Modeling of Localized Inelastic Deformation

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General outline:

A. Introduction
B. Elastoplasticity
C. Damage mechanics
D. Strain localization
E. Regularized continuum models
F. **Strong discontinuity models**

F. Strong discontinuity models

F.1 Fundamentals of fracture mechanics

F.2 Finite elements with discontinuities - introduction

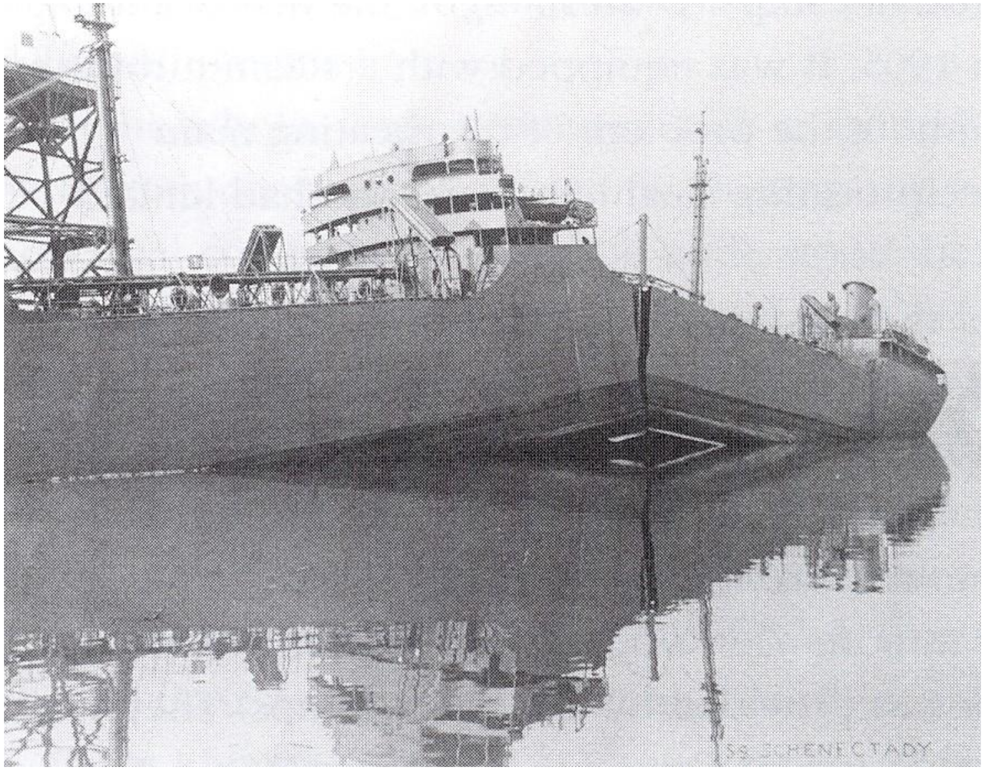
F.3 Embedded discontinuities (EED-EAS)

F.4 Extended finite elements (XFEM-PUM)

F.5 Comparative evaluation

F.6 Regularized continua with strong discontinuities

Failure of Liberty (and other) ships during WW II



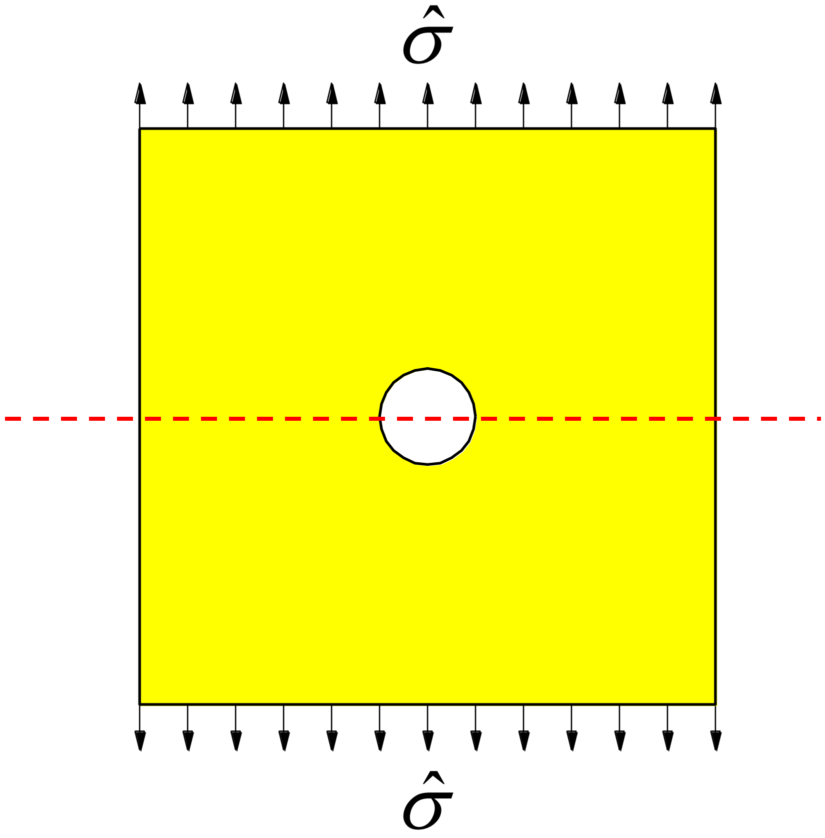
reason:

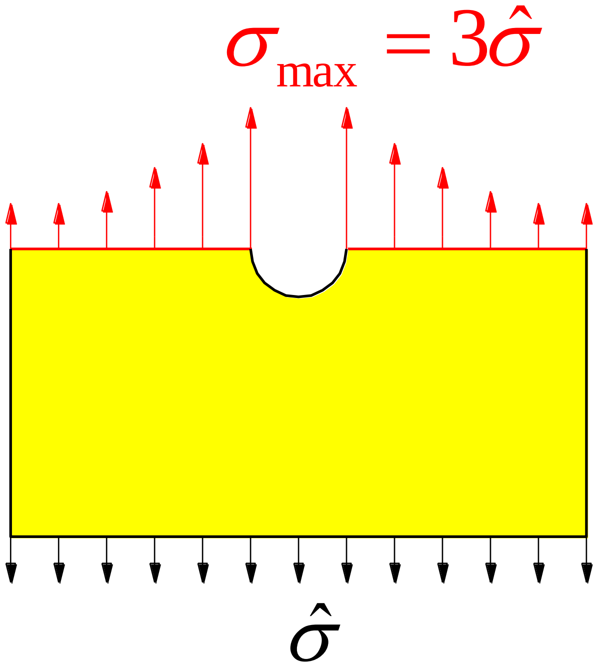
brittle fracture

19 ships broke in half without warning

Stress concentration near defects

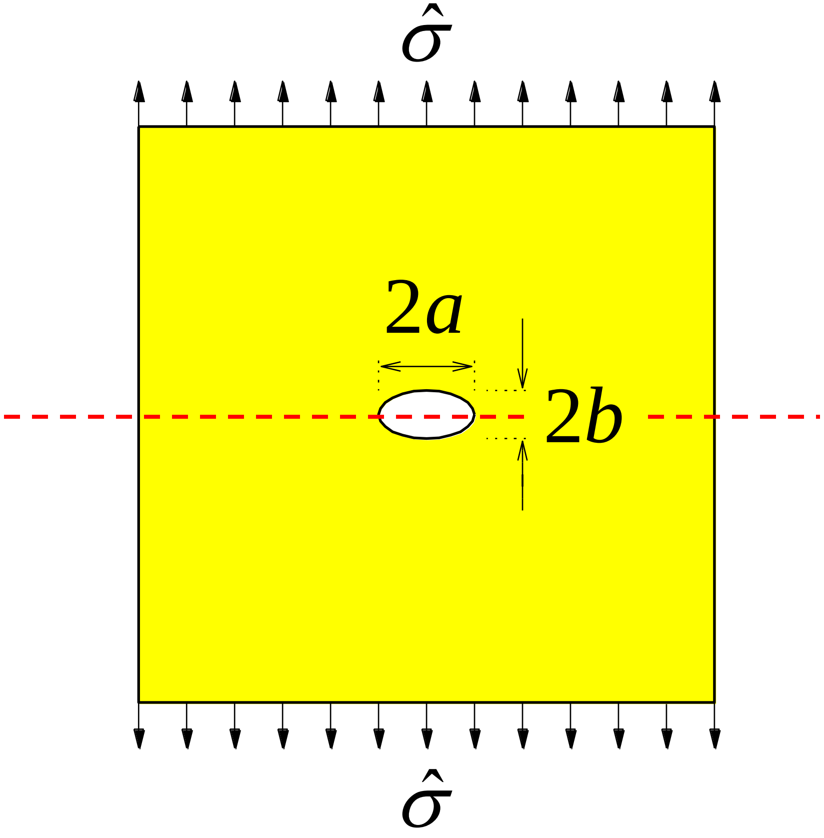
panel weakened by a spherical hole



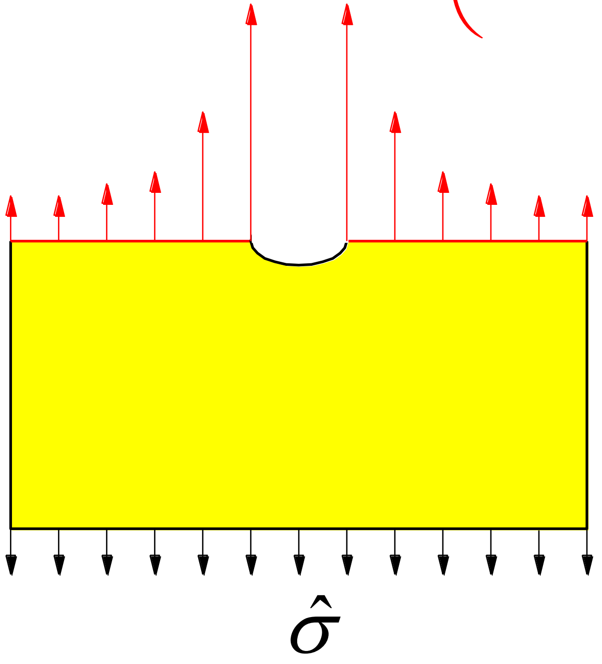


Stress concentration near defects

panel weakened by an eliptical hole

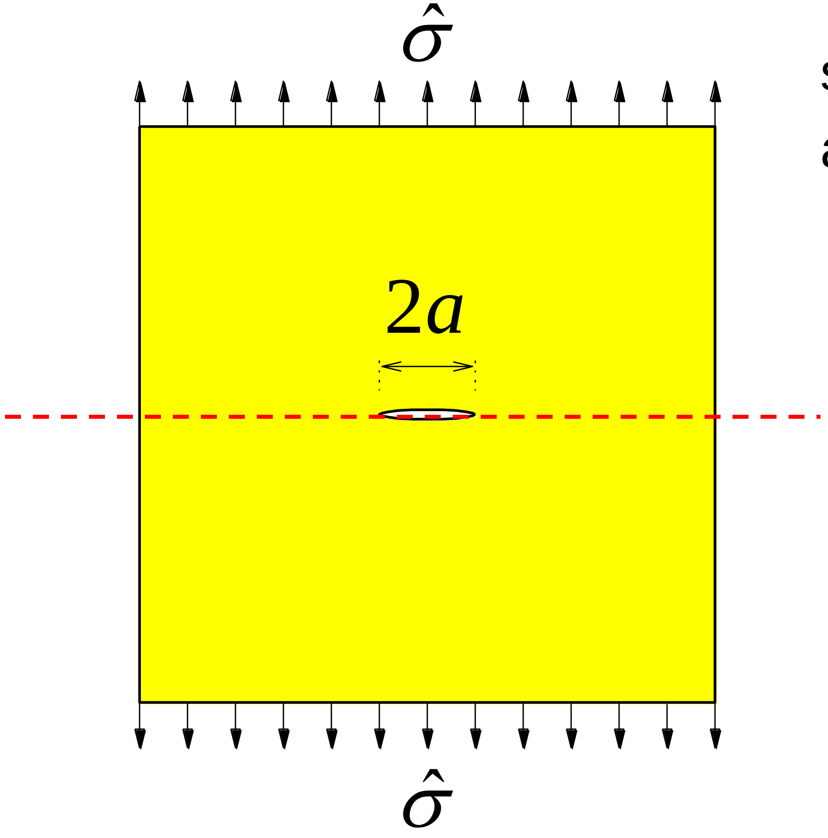


$$\sigma_{\max} = \left(1 + \frac{2a}{b}\right)\hat{\sigma}$$

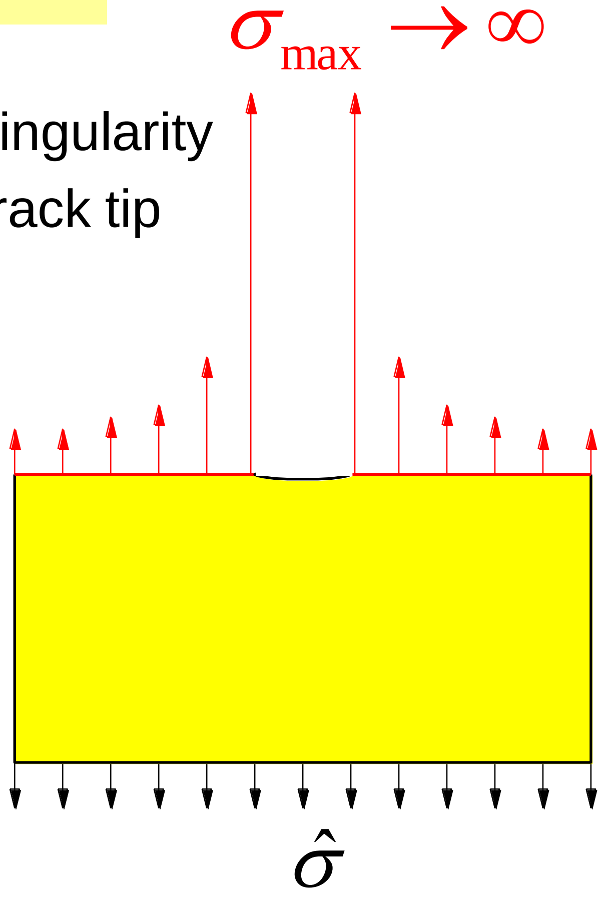


Stress concentration near defects

panel weakened by a crack



stress singularity
at the crack tip

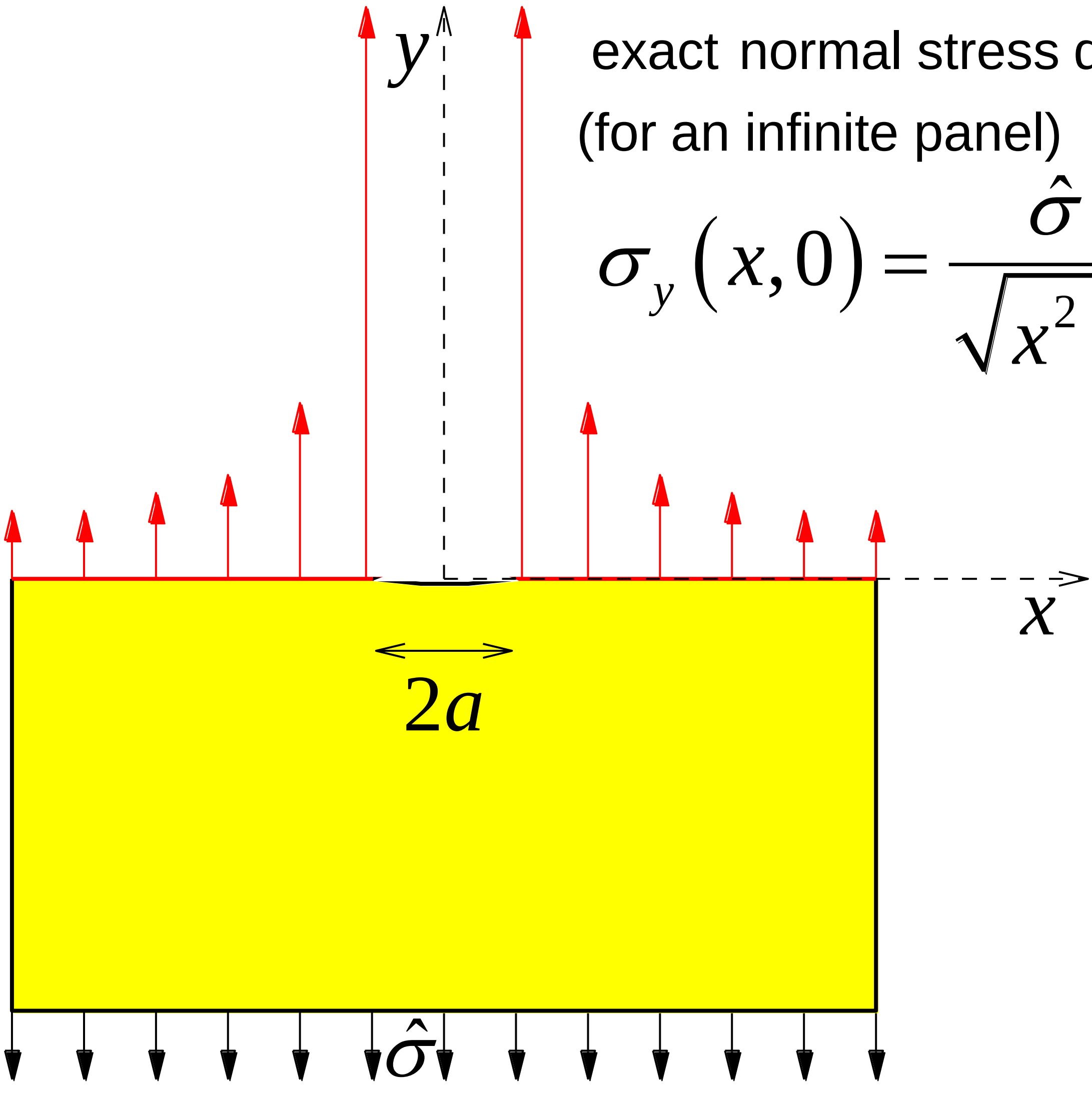


Stress concentration near defects

exact normal stress distribution

(for an infinite panel)

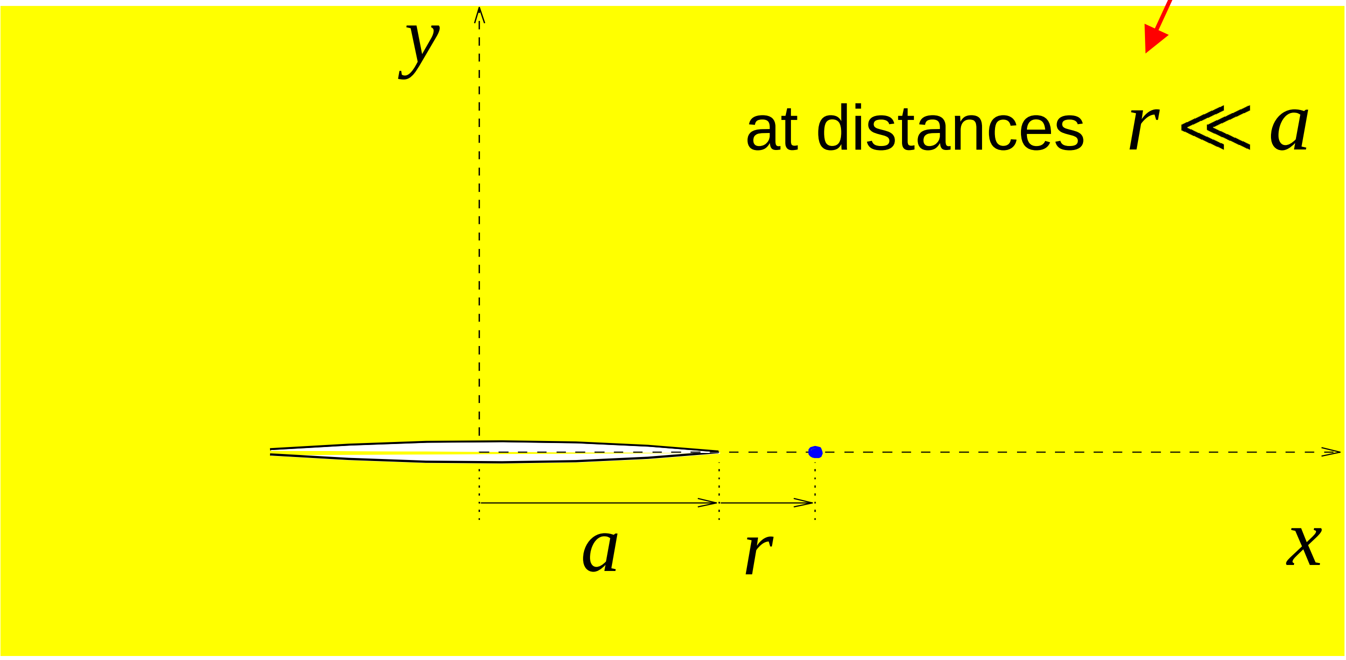
$$\sigma_y(x,0) = \frac{\hat{\sigma} \cdot x}{\sqrt{x^2 - a^2}} \quad \text{for } x > a$$



Singular stress field near the crack tip

exact approximation near the tip

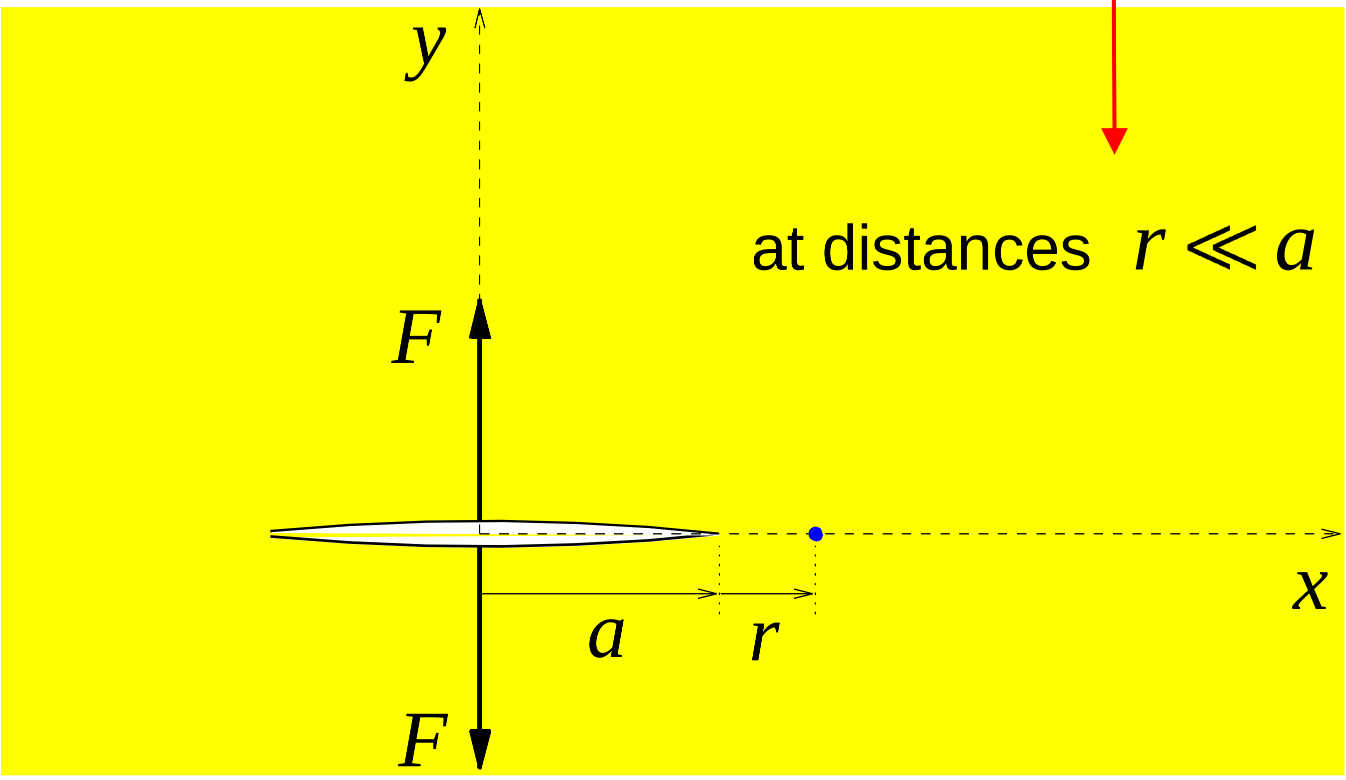
$$\sigma_y(x,0) = \frac{\hat{\sigma}\cdot x}{\sqrt{x^2-a^2}} = \frac{\hat{\sigma}\cdot(a+r)}{\sqrt{(a+r)^2-a^2}} \approx \frac{\hat{\sigma}\cdot a}{\sqrt{2ar}} = \hat{\sigma}\sqrt{\frac{a}{2}}\cdot\frac{1}{\sqrt{r}}$$



stress is inversely proportional to the square root of distance from the crack tip

