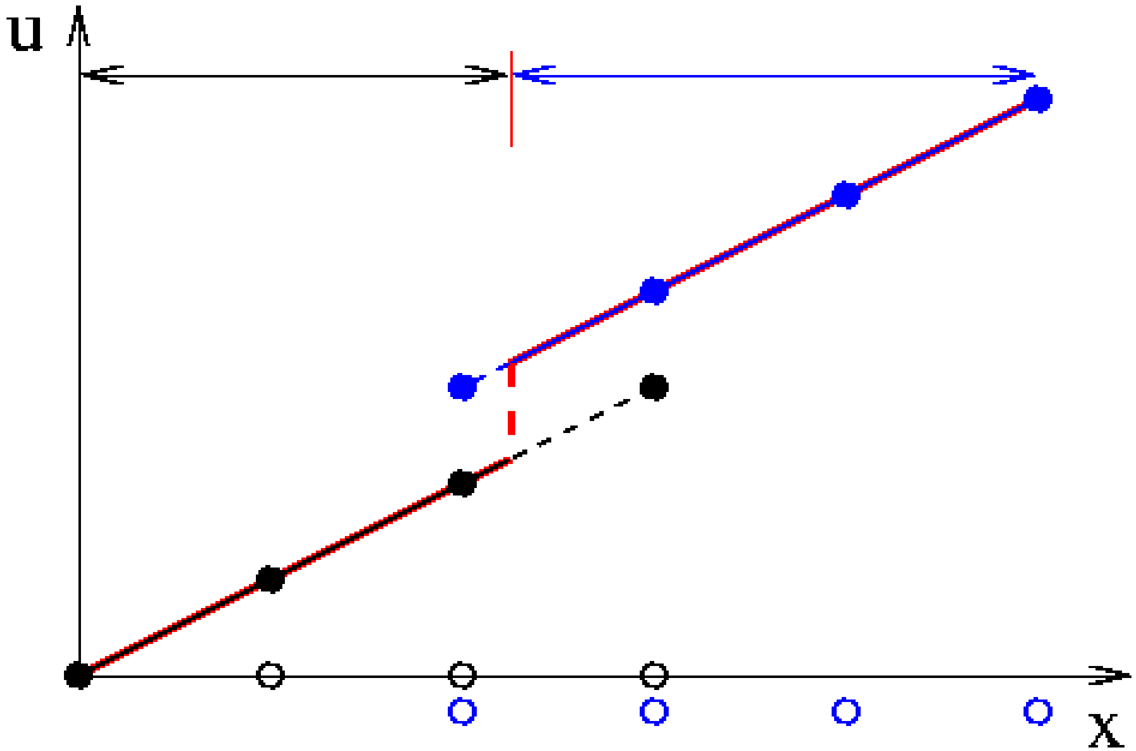
Embedded discontinuity (enhanced assumed strain)



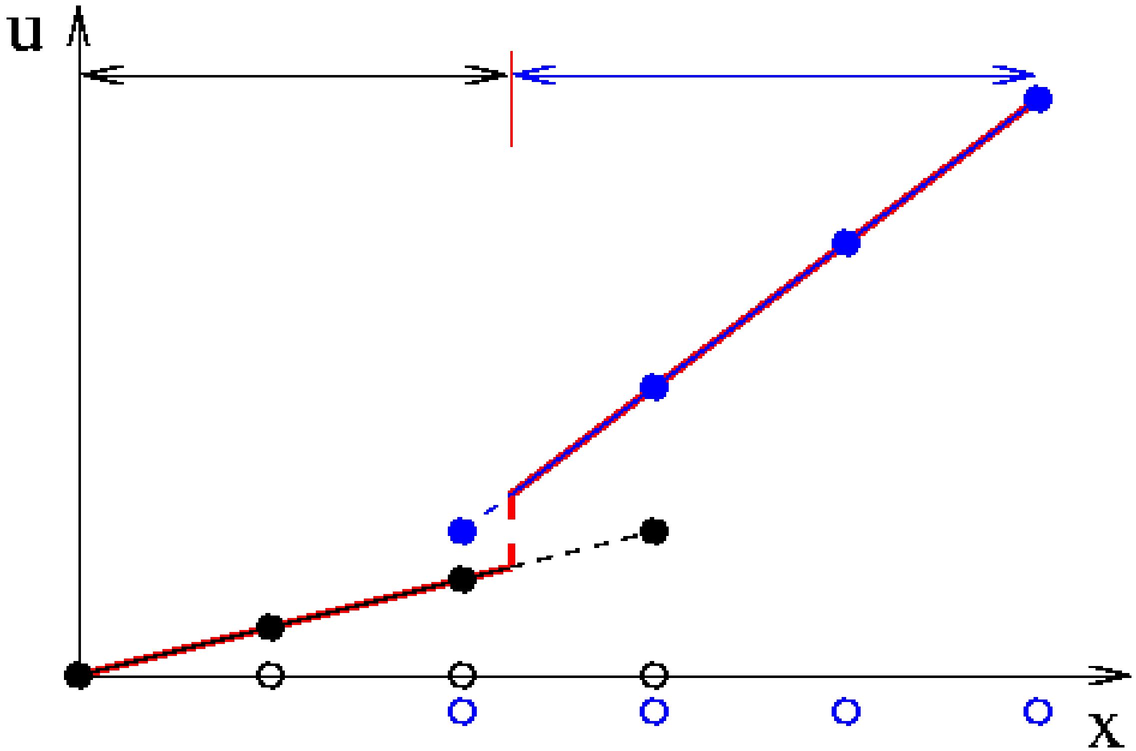
Embedded discontinuity (enhanced assumed strain)



Approximation on two overlapping meshes (XFEM)

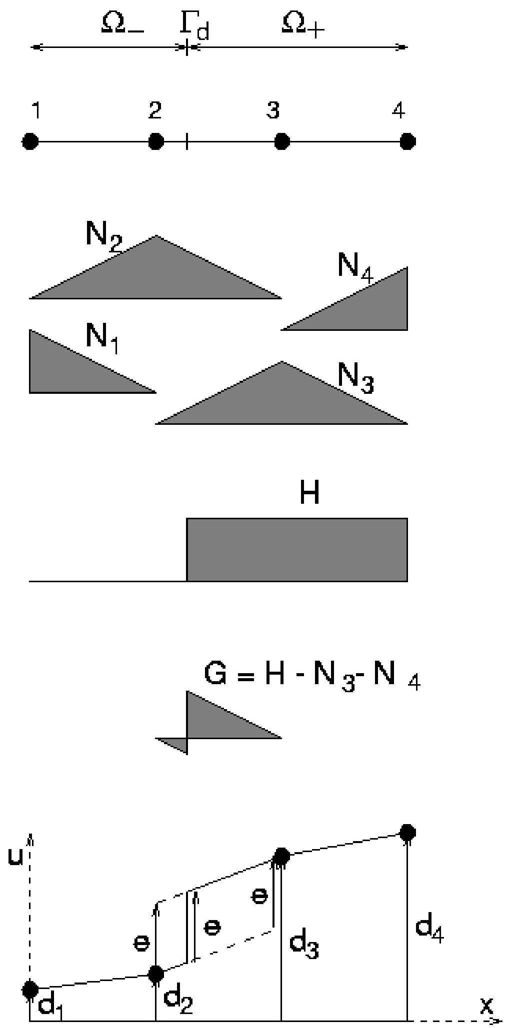


Approximation on two overlapping meshes (XFEM)



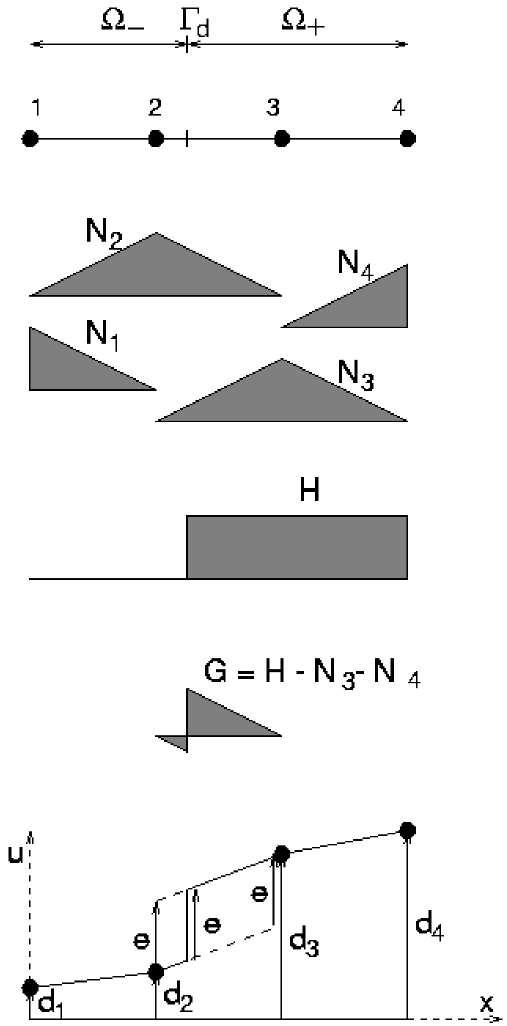
Enrichment of interpolation functions in one dimension

EED-EAS

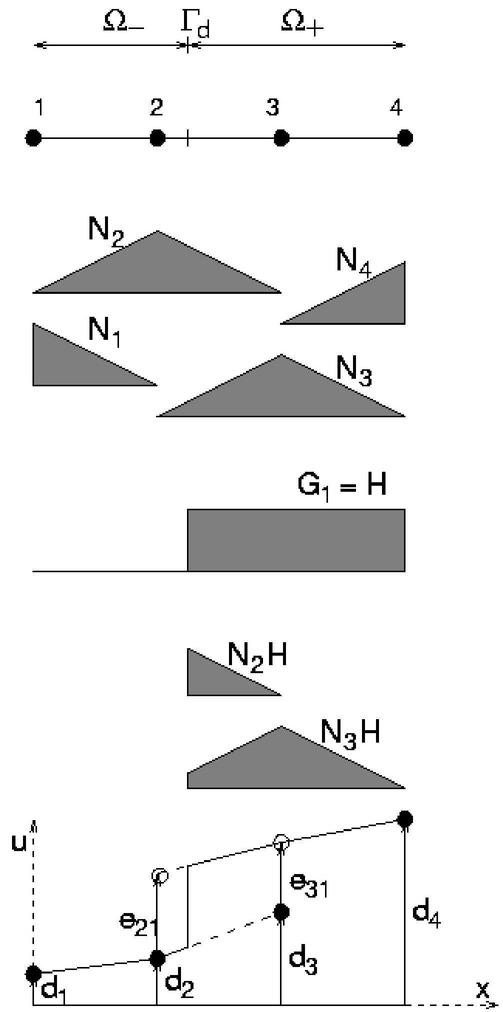


Enrichment of interpolation functions in one dimension

EED-EAS

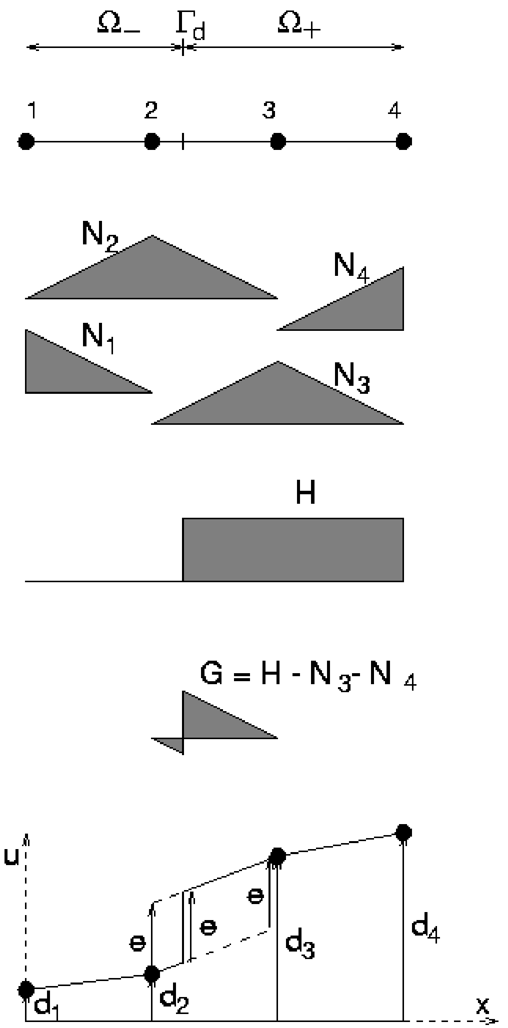


XFEM-PUM

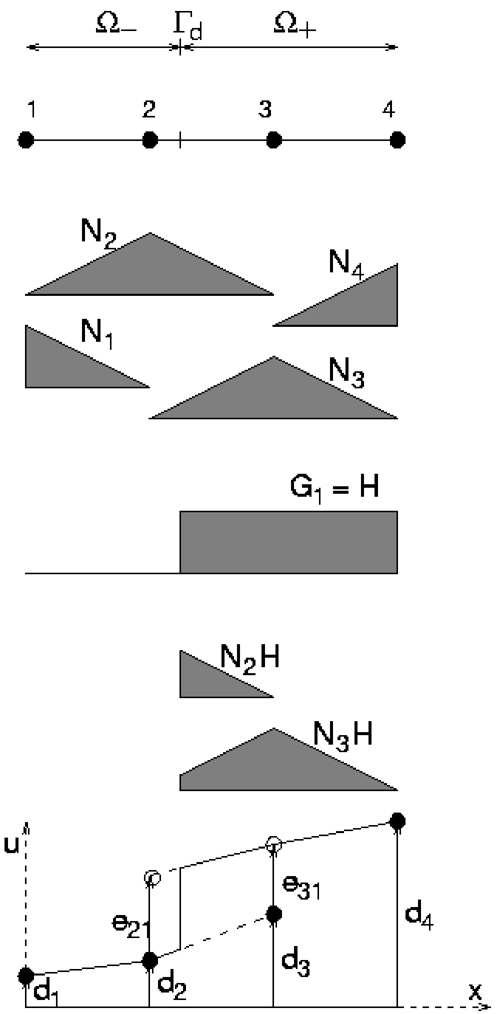


Enrichment of interpolation functions in one dimension

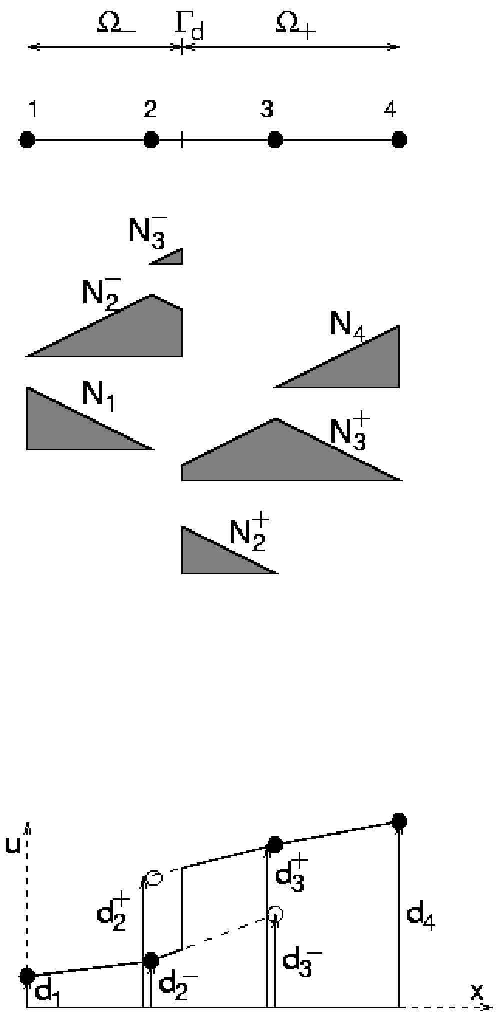
EED-EAS



XFEM-PUM



XFEM-PUM



F.3

Elements with Embedded Discontinuities (EAS)

Elements with embedded discontinuities

$$\mathbf{d}$$

$$\downarrow \quad \boldsymbol{\varepsilon} = \mathbf{B}\mathbf{d}$$

$$\boldsymbol{\varepsilon}$$

$$\downarrow \quad \boldsymbol{\sigma} = \tilde{\boldsymbol{\sigma}}(\boldsymbol{\varepsilon}, \ldots)$$

$$\boldsymbol{\sigma}$$

$$\downarrow \quad \mathbf{f}_{\text{int}} = \int_V \mathbf{B}^T \boldsymbol{\sigma} \, \mathrm{d}V$$

$$\mathbf{f}_{\text{int}}$$

Elements with embedded discontinuities

$\mathbf{d}$

$\boldsymbol{\varepsilon}$ $\mathbf{e}$ … new degrees of freedom characterizing separation (displacement jump)

$\boldsymbol{\sigma}$ $\mathbf{t}$ … traction

$\mathbf{f}_{\mathrm{int}}$

Elements with embedded discontinuities

$\mathbf{d}$

$\boldsymbol{\varepsilon}$ $\qquad$ $\mathbf{e}$

$\downarrow$ material $\downarrow$

$\boldsymbol{\sigma}$ $\qquad$ $\mathbf{t}$

$\mathbf{f}_{\text{int}}$

Elements with embedded discontinuities

$\mathbf{d}$

? kinematics ?

$\boldsymbol{\varepsilon}$ $\quad\quad$ $\mathbf{e}$

$\downarrow$ material $\downarrow$

$\boldsymbol{\sigma}$ $\quad\quad$ $\mathbf{t}$

? equilibrium ?

$\mathbf{f}_{\mathrm{int}}$

Elements with embedded discontinuities

$\mathbf{d}$

kinematics

$\boldsymbol{\varepsilon}$ $\mathbf{e}$

$\downarrow$ material $\downarrow$

$\boldsymbol{\sigma}$ $\mathbf{t}$

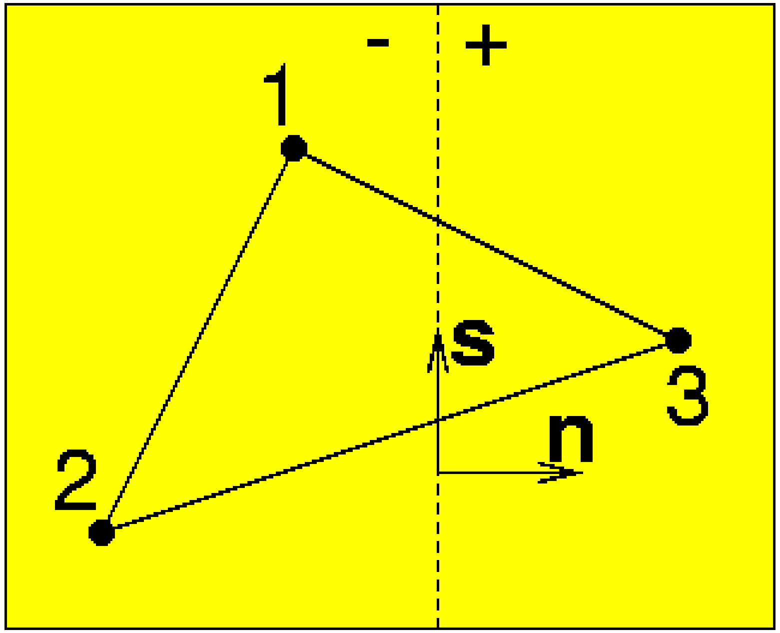
equilibrium

$\mathbf{f}_{\mathrm{int}}$

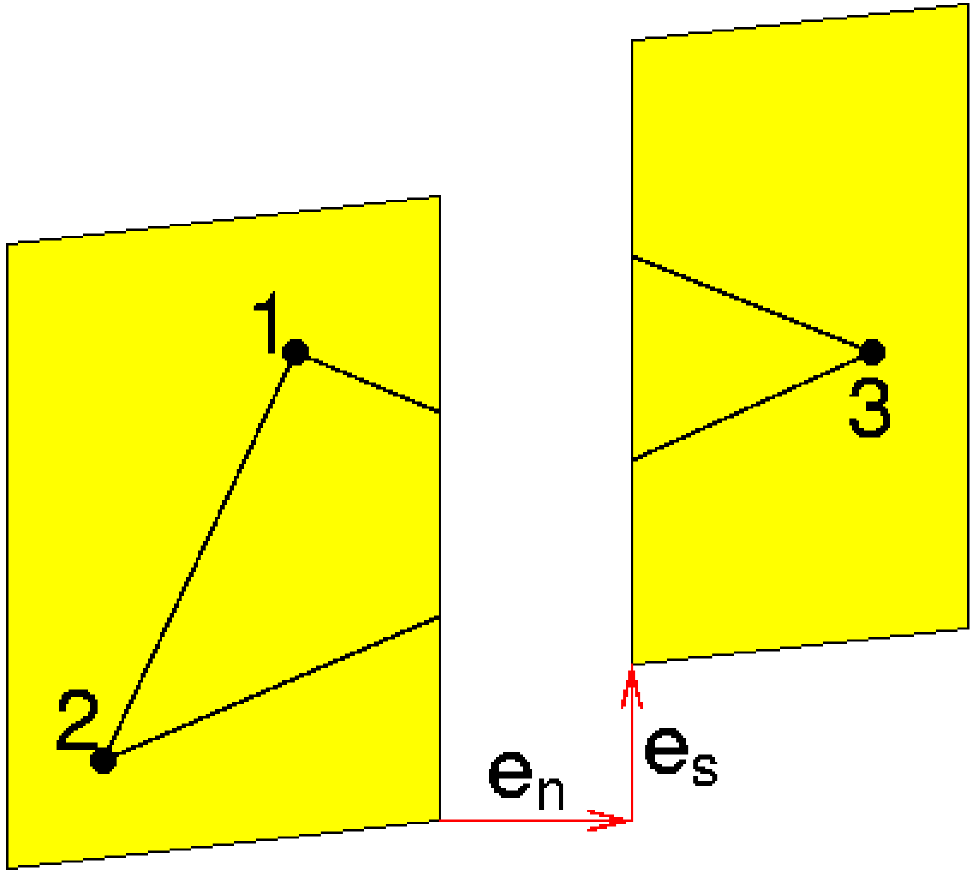
Three types of formulations:

- KOS … kinematically optimal symmetric
- SOS … statically optimal symmetric
- **SKON … kinematically and statically optimal nonsymmetric**