Singular stress field near the crack tip

$$\sigma_y(x,0) = \frac{Fa}{t\pi x\sqrt{x^2 - a^2}} \approx \frac{F}{t\pi\sqrt{2ar}} = \frac{F}{t\pi\sqrt{2a}} \cdot \frac{1}{\sqrt{r}}$$



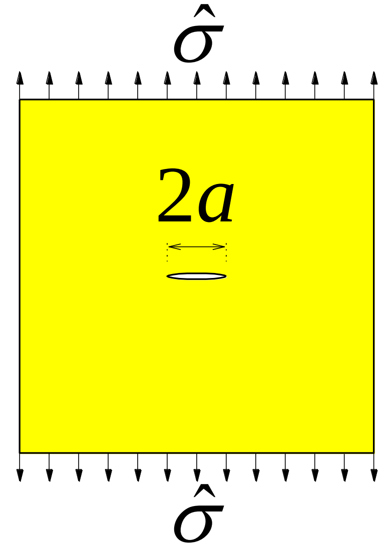
at distances $r \ll a$

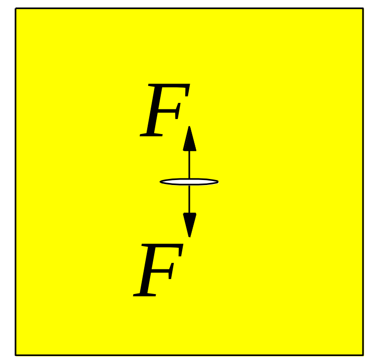
stress is inversely proportional to the square root of distance from the crack tip

Singular stress field near the crack tip

$2a = 20 \text{ mm}$

$\hat{\sigma} = 10 \text{ MPa}$





$2a = 30 \text{ mm}$

$t = 5 \text{ mm}$

$F = 1.9238 \text{ kN}$

near-tip stress approximation

$$\boxed{\hat{\sigma}\sqrt{\frac{a}{2}}} \cdot \frac{1}{\sqrt{r}} \qquad \boxed{\frac{F}{t\pi\sqrt{2a}}} \cdot \frac{1}{\sqrt{r}}$$

same value of this factor

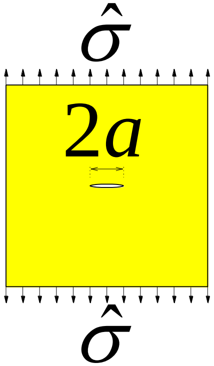
same stress concentration near the tip

Singular stress field near the crack tip

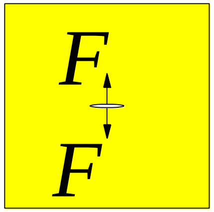
general expression for the singular part of stress field
that dominates near the crack tip

$$\boxed{\sigma_y(x,0) \approx \frac{K_I}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{r}}}$$

K_I ... stress intensity factor



$$\sigma_y(x,0) \approx \hat{\sigma}\sqrt{\frac{a}{2}} \cdot \frac{1}{\sqrt{r}} \quad \ldots \quad K_I = \hat{\sigma}\sqrt{\pi a}$$

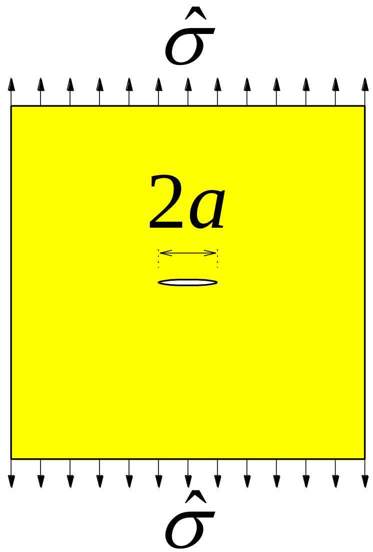


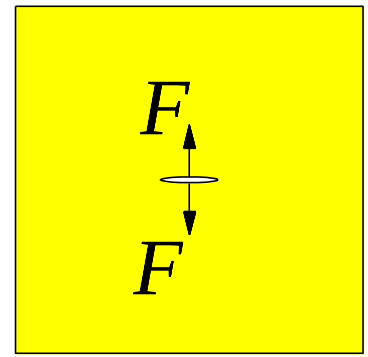
$$\sigma_y(x,0) \approx \frac{F}{t\pi\sqrt{2a}} \cdot \frac{1}{\sqrt{r}} \quad \ldots \quad K_I = \frac{F}{t\sqrt{\pi a}}$$

Singular stress field near the crack tip

$$2a = 20\ \text{mm}$$

$$\hat{\sigma} = 10\ \text{MPa}$$





$$2a = 30\ \text{mm}$$

$$t = 5\ \text{mm}$$

$$F = 1.9238\ \text{kN}$$

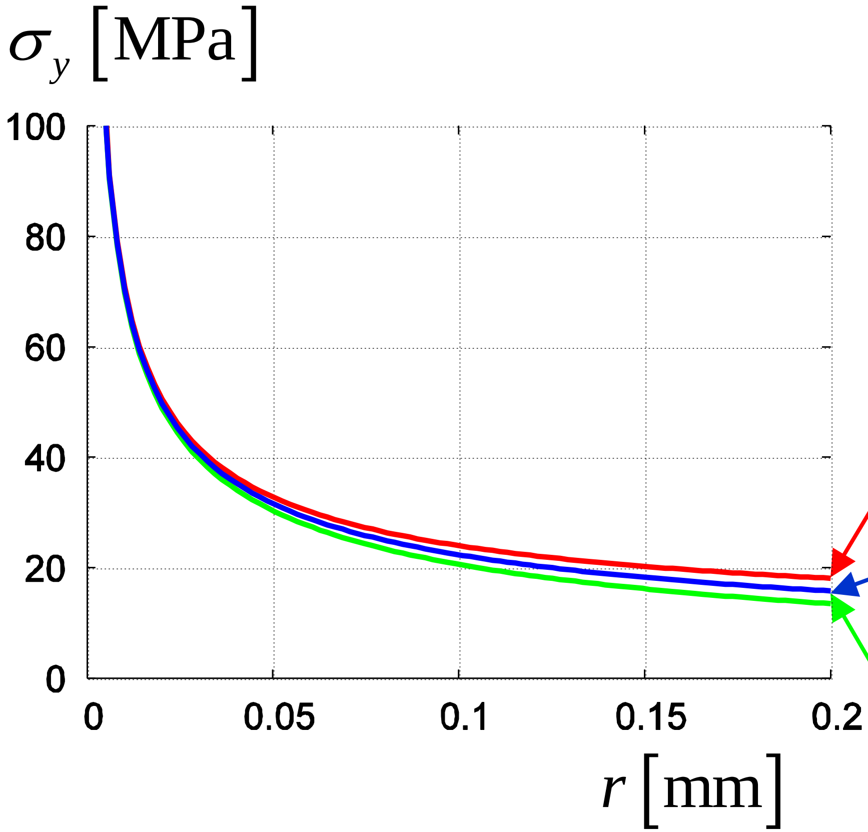
$$K_I = \hat{\sigma}\sqrt{\pi a} = 1{,}772 \cdot 10^6\ \text{Nm}^{-3/2}$$

$$K_I = \frac{F}{t\sqrt{\pi a}} = 1{,}772 \cdot 10^6\ \text{Nm}^{-3/2}$$

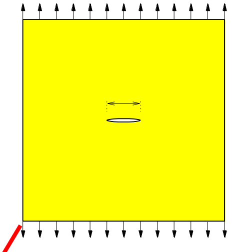
same stress intensity factor

⇒ same stress concentration near the tip

Singular stress field near the crack tip



$a_1 = 10$ mm, $\hat{\sigma} = 10$ MPa

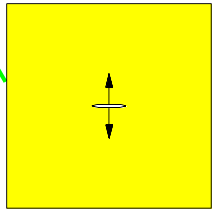


$$\frac{\hat{\sigma} \cdot (a_1 + r)}{\sqrt{(a_1 + r)^2 - a_1^2}}$$

asymptotic stress field

$$\frac{K_I}{\sqrt{2\pi r}}$$

$K_I = 1.772 \text{ MNm}^{-3/2}$



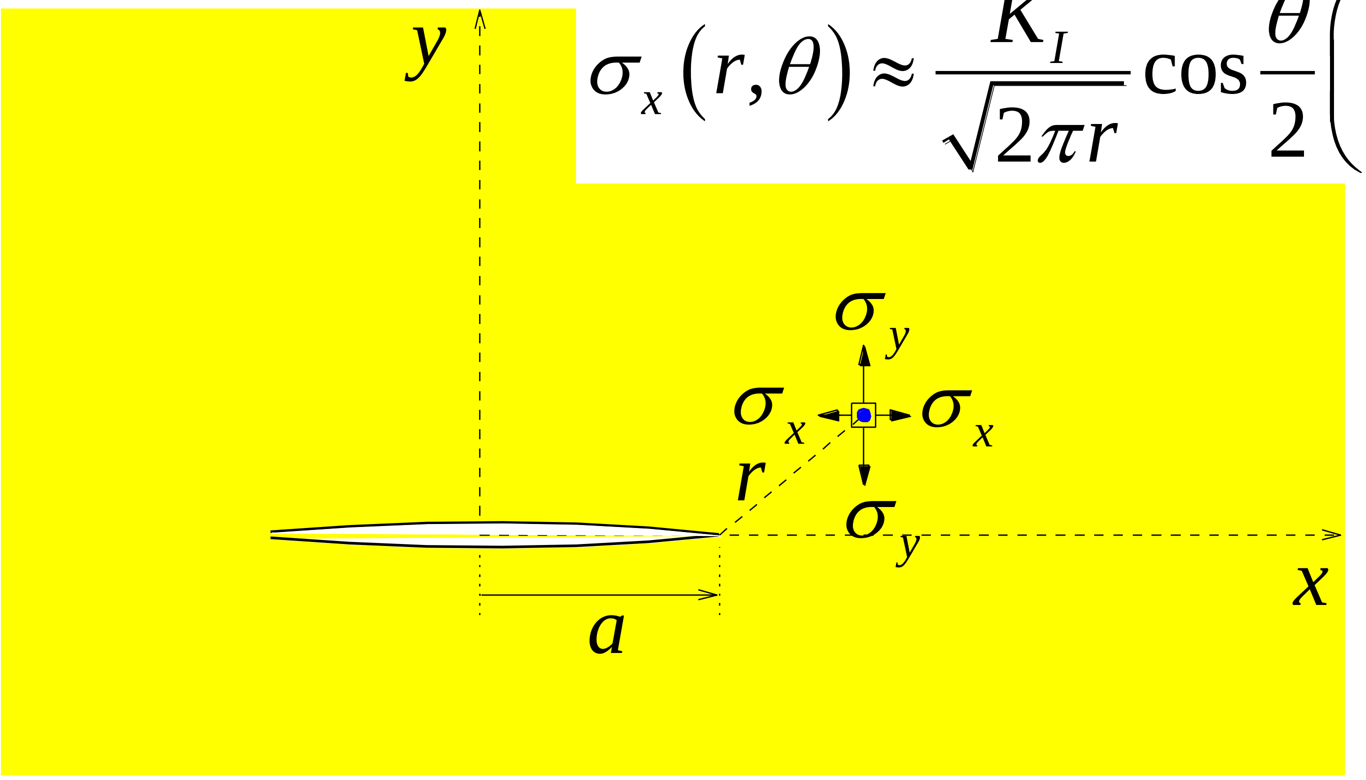
$$\frac{Fa_2}{t\pi (a_2 + r)\sqrt{(a_2 + r)^2 - a_2^2}}$$

$a_2 = 15$ mm, $F/t = 385$ kN/m

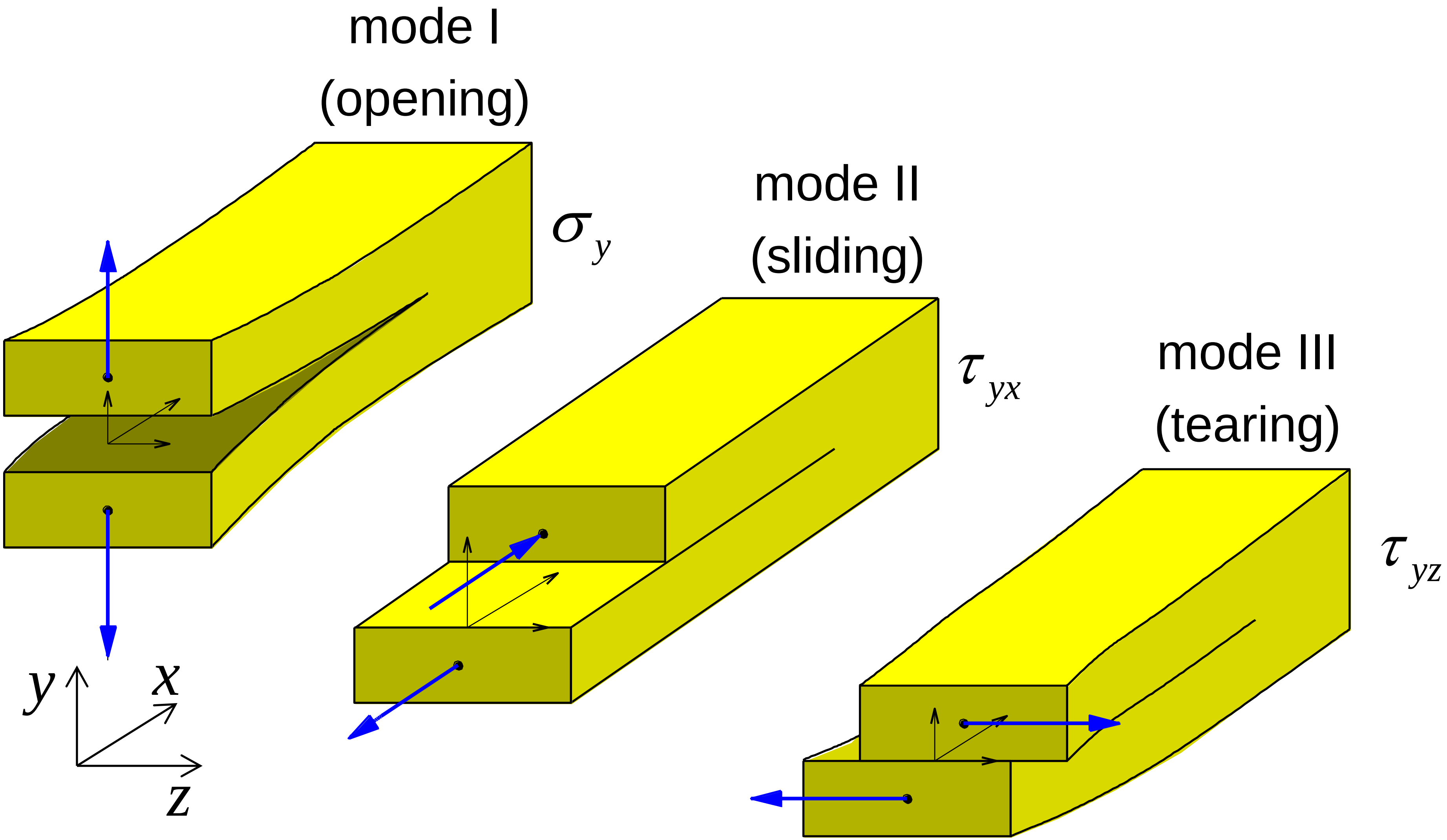
Singular stress field near the crack tip

$$\sigma_y(r,\theta) \approx \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1+\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)$$

$$\sigma_x(r,\theta) \approx \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)$$



Basic fracture modes



Near-tip asymptotic fields

crack loaded in a mixed mode (combination of modes I and II):

$$\sigma_y(r,\theta) \approx \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1+\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) - \frac{K_{II}}{\sqrt{2\pi r}} \sin\frac{\theta}{2}\left(2+\cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right)$$

$$\sigma_x(r,\theta) \approx \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) + \frac{K_{II}}{\sqrt{2\pi r}} \sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2}$$

$$\tau_{xy}(r,\theta) \approx \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{3\theta}{2} + \frac{K_{II}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)$$

Crack propagation – Irwin (local) criterion

A crack loaded in mode I propagates
if the stress intensity factor at its tip attains a critical value:

$$K_I = K_c$$

stress intensity factor
(depends on loading,
shape and dimensions
of the body
and on the crack size)

fracture toughness
(material property)

$$\left[\mathrm{Nm}^{-3/2} \right]$$

Crack propagation – Griffith (global) criterion

A crack loaded in mode I propagates
if its propagation releases a critical amount of energy:

$$\boxed{\mathcal{G} = G_f}$$

energy release rate
(depends on loading,
shape and dimensions
of the body
and on the crack size)

fracture energy
(material property)

$$\left[\mathrm{J/m^2} \equiv \mathrm{N\,/\,m} \right]$$

Crack propagation criteria

crack propagates if

$$K_I = K_c \qquad\qquad \mathcal{G} = G_f$$

local (Irwin) criterion

global (Griffith) criterion

for plane stress and mode I loading it can be shown that

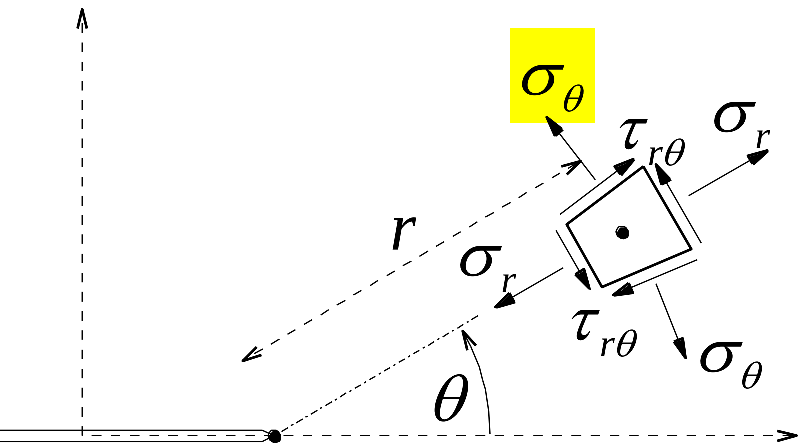
$$\boxed{\mathcal{G} = \frac{K_I^2}{E}}$$

the above criteria are then equivalent and the fracture tougness and fracture energy are linked by

$$G_f = \frac{K_c^2}{E} \qquad\qquad K_c = \sqrt{EG_f}$$

Direction of crack propagation

for mode I loading, the crack can be expected to propagate straight ahead, but for general mixed-mode loading we need a criterion for the crack direction



maximum circumferential stress criterion (maximum hoop stress criterion):

crack propagates in the direction perpendicular to the maximum circumferential stress

(evaluated on a circle of a small diameter centered at the tip)

the direction of propagation is given by the angle θ_c for which

$$\sigma_\theta(r, \theta_c) = \max_{-\pi<\theta<\pi} \sigma_\theta(r, \theta)$$

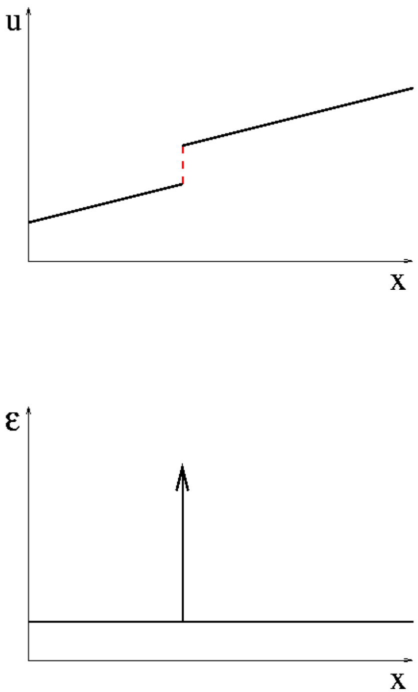
F.2

Finite elements with discontinuities:

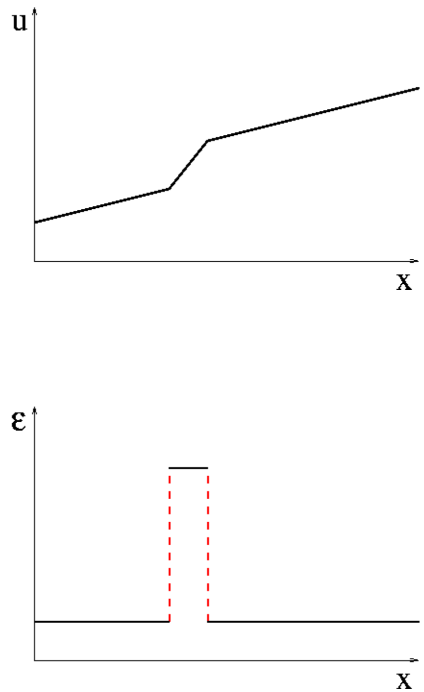
Introduction

Classification of models: kinematic aspects

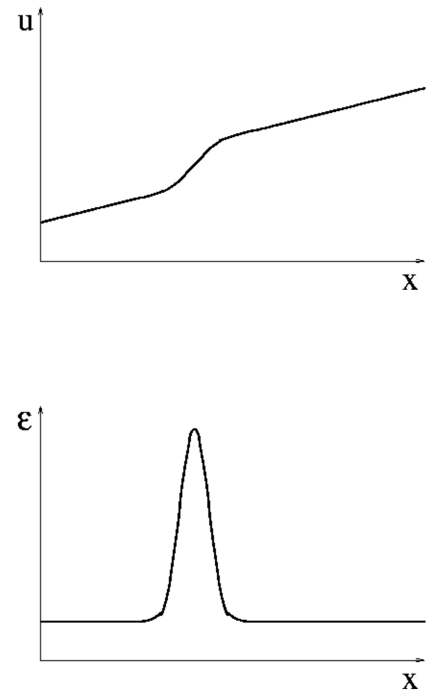
Strong
discontinuity



Weak
discontinuity

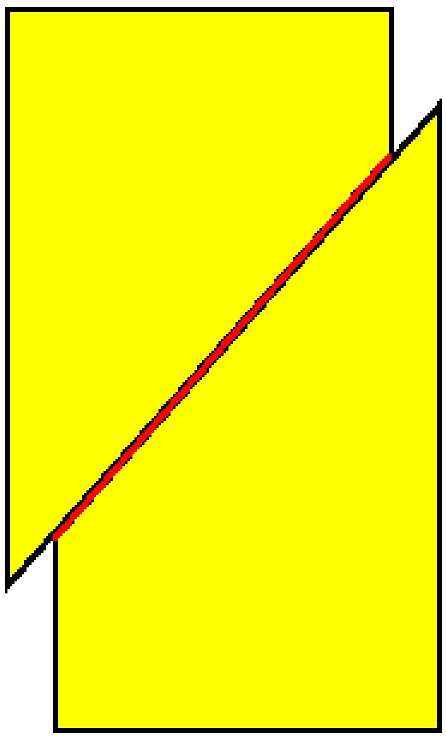


Regularized
localization zone

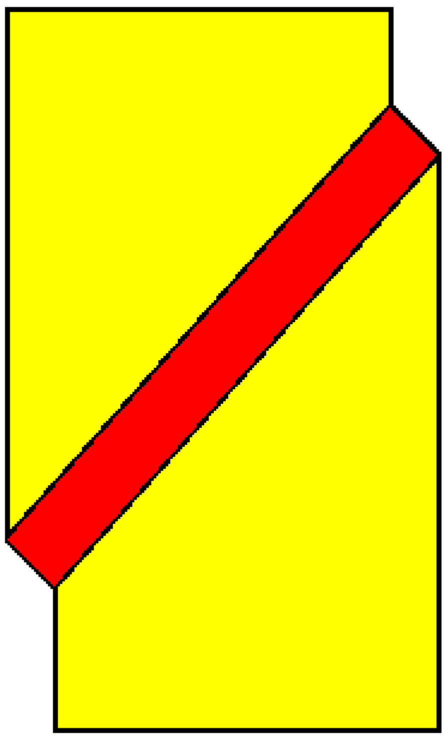


Classification of models: kinematic aspects

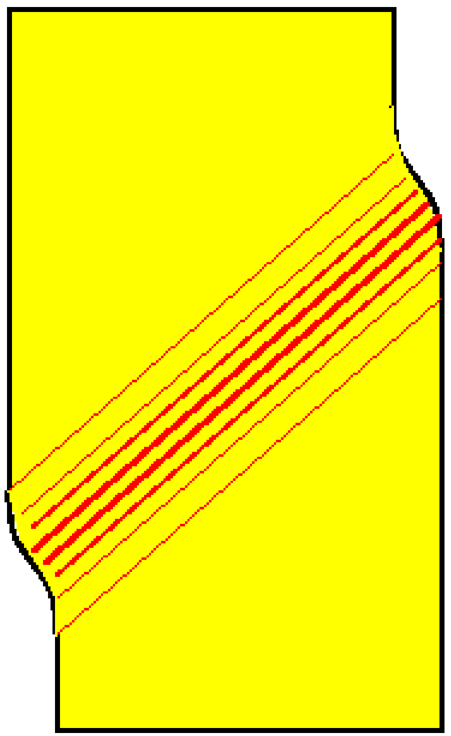
Strong
discontinuity



Weak
discontinuity

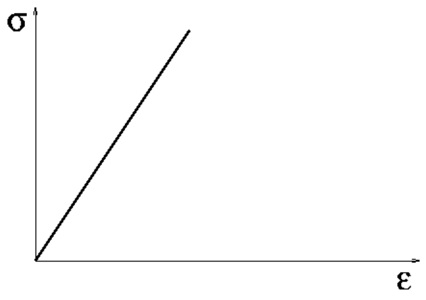


Regularized
localization zone

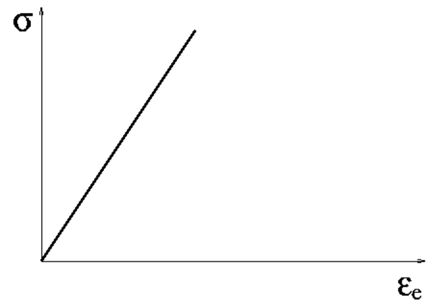


Classification of models: material laws

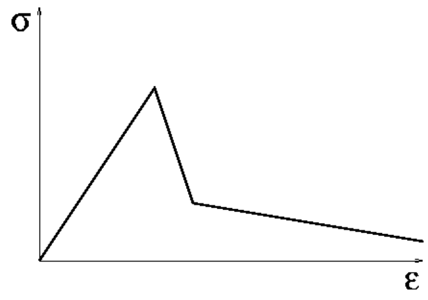
Stress-strain law



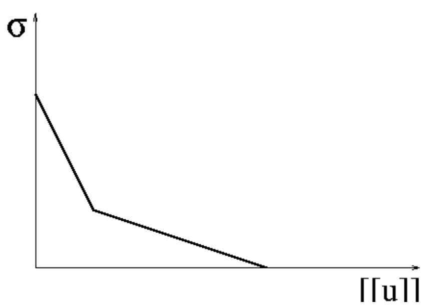
Stress-strain law
(pre-localization part)



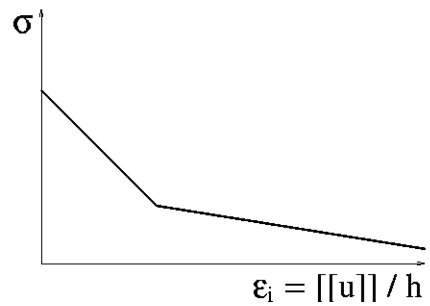
Stress-strain law



Traction-separation law



Stress-strain law
(post-localization part)

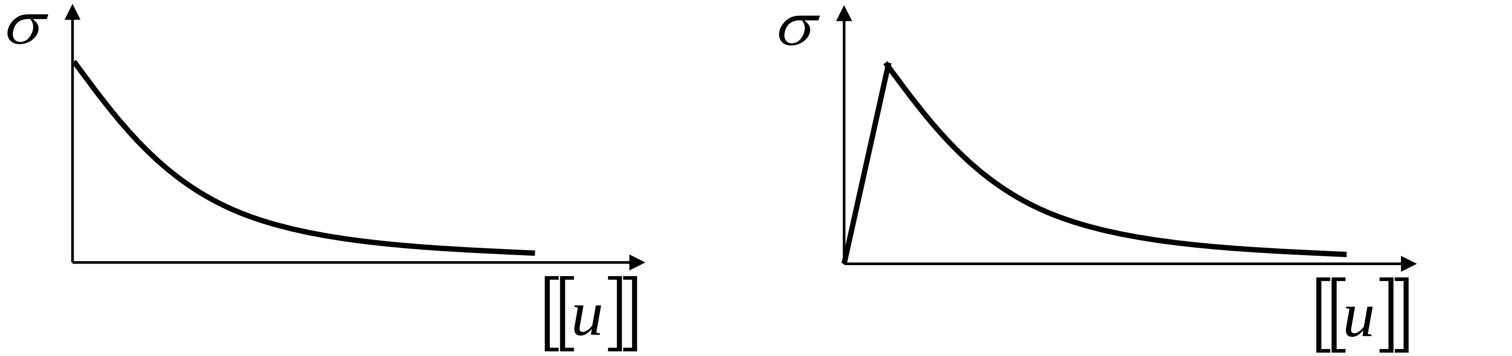


Enrichment acting
as localization limiter:

- nonlocal
- gradient
- Cosserat
- viscosity

Traction-separation laws

1) Formulated directly in the traction-separation space
   a) with nonzero elastic compliance (elasto-plastic, …)
   b) with zero elastic compliance (rigid-plastic, …)



For general applications, we need a link between the separation **vector** (displacement jump vector) and the traction **vector**:

$$[\![\mathbf{u}]\!] \longrightarrow \mathbf{t}$$

Traction-separation laws

2) "Derived" from a stress-strain law (softening continuum) using the strong discontinuity approach

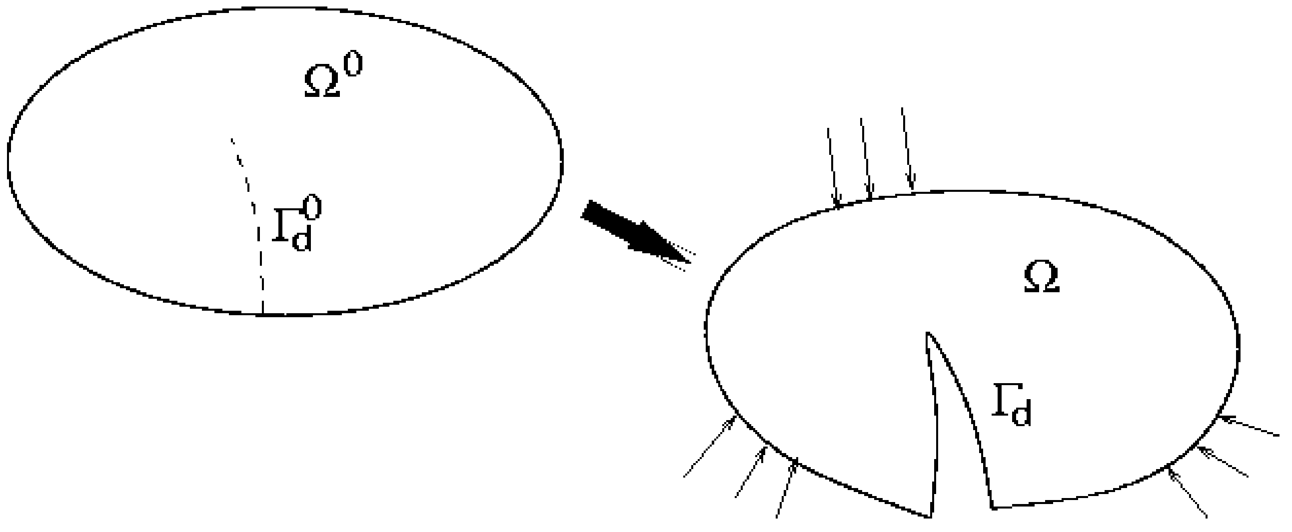
$$[\![\mathbf{u}]\!]$$

$$\mathbf{t} = \mathbf{n} \cdot \boldsymbol{\sigma}$$

$$\boldsymbol{\varepsilon} = \frac{1}{h}\left(\left[\![\mathbf{u}]\!\right] \otimes \mathbf{n}\right)_{sym}$$

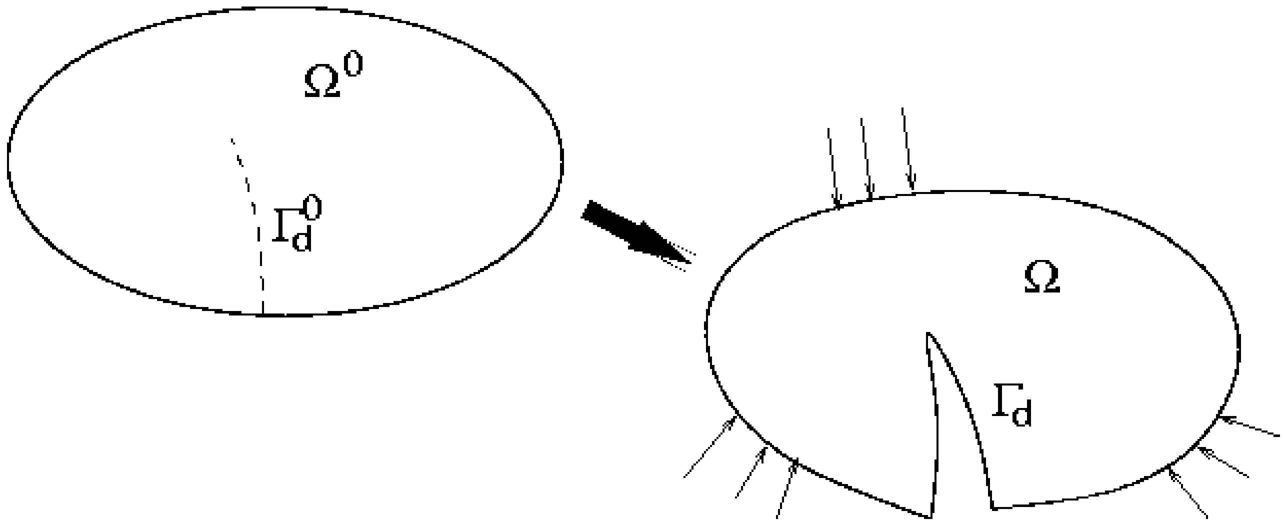
$$\boldsymbol{\sigma} = \tilde{\boldsymbol{\sigma}}\left(\boldsymbol{\varepsilon}, \ldots; h\right)$$

Finite element representation of strong discontinuities



1) Discontinuities at element interfaces:
   a) Remeshing
   b) Interspersed potential discontinuities

Finite element representation of strong discontinuities



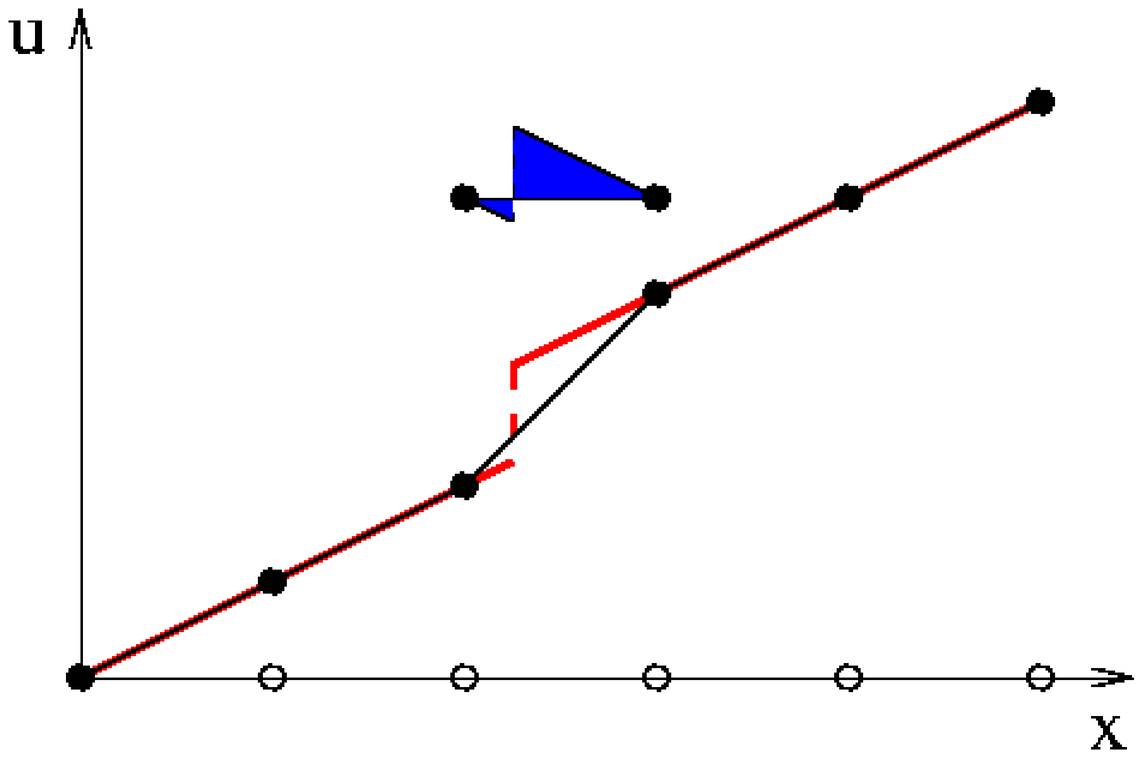
2) Arbitrary discontinuities across elements:

   a) Elements with embedded discontinuities using the enhanced assumed strain formulation (EED-EAS)

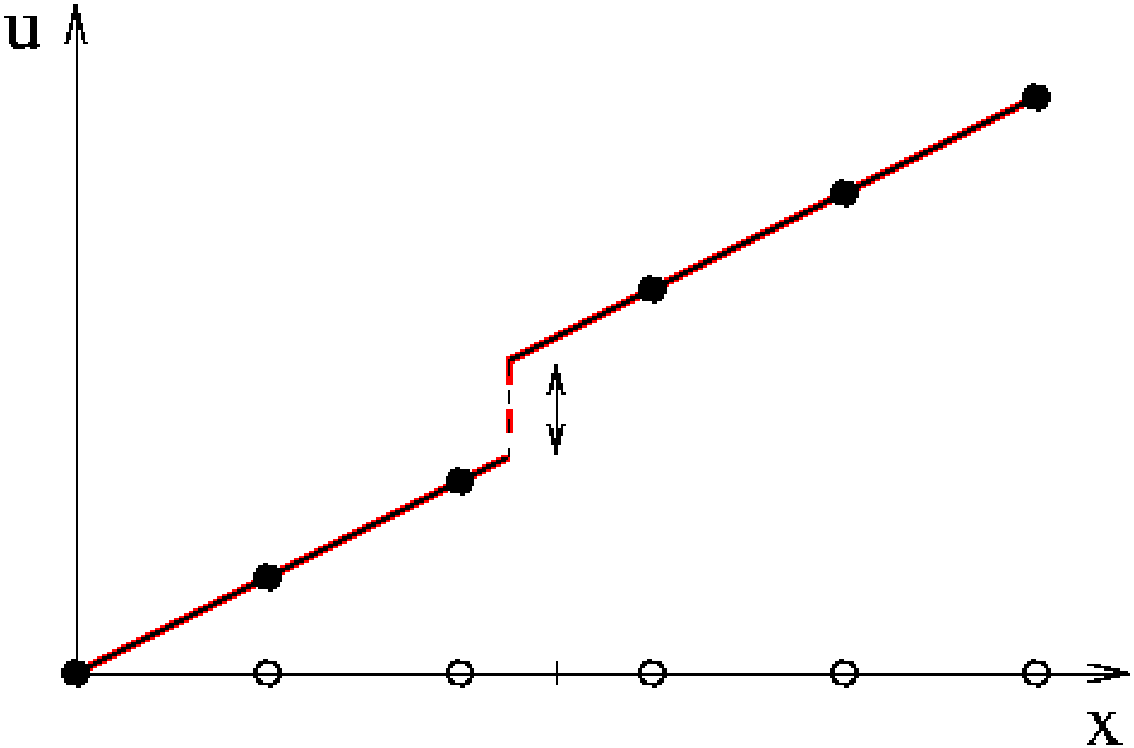
   aka EFEM, SDA, GSDA, …

   b) Extended finite elements based on the partition-of-unity concept (XFEM-PUM) aka GFEM, …

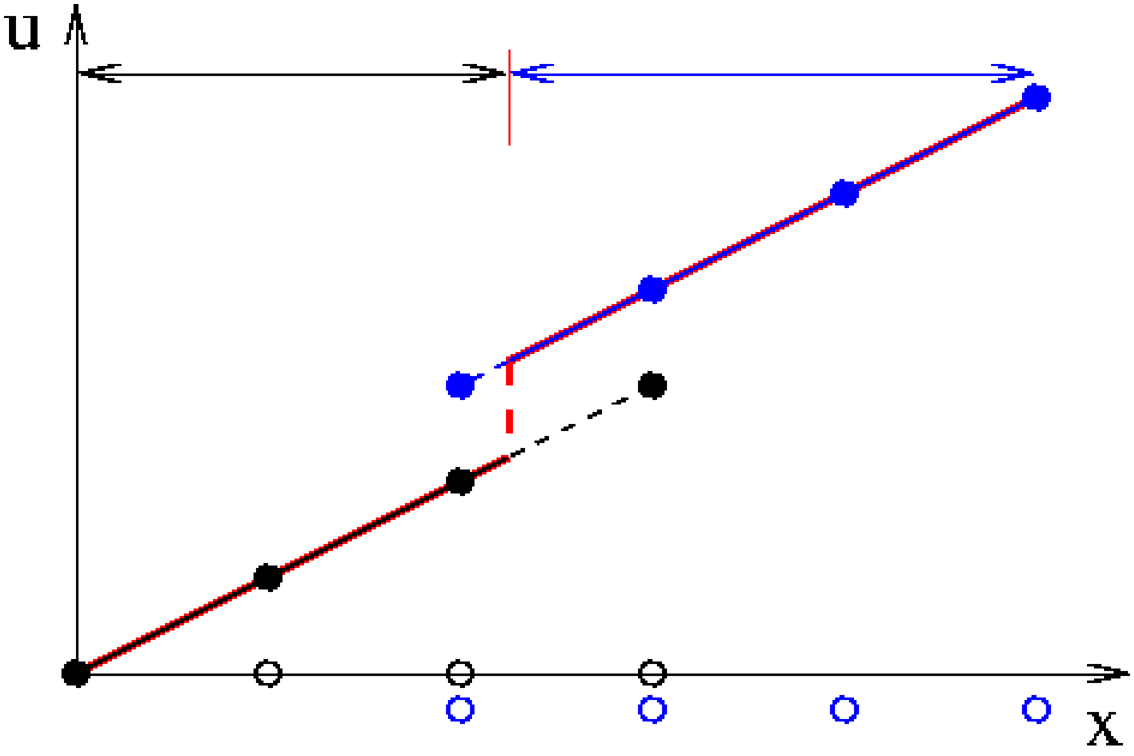
Embedded discontinuity (enhanced assumed strain)



Embedded discontinuity (enhanced assumed strain)



Approximation on two overlapping meshes (XFEM)



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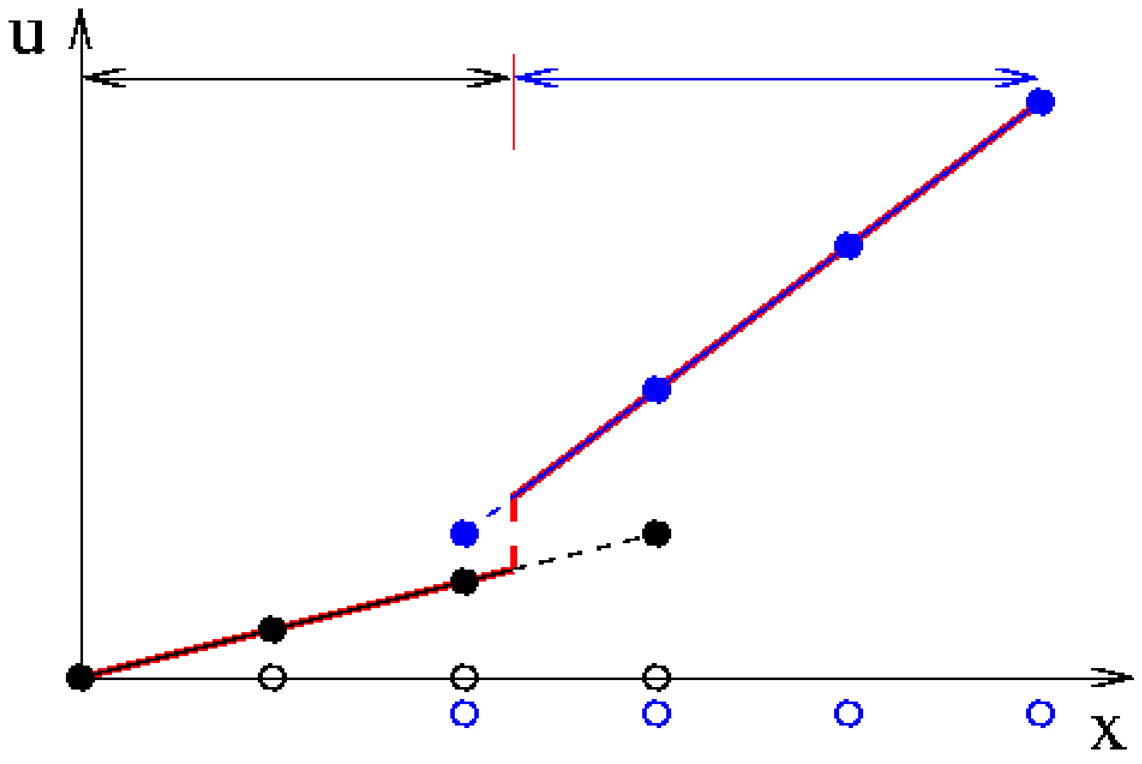
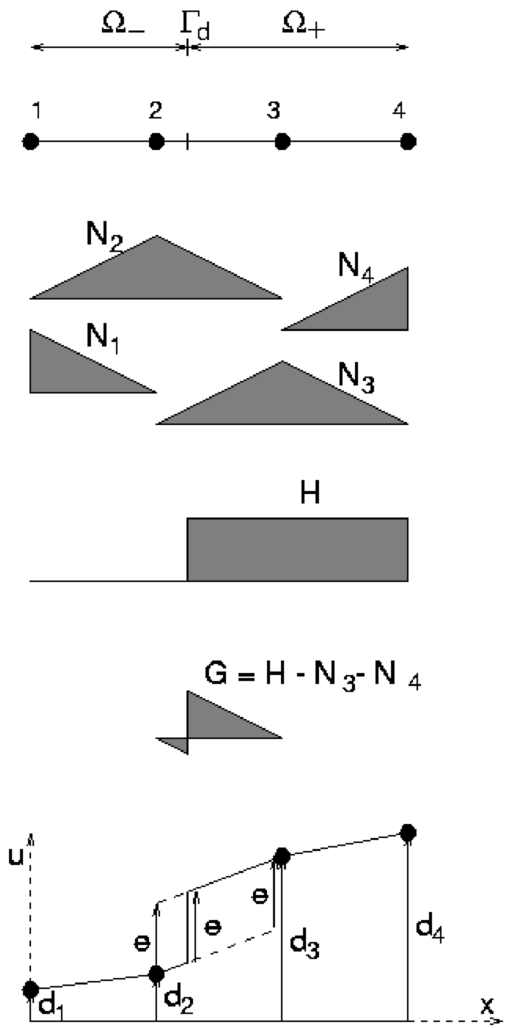