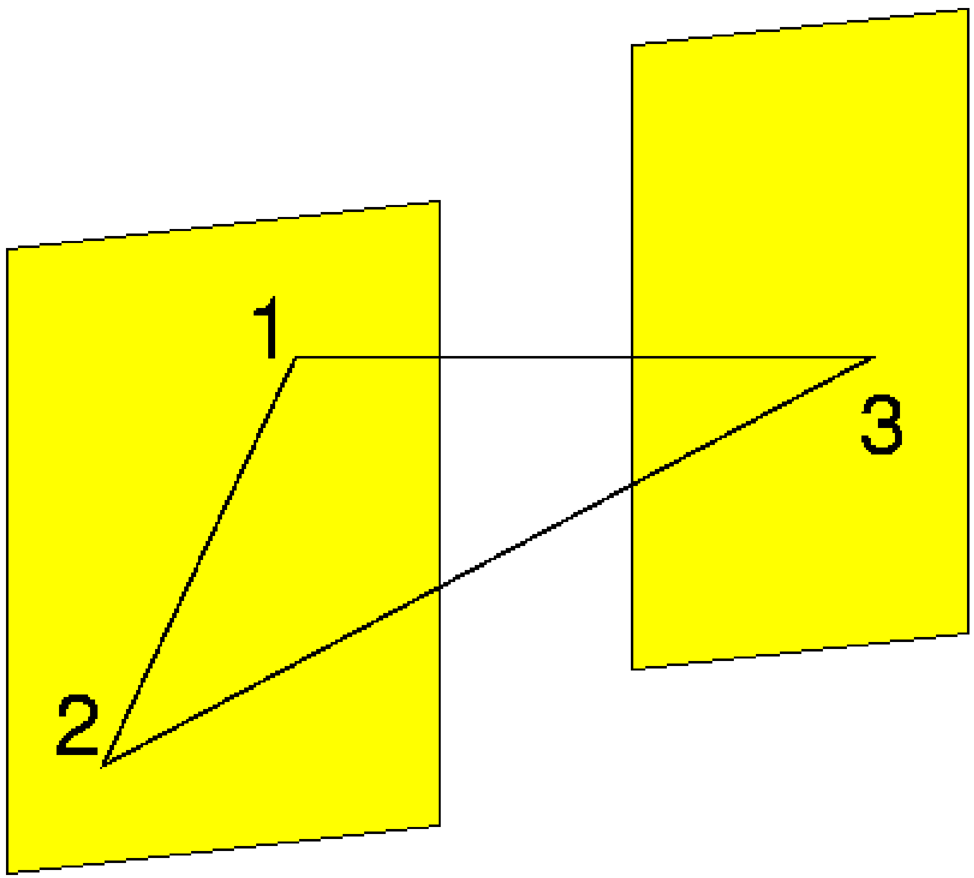
Elements with embedded discontinuities



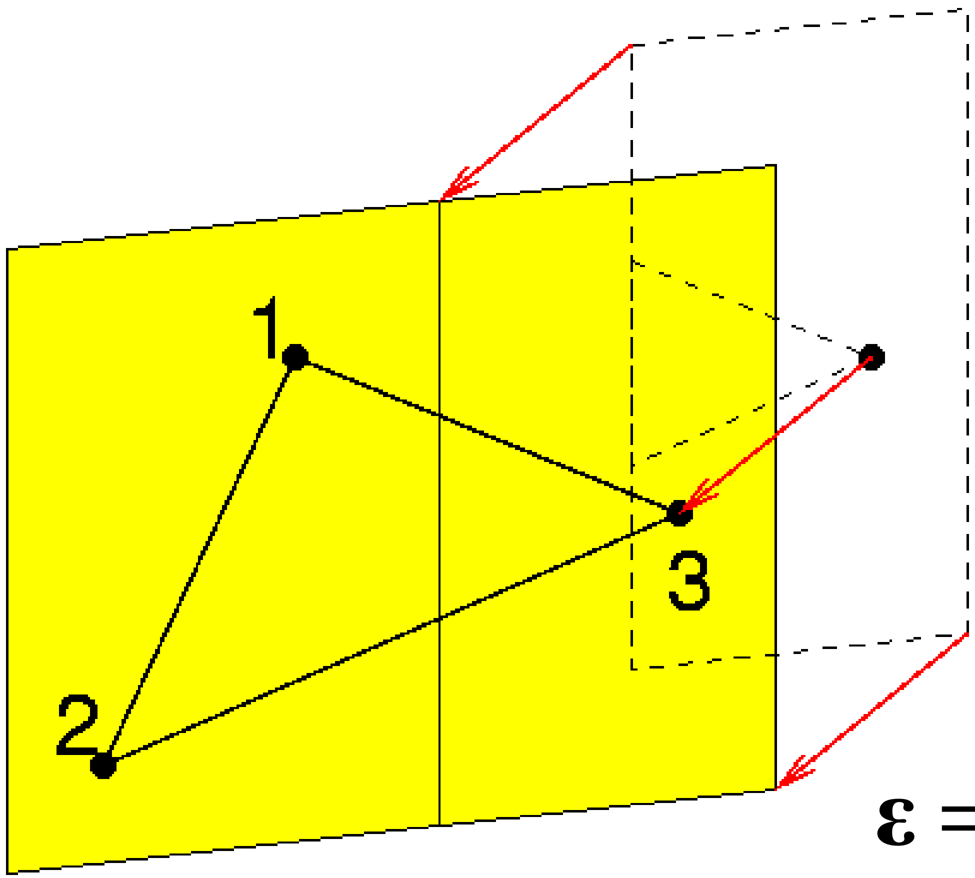
Elements with embedded discontinuities



Elements with embedded discontinuities



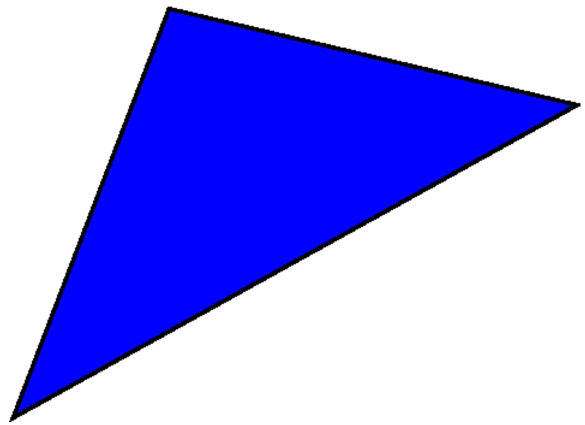
Elements with embedded discontinuities



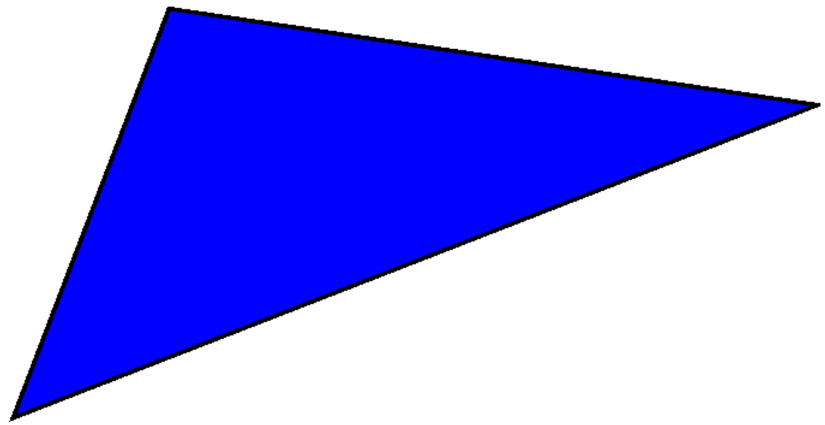
$$\boldsymbol{\varepsilon} = \mathbf{B}\left(\mathbf{d} - \mathbf{H}\mathbf{e}\right)$$

$$\mathbf{t} = \mathbf{P}^T \boldsymbol{\sigma}$$

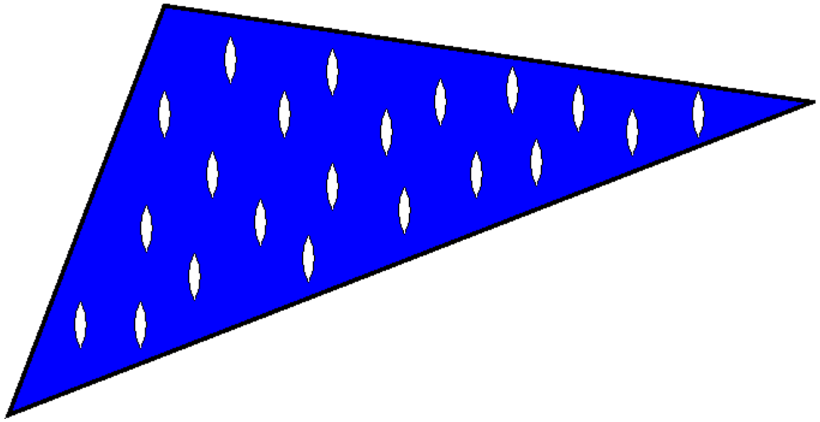
Smeared crack



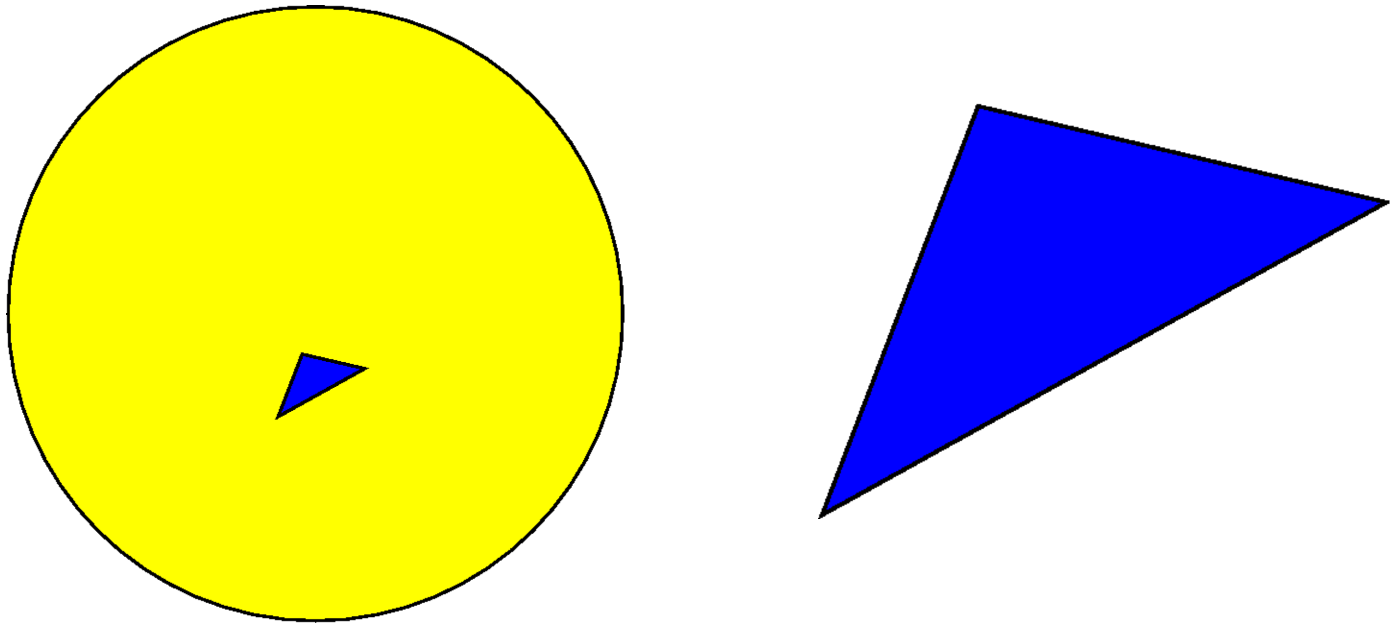
Smeared crack



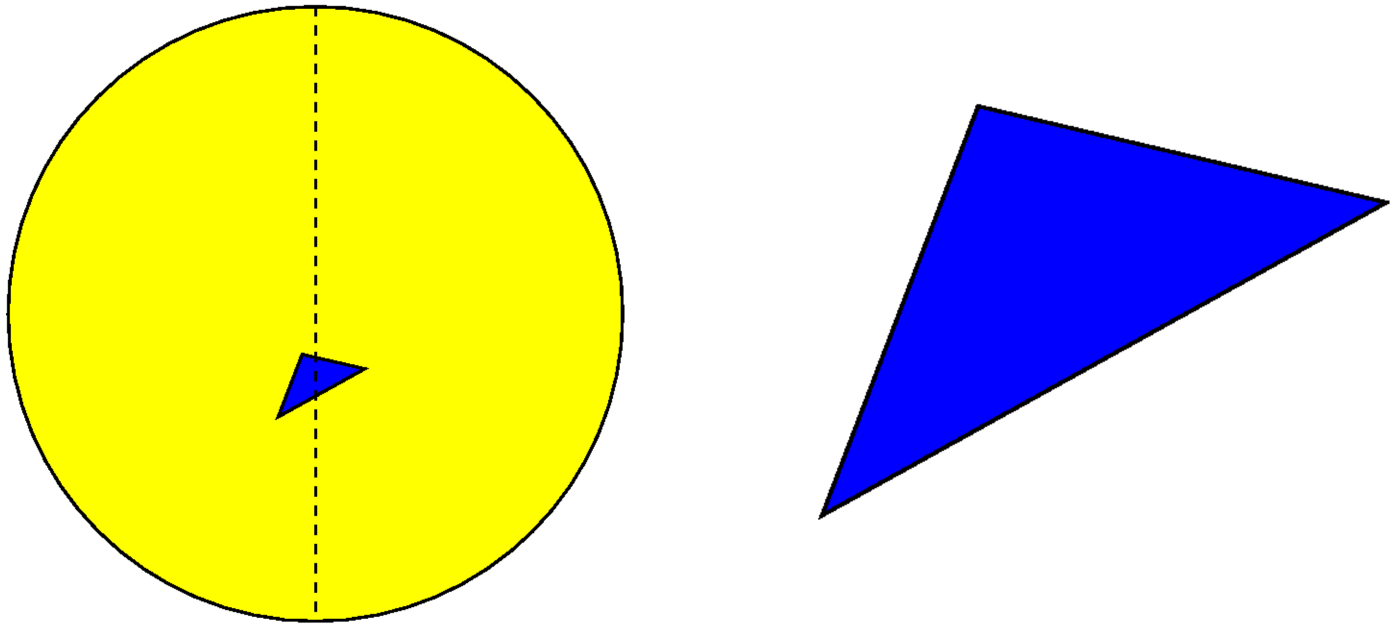
Smeared crack



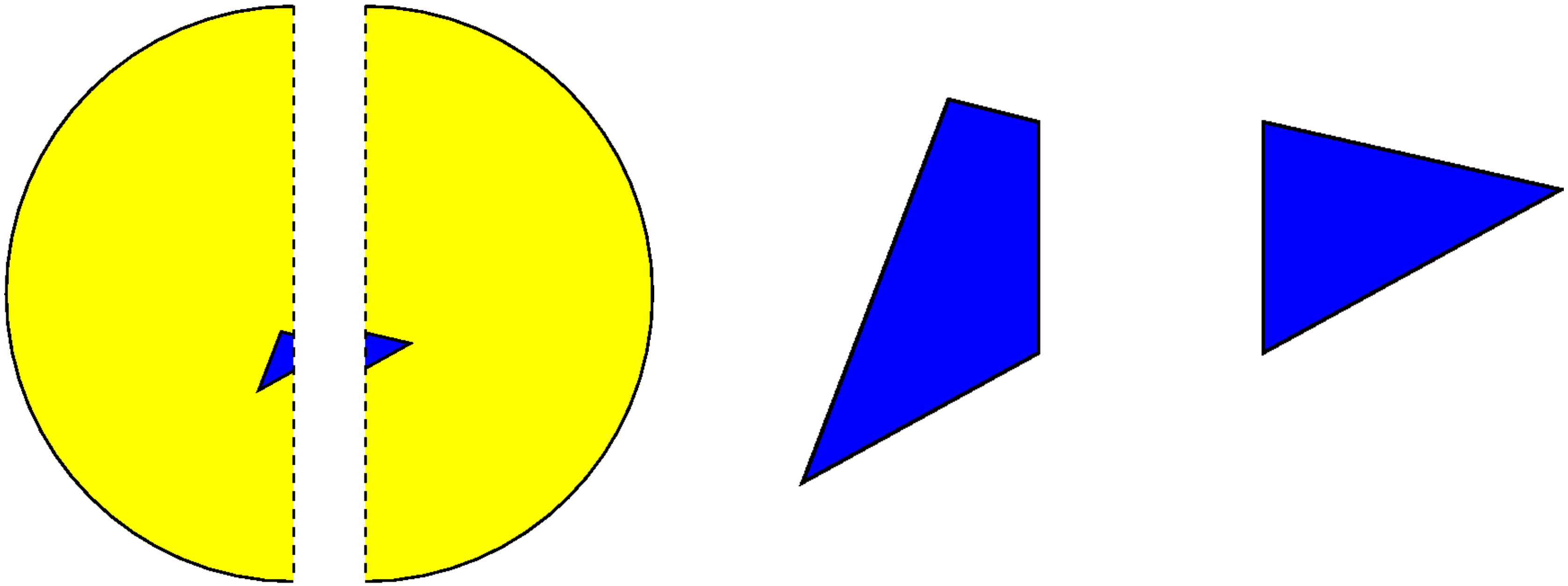
Smeared crack



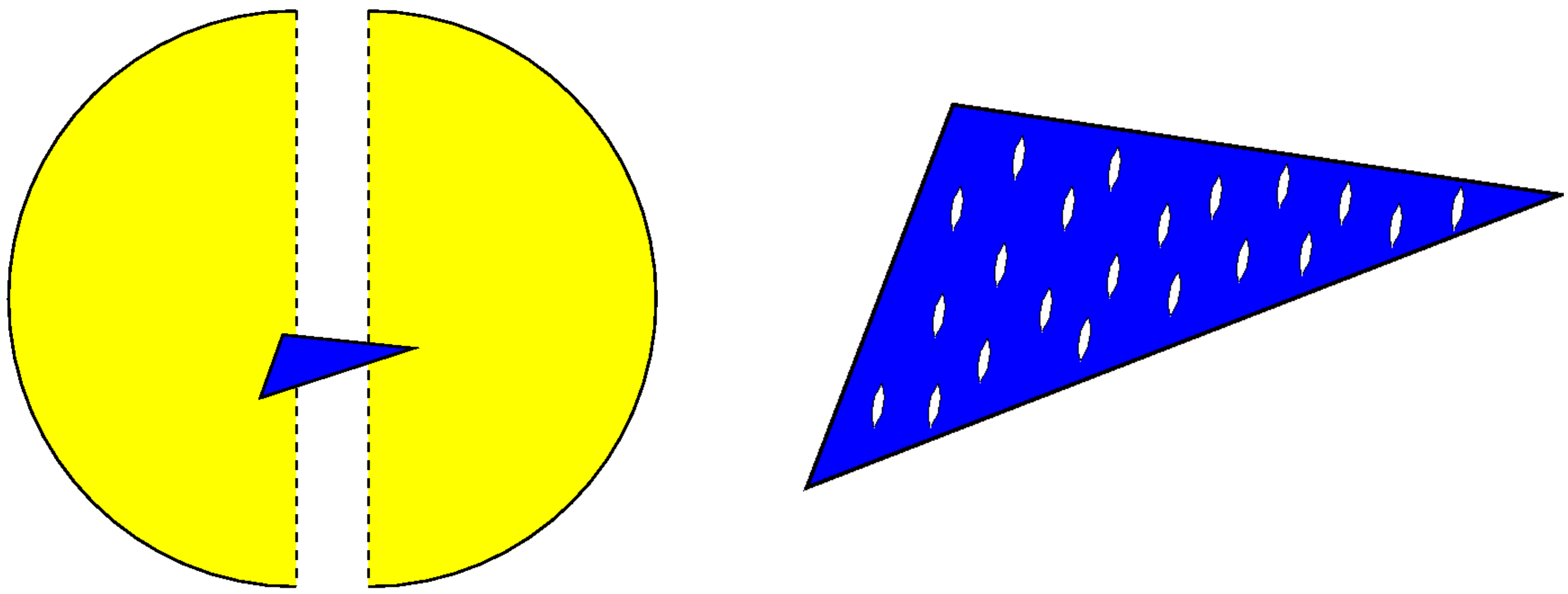
Smeared crack



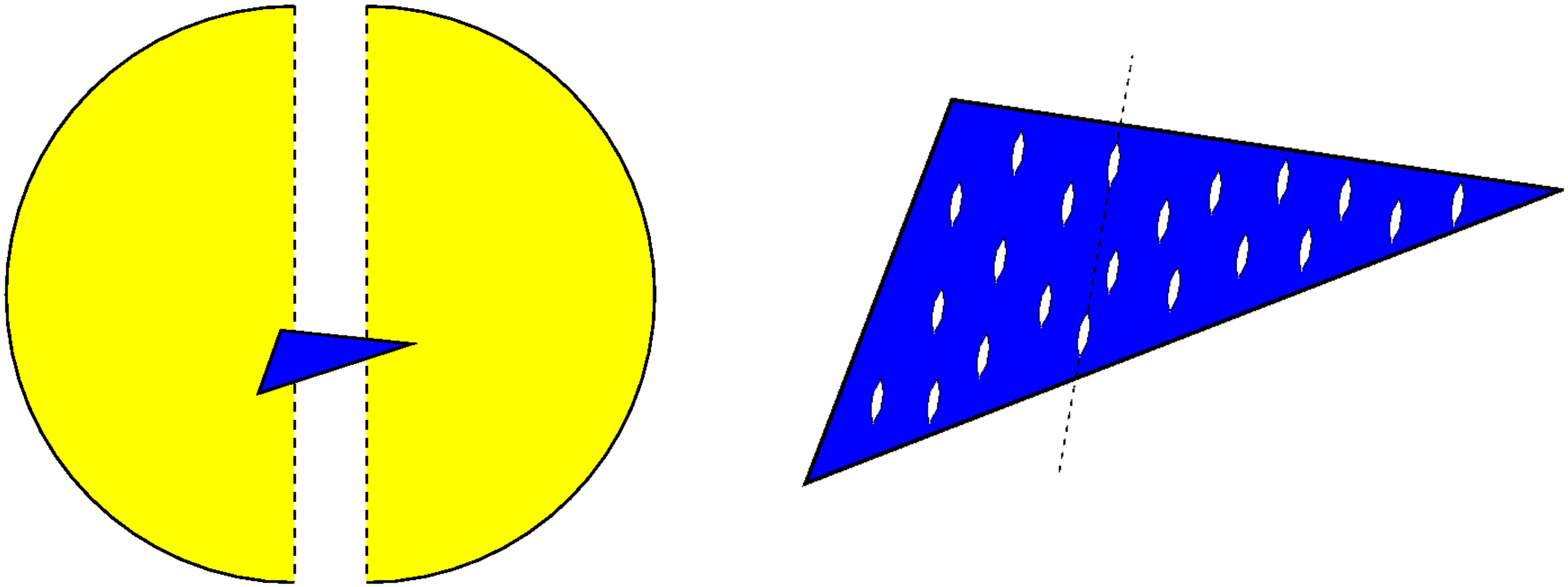
Smeared crack



Smeared crack

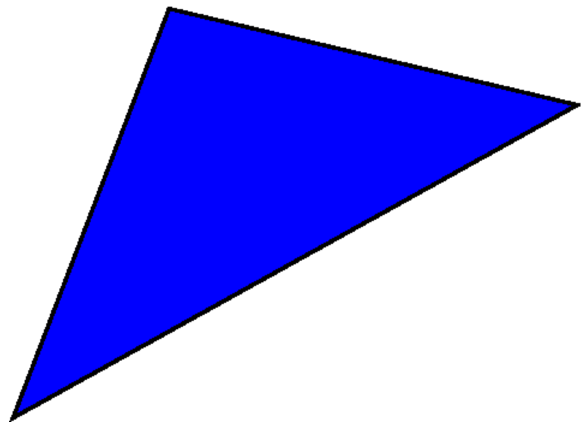


Smeared crack

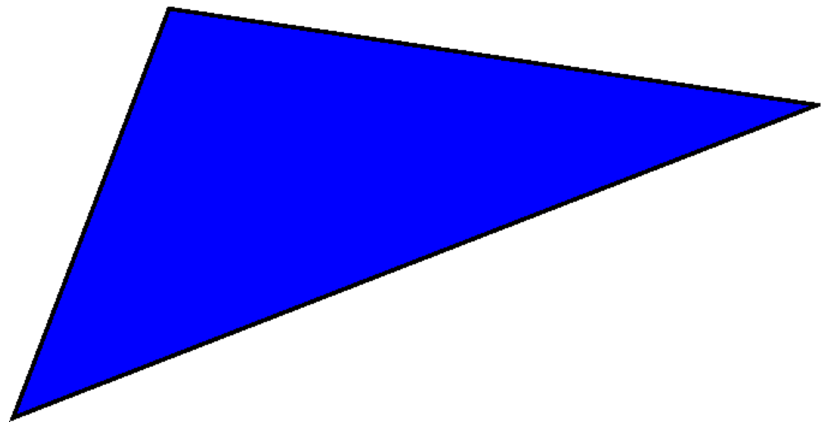


- Misalignment between crack and element
- Distorted principal directions
- Stress locking

Embedded crack (EAS approach)



Embedded crack (EAS approach)



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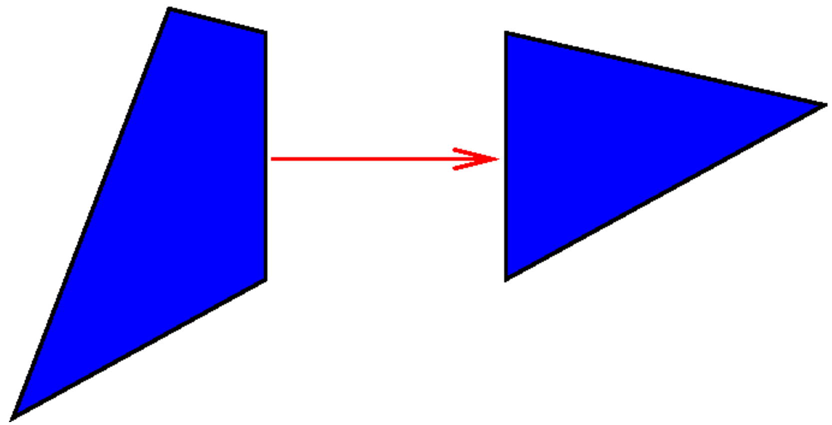
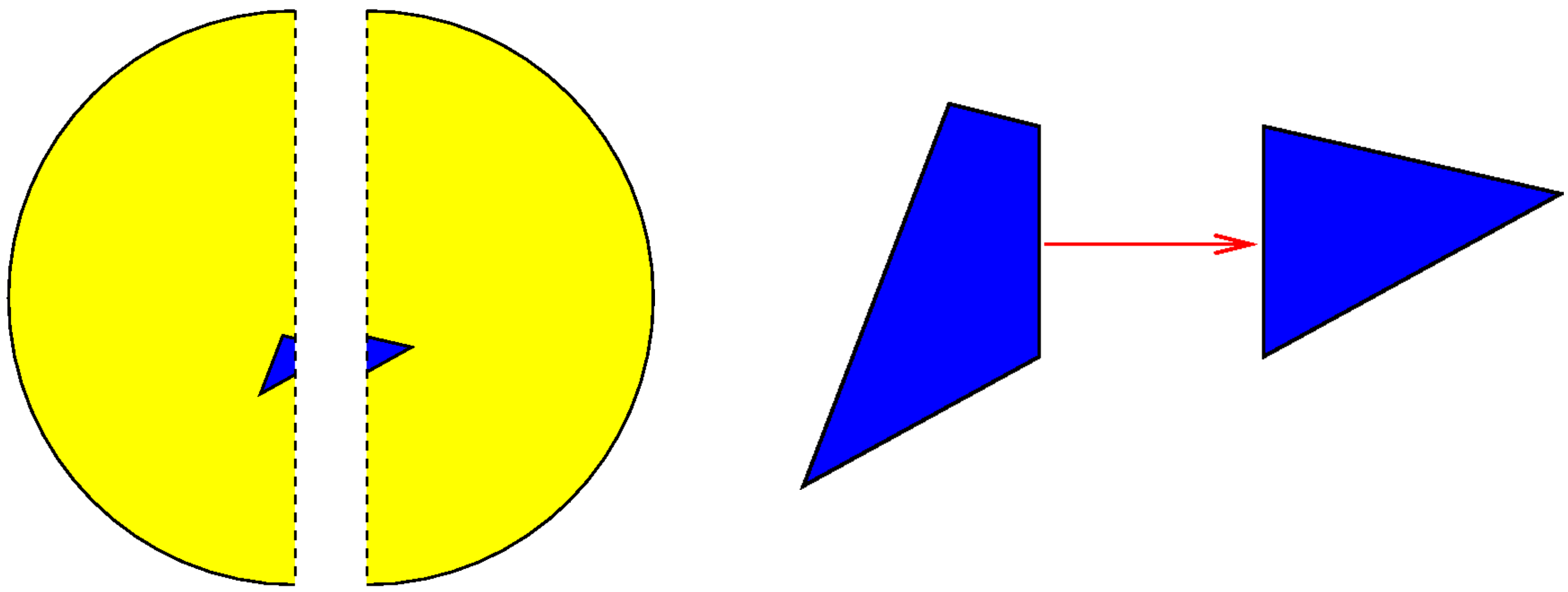
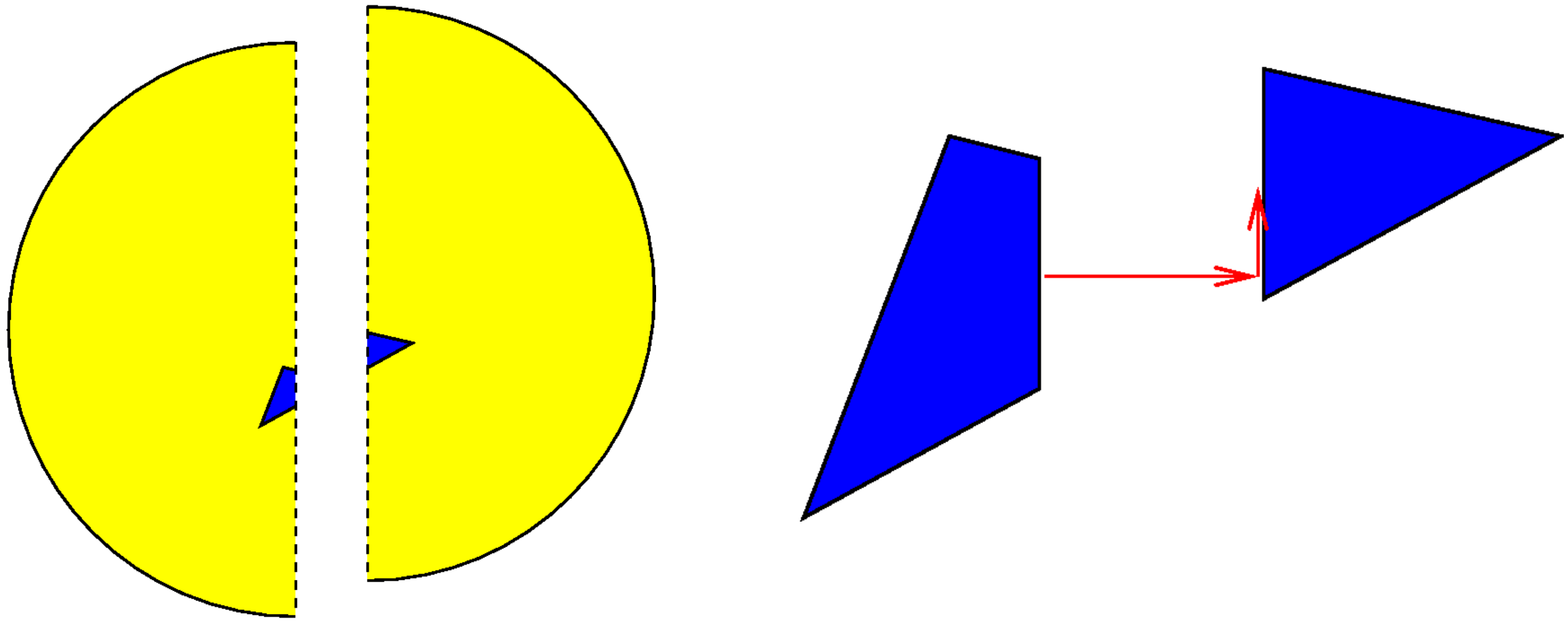


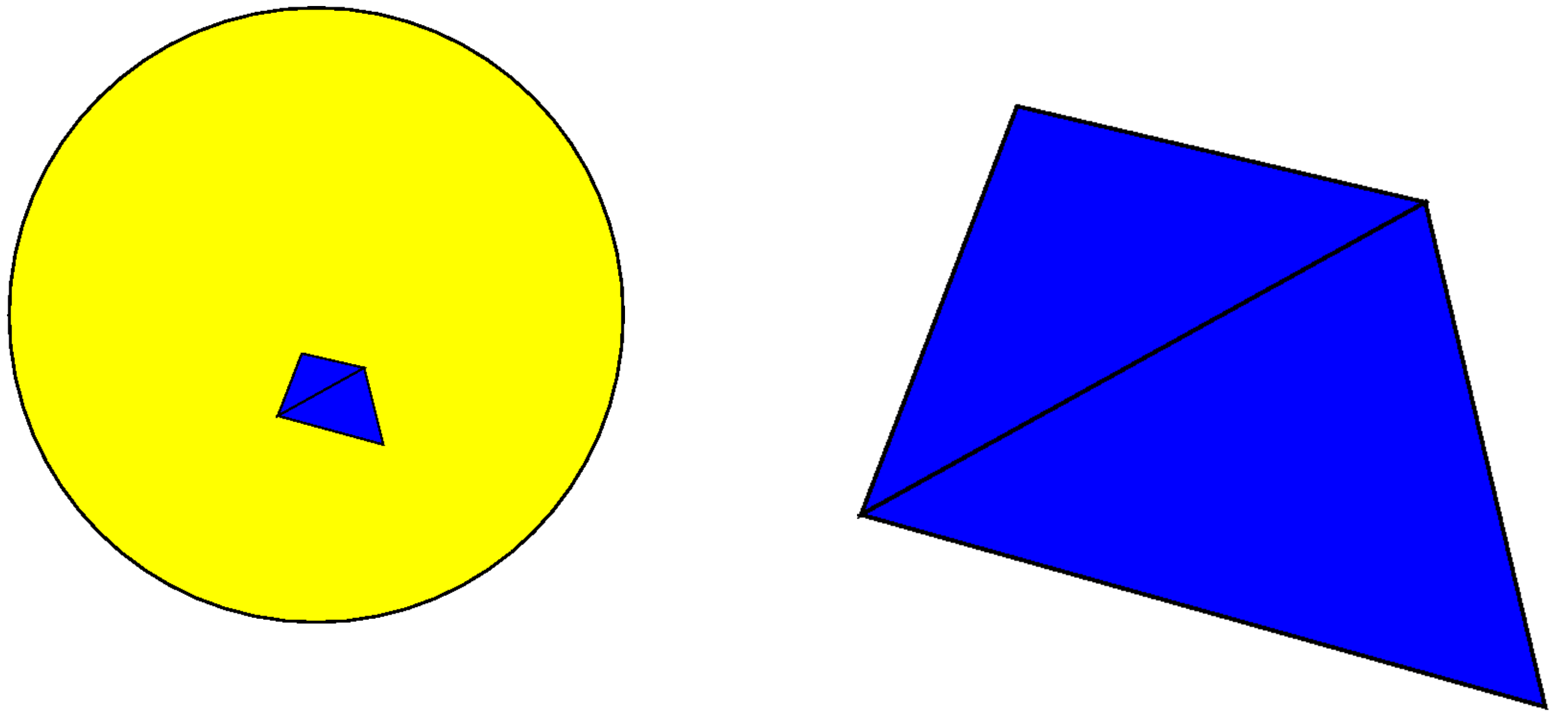
Embedded crack (EAS approach)



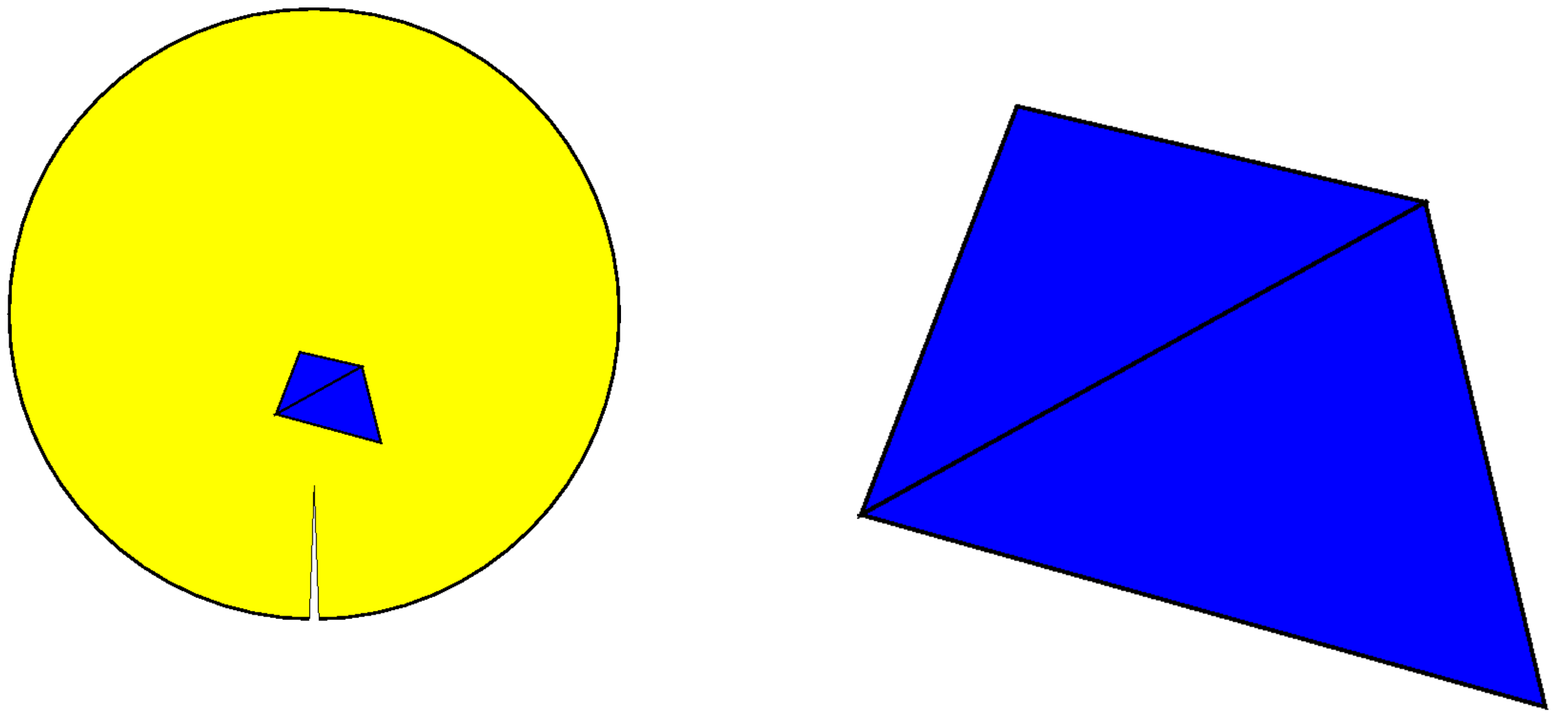
Embedded crack (EAS approach)



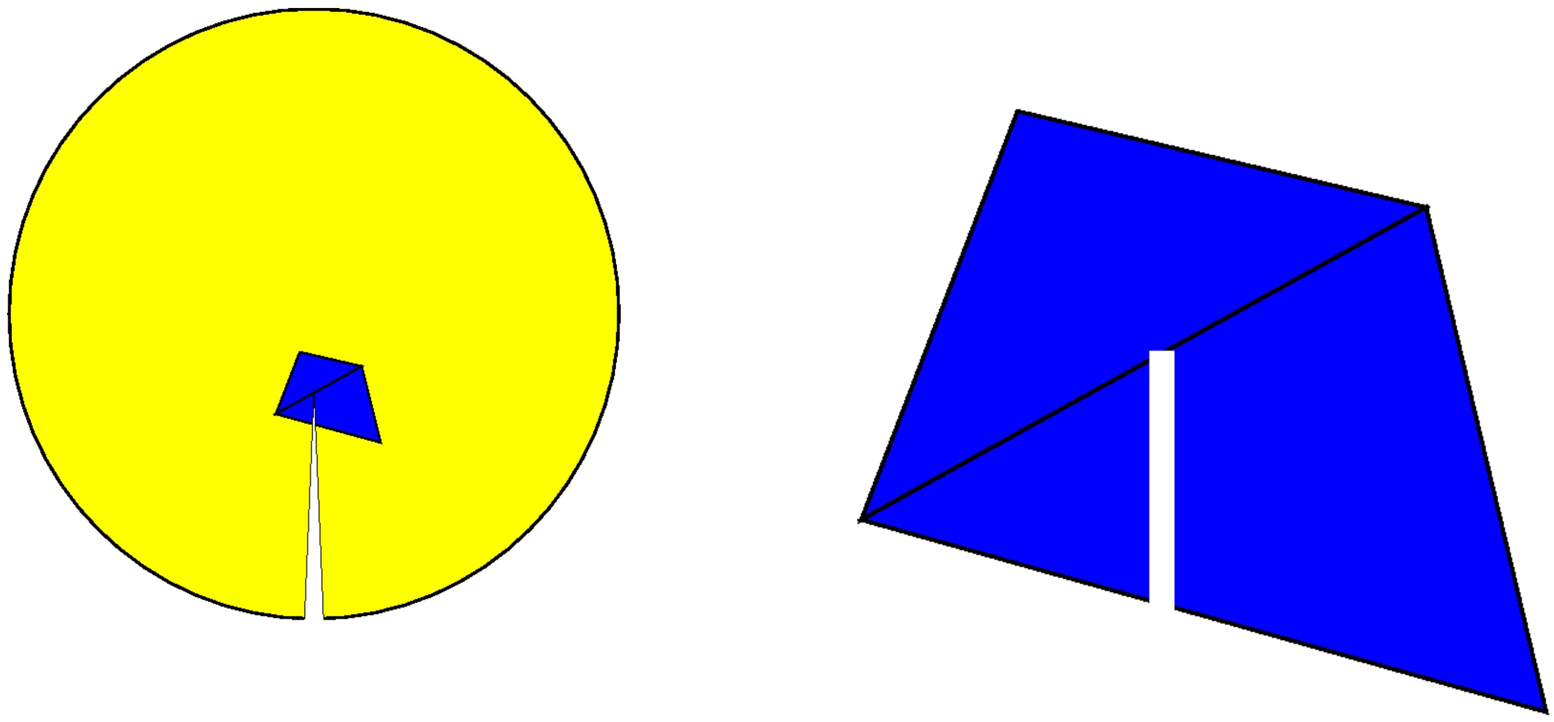
EED-EAS approach: discontinuous interpolation



EED- EAS approach: discontinuous interpolation



EED- EAS approach: discontinuous interpolation



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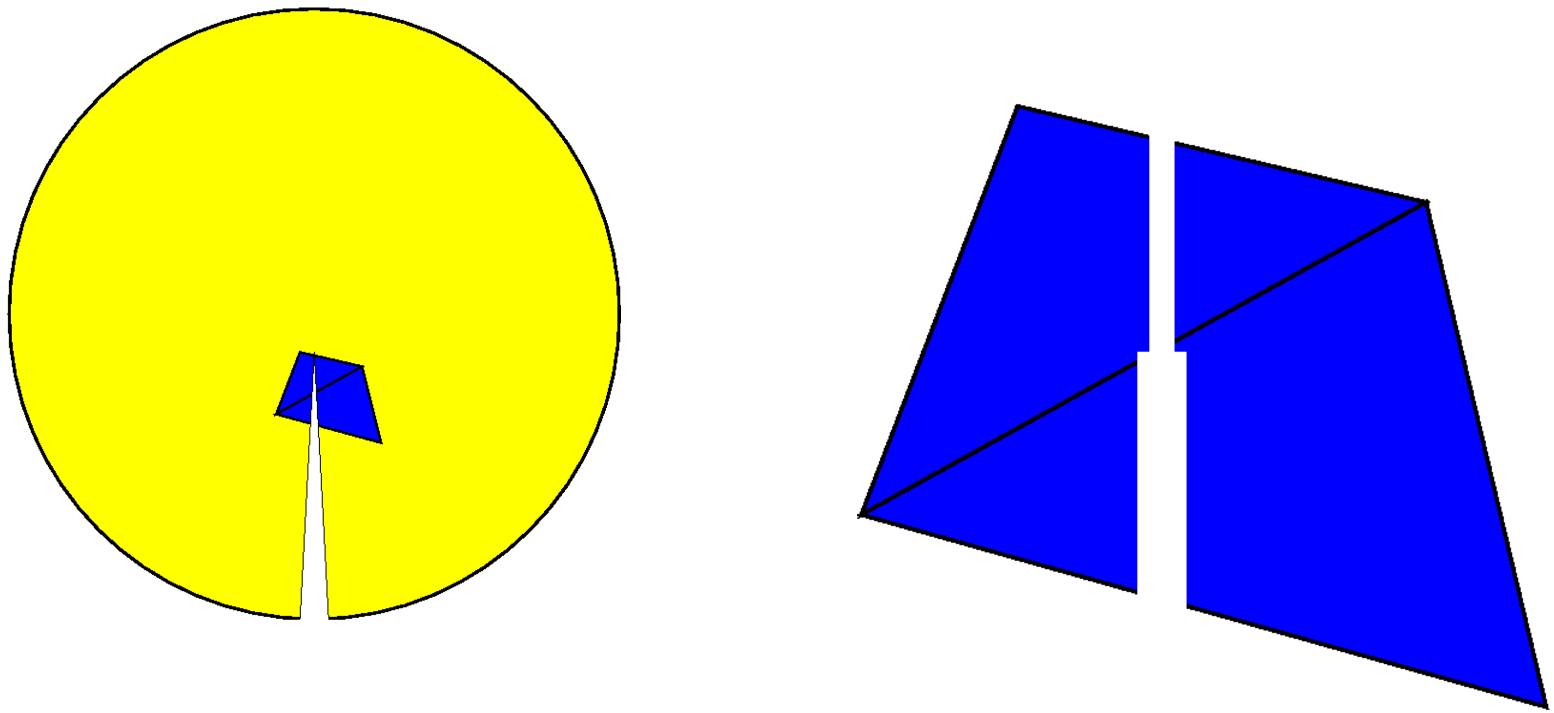

F.4

Extended Finite Elements (XFEM)

Based on Partition of Unity

Partition of Unity Method

Standard finite element approximation:

$$\mathbf{u}(\mathbf{x}) = \sum_{I=1}^{Nnod} N_I(\mathbf{x})\,\mathbf{d}_I$$

The shape functions are a partition of unity:

$$\sum_{I=1}^{Nnod} N_I(\mathbf{x}) = 1$$

Partition of Unity Method

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The shape functions are a partition of unity:

$$\sum_{I=1}^{Nnod} N_I(\mathbf{x}) = 1$$

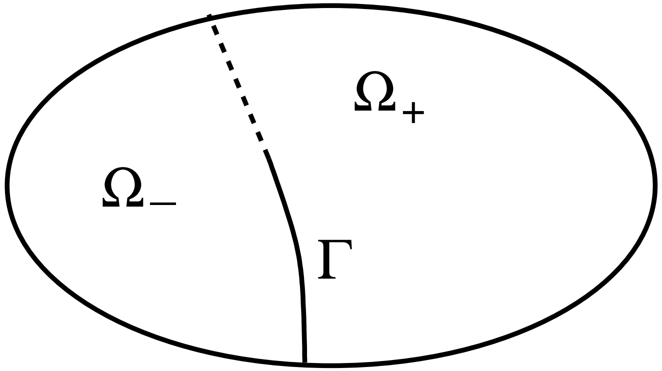
Enriched approximation:

$$\mathbf{u}(\mathbf{x}) = \sum_{I=1}^{Nnod} N_I(\mathbf{x}) \left[\mathbf{d}_I + \sum_{i \in L_I} G_i(\mathbf{x}) \mathbf{e}_{iI} \right]$$

selected enrichment functions

Partition of Unity Method – eXtended Finite Elements

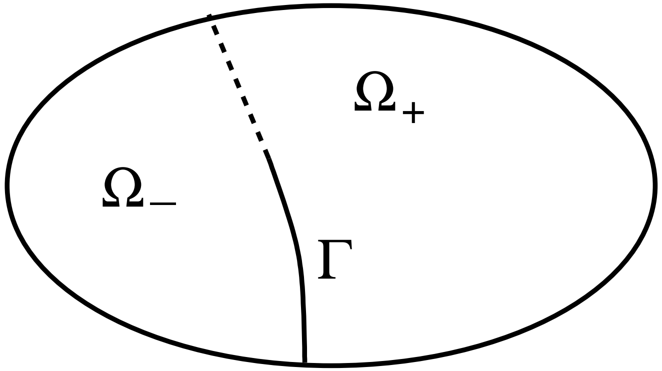
Enrichment by Heaviside function:



$$H_\Gamma(\mathbf{x}) = \begin{cases} 1 & \text{for } x \in \Omega^+ \\ 0 & \text{for } x \in \Omega^- \end{cases}$$

Partition of Unity Method – eXtended Finite Elements

Enrichment by Heaviside function:



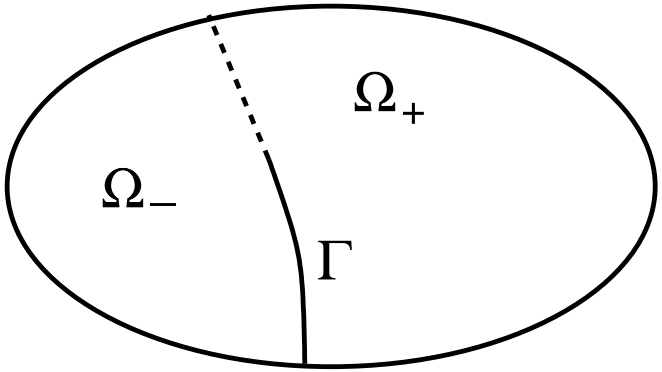
$$H_\Gamma(\mathbf{x}) = \begin{cases} 1 & \text{for } x \in \Omega^+ \\ 0 & \text{for } x \in \Omega^- \end{cases}$$

$$\mathbf{u}(\mathbf{x}) = \sum_{I=1}^{Nnod} N_I(\mathbf{x})\left[\mathbf{d}_I + H_\Gamma(\mathbf{x})\mathbf{e}_I\right] =$$

$$= \sum_{I=1}^{Nnod} N_I(\mathbf{x})\mathbf{d}_I + \sum_{I=1}^{Nnod} N_I(\mathbf{x}) H_\Gamma(\mathbf{x})\mathbf{e}_I$$