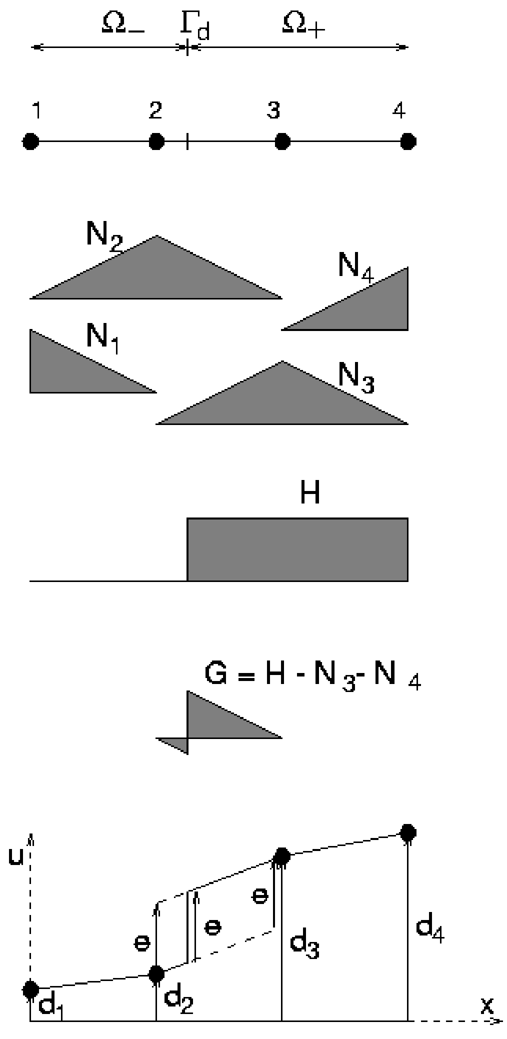
Enrichment of interpolation functions in one dimension

EED-EAS

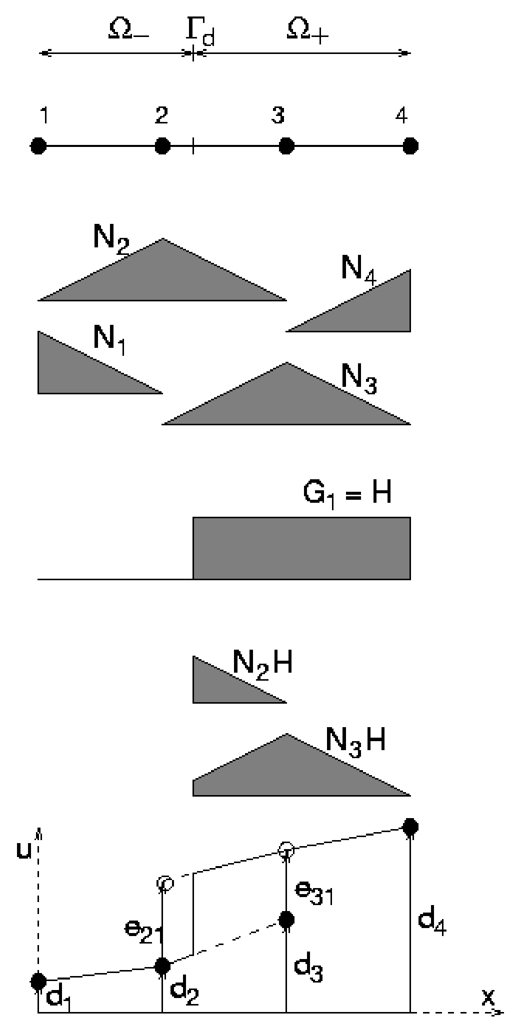


Enrichment of interpolation functions in one dimension

EED-EAS

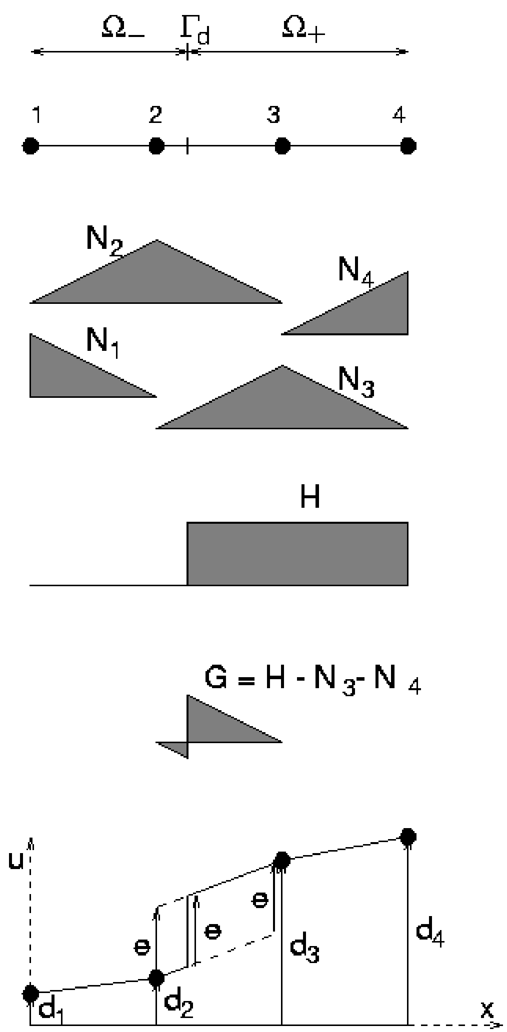


XFEM-PUM

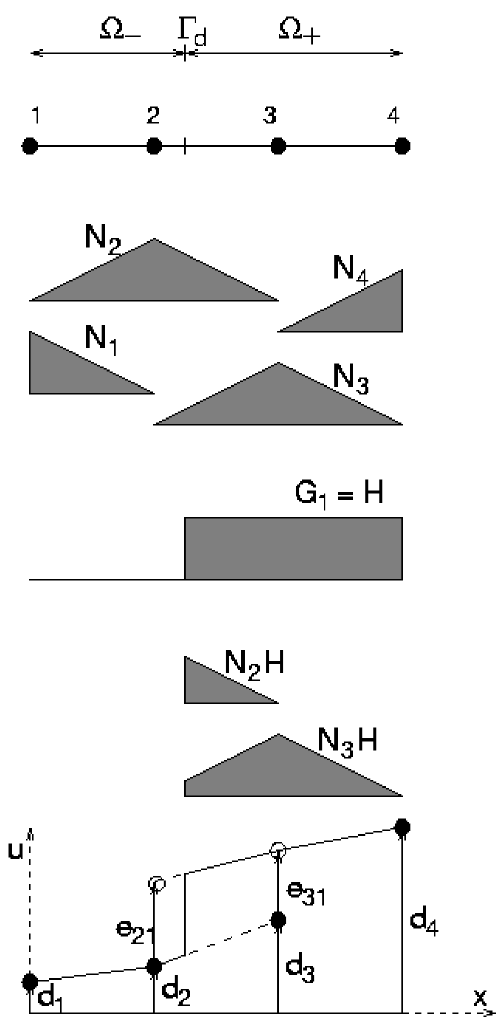


Enrichment of interpolation functions in one dimension

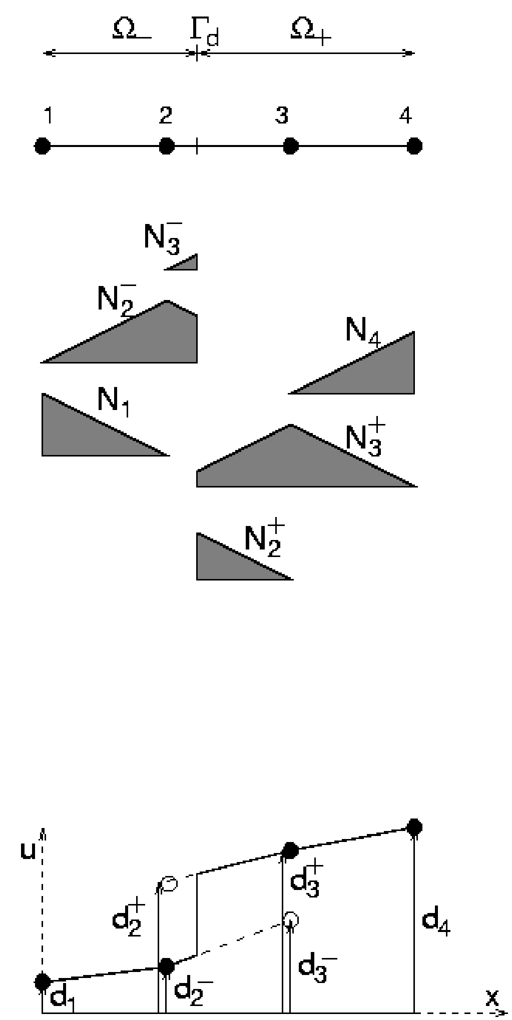
EED-EAS



XFEM-PUM



XFEM-PUM



F.3

Elements with Embedded Discontinuities (EAS)

Elements with embedded discontinuities

$$\begin{array}{l} \mathbf{d} \\ \Big\downarrow \quad \boldsymbol{\varepsilon} = \mathbf{B}\mathbf{d} \\ \boldsymbol{\varepsilon} \\ \Big\downarrow \quad \boldsymbol{\sigma} = \tilde{\boldsymbol{\sigma}}(\boldsymbol{\varepsilon}, \ldots) \\ \boldsymbol{\sigma} \\ \Big\downarrow \quad \mathbf{f}_{\mathrm{int}} = \int_V \mathbf{B}^T \boldsymbol{\sigma} \, \mathrm{d}V \\ \mathbf{f}_{\mathrm{int}} \end{array}$$

Elements with embedded discontinuities

$\mathbf{d}$

$\boldsymbol{\varepsilon}$ $\mathbf{e}$ … new degrees of freedom
characterizing separation (displacement jump)

$\boldsymbol{\sigma}$ $\mathbf{t}$ … traction

$\mathbf{f}_{\text{int}}$

Elements with embedded discontinuities

$$\mathbf{d}$$

$$\begin{tikzcd}
\boldsymbol{\varepsilon} \arrow[d] \arrow[r, phantom, "\text{material}" below] & \mathbf{e} \arrow[d] \\
\boldsymbol{\sigma} & \mathbf{t}
\end{tikzcd}$$

$$\mathbf{f}_{\text{int}}$$

Elements with embedded discontinuities

$$\mathbf{d}$$

? kinematics ?

$$\begin{array}{ccc} \boldsymbol{\varepsilon} & & \mathbf{e} \\ \Big\downarrow & \text{material} & \Big\downarrow \\ \boldsymbol{\sigma} & & \mathbf{t} \end{array}$$

? equilibrium ?

$$\mathbf{f}_{\mathrm{int}}$$

Elements with embedded discontinuities

$\mathbf{d}$

kinematics

$\boldsymbol{\varepsilon}$ $\qquad$ $\mathbf{e}$

material

$\boldsymbol{\sigma}$ $\qquad$ $\mathbf{t}$

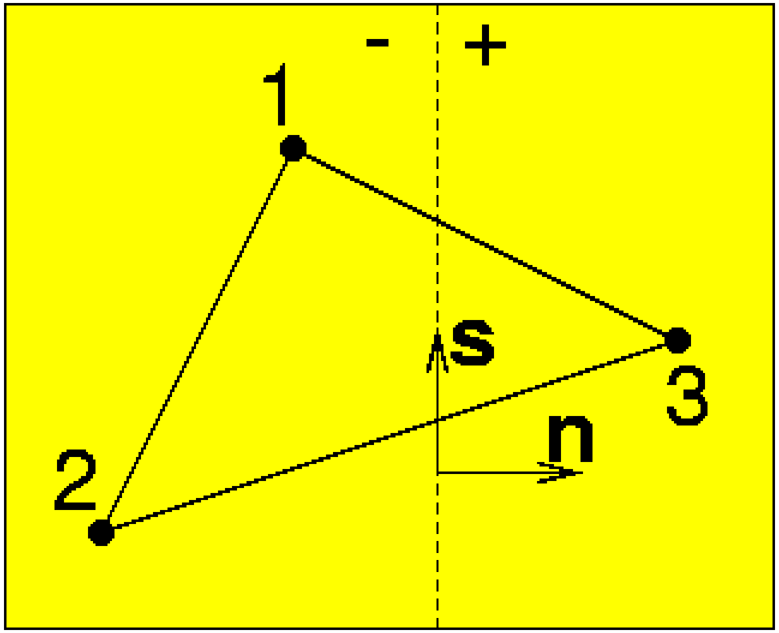
equilibrium

$\mathbf{f}_{\mathrm{int}}$

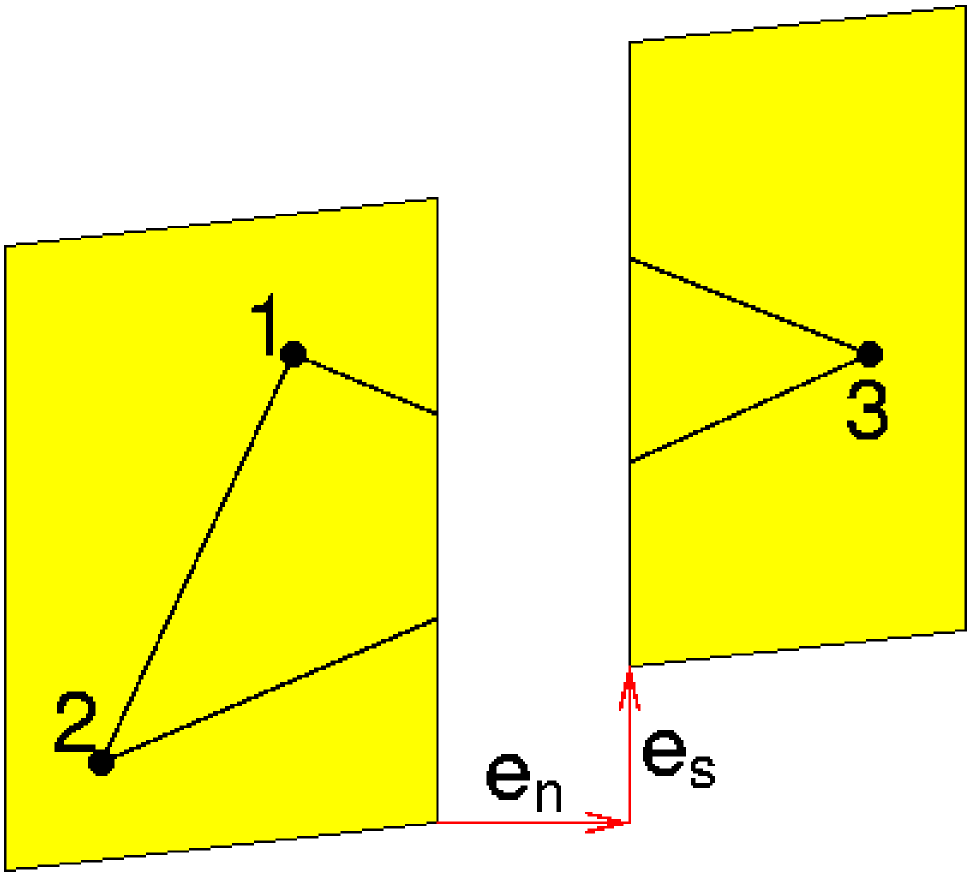
Three types of formulations:

- KOS … kinematically optimal symmetric
- SOS … statically optimal symmetric
- **SKON … kinematically and statically optimal nonsymmetric**

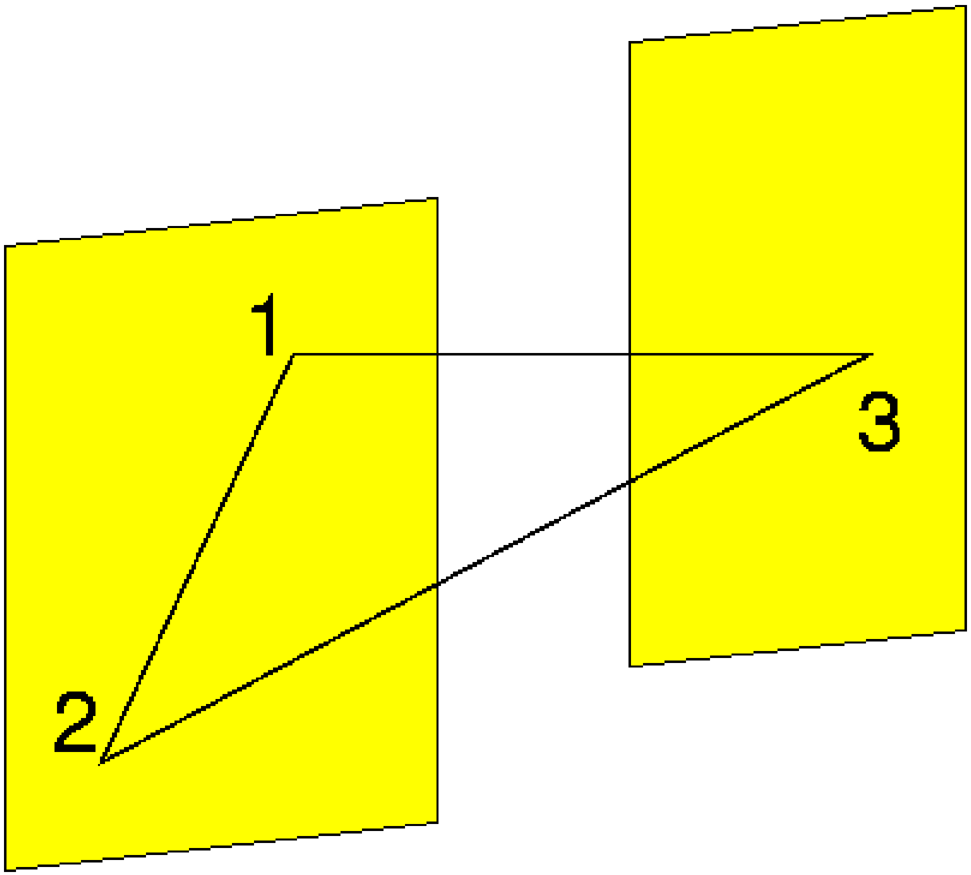
Elements with embedded discontinuities



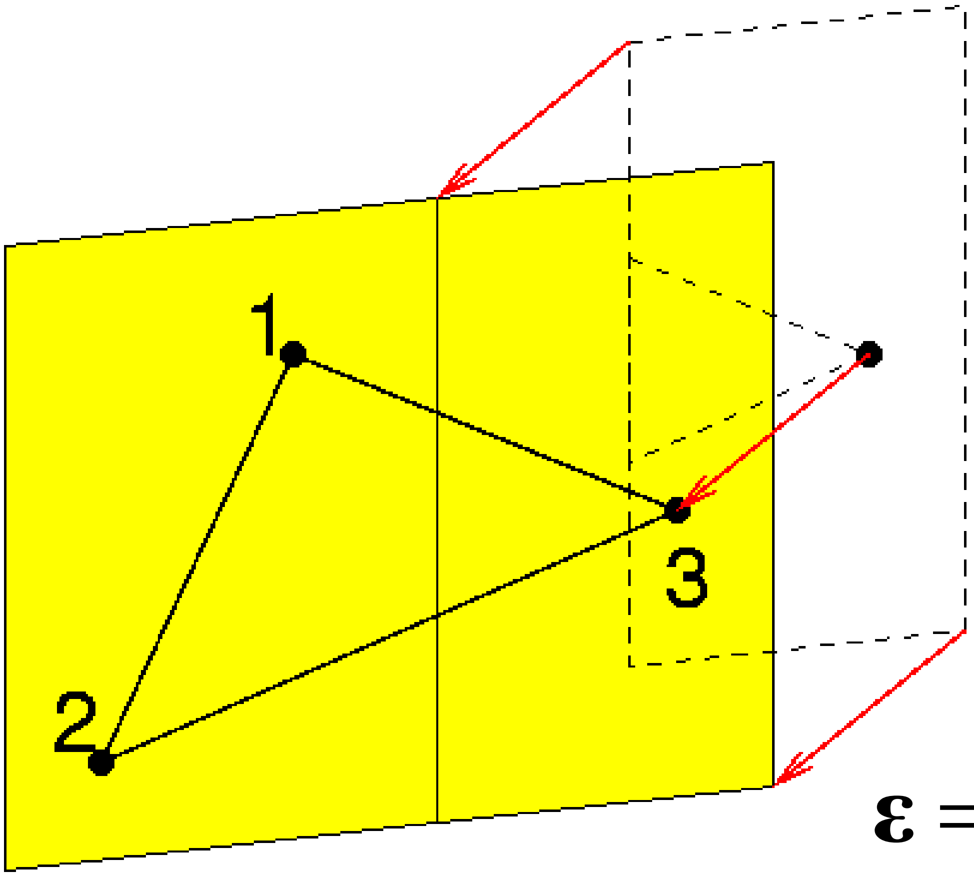
Elements with embedded discontinuities



Elements with embedded discontinuities



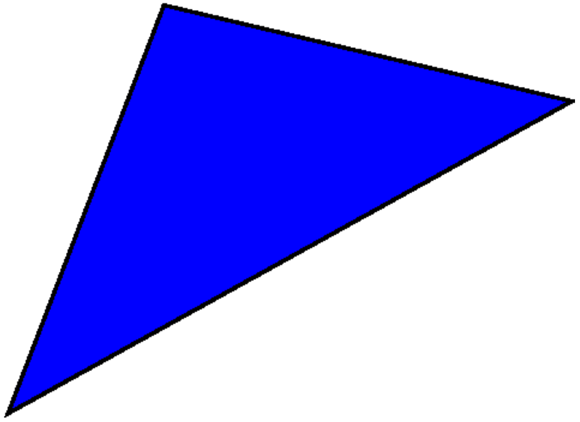
Elements with embedded discontinuities



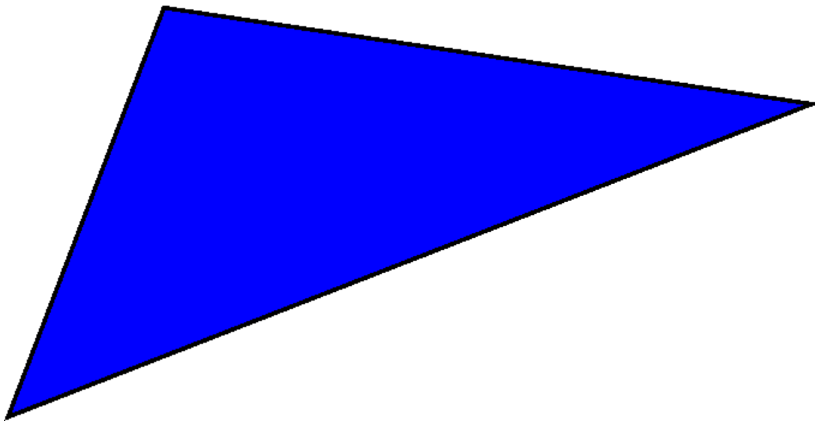
$$\boldsymbol{\varepsilon} = \mathbf{B}(\mathbf{d} - \mathbf{He})$$

$$\mathbf{t} = \mathbf{P}^T \boldsymbol{\sigma}$$

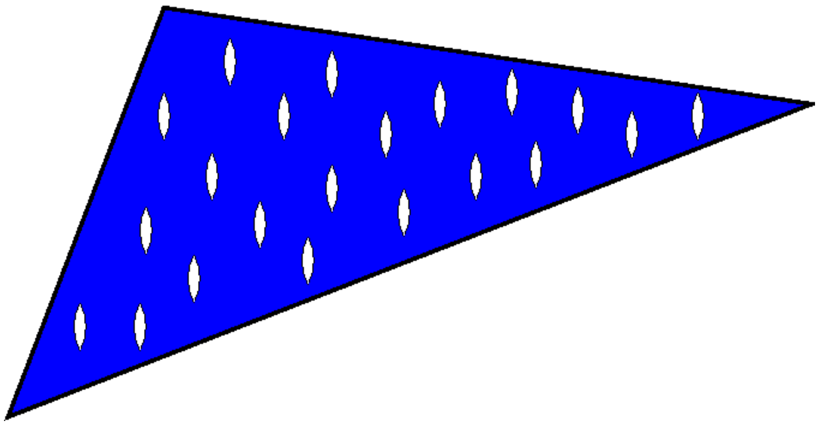
Smeared crack



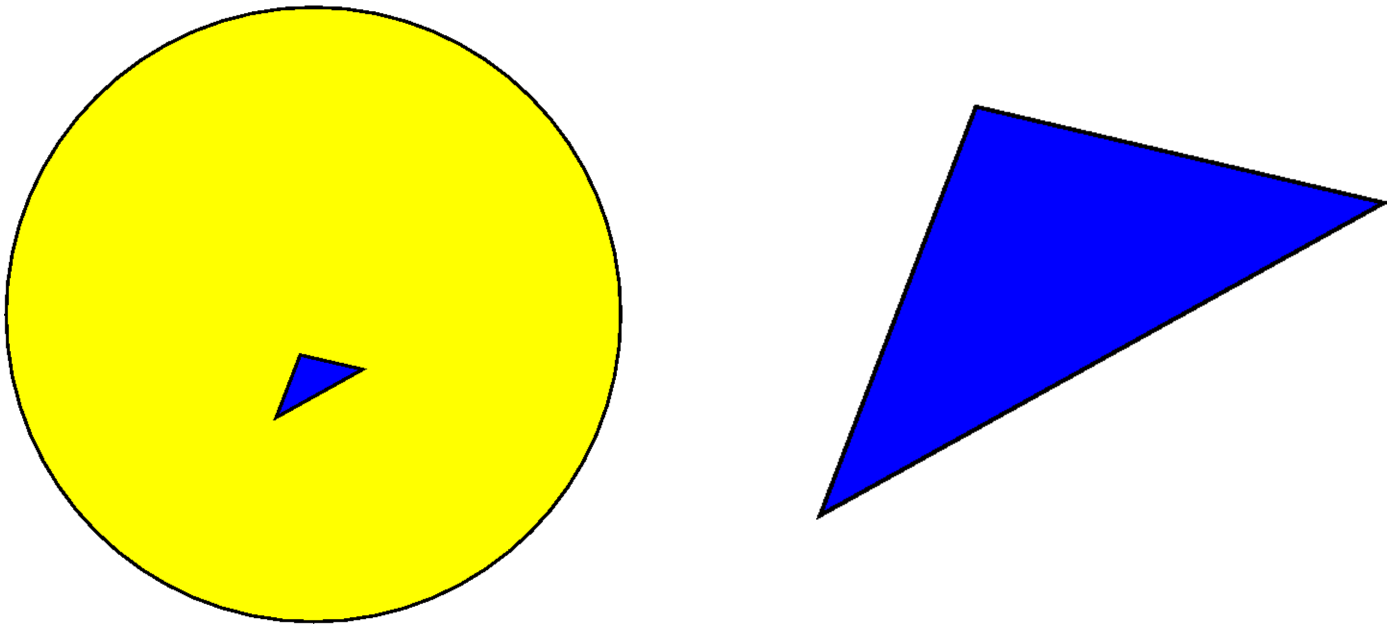
Smeared crack



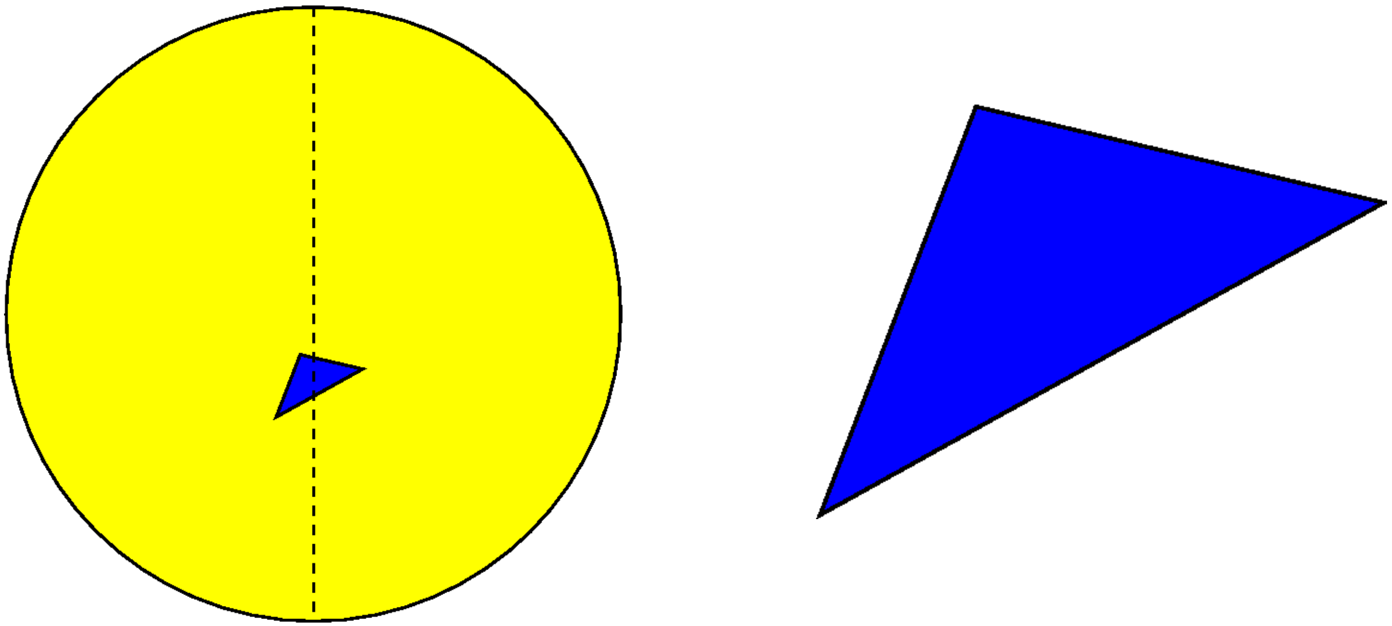
Smeared crack



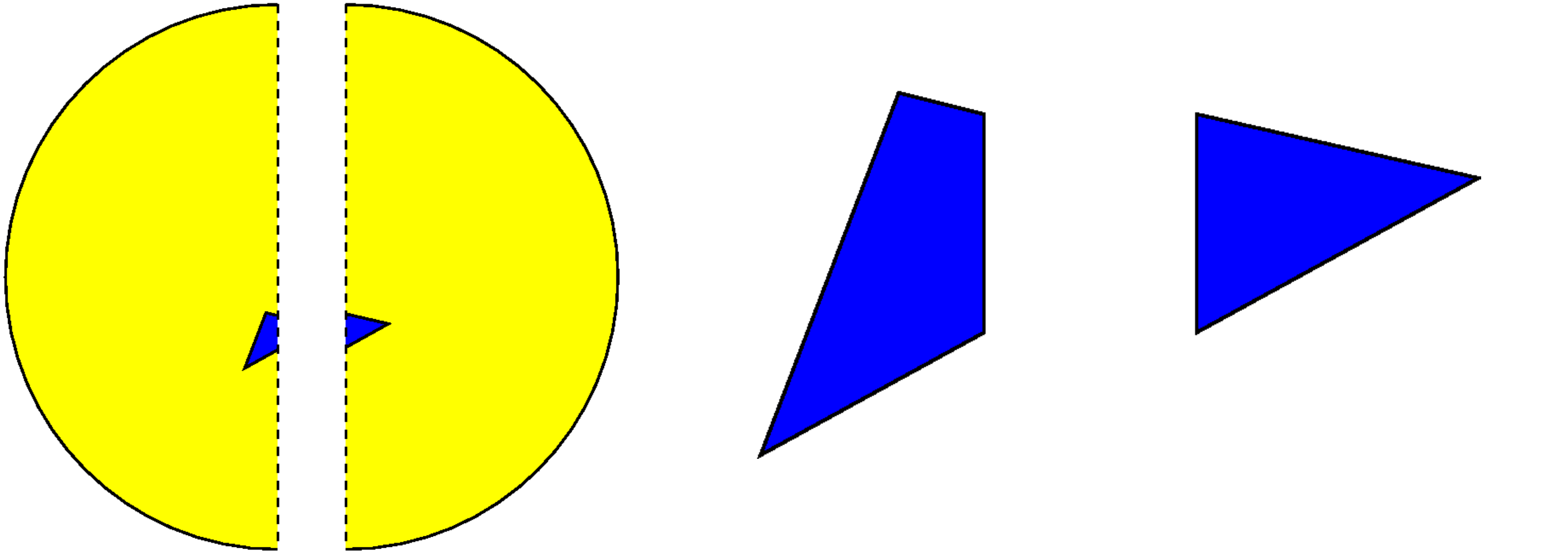
Smeared crack



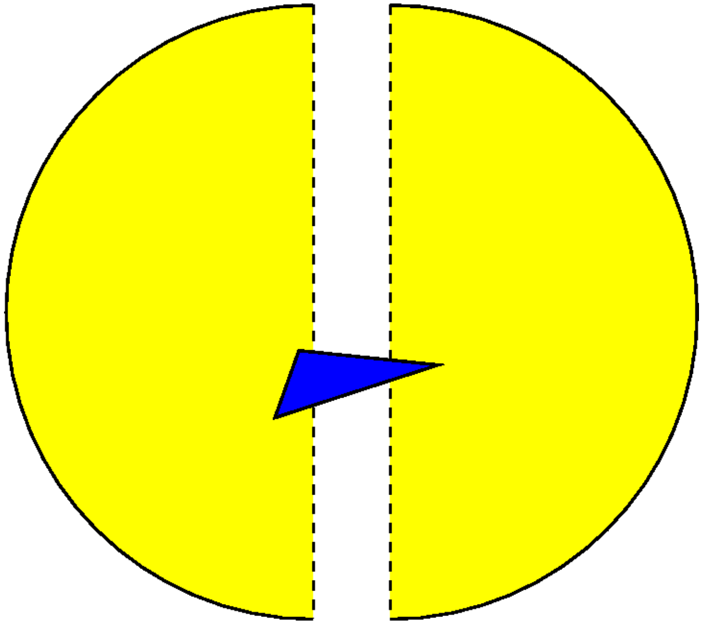
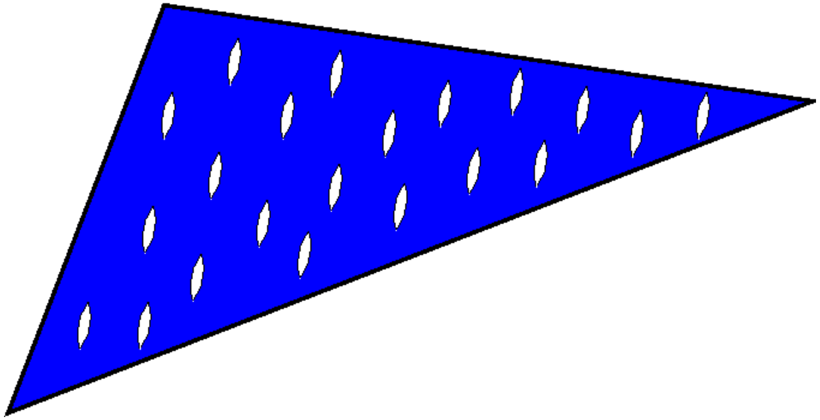
Smeared crack



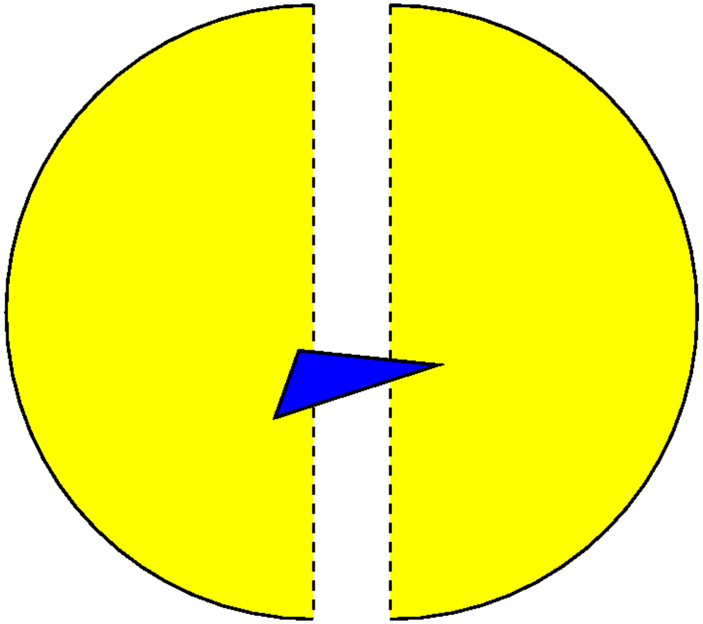
Smeared crack

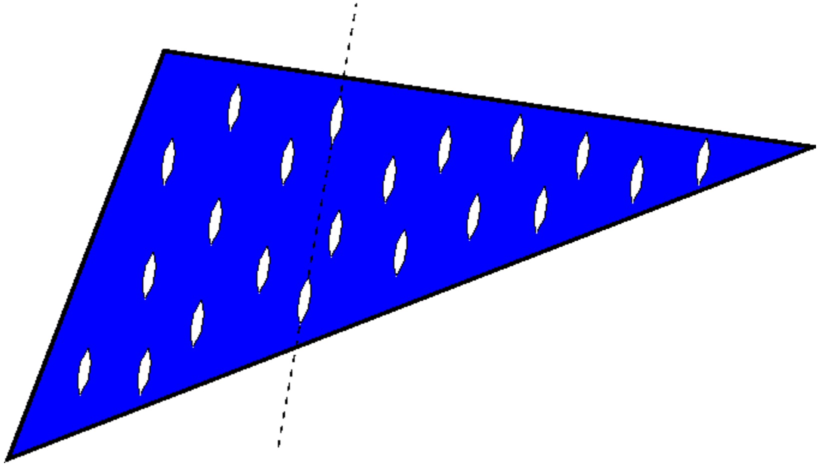


Smeared crack

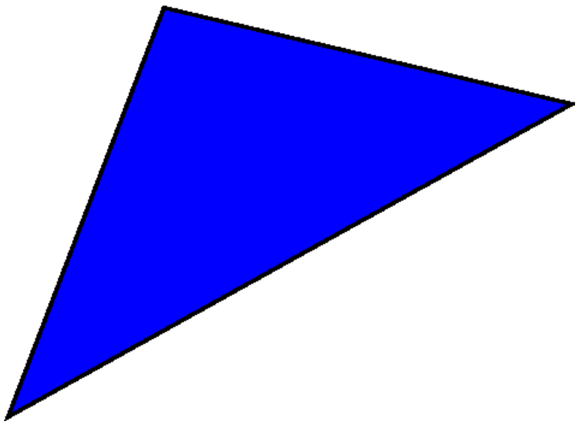
Smeared crack





- Misalignment between crack and element
- Distorted principal directions
- Stress locking

Embedded crack (EAS approach)



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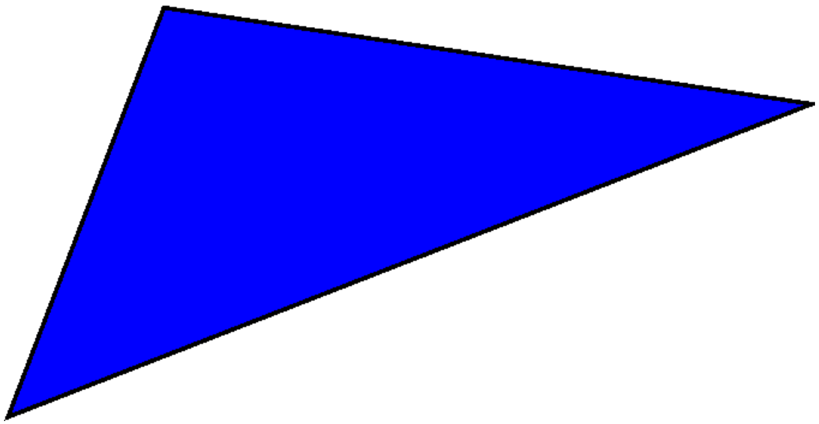
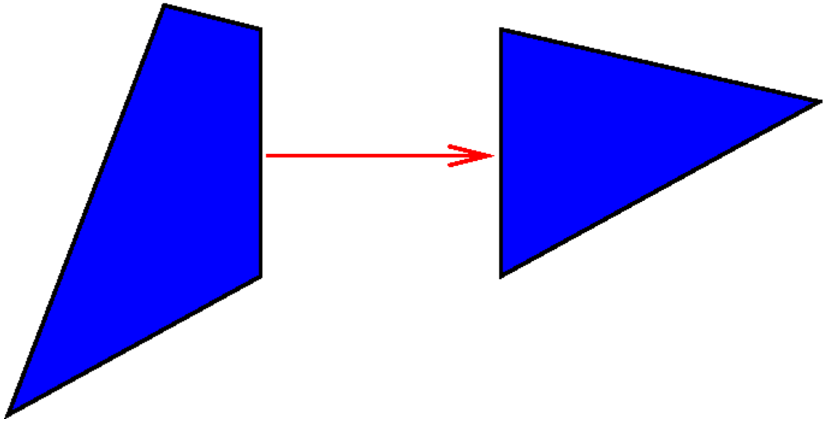
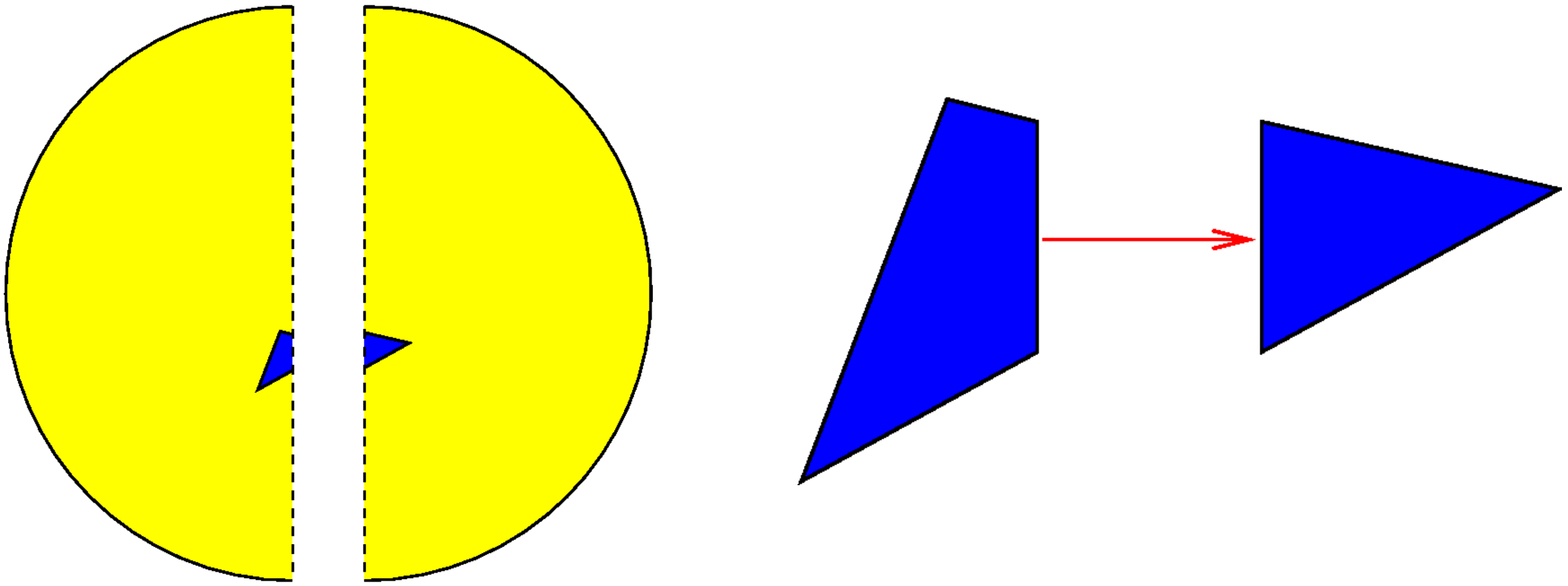