Partition of Unity Method – eXtended Finite Elements

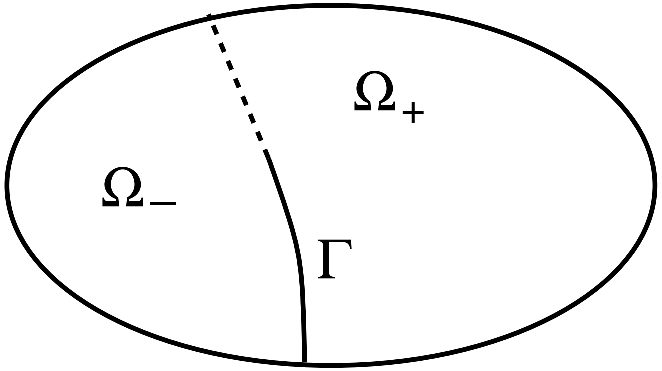
If the support of N_I is contained in Ω^+, then $N_I H_\Gamma = N_I$



If the support of N_I is contained in Ω^-, then $N_I H_\Gamma = 0$

Partition of Unity Method – eXtended Finite Elements

If the support of N_I is contained in Ω^+, then $N_I H_\Gamma = N_I$



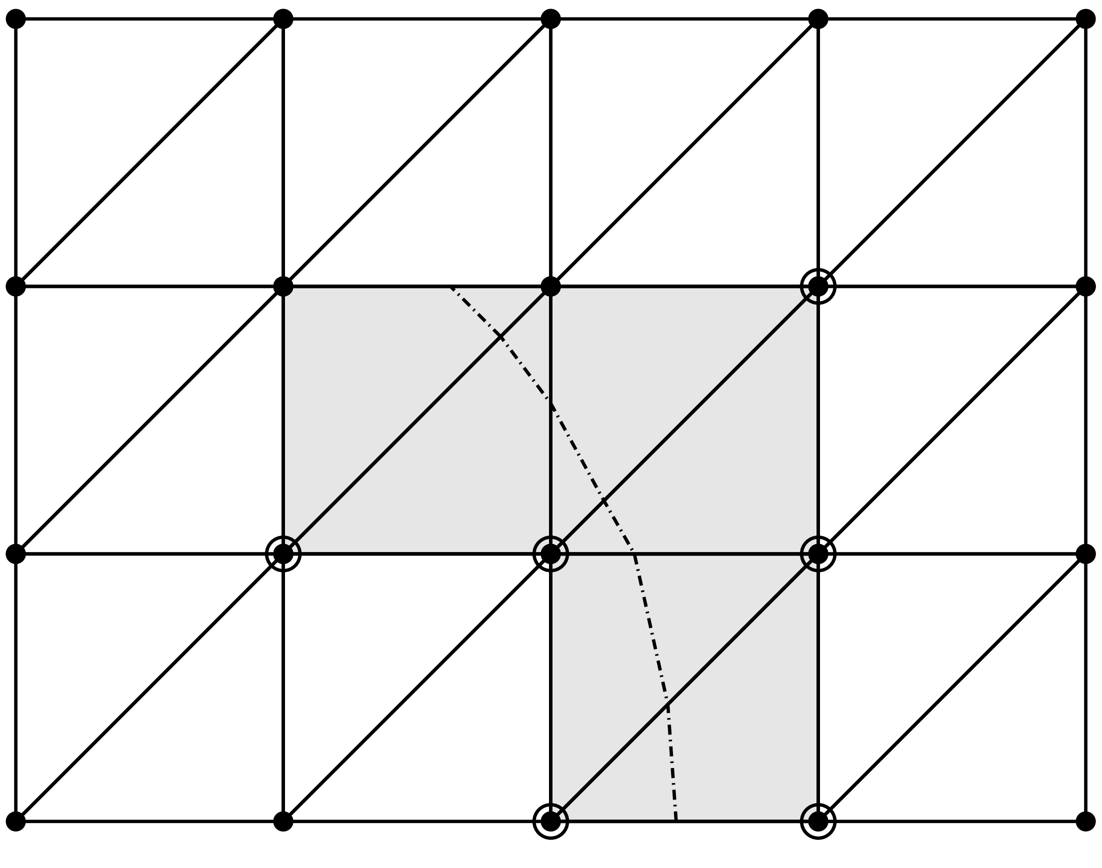
If the support of N_I is contained in Ω^-, then $N_I H_\Gamma = 0$

Only if the support of N_I is cut by Γ ,
then the function $N_I H_\Gamma$ really enriches the basis.

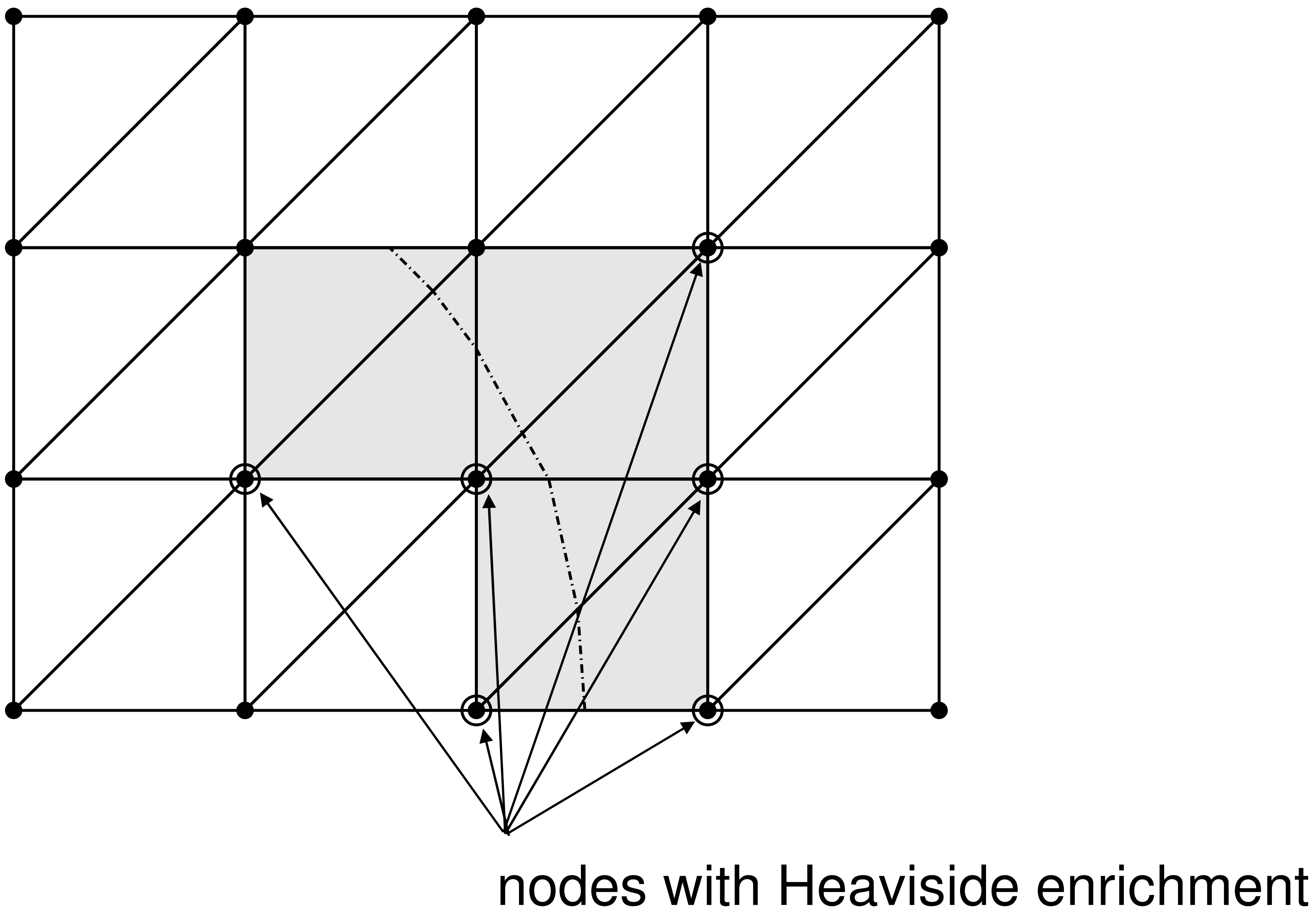
$$\mathbf{u}(\mathbf{x}) = \sum_{I=1}^{Nnod} N_I(\mathbf{x})\mathbf{d}_I + \sum_{I \in S_H} N_I(\mathbf{x}) H_\Gamma(\mathbf{x})\mathbf{e}_I$$

set of nodes with Heaviside enrichment

Partition of Unity Method – eXtended Finite Elements



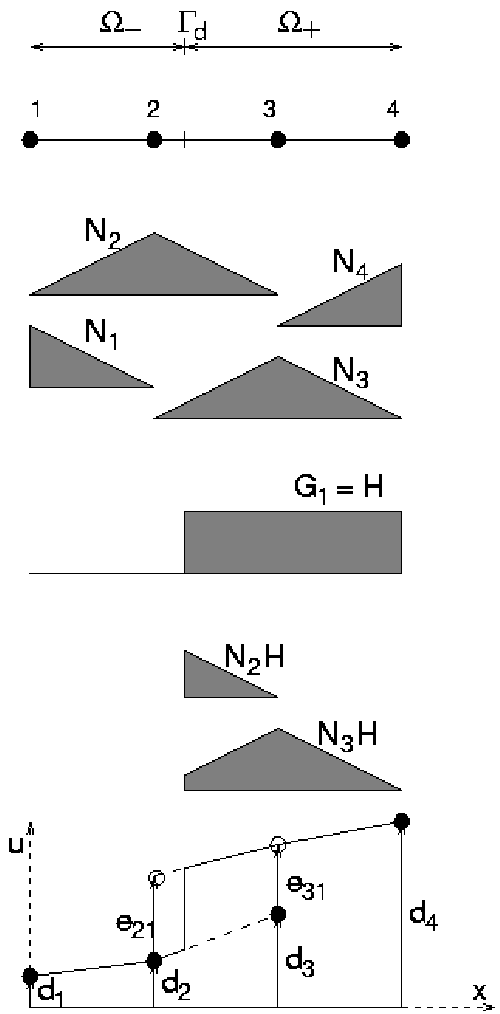
Partition of Unity Method – eXtended Finite Elements



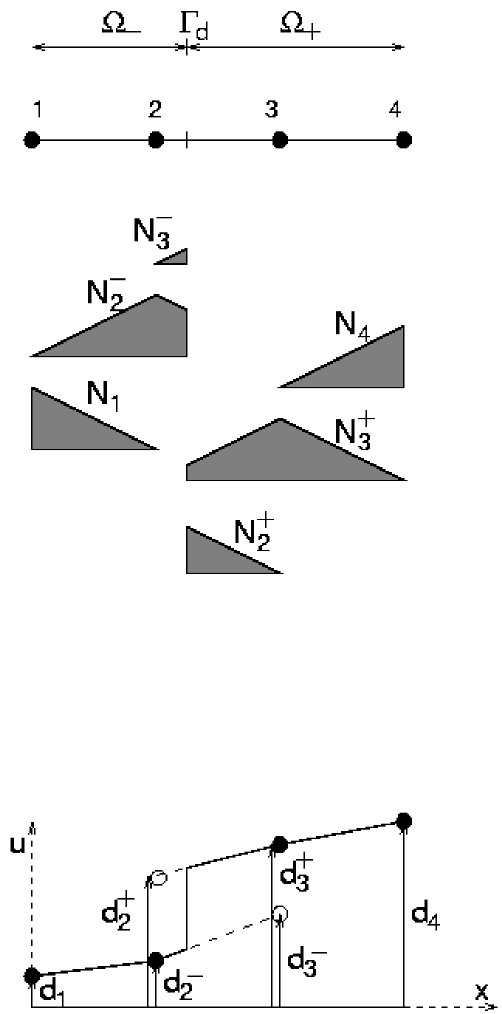
Partition of Unity Method – eXtended Finite Elements

The enriched approximation can be rearranged
to give better physical meaning to the degrees of freedom:

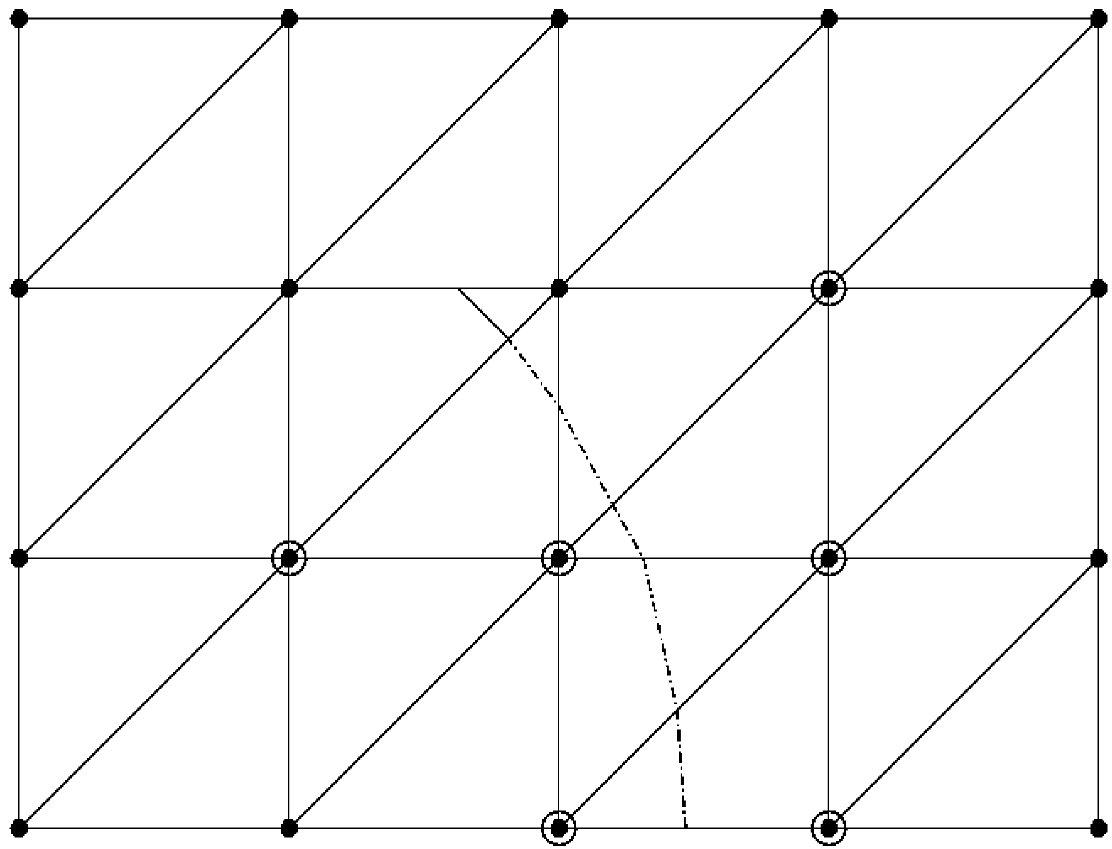
XFEM-PUM



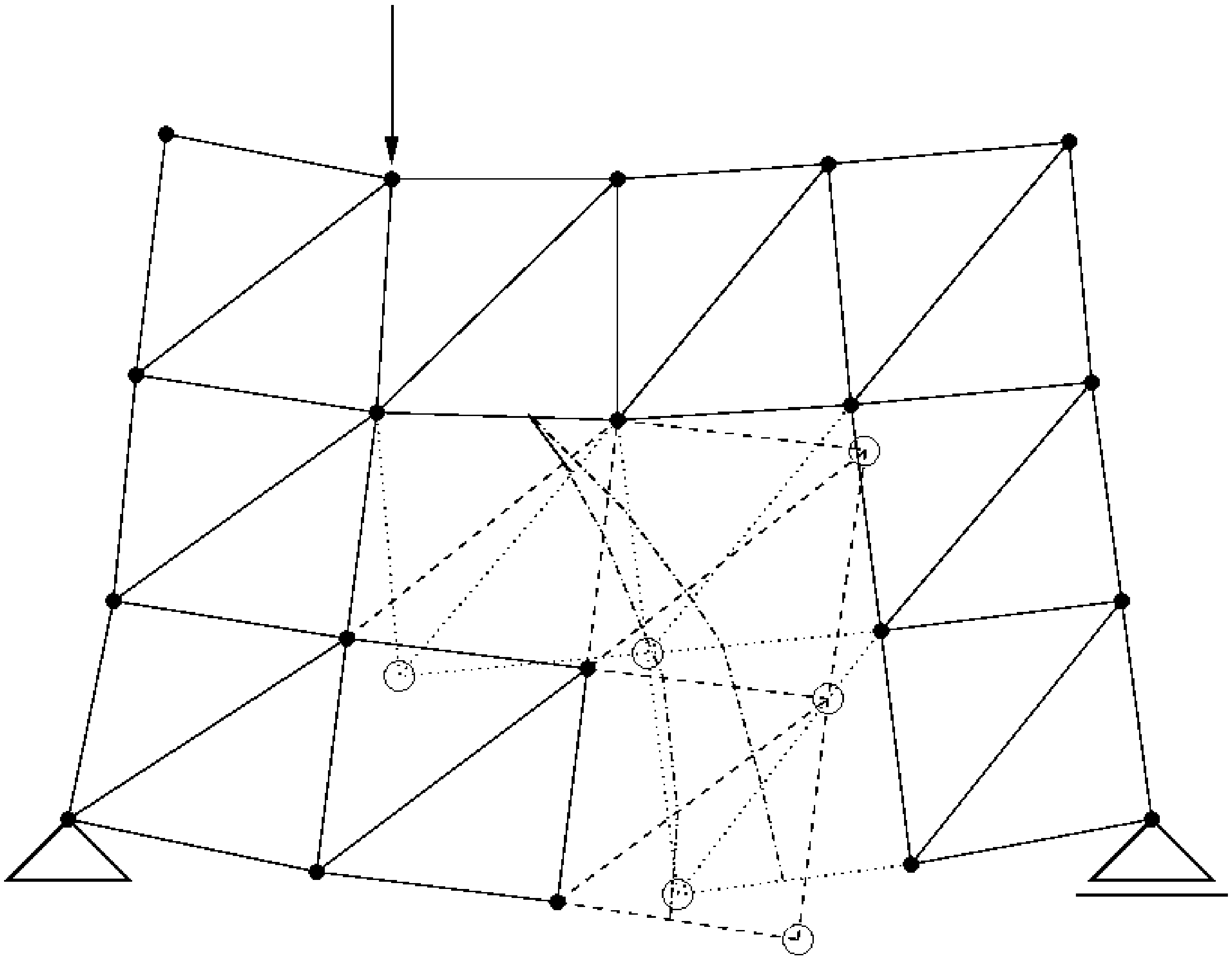
XFEM-PUM



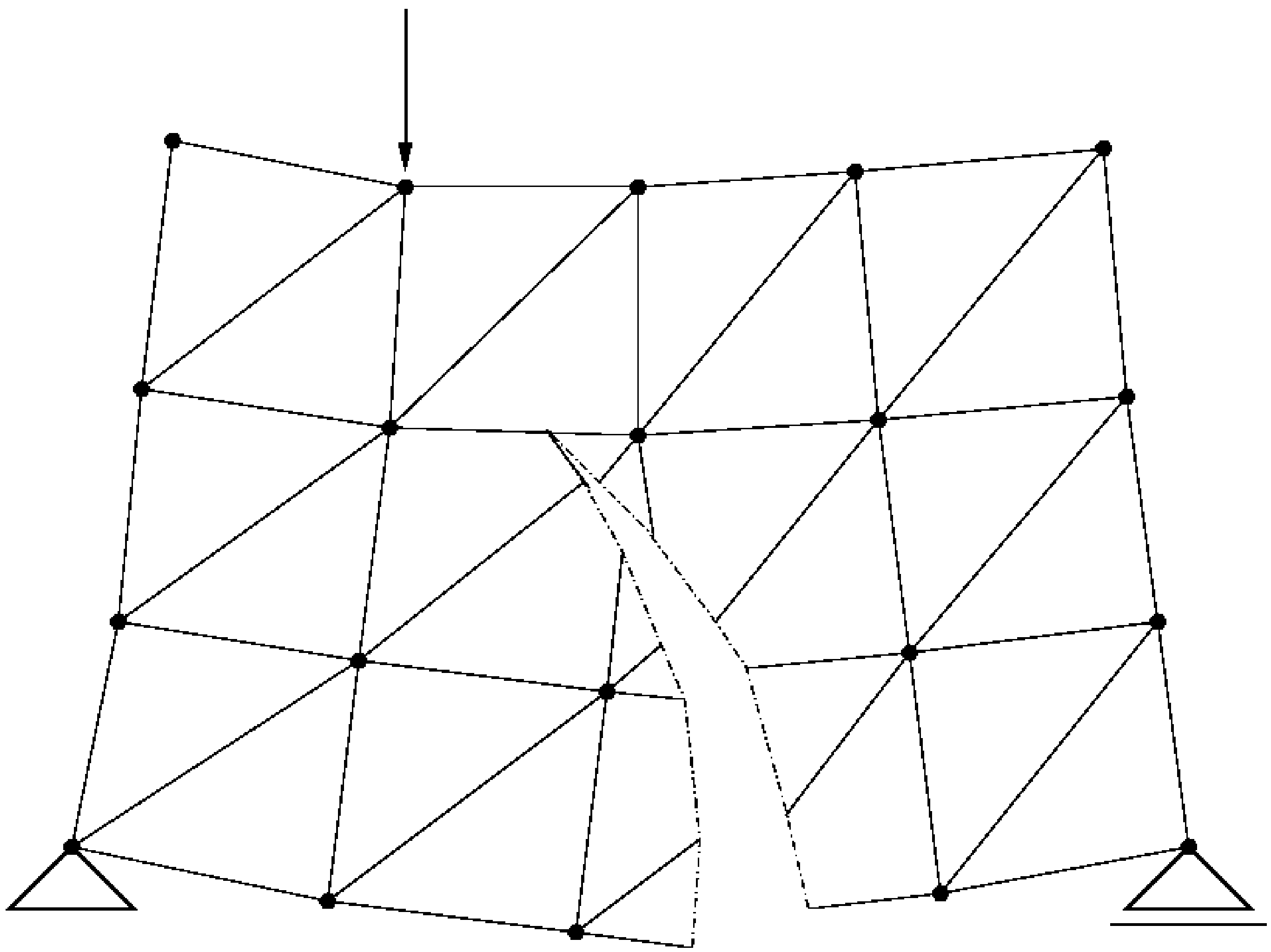
XFEM – enrichment by step function



XFEM – enrichment by step function

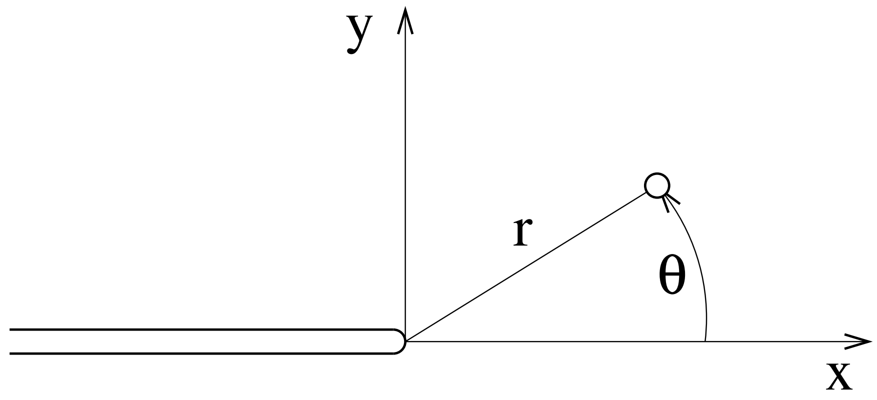


XFEM – enrichment by step function



XFEM – tip enrichment

Additional enrichment improving the approximation around the crack tip:



Functions that appear in the analytical near-tip solution:

$$B_1(r,\theta) = \sqrt{r}\sin\frac{\theta}{2} \qquad B_3(r,\theta) = \sqrt{r}\sin\frac{\theta}{2}\sin\theta$$

$$B_2(r,\theta) = \sqrt{r}\cos\frac{\theta}{2} \qquad B_4(r,\theta) = \sqrt{r}\cos\frac{\theta}{2}\sin\theta$$

XFEM – tip enrichment

Additional enrichment improving the approximation around the crack tip:

$$\mathbf{u}(\mathbf{x}) = \sum_{I=1}^{Nnod} N_I(\mathbf{x})\mathbf{d}_I + \sum_{I\in S_H} N_I(\mathbf{x})H_\Gamma(\mathbf{x})\mathbf{e}_{0I} +$$

$$+ \sum_{I\in S_B}\sum_{i=1}^{4} N_I(\mathbf{x})B_i(r(\mathbf{x}),\theta(\mathbf{x}))\mathbf{e}_{iI}$$

Functions that appear in the analytical near-tip solution:

$$B_1(r,\theta) = \sqrt{r}\sin\frac{\theta}{2} \qquad B_3(r,\theta) = \sqrt{r}\sin\frac{\theta}{2}\sin\theta$$

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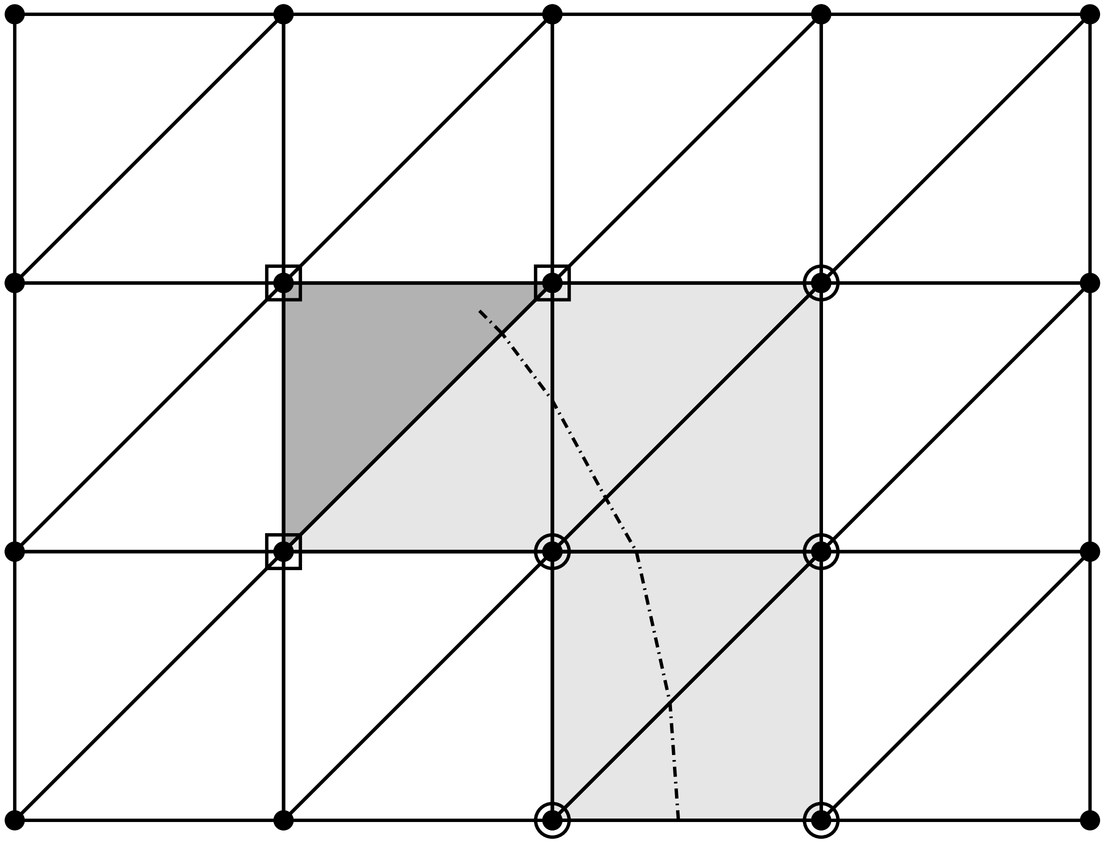
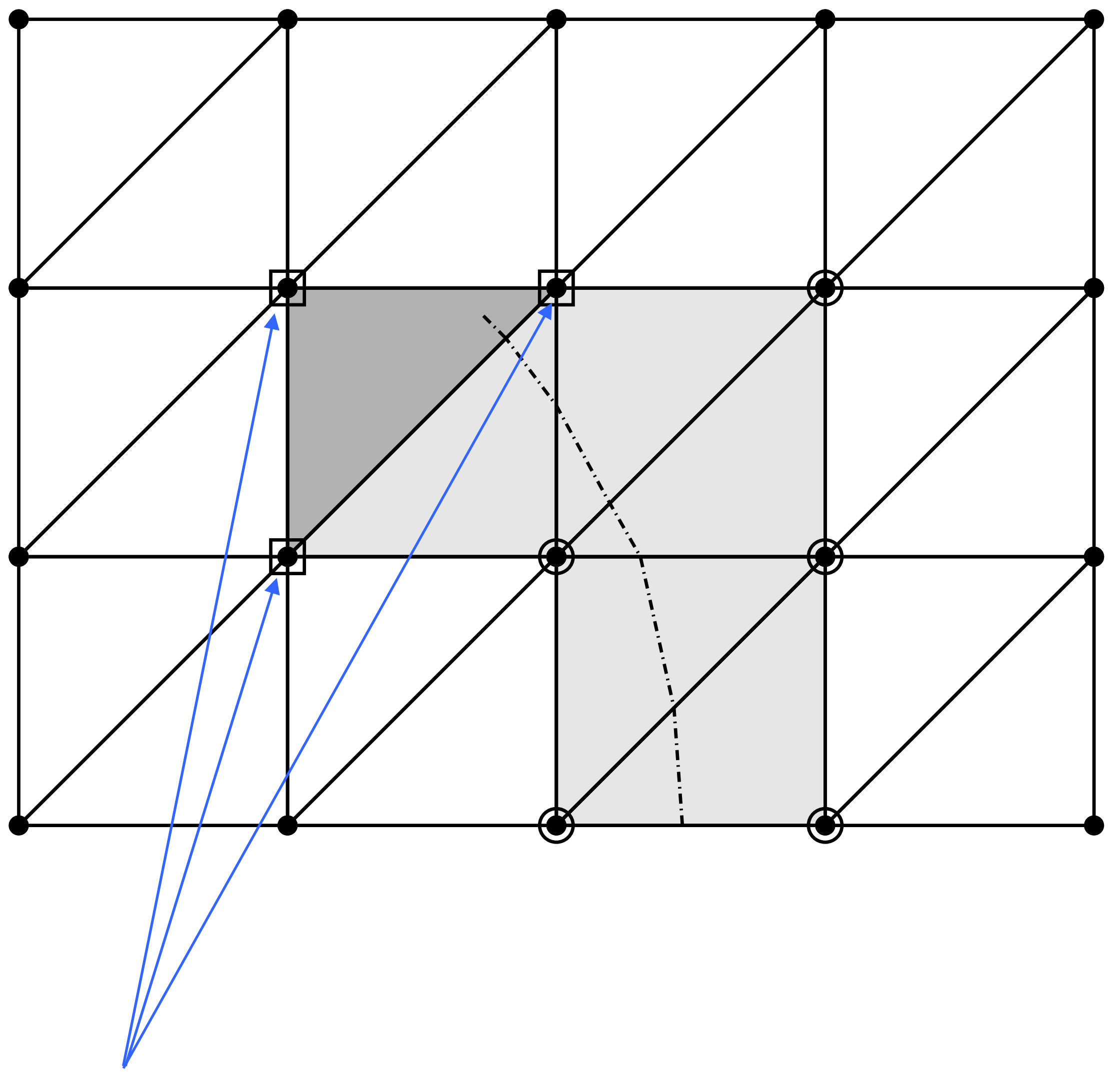
XFEM – tip enrichment

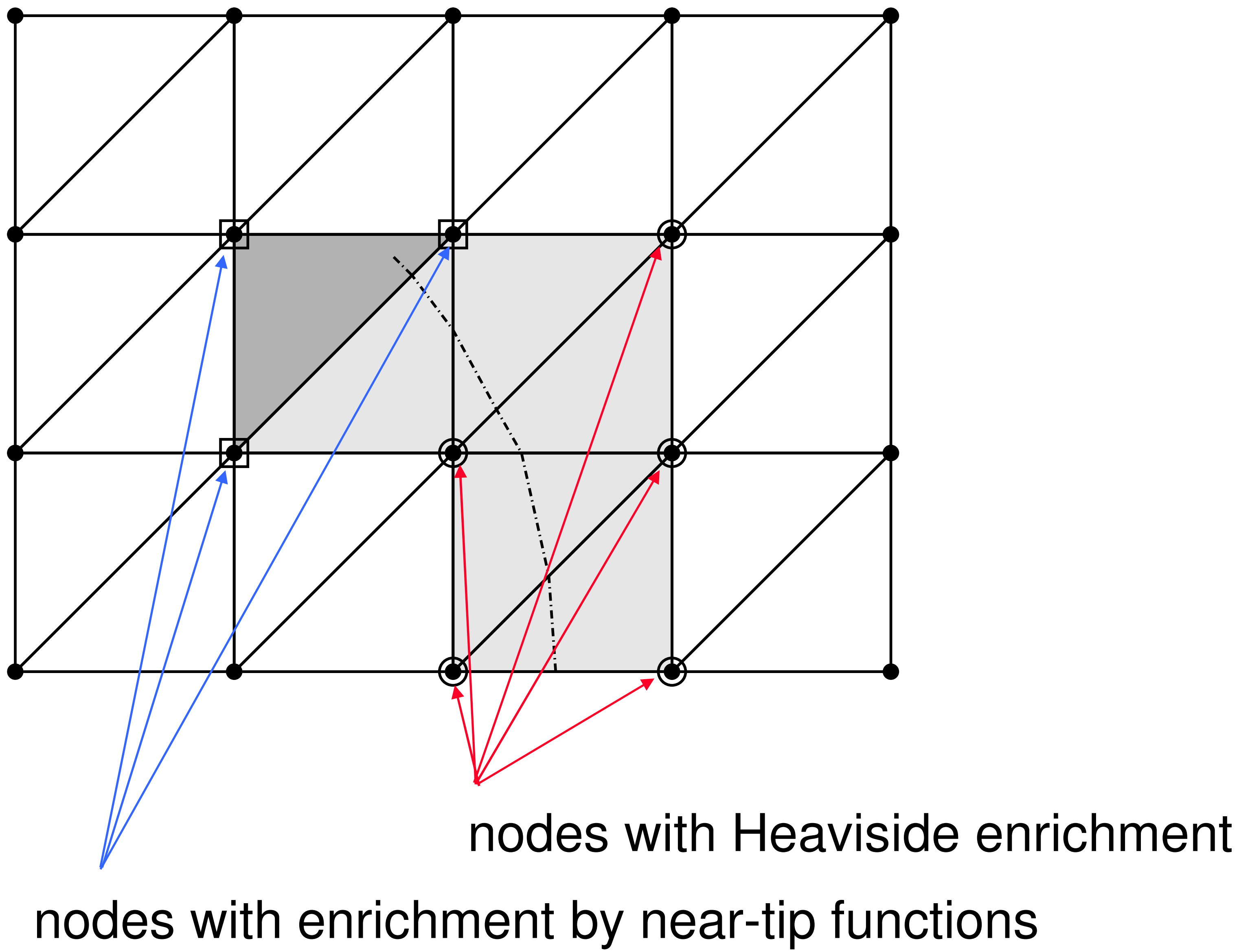


XFEM – tip enrichment

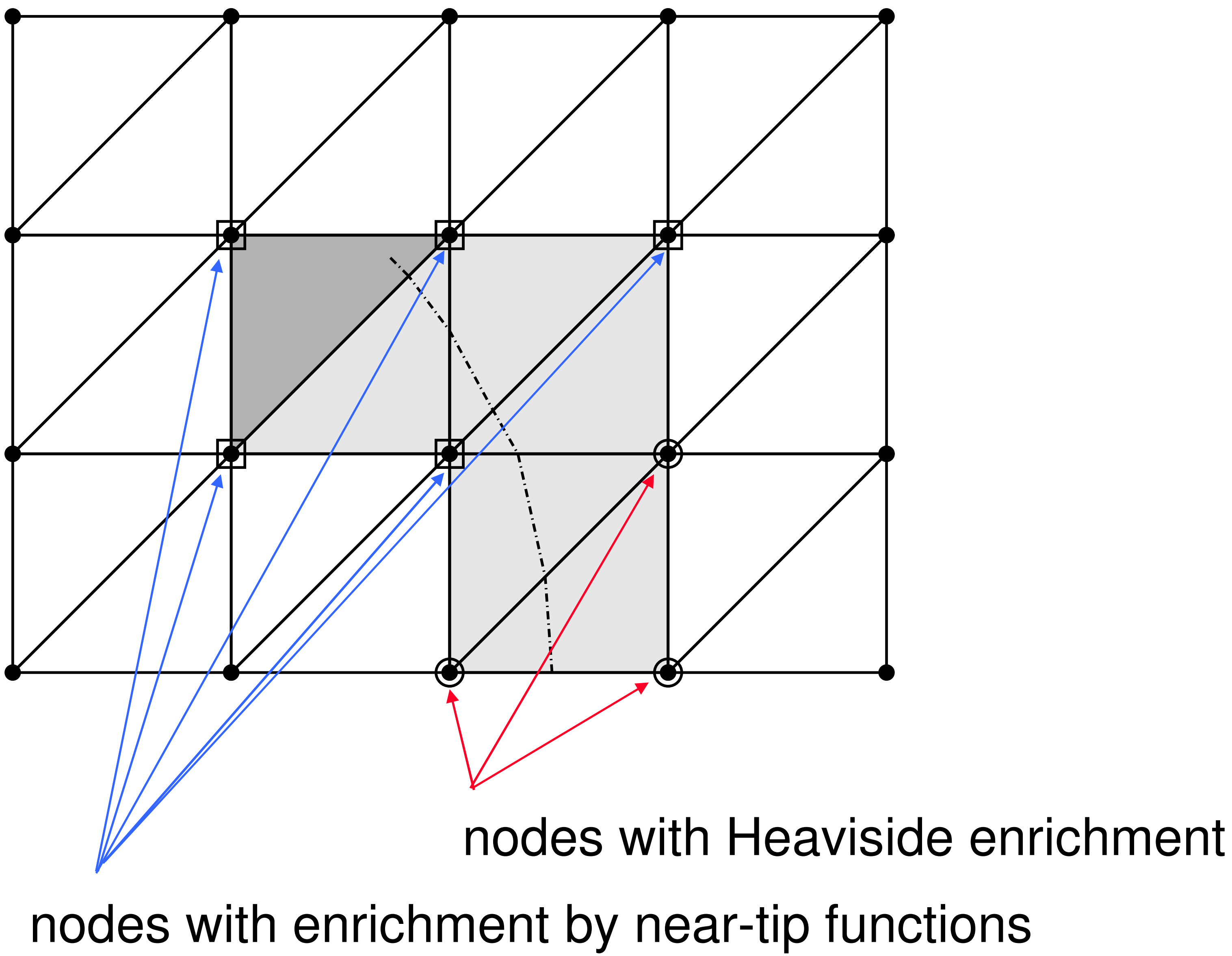


nodes with enrichment by near-tip functions

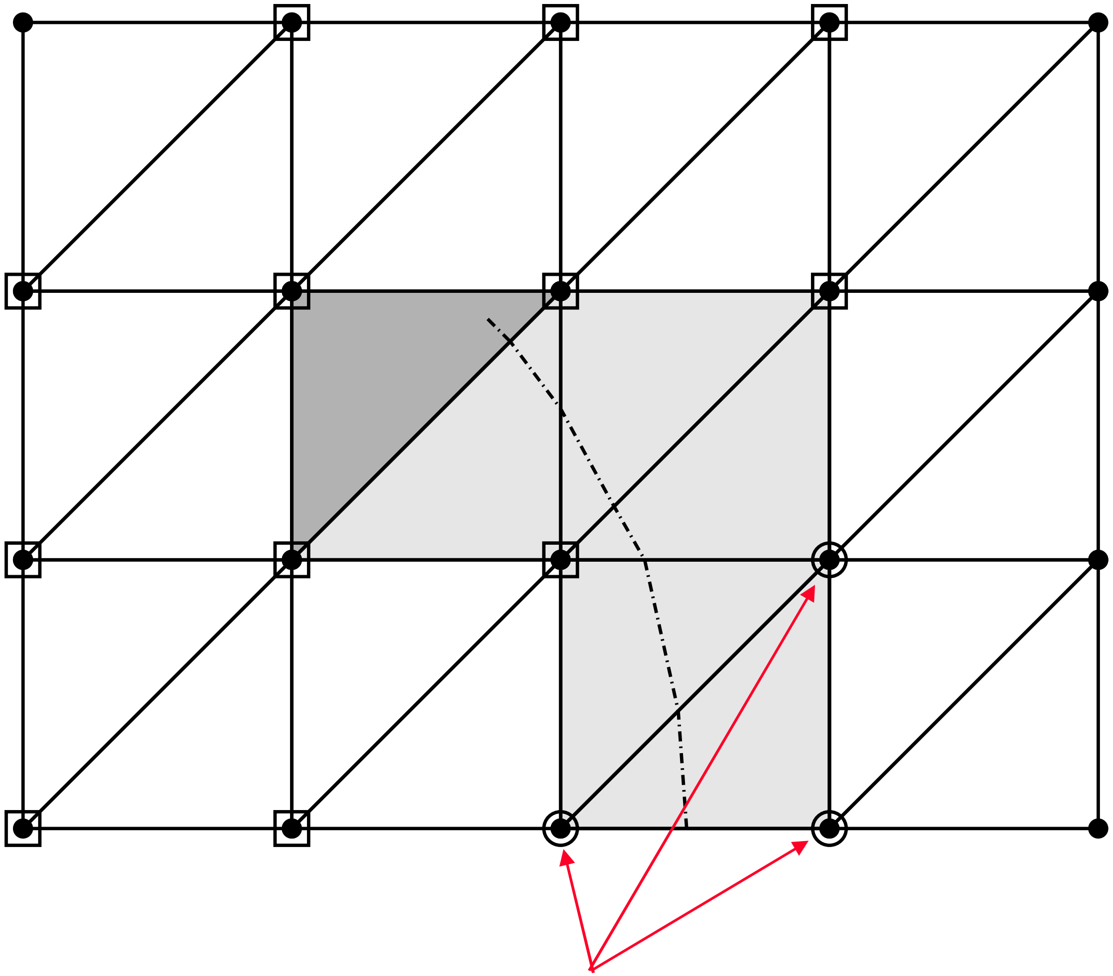
XFEM – tip enrichment



XFEM – tip enrichment



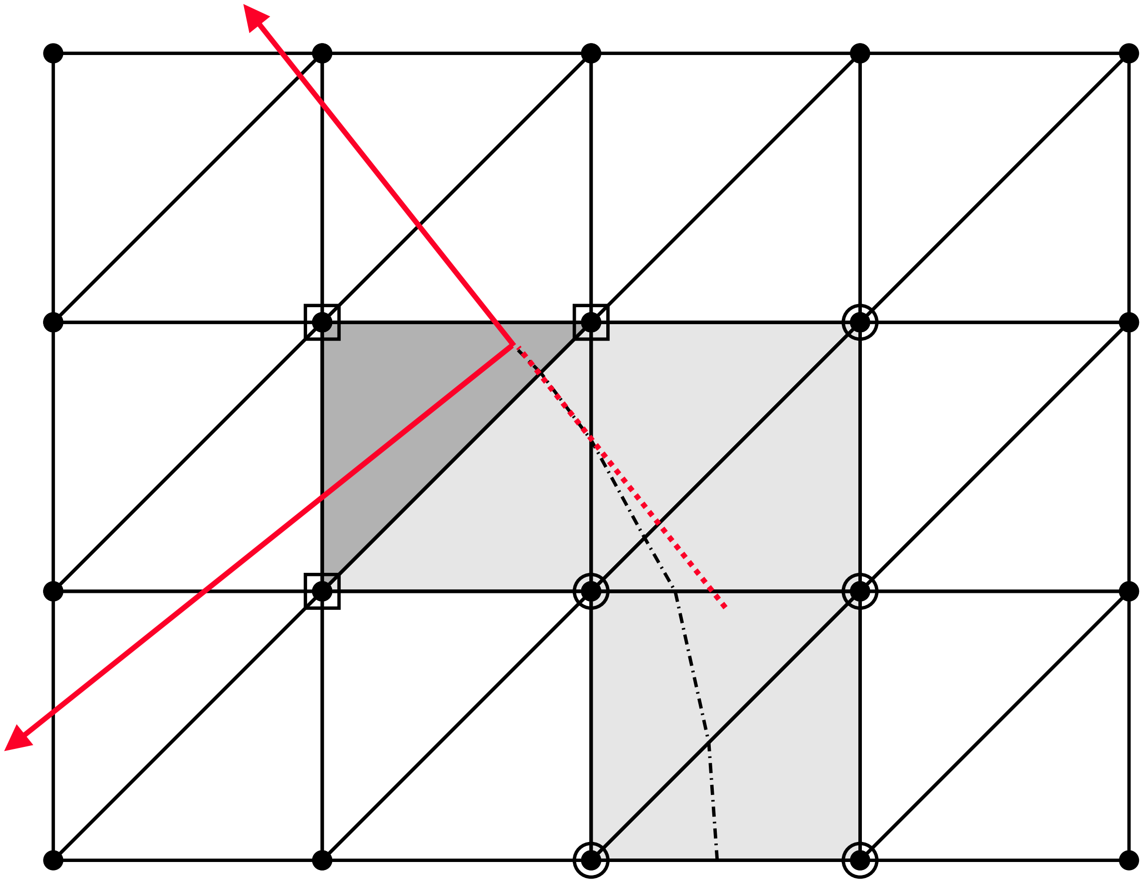
XFEM – tip enrichment



nodes with Heaviside enrichment

nodes with enrichment by near-tip functions

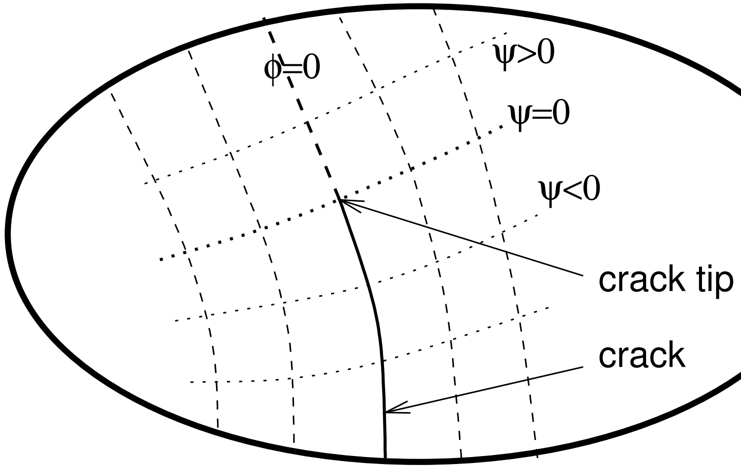
XFEM – tip enrichment



But if the crack is curved, we cannot define functions B_i in terms of the standard polar coordinates because B_1 would not be discontinuous across the crack but across the dotted line.