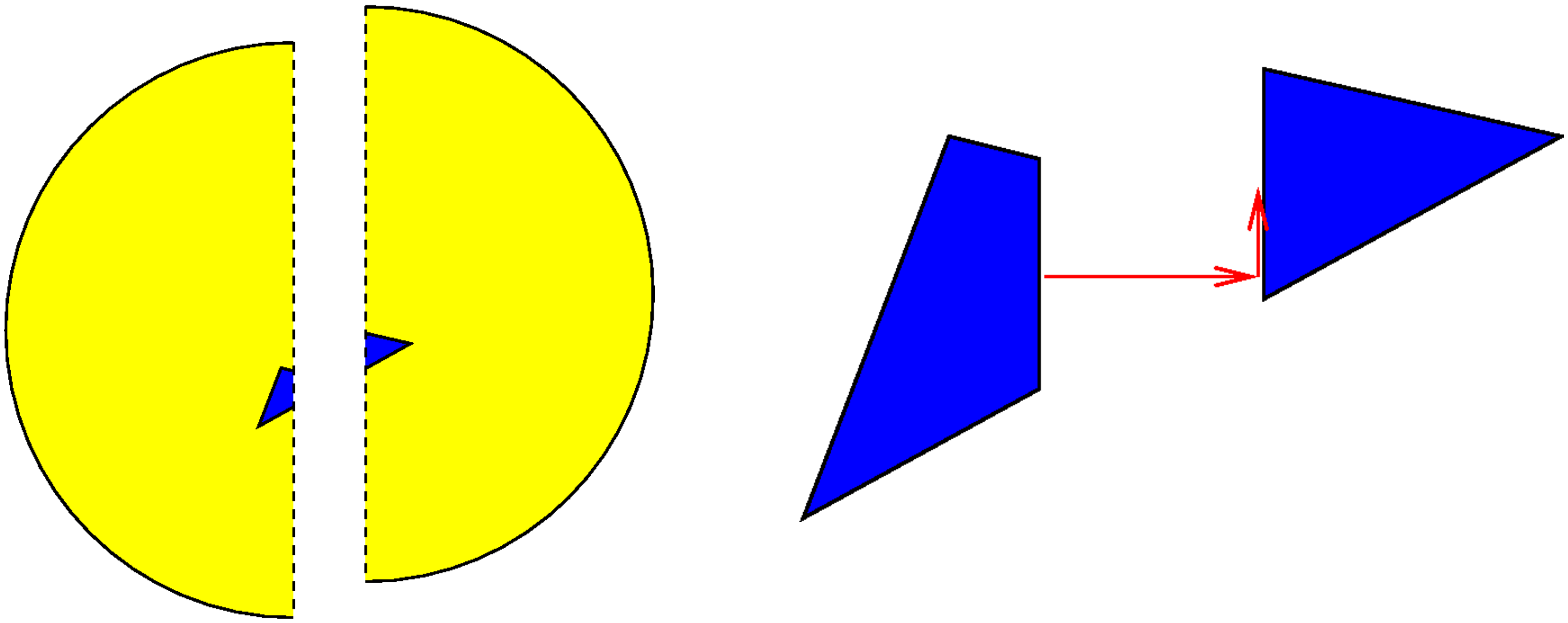
Embedded crack (EAS approach)



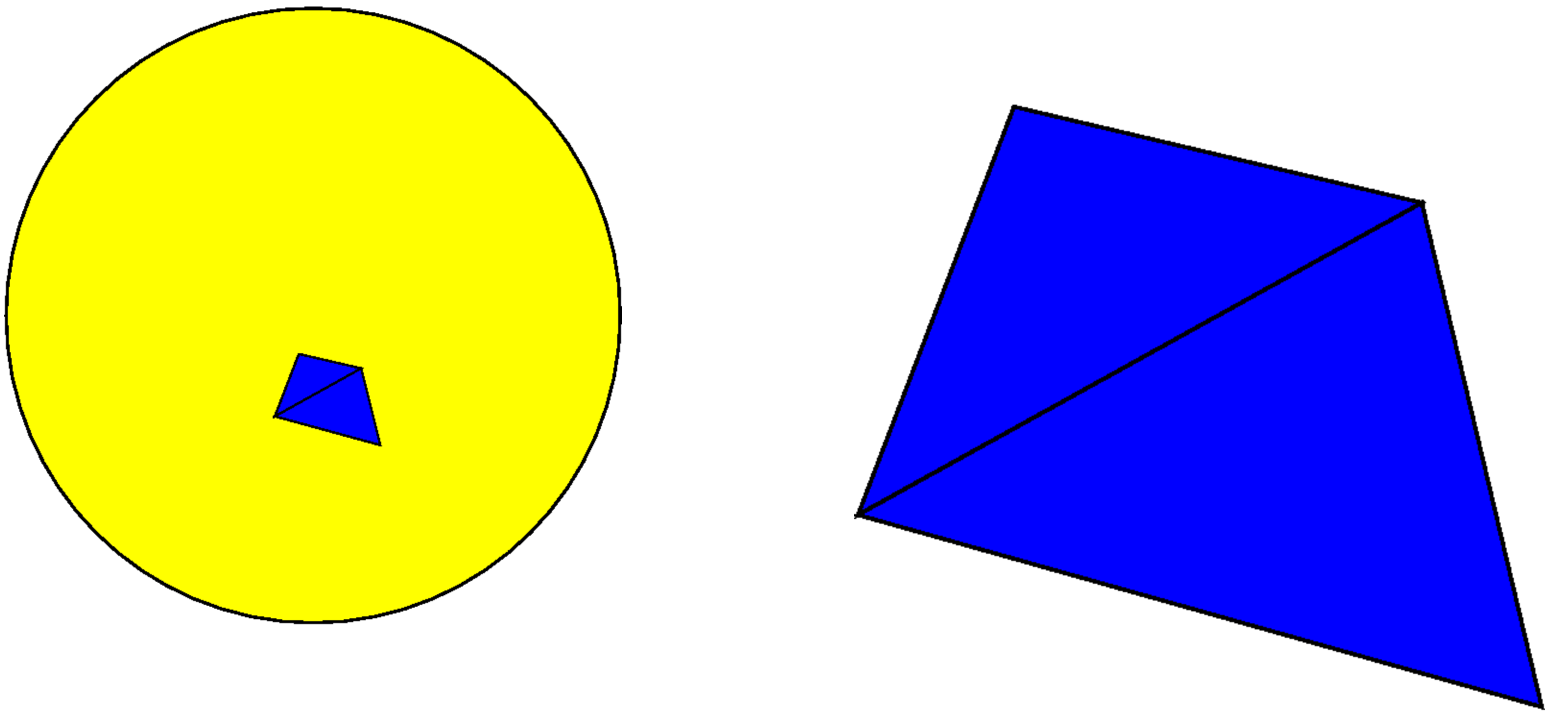
Embedded crack (EAS approach)



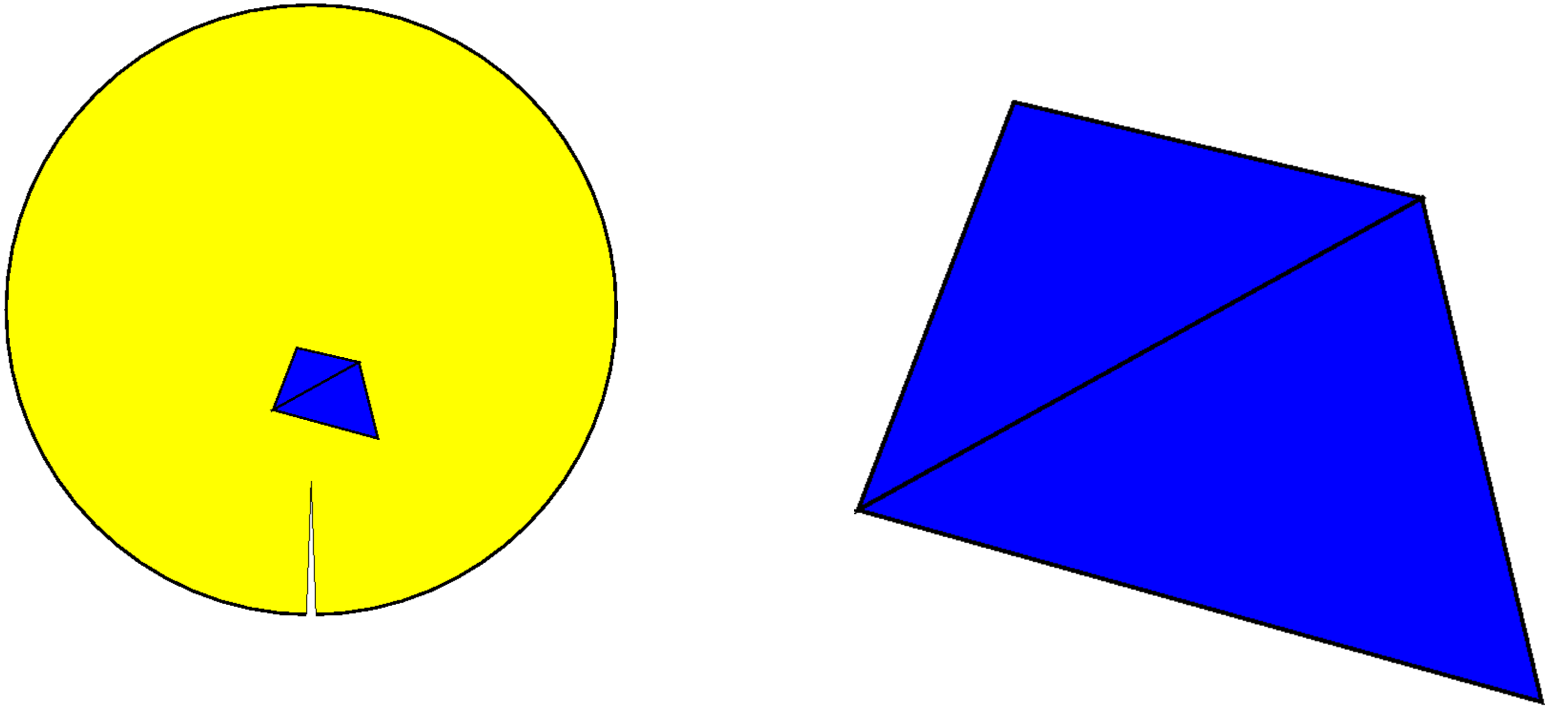
Embedded crack (EAS approach)



EED-EAS approach: discontinuous interpolation



EED- EAS approach: discontinuous interpolation



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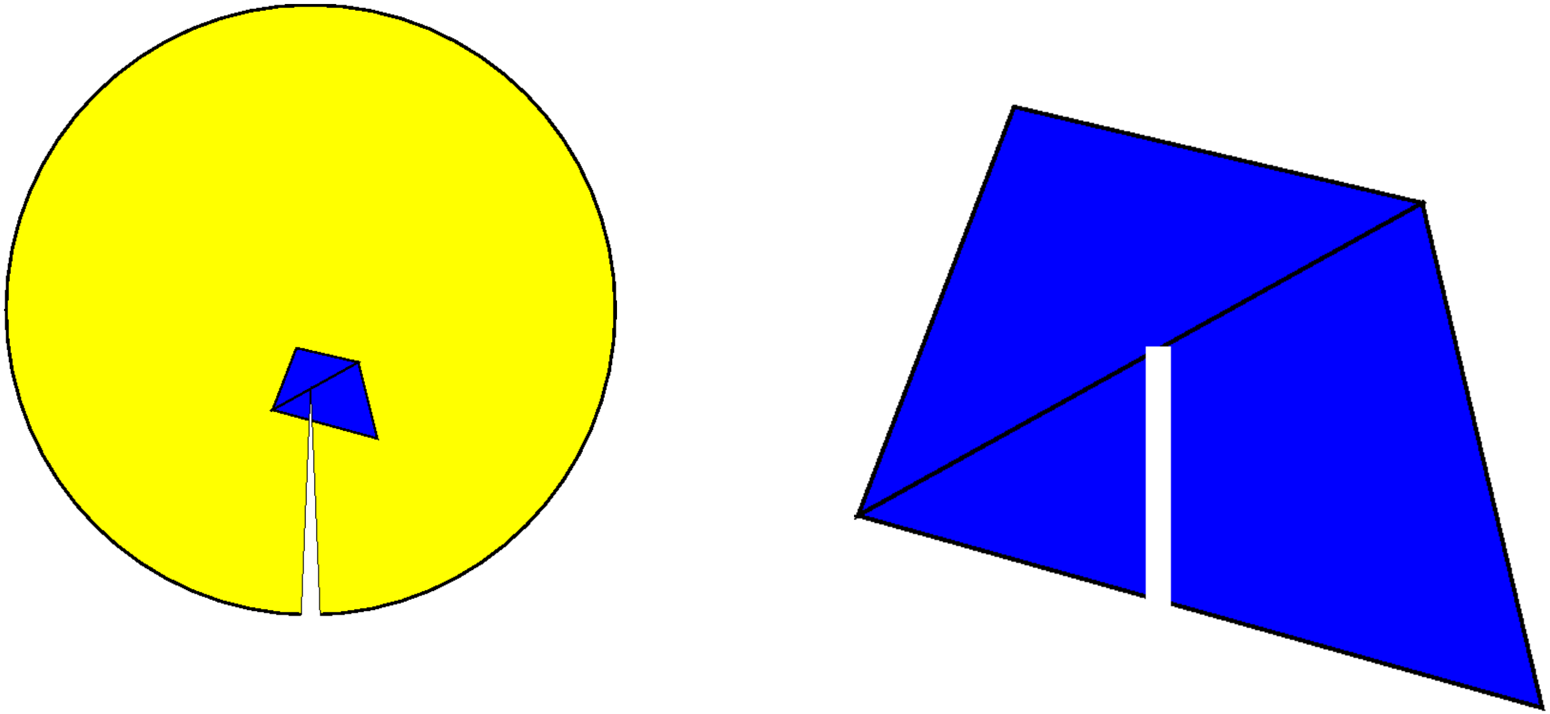
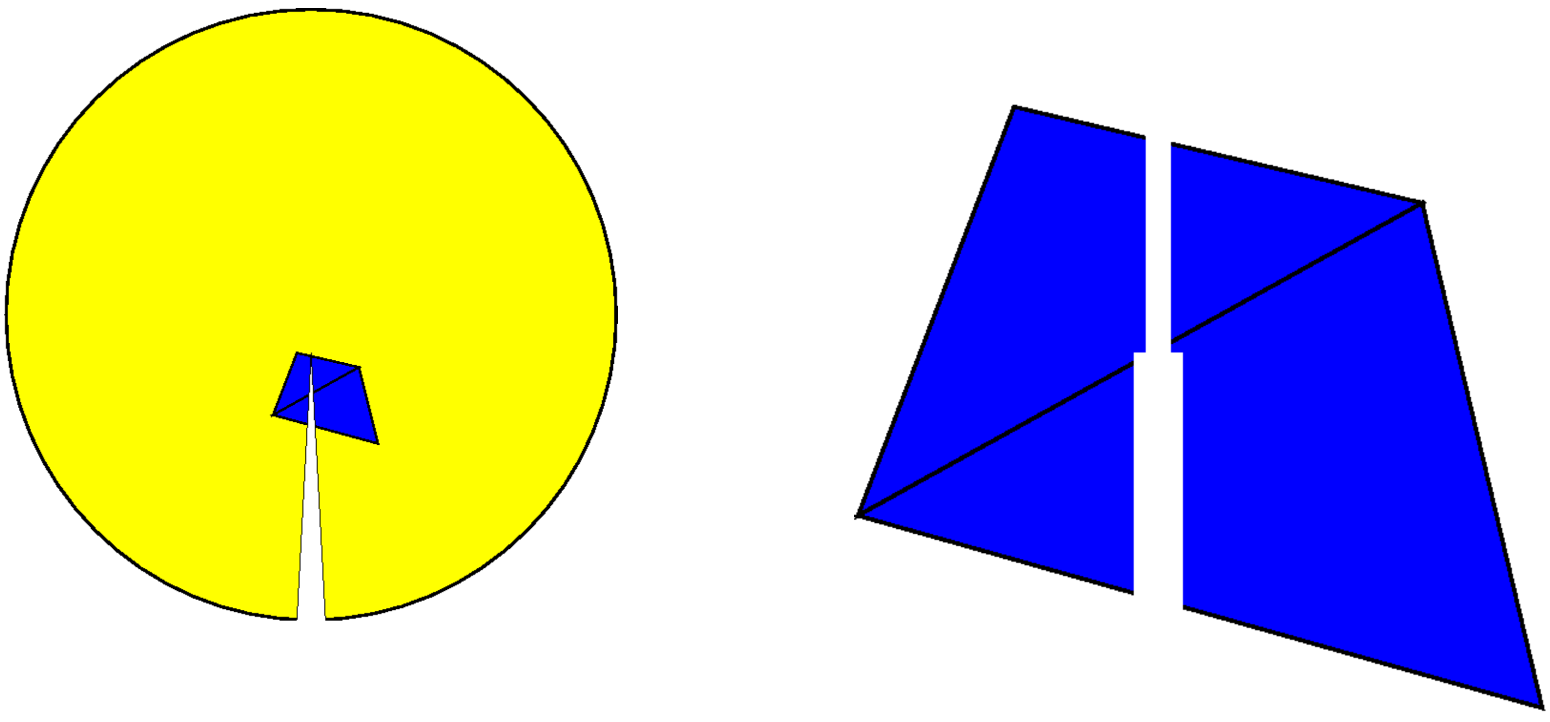


EED- EAS approach: discontinuous interpolation



F.4

Extended Finite Elements (XFEM)

Based on Partition of Unity

Partition of Unity Method

Standard finite element approximation:

$$\mathbf{u}(\mathbf{x}) = \sum_{I=1}^{Nnod} N_I(\mathbf{x})\,\mathbf{d}_I$$

The shape functions are a partition of unity:

$$\sum_{I=1}^{Nnod} N_I(\mathbf{x}) = 1$$

Partition of Unity Method

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The shape functions are a partition of unity:

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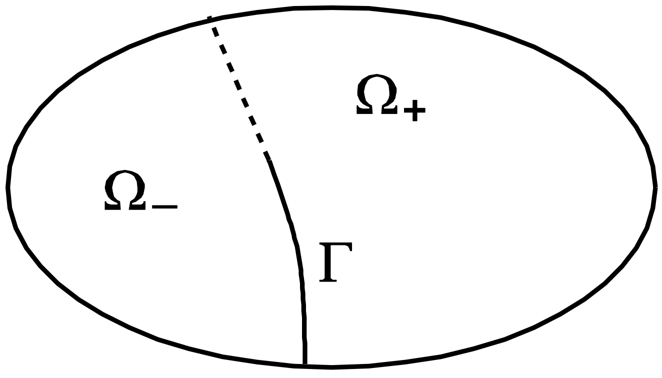
Enriched approximation:

$$\mathbf{u}(\mathbf{x}) = \sum_{I=1}^{Nnod} N_I(\mathbf{x})\left[\mathbf{d}_I + \sum_{i \in L_I} G_i(\mathbf{x})\mathbf{e}_{iI}\right]$$

selected enrichment functions

Partition of Unity Method – eXtended Finite Elements

Enrichment by Heaviside function:



$$H_\Gamma(\mathbf{x}) = \begin{cases} 1 & \text{for } x \in \Omega^+ \\ 0 & \text{for } x \in \Omega^- \end{cases}$$

Partition of Unity Method – eXtended Finite Elements

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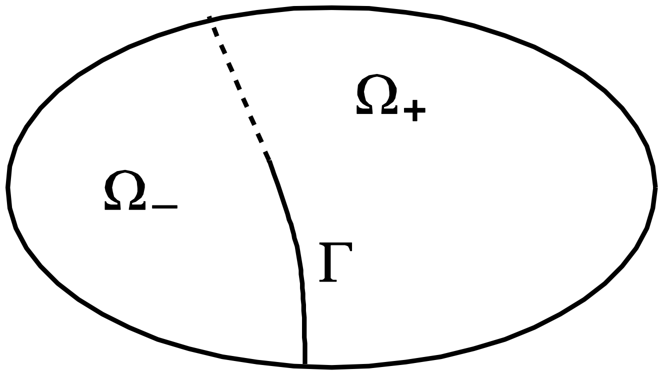


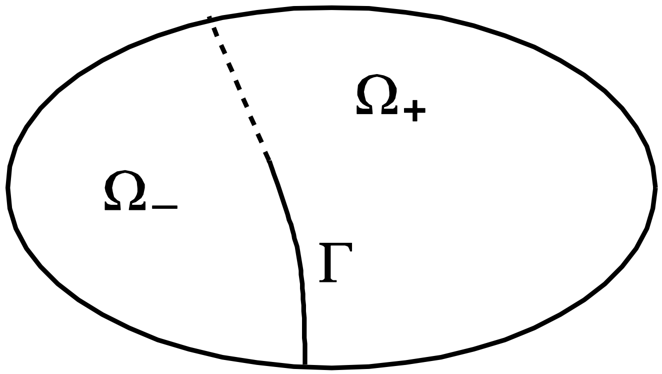
$$H_\Gamma(\mathbf{x}) = \begin{cases} 1 & \text{for } x \in \Omega^+ \\ 0 & \text{for } x \in \Omega^- \end{cases}$$

$$\mathbf{u}(\mathbf{x}) = \sum_{I=1}^{Nnod} N_I(\mathbf{x})\left[\mathbf{d}_I + H_\Gamma(\mathbf{x})\mathbf{e}_I\right] =$$

$$= \sum_{I=1}^{Nnod} N_I(\mathbf{x})\mathbf{d}_I + \sum_{I=1}^{Nnod} N_I(\mathbf{x}) H_\Gamma(\mathbf{x})\mathbf{e}_I$$

Partition of Unity Method – eXtended Finite Elements

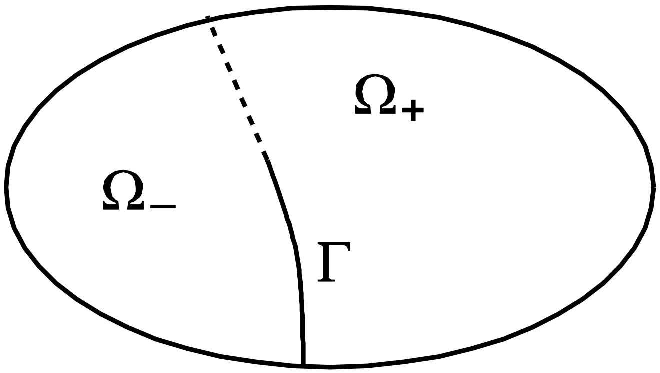
If the support of N_I is contained in Ω^+, then $N_I H_\Gamma = N_I$



If the support of N_I is contained in Ω^-, then $N_I H_\Gamma = 0$

Partition of Unity Method – eXtended Finite Elements

If the support of N_I is contained in Ω^+, then $N_I H_\Gamma = N_I$



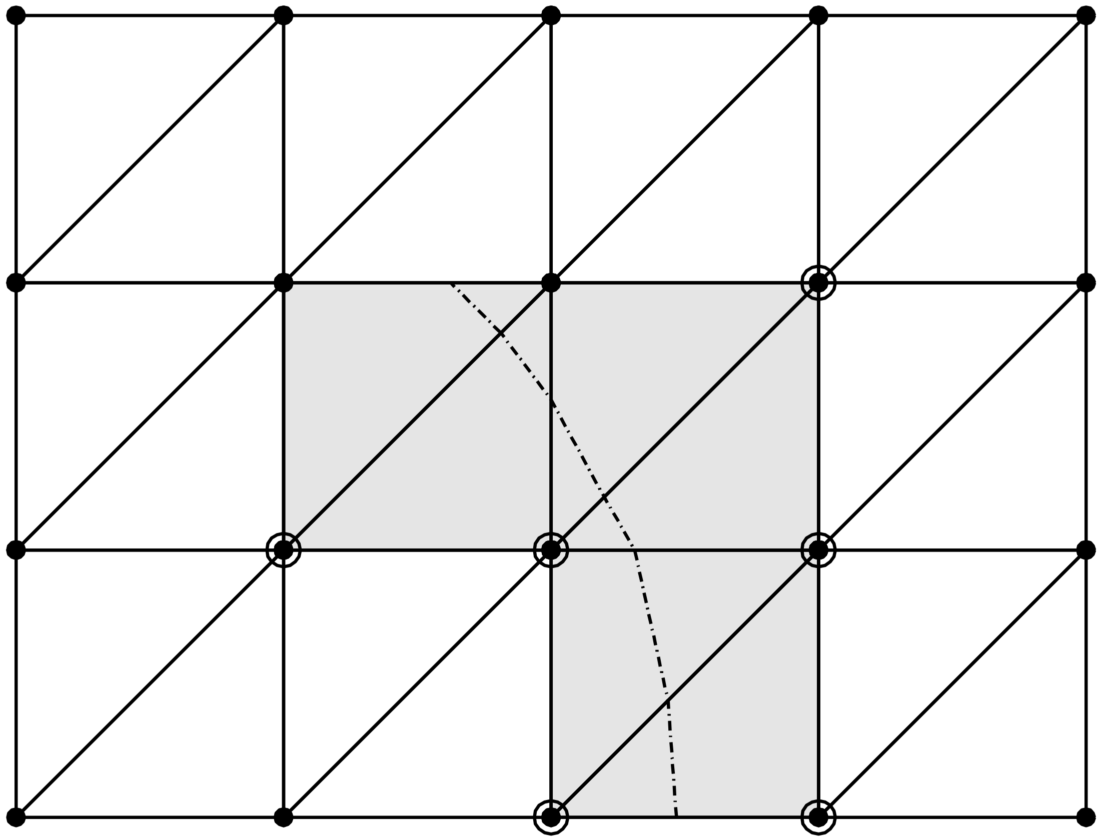
If the support of N_I is contained in Ω^-, then $N_I H_\Gamma = 0$

Only if the support of N_I is cut by Γ ,
then the function $N_I H_\Gamma$ really enriches the basis.

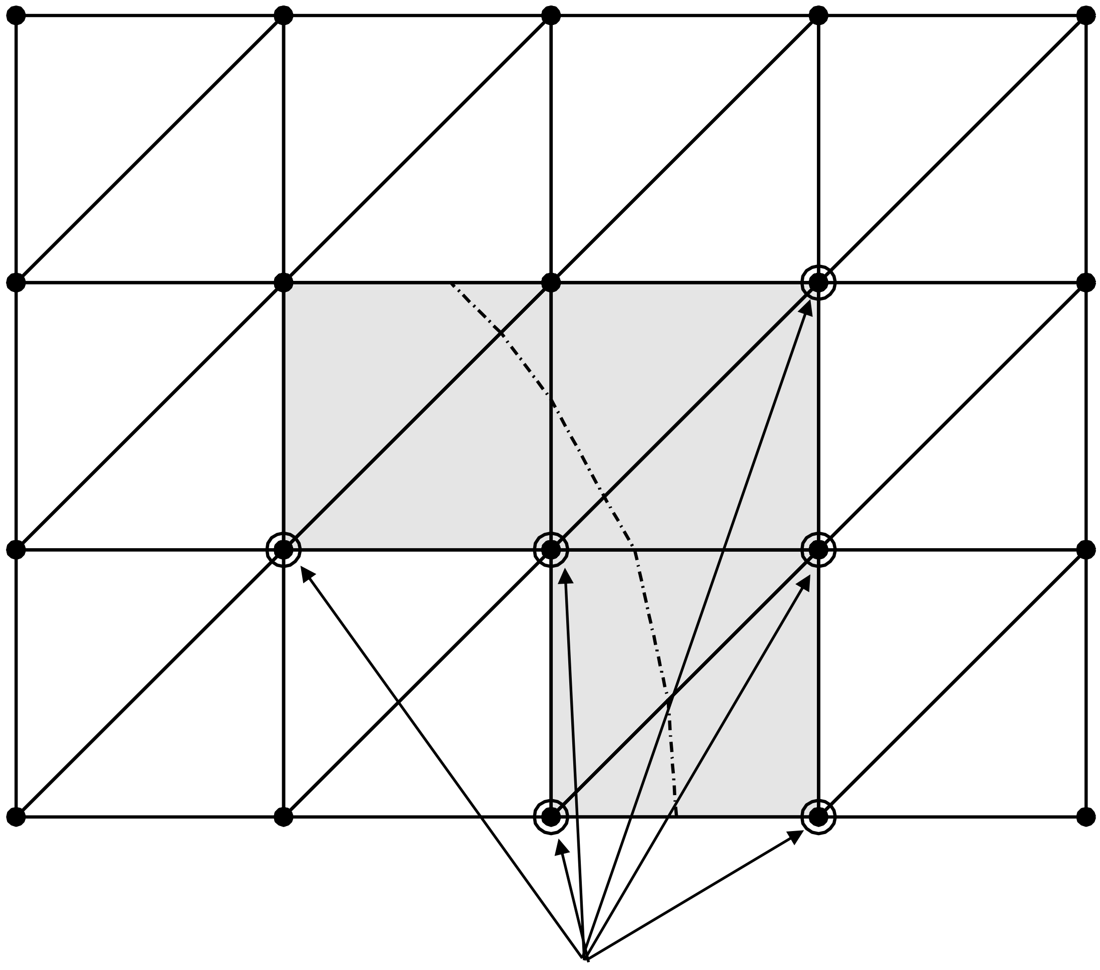
$$\mathbf{u}(\mathbf{x}) = \sum_{I=1}^{Nnod} N_I(\mathbf{x})\mathbf{d}_I + \sum_{I \in S_H} N_I(\mathbf{x}) H_\Gamma(\mathbf{x})\mathbf{e}_I$$

set of nodes with Heaviside enrichment

Partition of Unity Method – eXtended Finite Elements



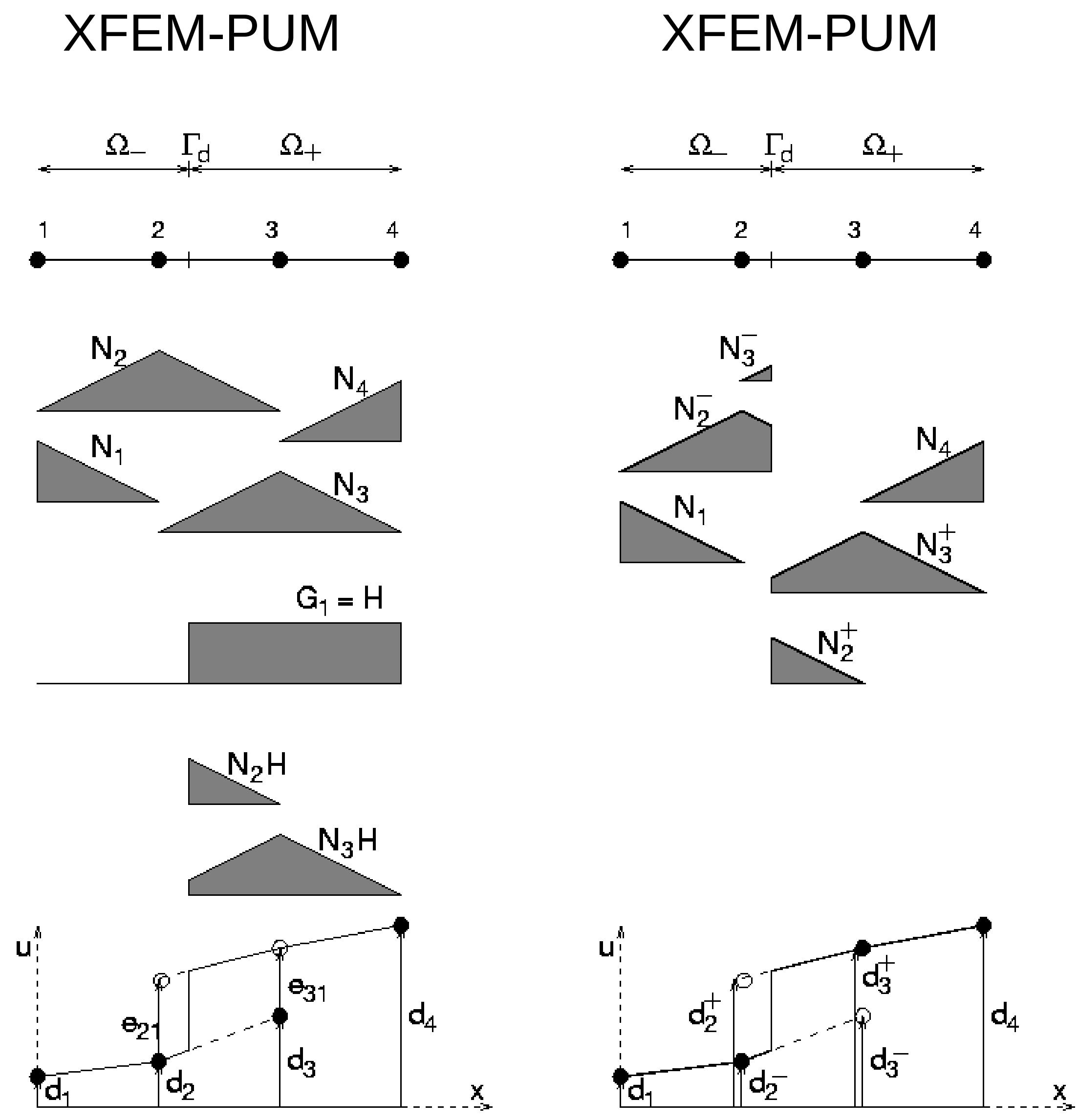
Partition of Unity Method – eXtended Finite Elements



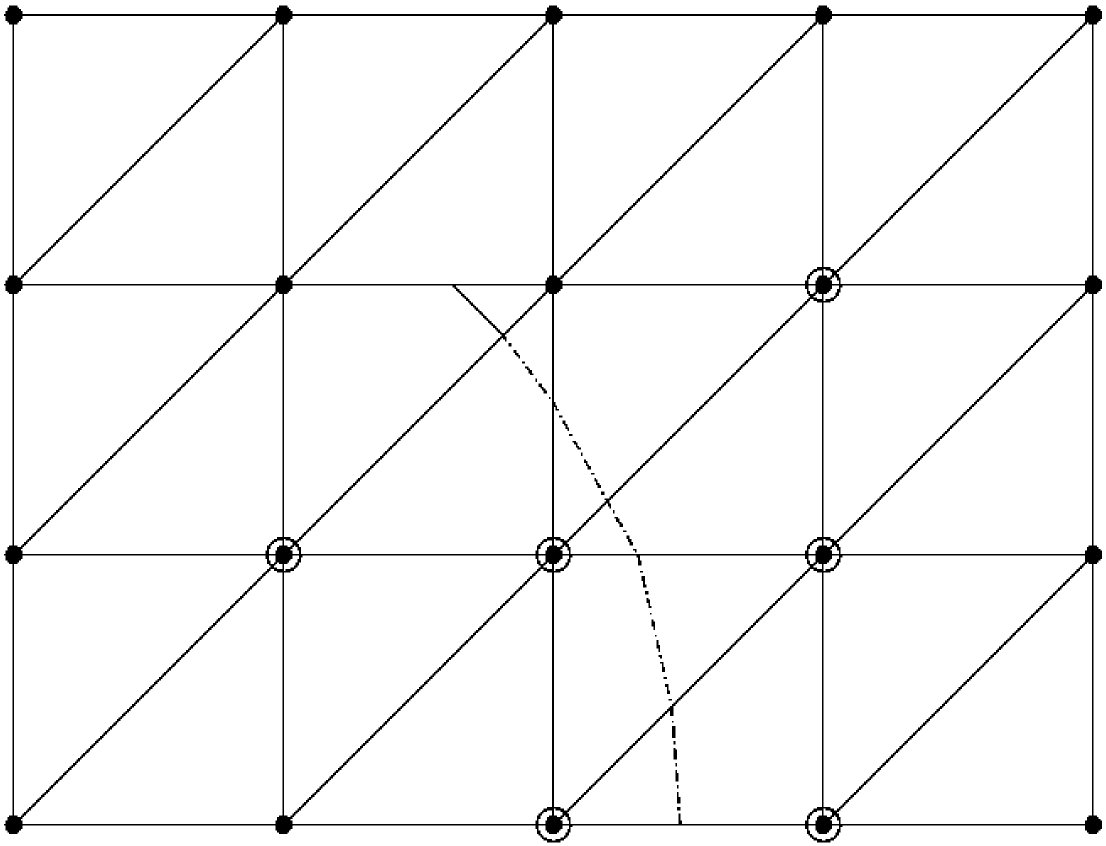
nodes with Heaviside enrichment

Partition of Unity Method – eXtended Finite Elements

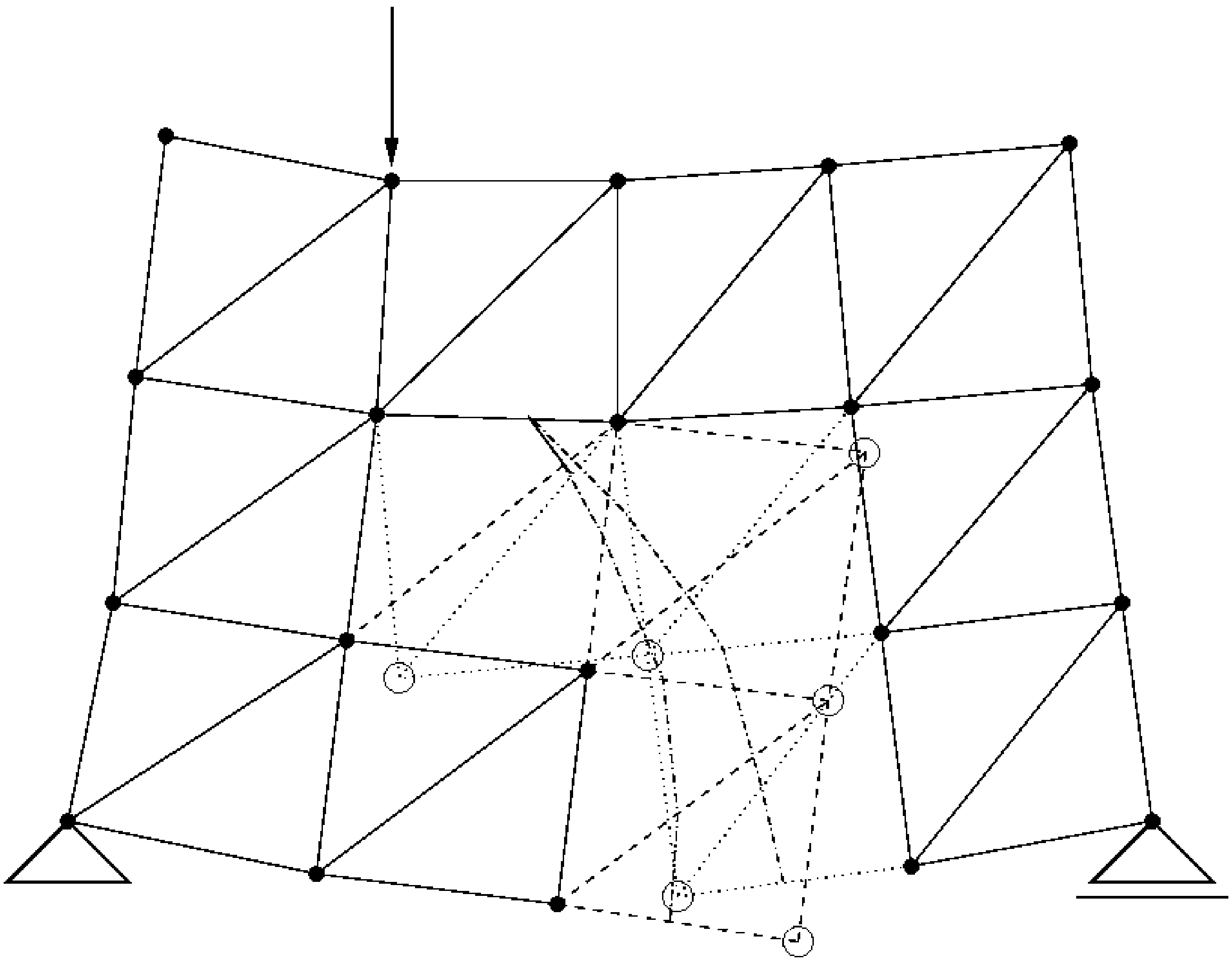
The enriched approximation can be rearranged
to give better physical meaning to the degrees of freedom:



XFEM – enrichment by step function



XFEM – enrichment by step function



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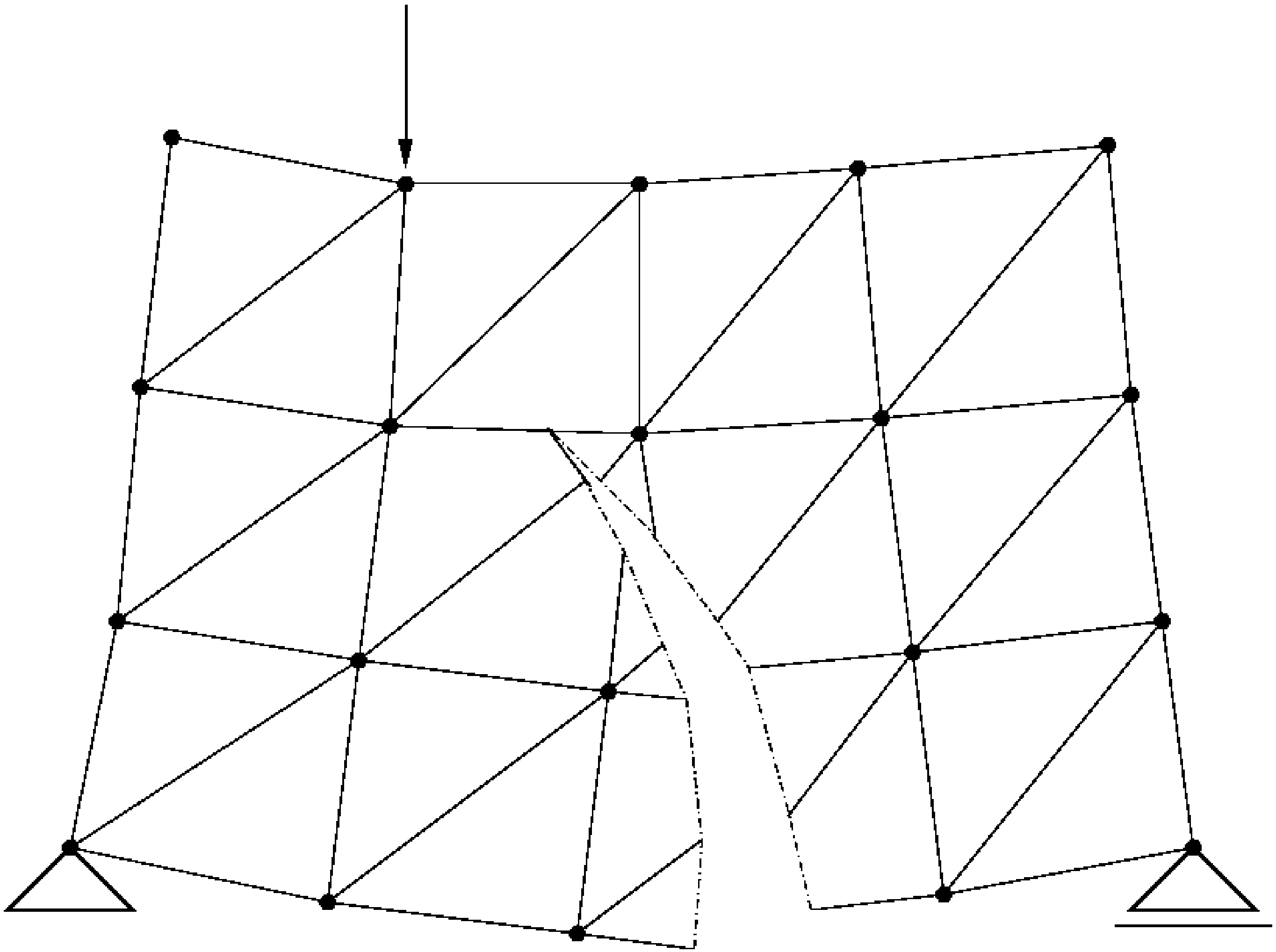
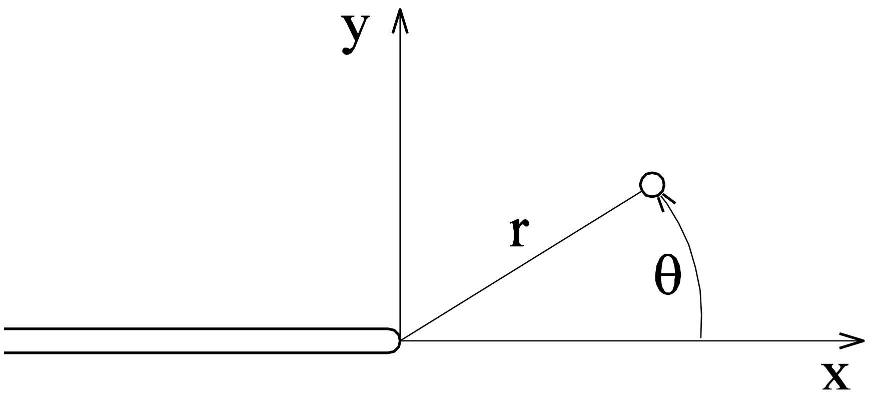


XFEM – tip enrichment

Additional enrichment improving the approximation around the crack tip:



Functions that appear in the analytical near-tip solution:

$$B_1(r,\theta) = \sqrt{r}\sin\frac{\theta}{2} \qquad B_3(r,\theta) = \sqrt{r}\sin\frac{\theta}{2}\sin\theta$$

$$B_2(r,\theta) = \sqrt{r}\cos\frac{\theta}{2} \qquad B_4(r,\theta) = \sqrt{r}\cos\frac{\theta}{2}\sin\theta$$

XFEM – tip enrichment

Additional enrichment improving the approximation around the crack tip:

$$\mathbf{u}(\mathbf{x}) = \sum_{I=1}^{Nnod} N_I(\mathbf{x})\mathbf{d}_I + \sum_{I \in S_H} N_I(\mathbf{x}) H_\Gamma(\mathbf{x})\mathbf{e}_{0I} +$$

$$+ \sum_{I \in S_B} \sum_{i=1}^{4} N_I(\mathbf{x}) B_i\left(r(\mathbf{x}), \theta(\mathbf{x})\right)\mathbf{e}_{iI}$$

Functions that appear in the analytical near-tip solution:

$$B_1(r,\theta) = \sqrt{r}\sin\frac{\theta}{2} \qquad B_3(r,\theta) = \sqrt{r}\sin\frac{\theta}{2}\sin\theta$$

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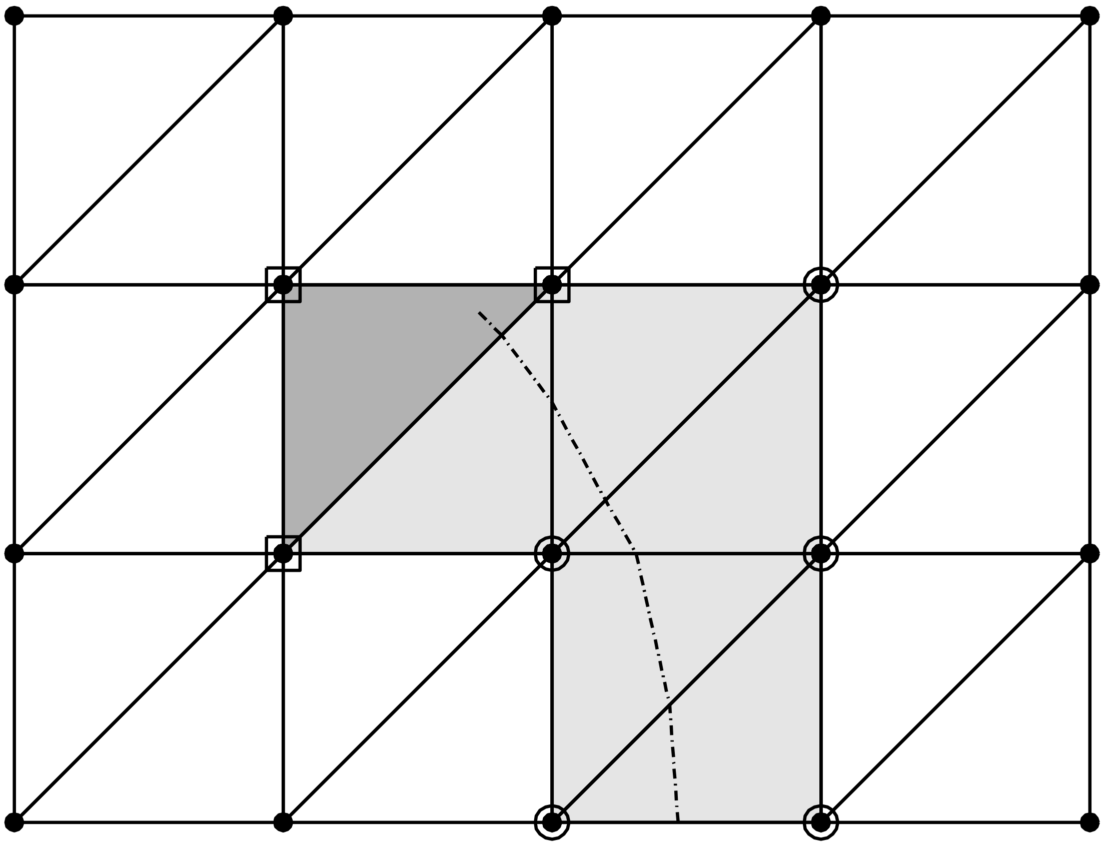
XFEM – tip enrichment