XFEM – level set functions

Remedy:
Construct curvilinear coordinates φ and ψ such that the crack is characterized by $\varphi = 0$ and $\psi \leq 0$



XFEM – level set functions

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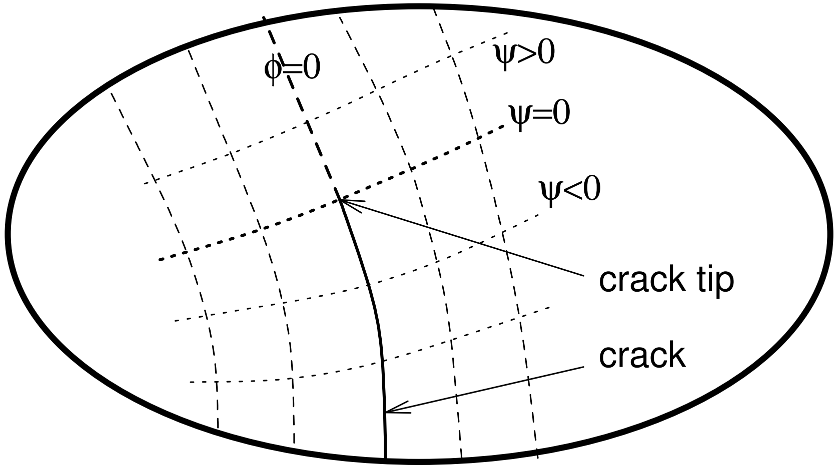


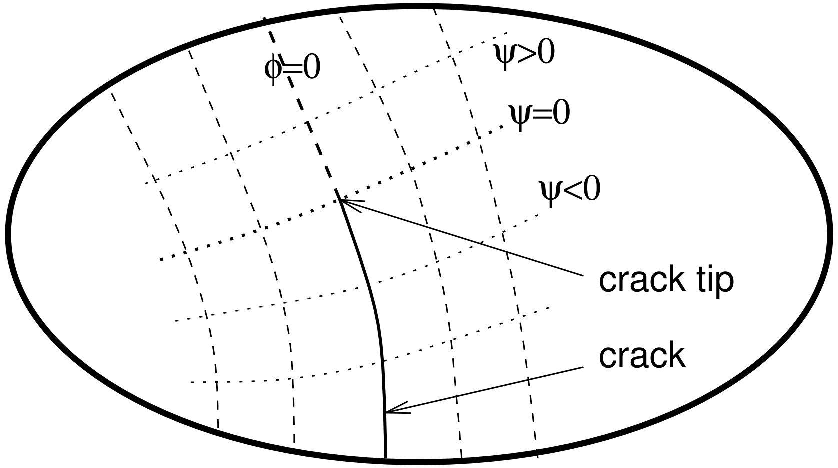
and define B_i in terms of the pseudo-polar coordinates

$$r(\psi,\varphi) = \sqrt{\psi^2 + \varphi^2}$$

$$\theta(\psi,\varphi) = \operatorname{sgn}(\varphi)\arccos\frac{\psi}{\sqrt{\psi^2+\varphi^2}}$$

XFEM – level set functions

Functions φ and ψ are the so-called **level set functions.**

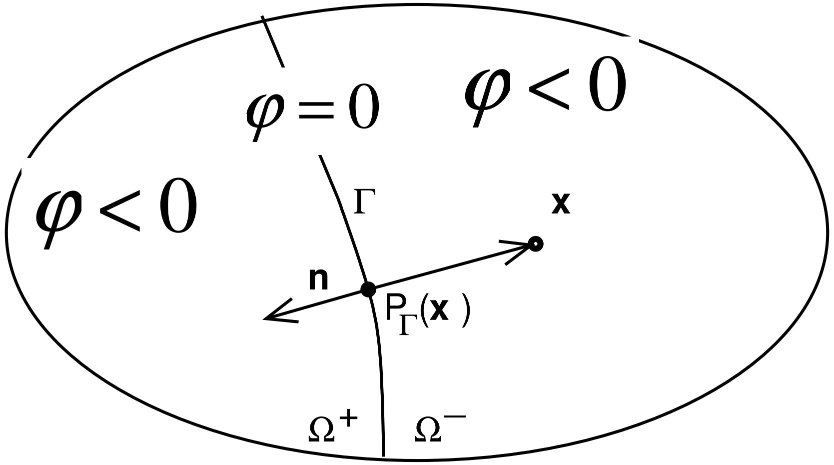


They are defined by their values at nodes around the crack and interpolated using the standard shape functions:

$$\varphi(\mathbf{x}) = \sum_I N_I(\mathbf{x})\,\varphi_I, \quad \psi(\mathbf{x}) = \sum_I N_I(\mathbf{x})\,\psi_I$$

XFEM – level set functions

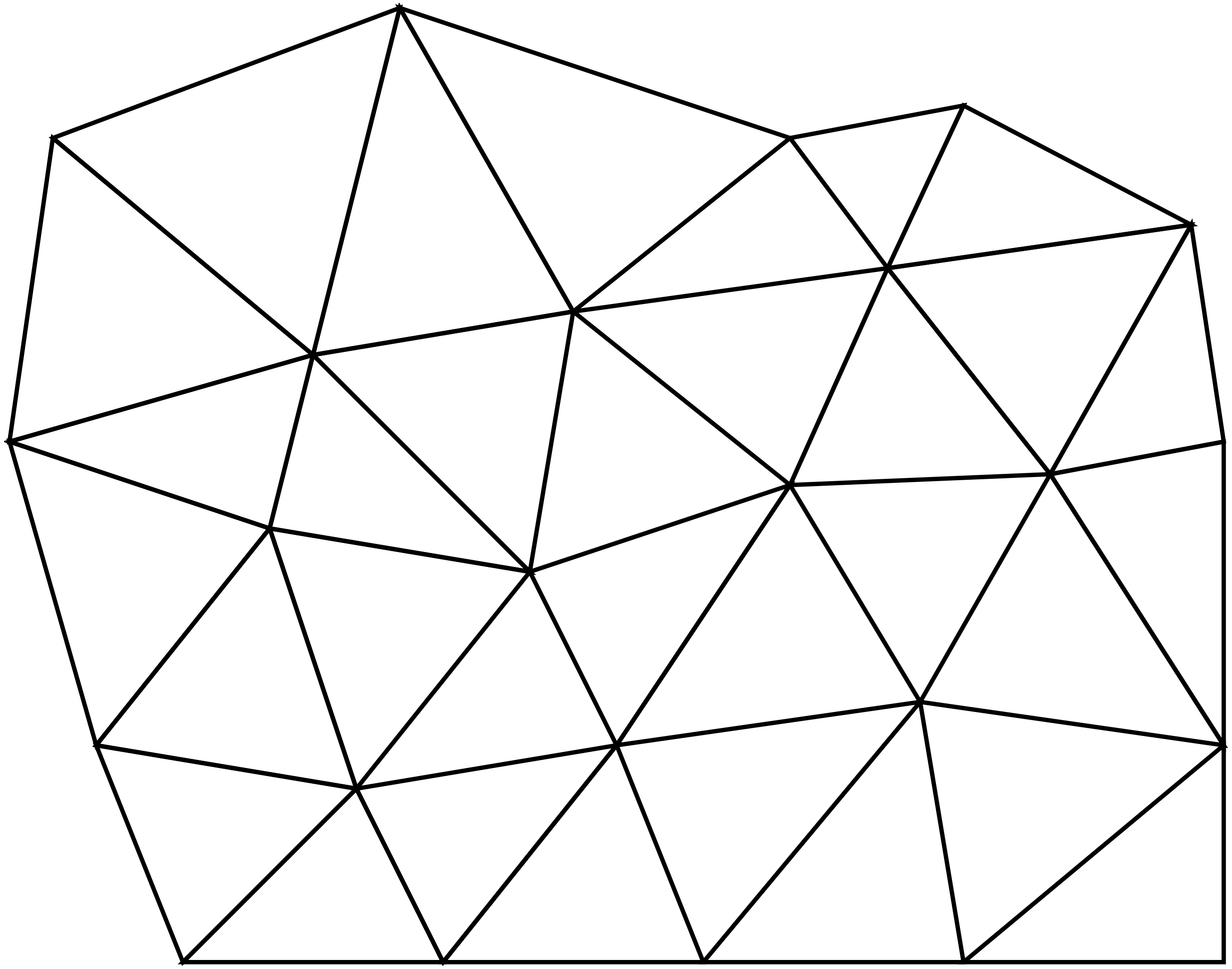
For an existing crack, function φ can be constructed as the signed distance function:



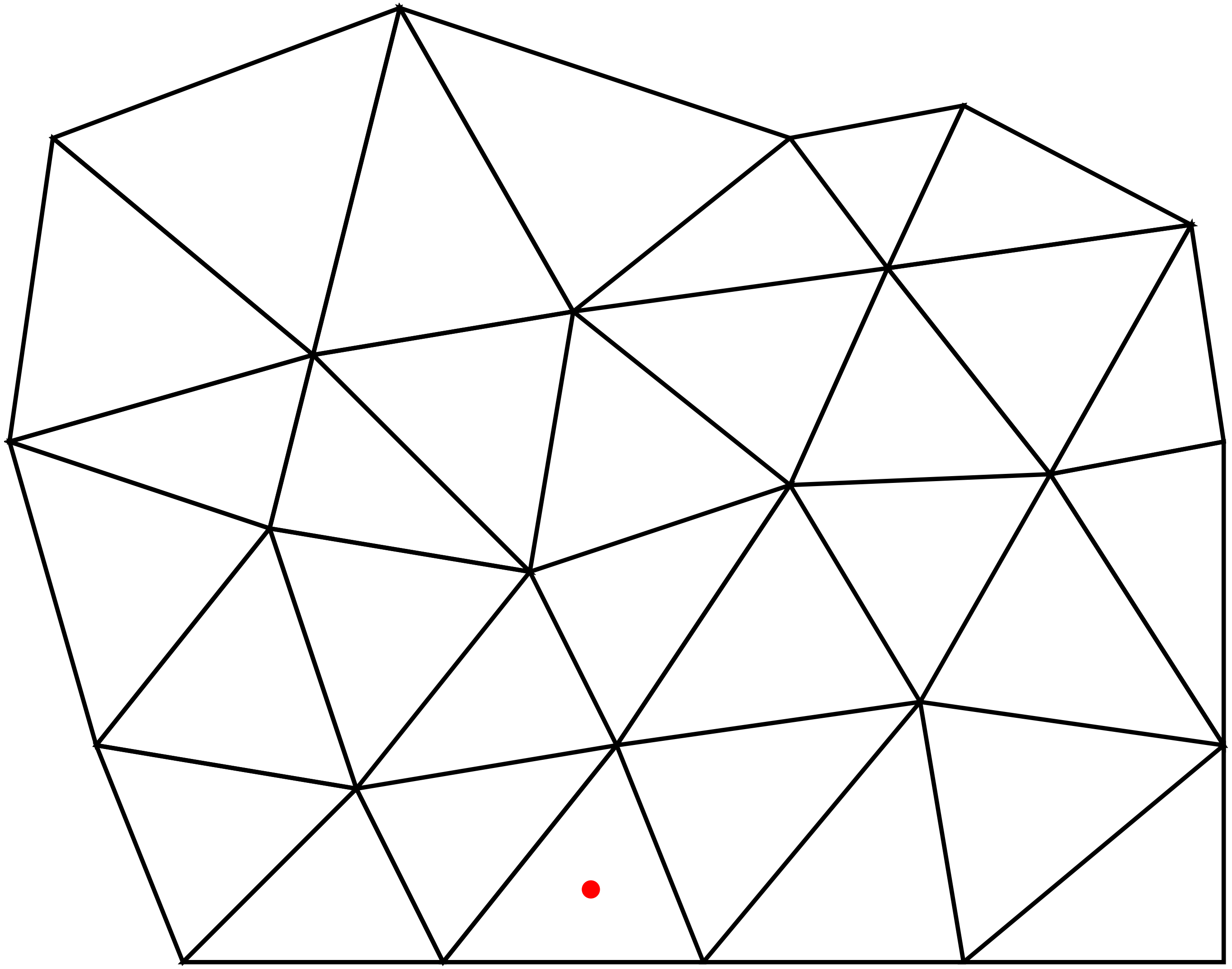
$$\varphi(\mathbf{x}) = \left\|\mathbf{x} - P_\Gamma(\mathbf{x})\right\| \operatorname{sgn}\left[(\mathbf{x} - P_\Gamma(\mathbf{x})) \cdot \mathbf{n}(P_\Gamma(\mathbf{x}))\right]$$

Criteria for Direction of Crack Propagation

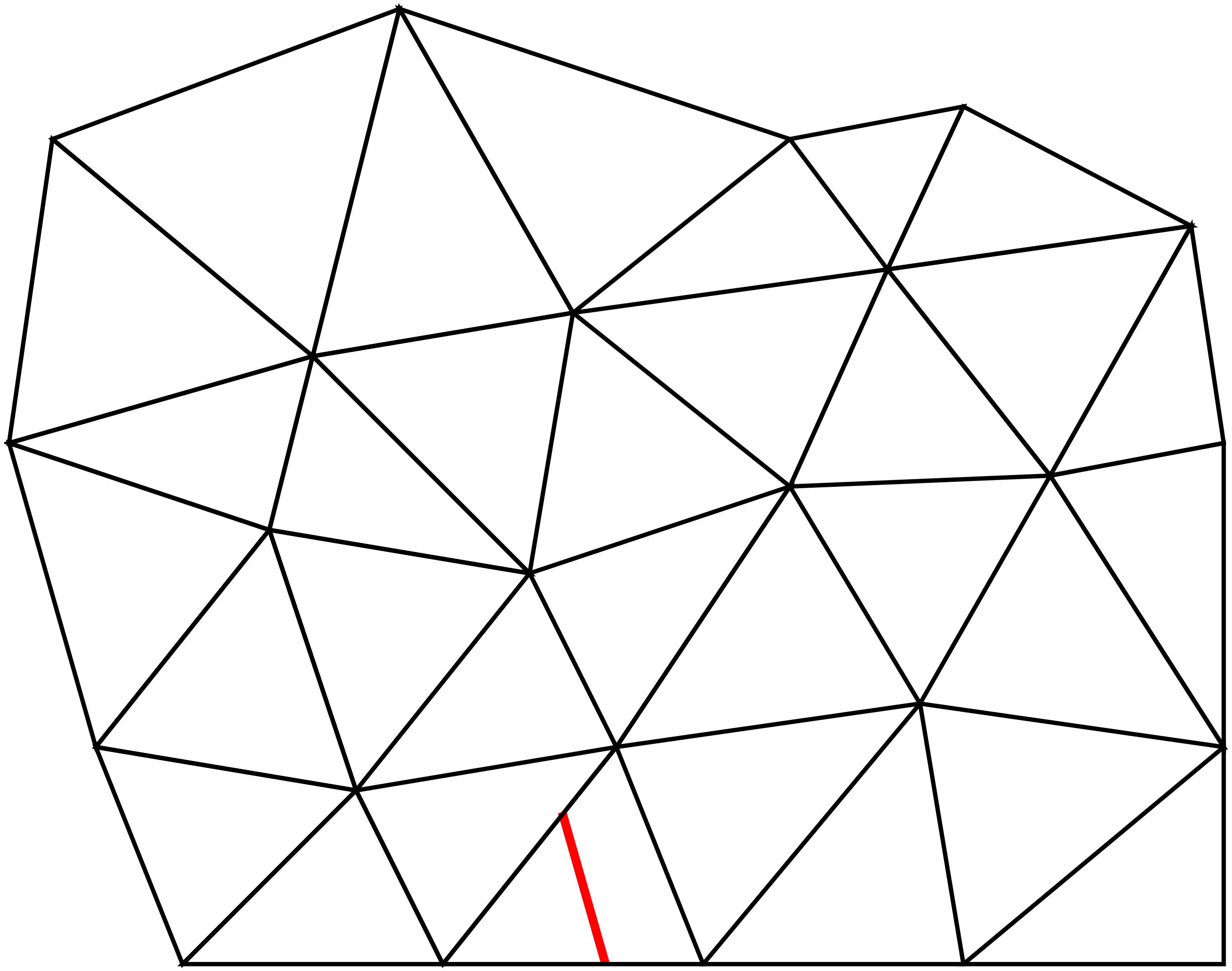
Tracking of a propagating crack



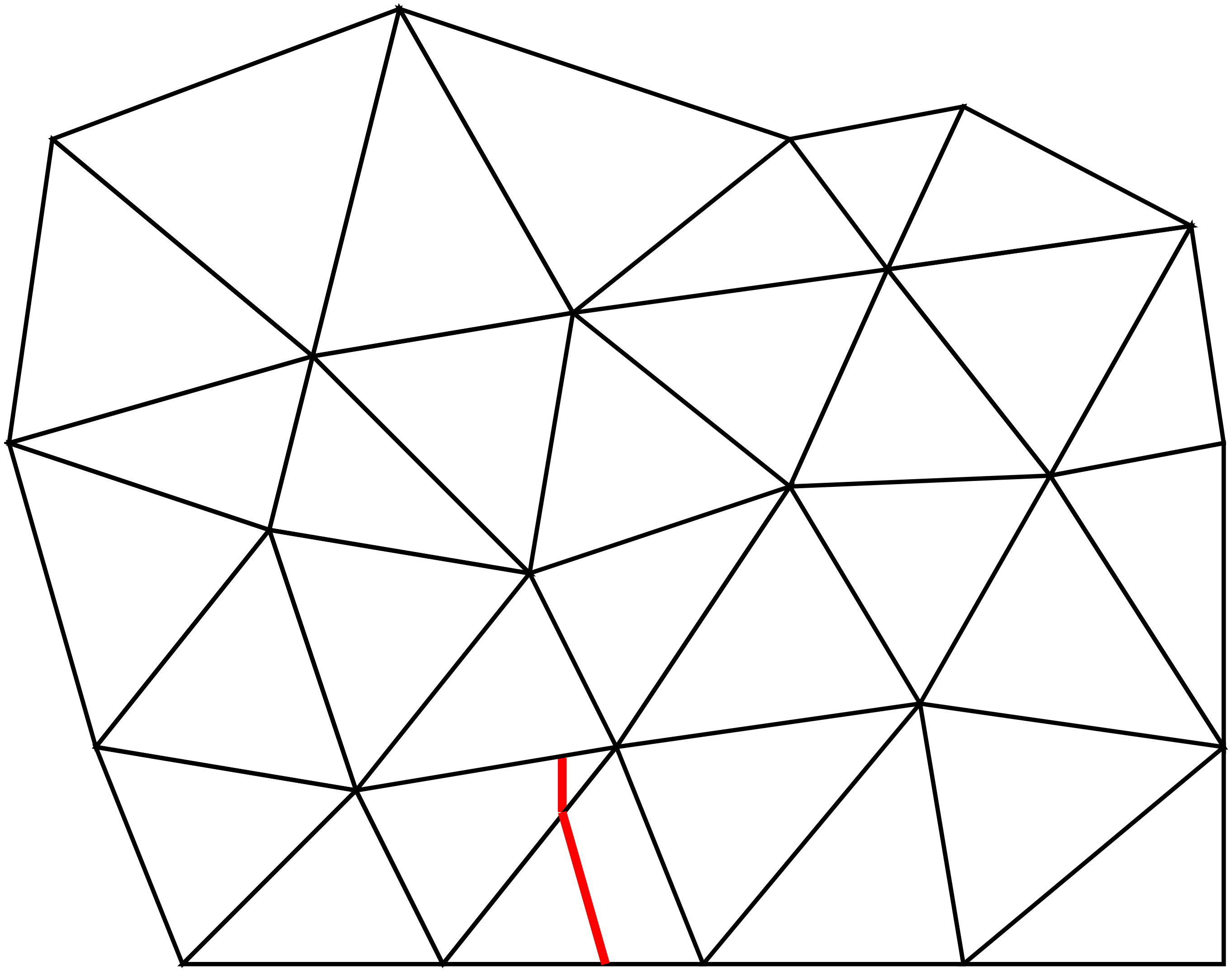
Tracking of a propagating crack



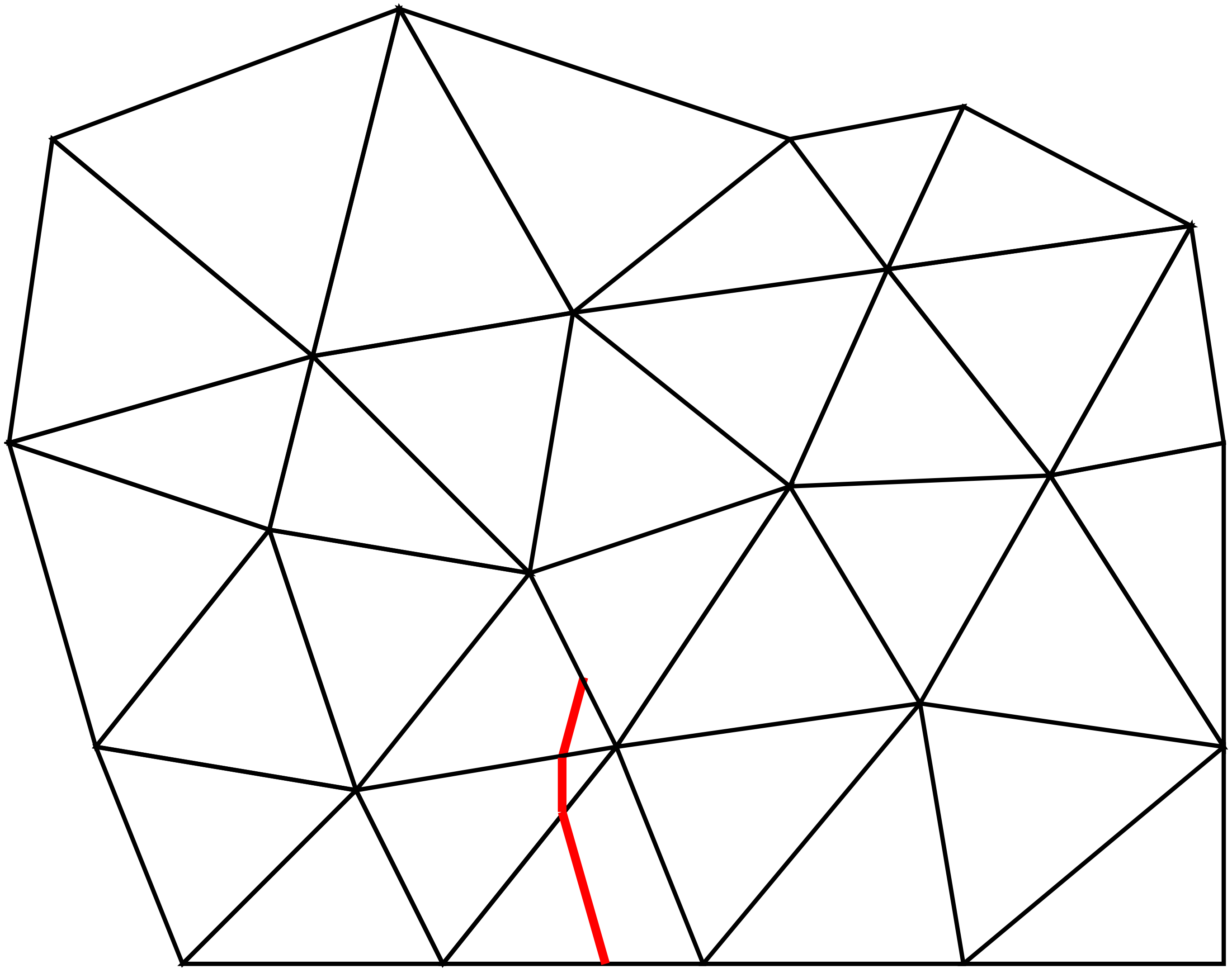
Tracking of a propagating crack



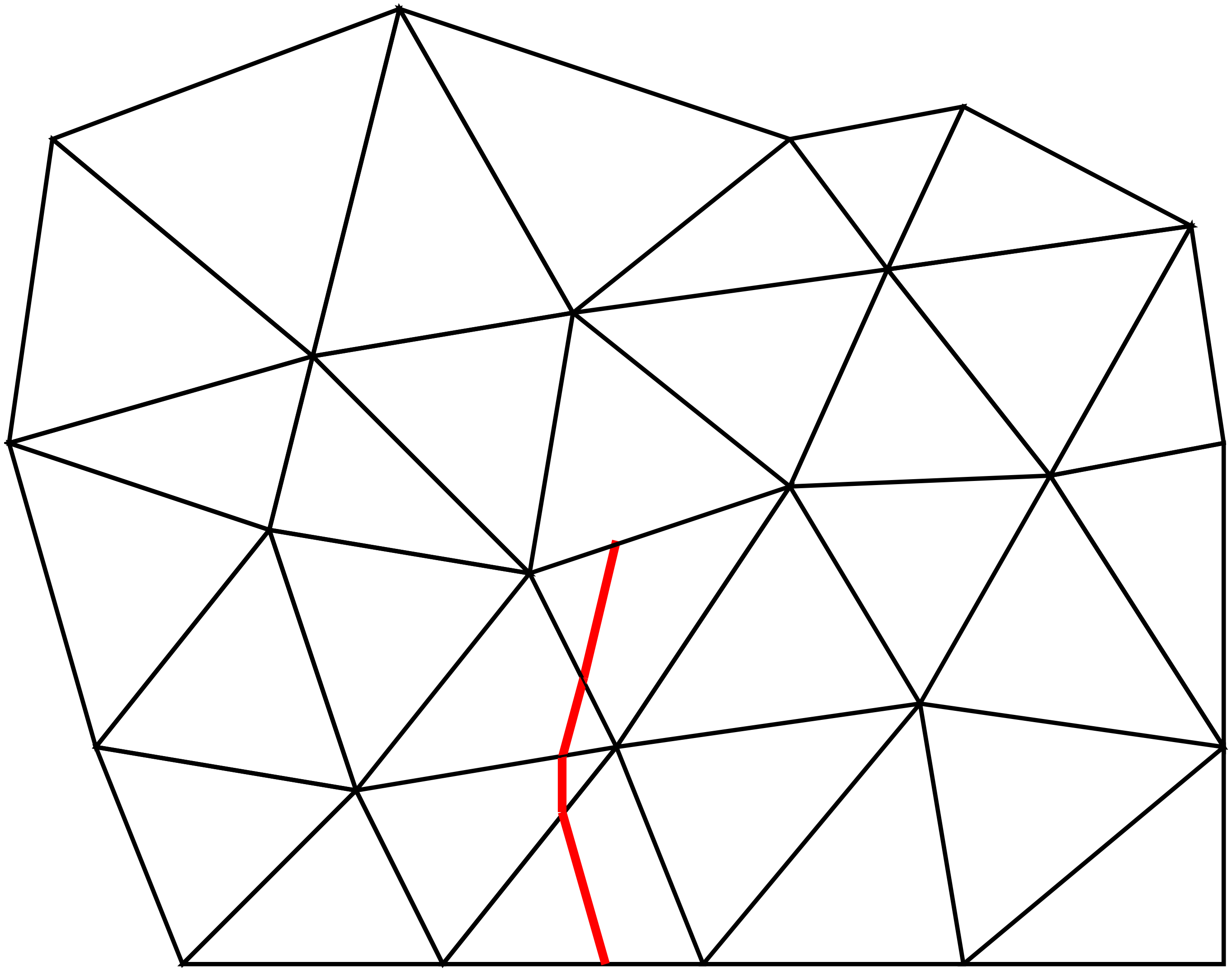
Tracking of a propagating crack



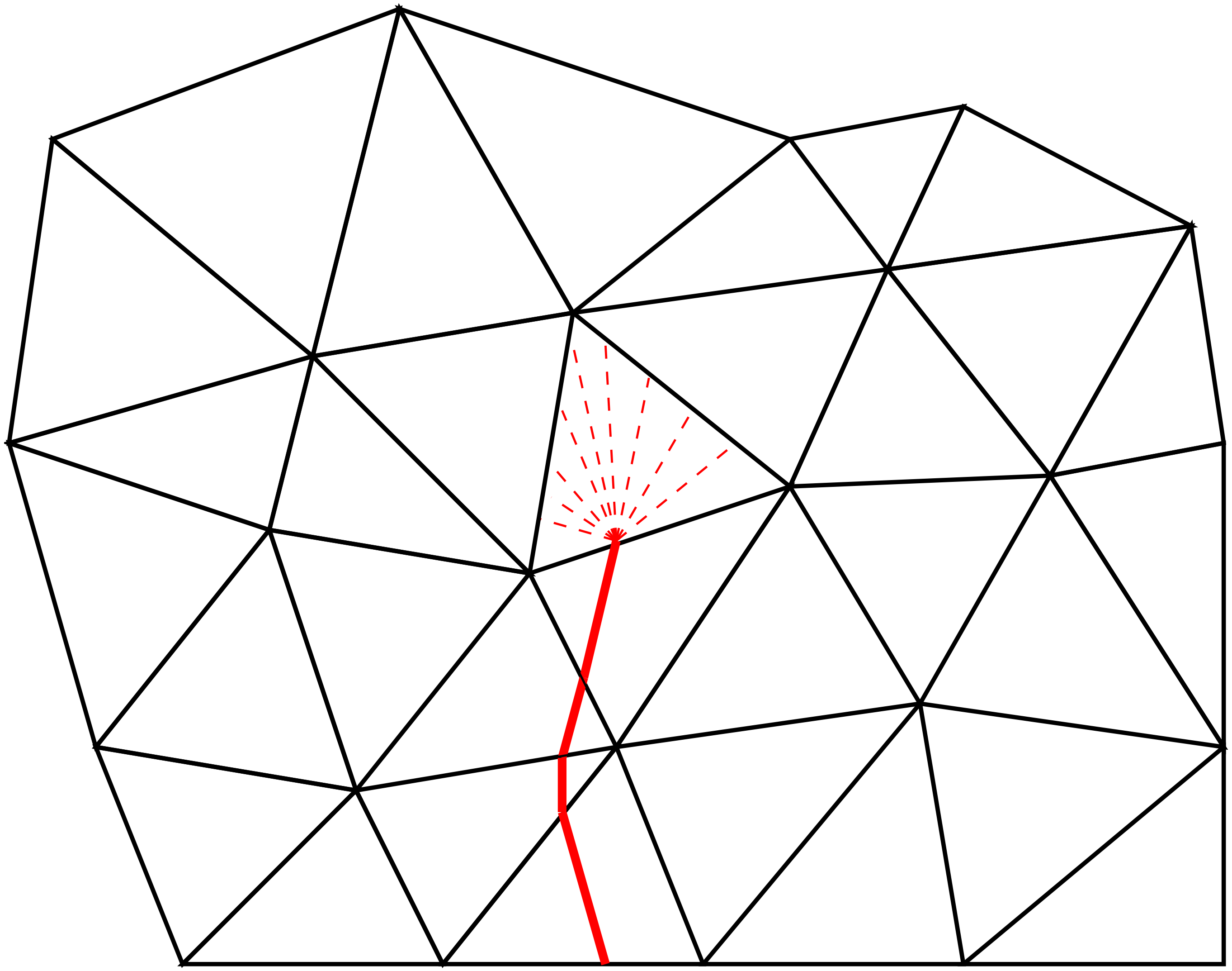
Tracking of a propagating crack



Tracking of a propagating crack



Tracking of a propagating crack



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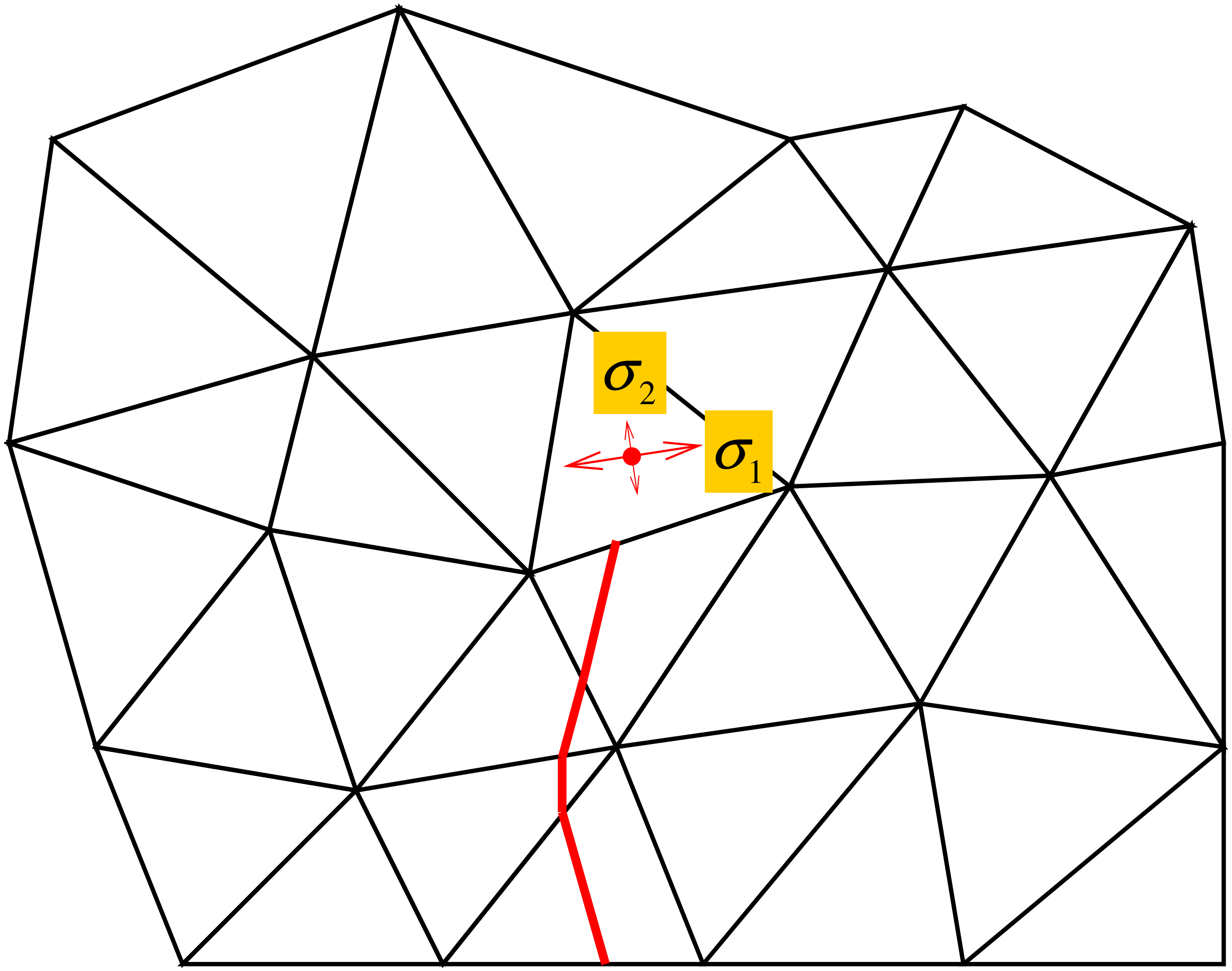
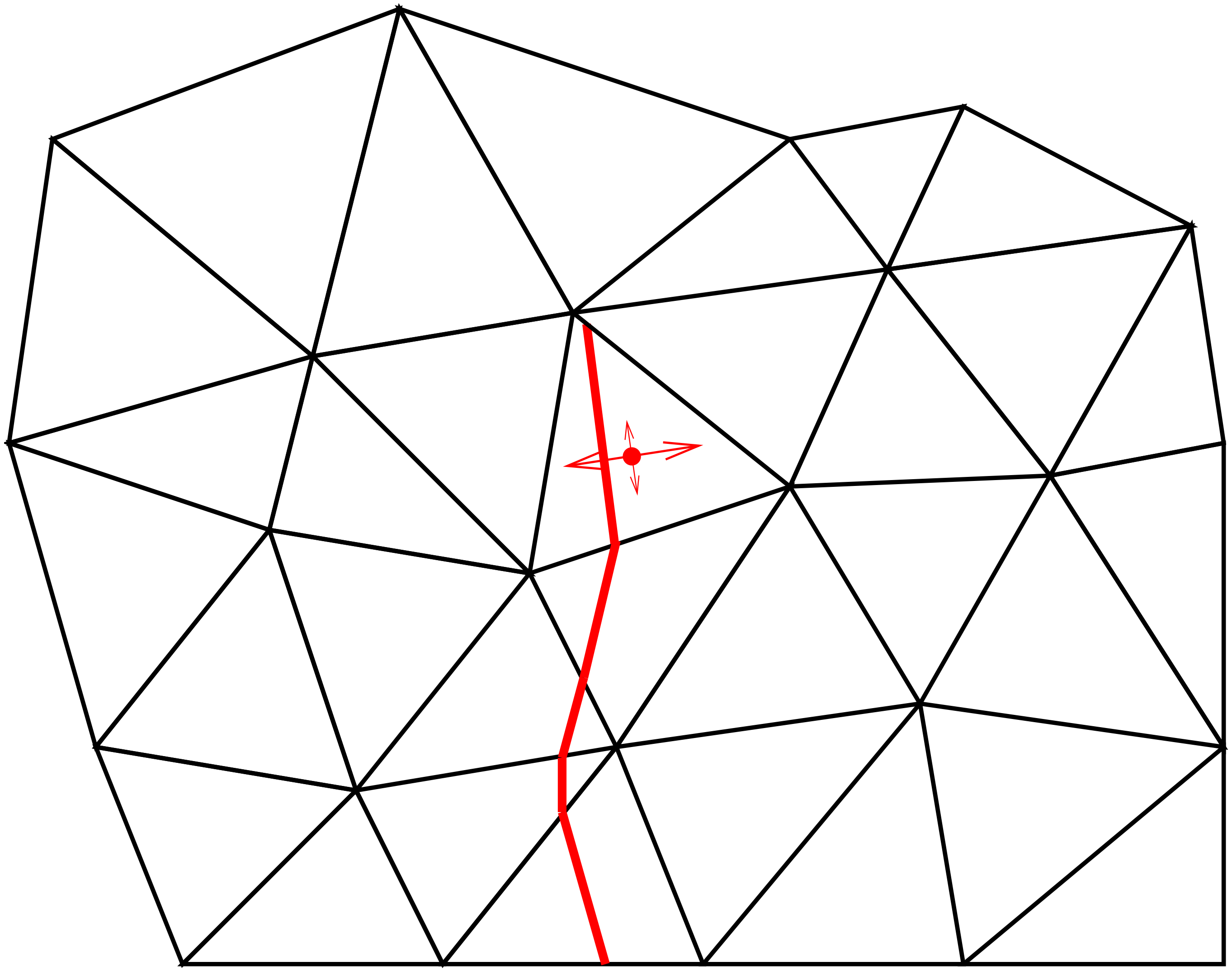
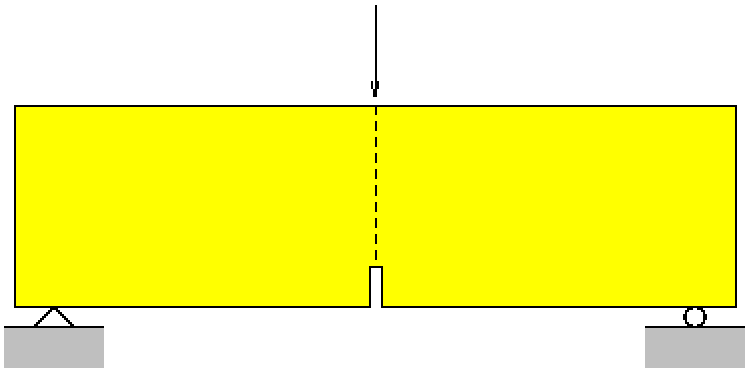


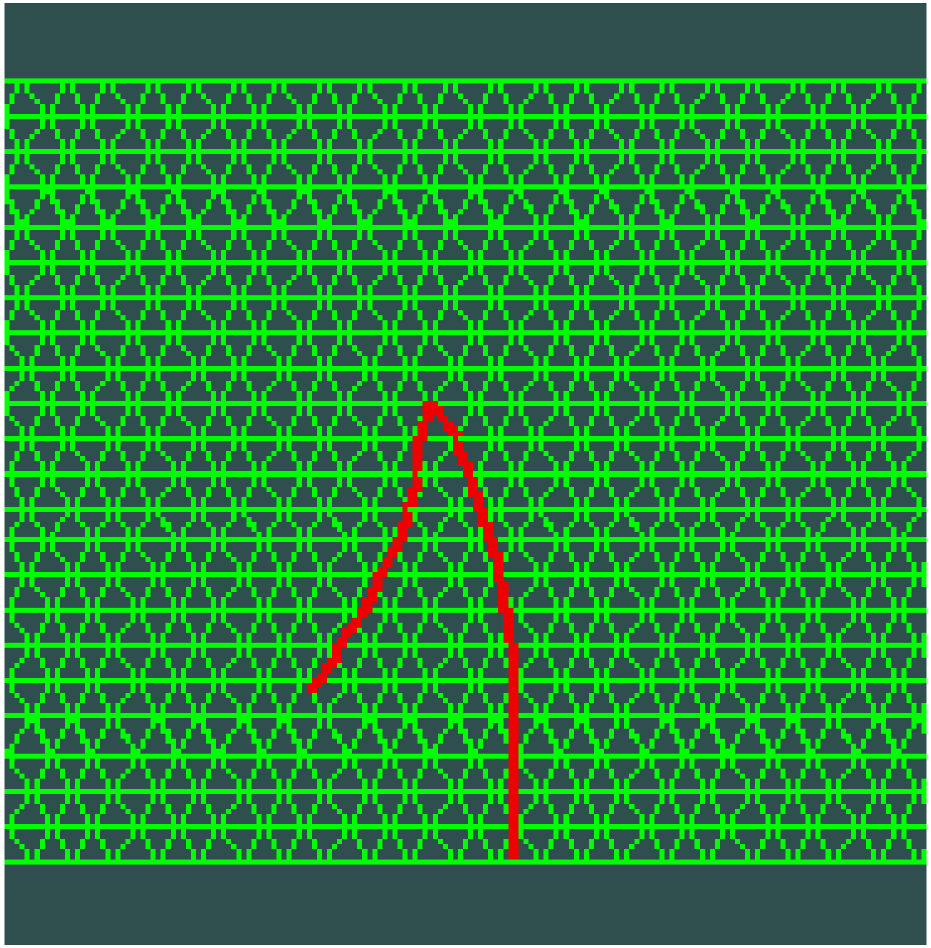
Tracking of a propagating crack



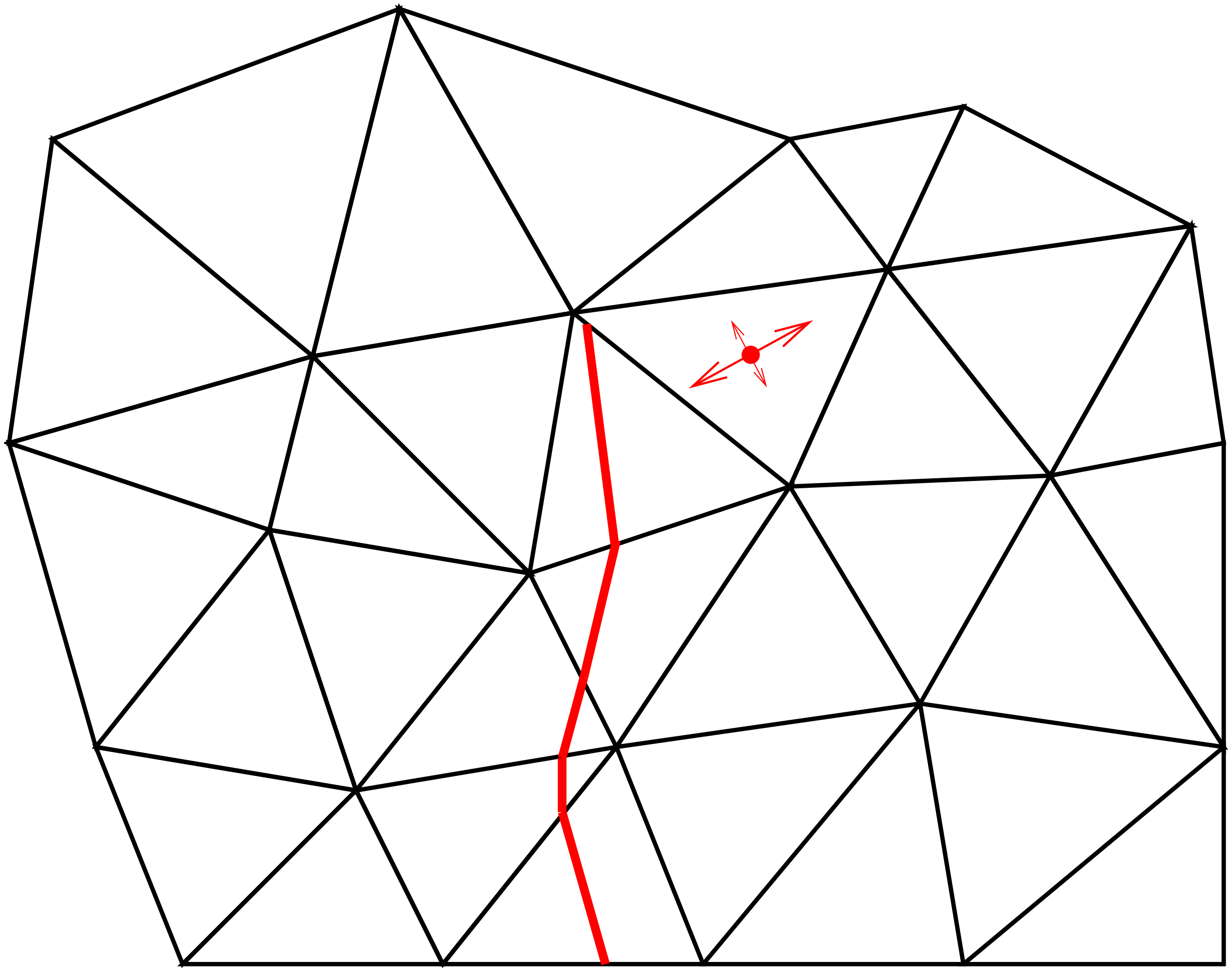
Tracking of a propagating crack

Crack direction = normal to the maximum principal stress direction

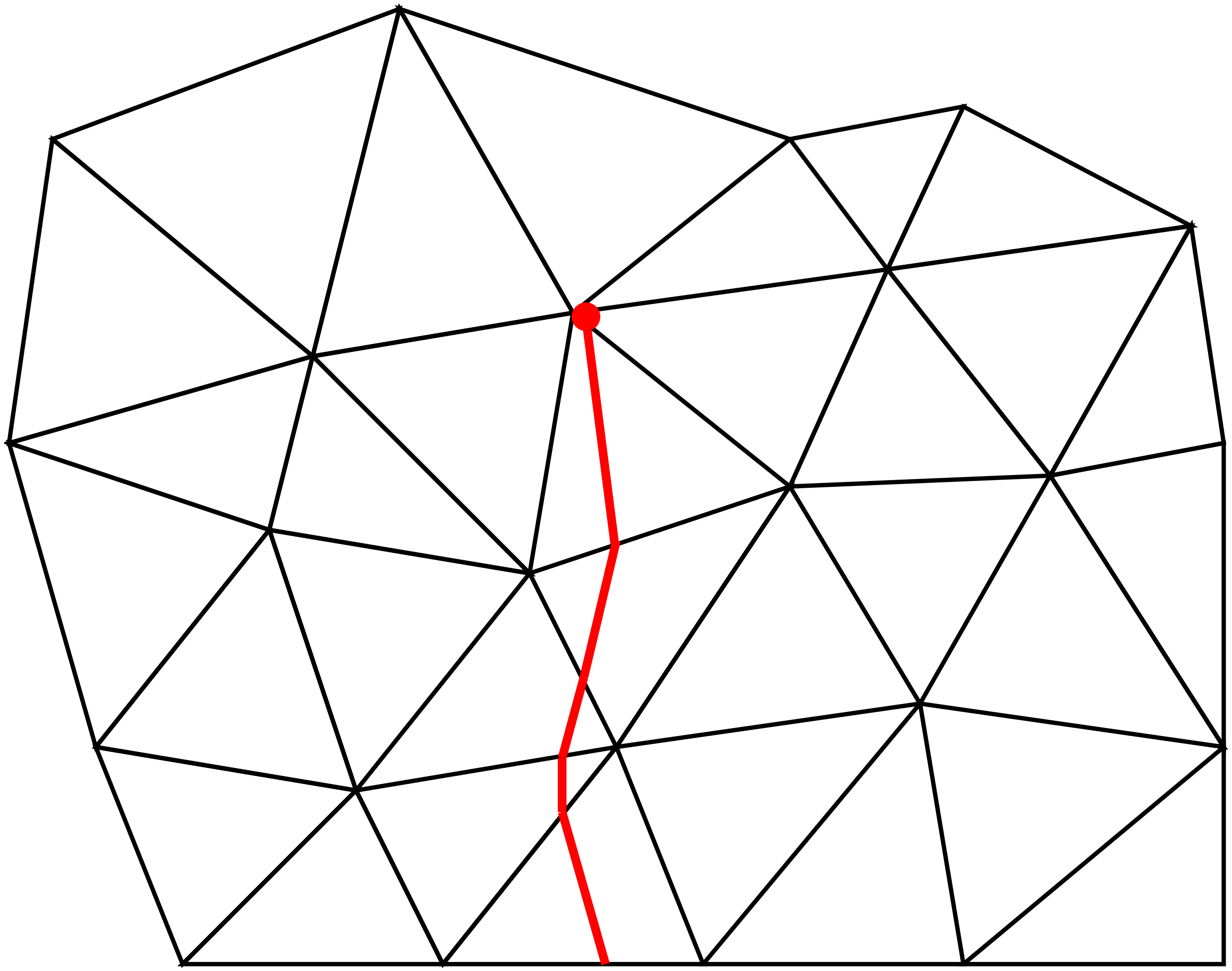




Tracking of a propagating crack

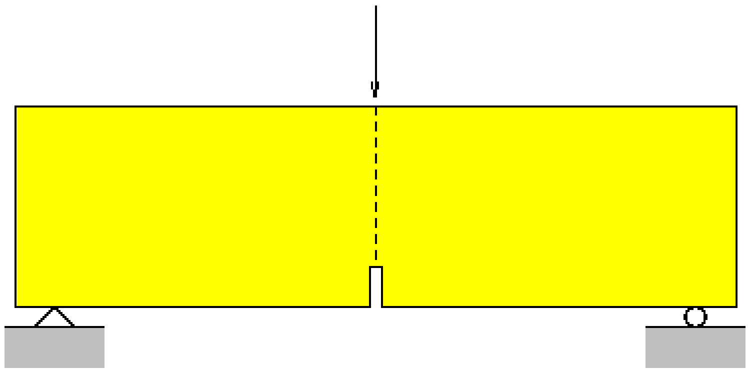


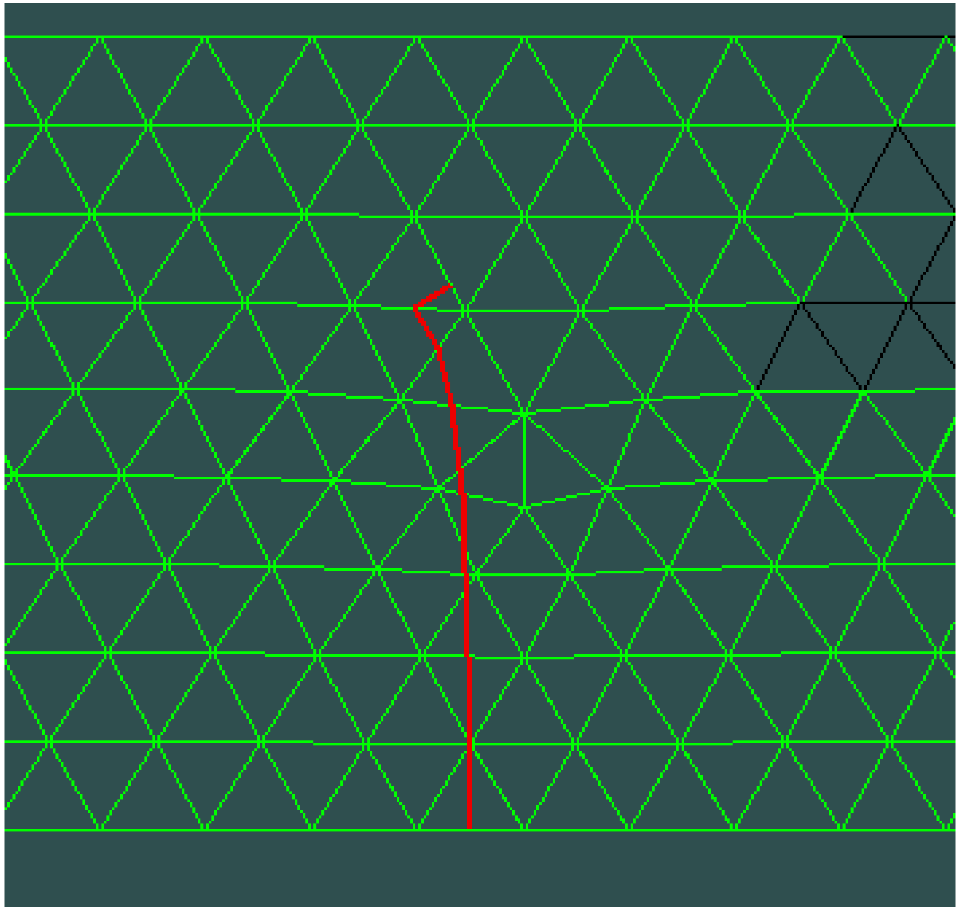
Tracking of a propagating crack



Tracking of a propagating crack

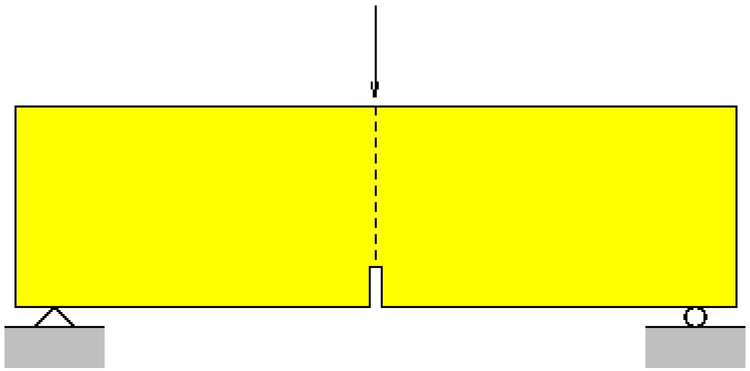
Crack direction = normal to the direction of maximum principal **nonlocal** stress (or strain)



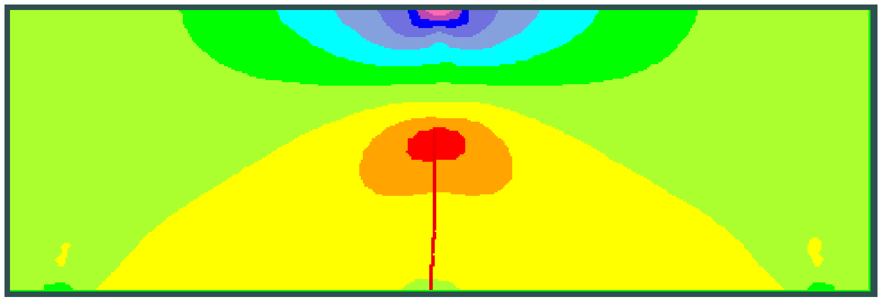


Tracking of a propagating crack

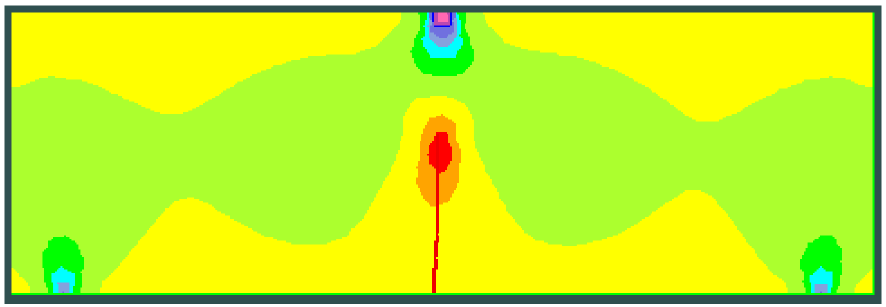
Stress state around the tip of a cohesive crack
is very close to equibiaxial tension



σ_x

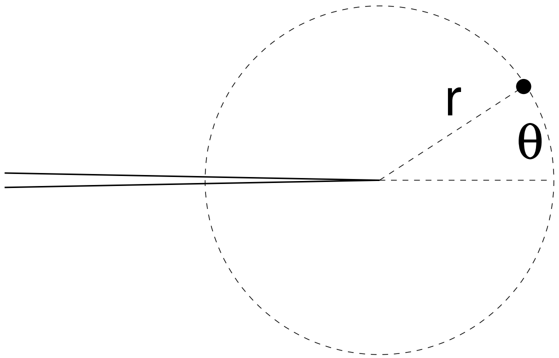


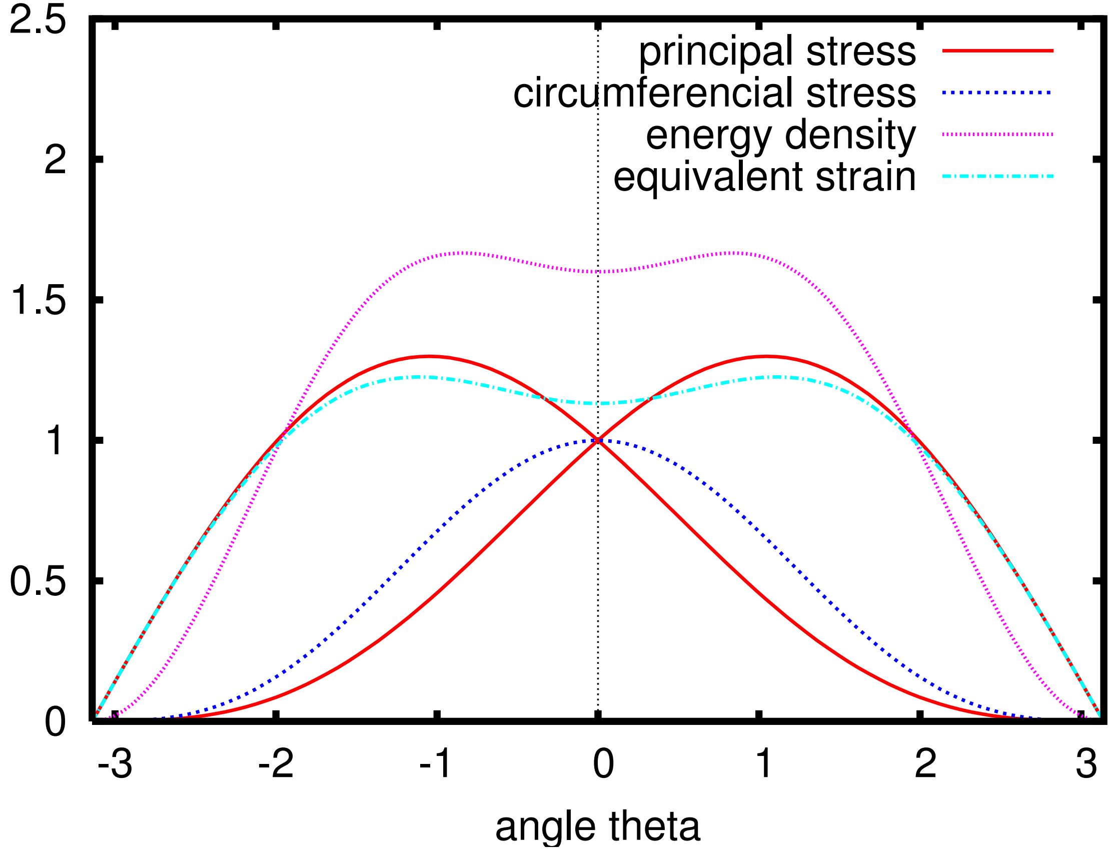
σ_y



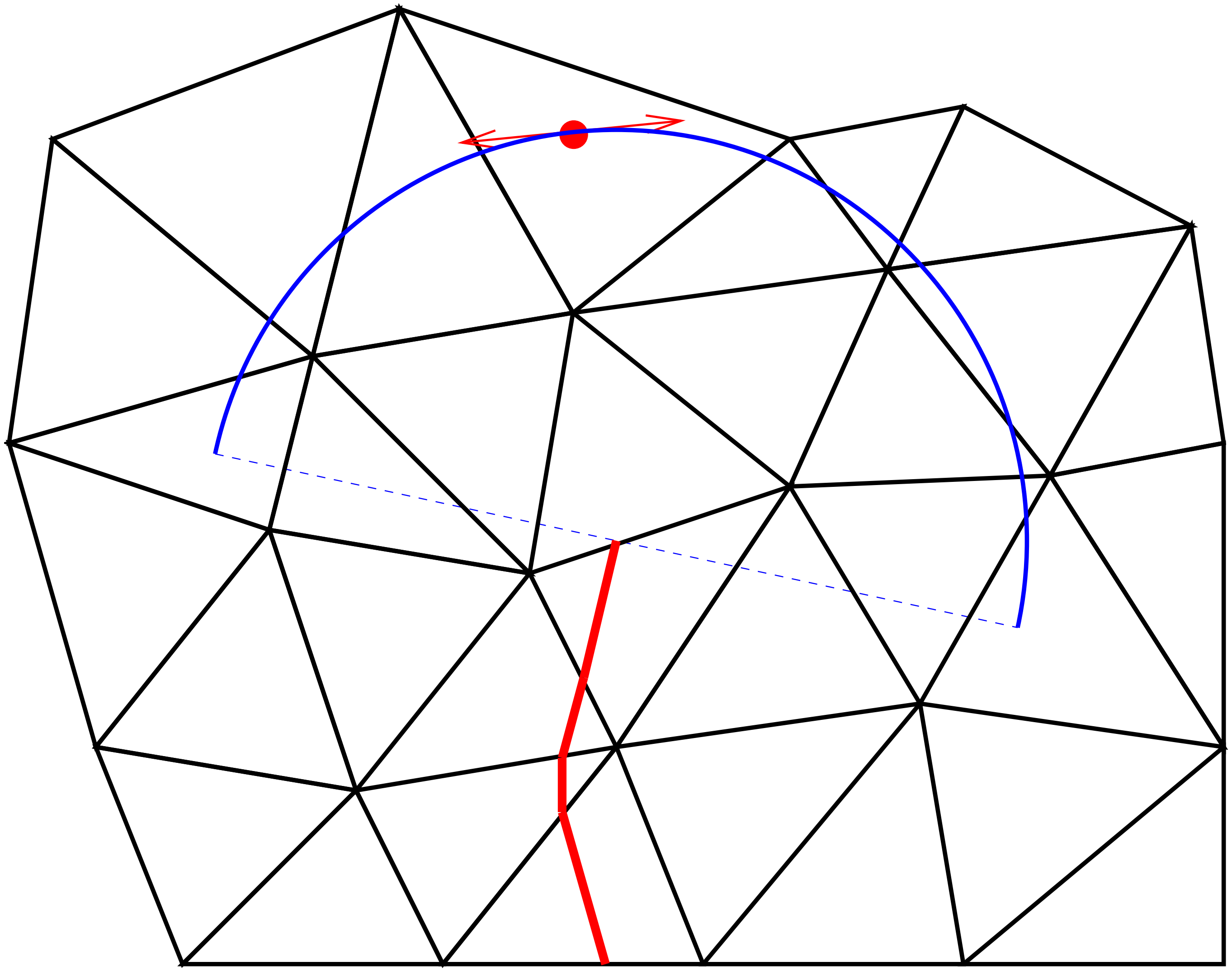
Tracking of a propagating crack

Stress distribution at constant distance from the tip of a stress-free crack

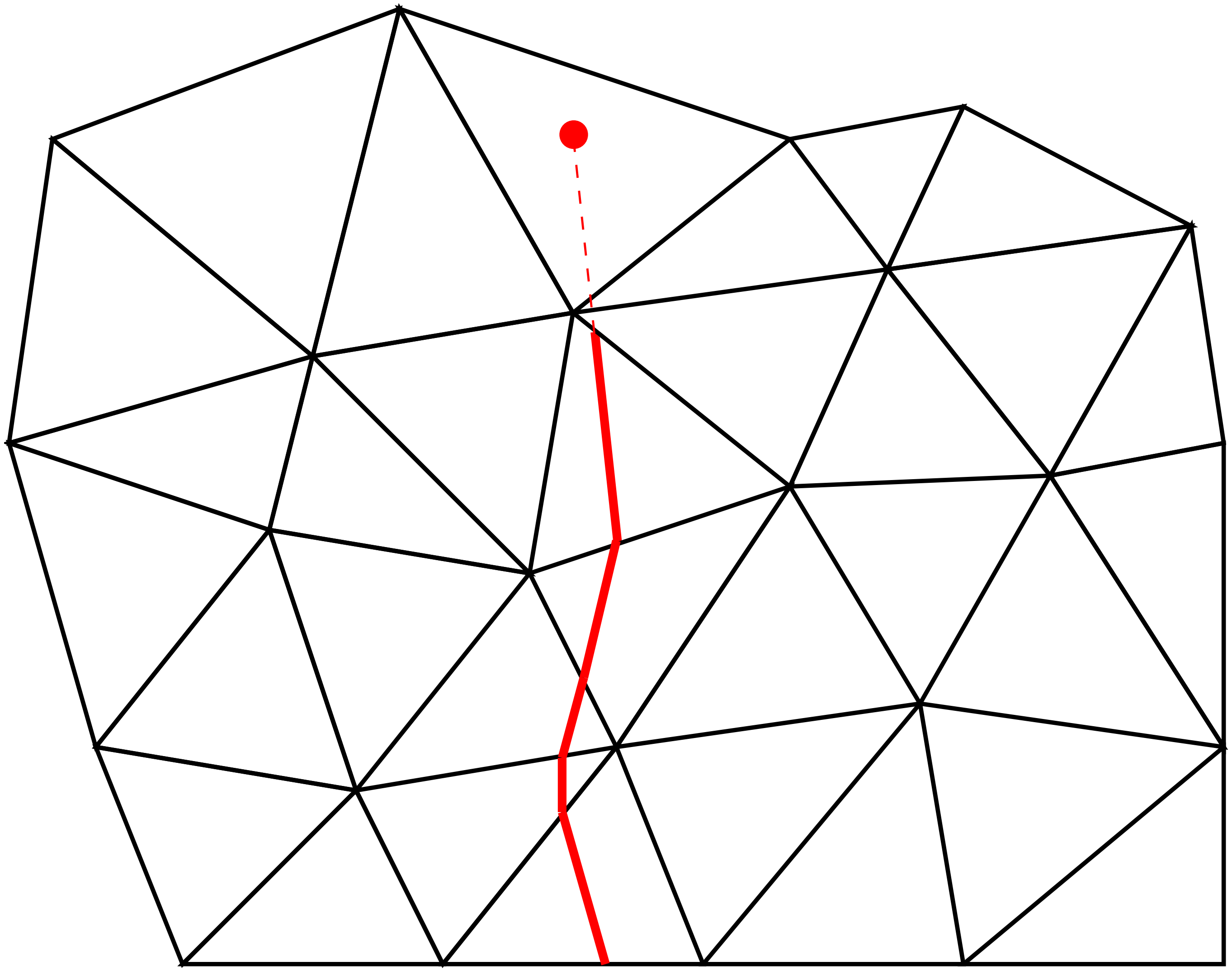




Tracking of a propagating crack

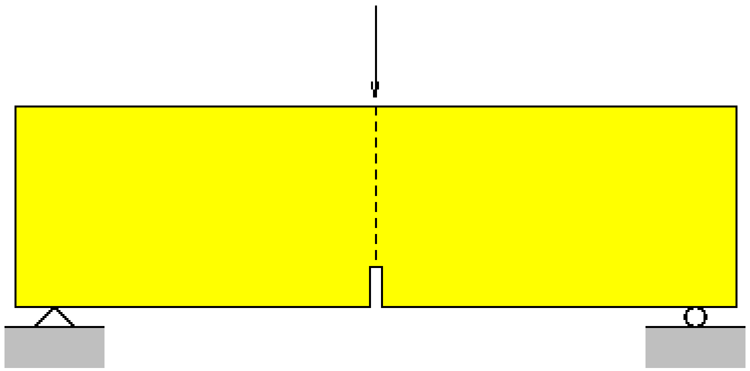


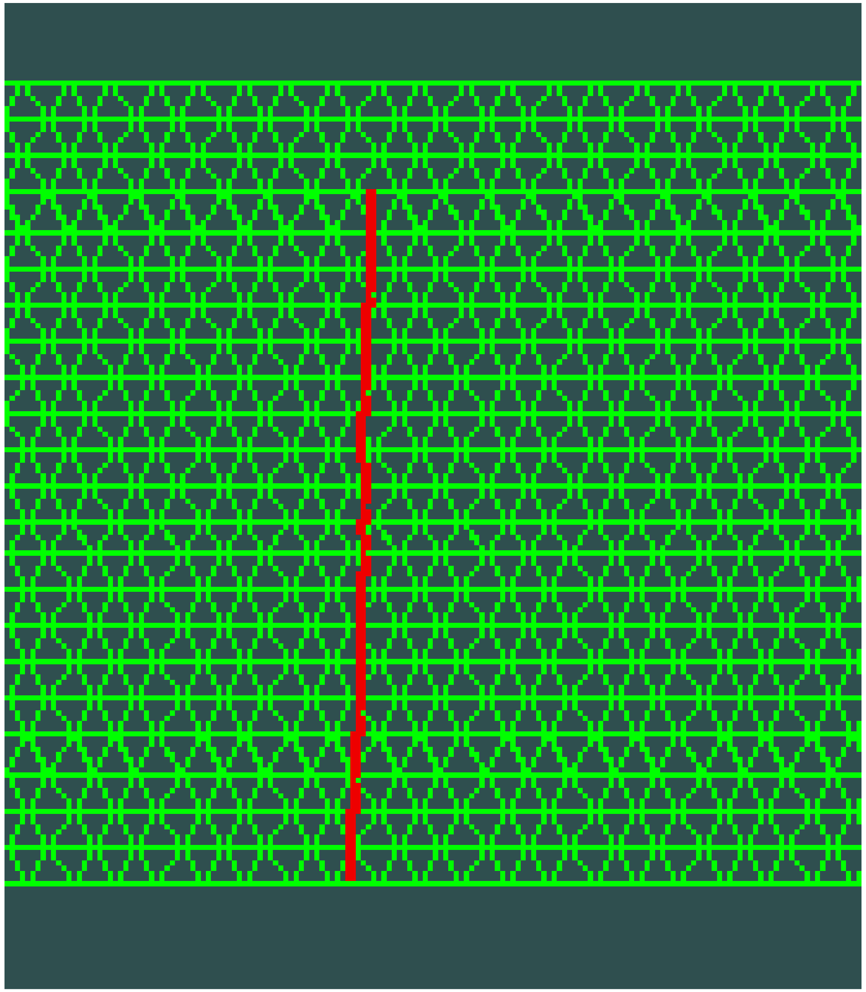
Tracking of a propagating crack



Tracking of a propagating crack

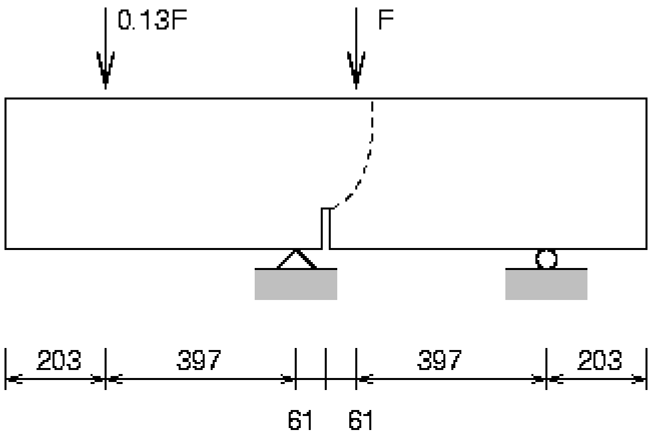
Crack direction = normal to the direction of maximum circumferential stress

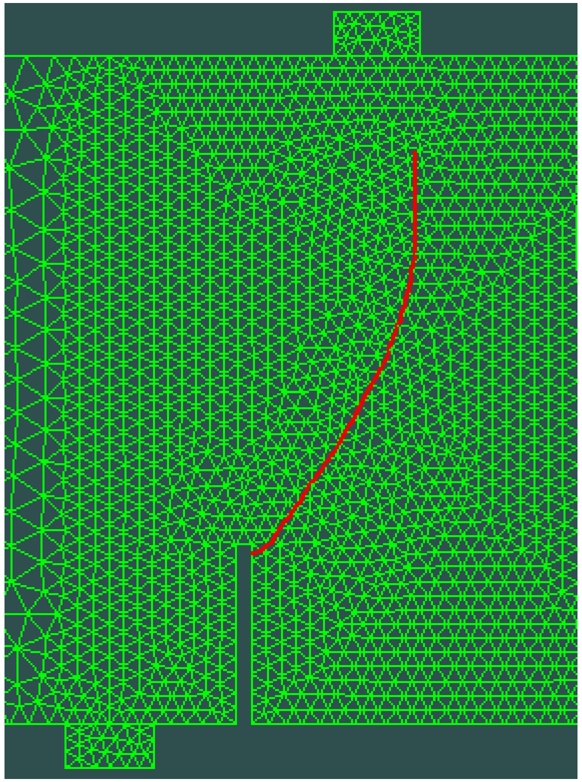




Tracking of a propagating crack

Crack direction = normal to the direction of
maximum circumferential stress





Tracking of a propagating crack

Crack direction = normal to the direction of maximum circumferential stress