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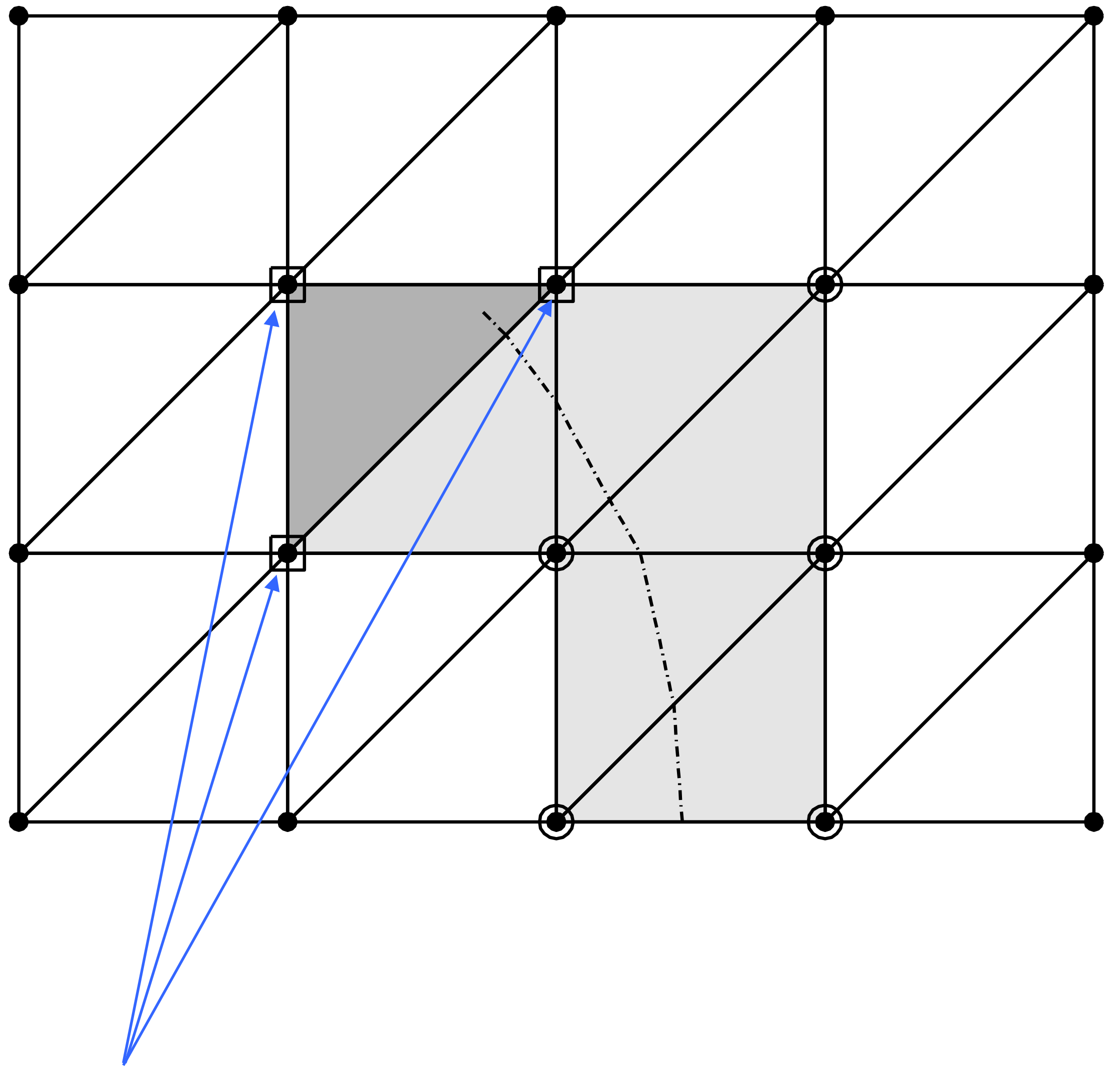
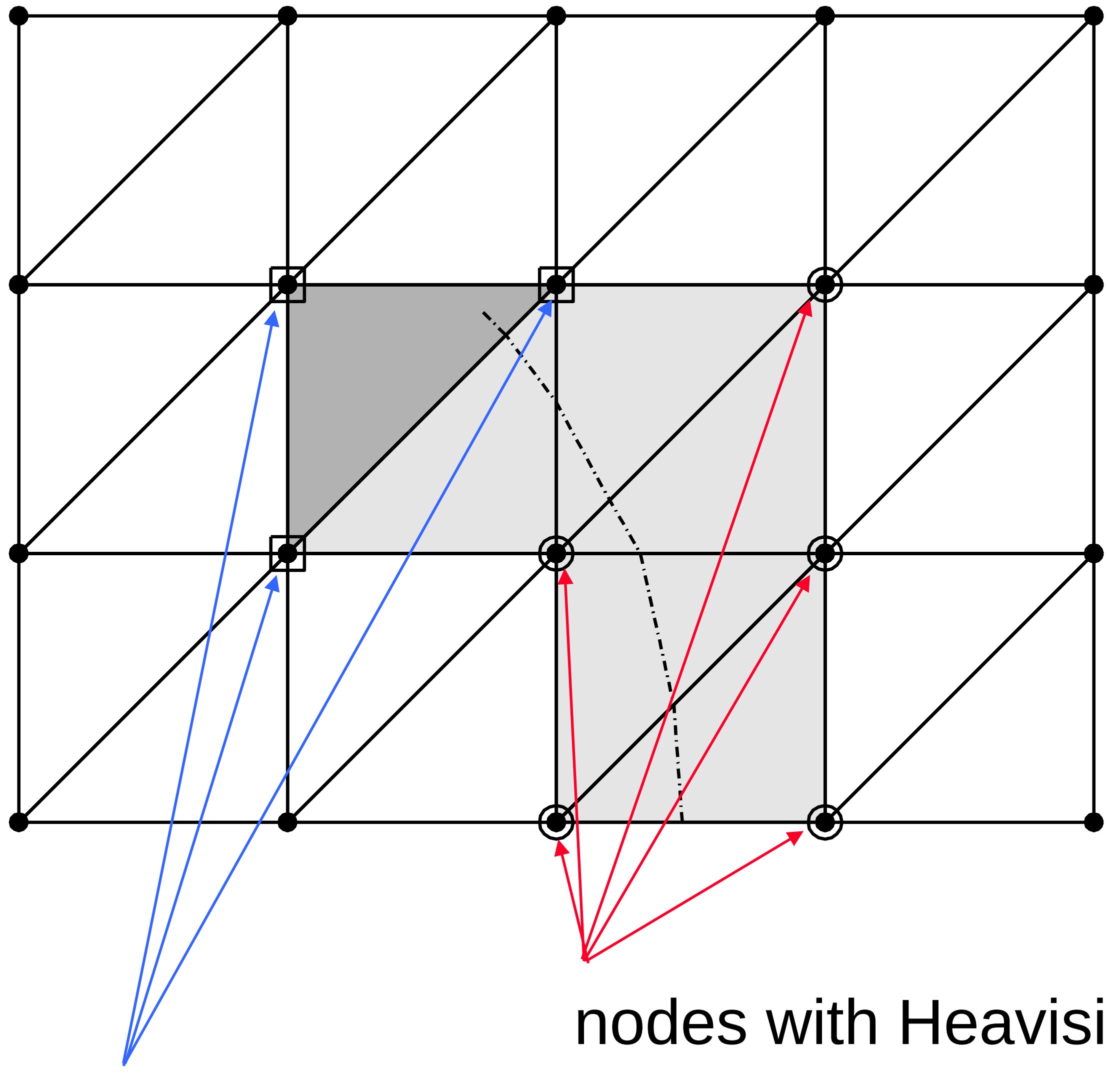


nodes with enrichment by near-tip functions

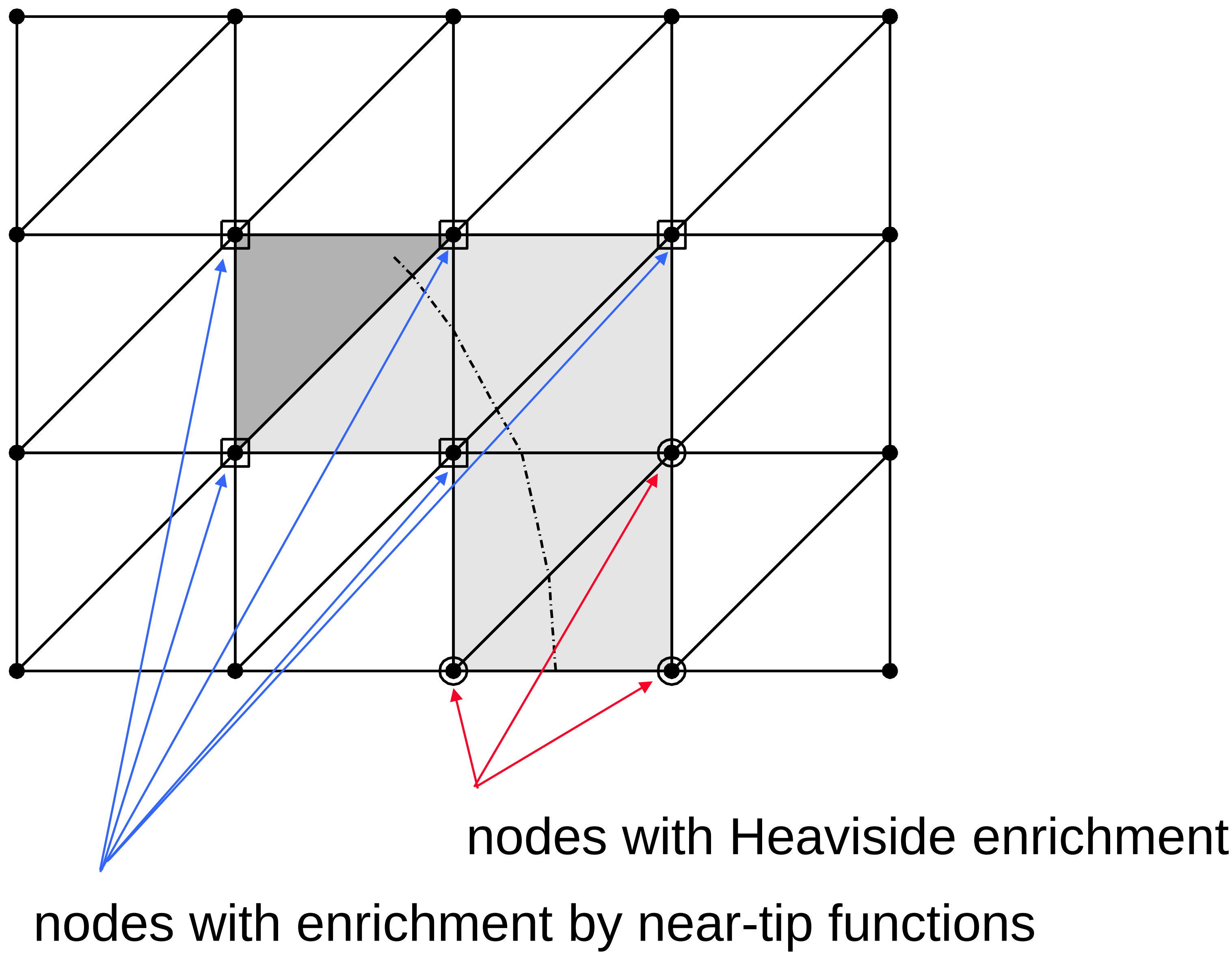
XFEM – tip enrichment



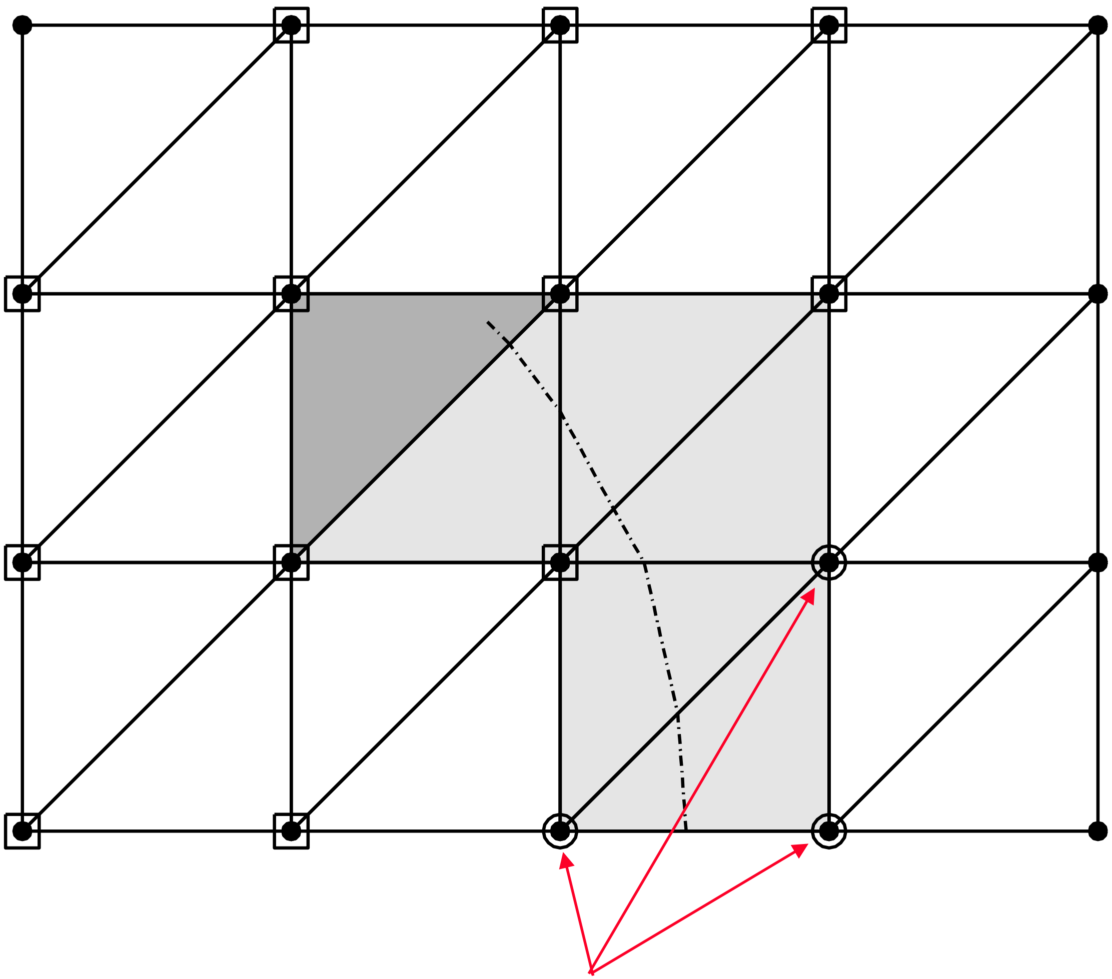
nodes with Heaviside enrichment

nodes with enrichment by near-tip functions

XFEM – tip enrichment



XFEM – tip enrichment



nodes with Heaviside enrichment

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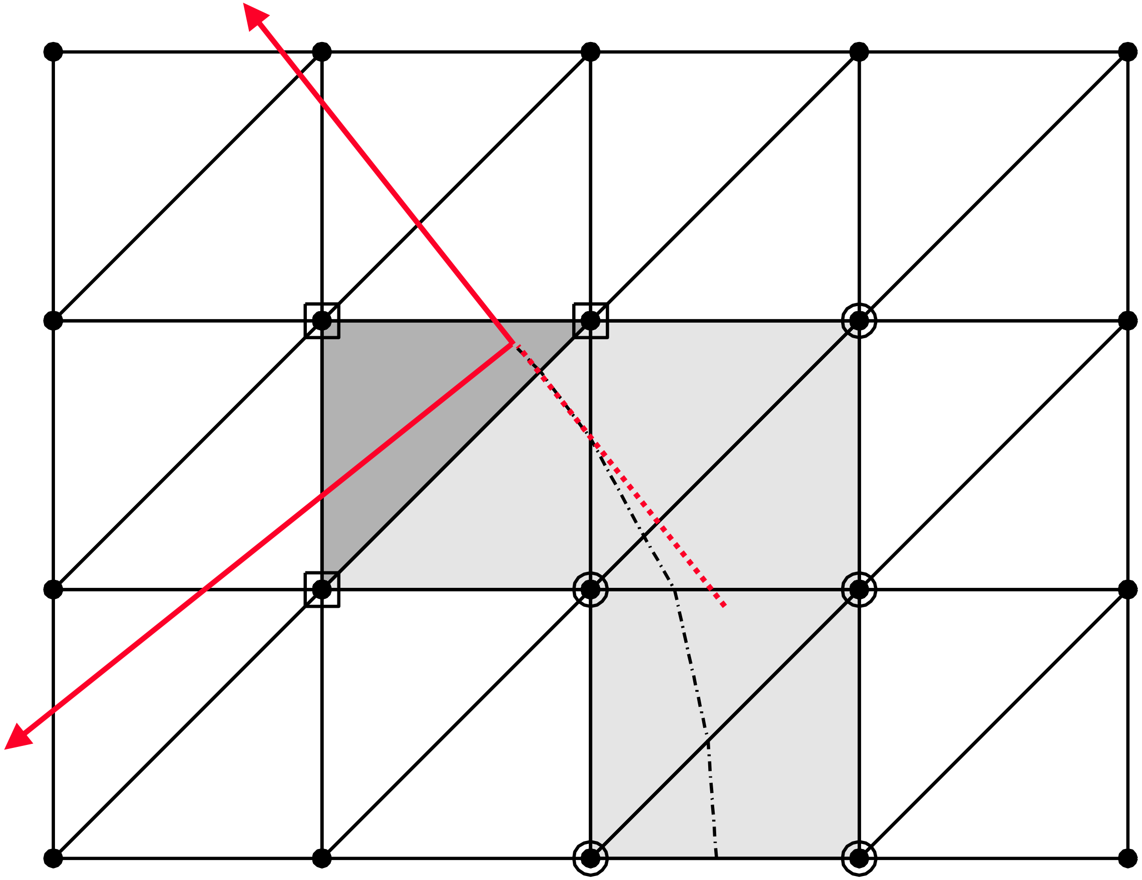
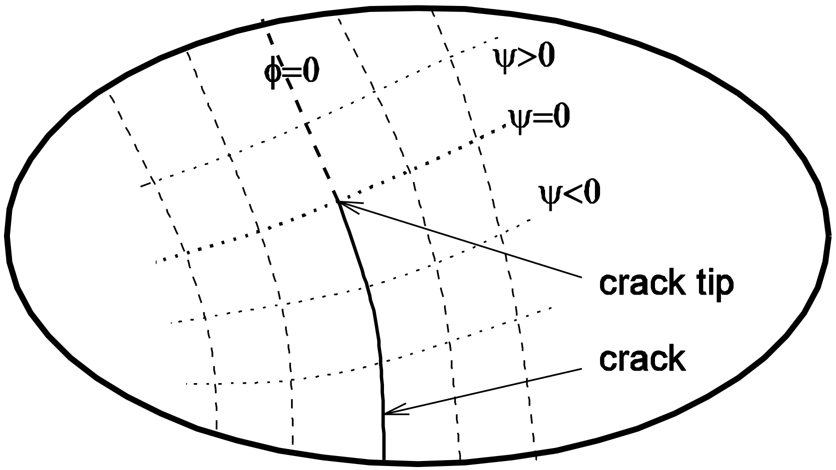
XFEM – tip enrichment



But if the crack is curved, we cannot define functions B_i in terms of the standard polar coordinates because B_1 would not be discontinuous across the crack but across the dotted line.

XFEM – level set functions

Remedy:
Construct curvilinear coordinates φ and ψ such that the crack is characterized by $\varphi = 0$ and $\psi \leq 0$



XFEM – level set functions

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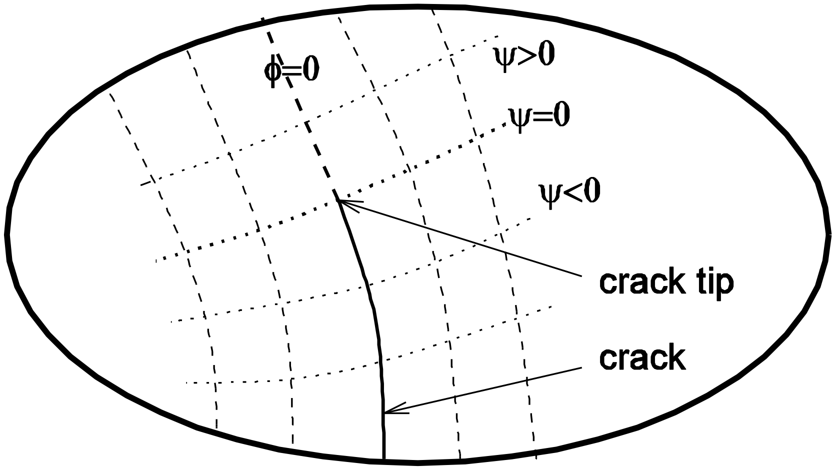


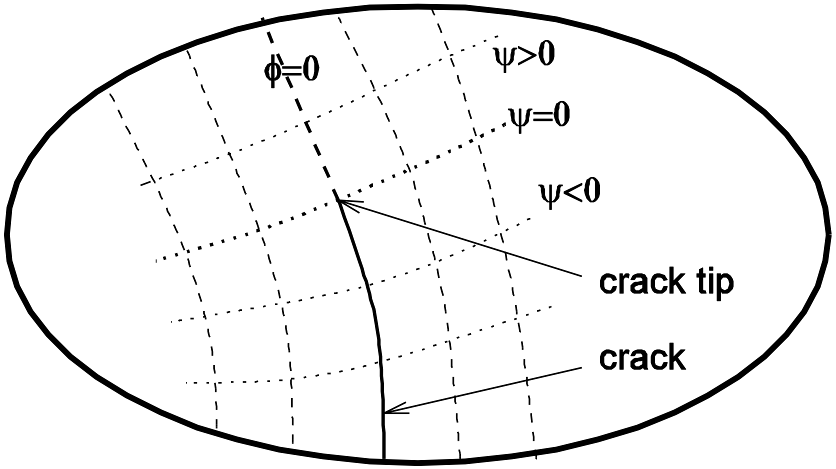
and define B_i in terms of the pseudo-polar coordinates

$$r(\psi,\varphi) = \sqrt{\psi^2 + \varphi^2}$$

$$\theta(\psi,\varphi) = \operatorname{sgn}(\varphi)\arccos\frac{\psi}{\sqrt{\psi^2 + \varphi^2}}$$

XFEM – level set functions

Functions φ and ψ are the so-called **level set functions.**

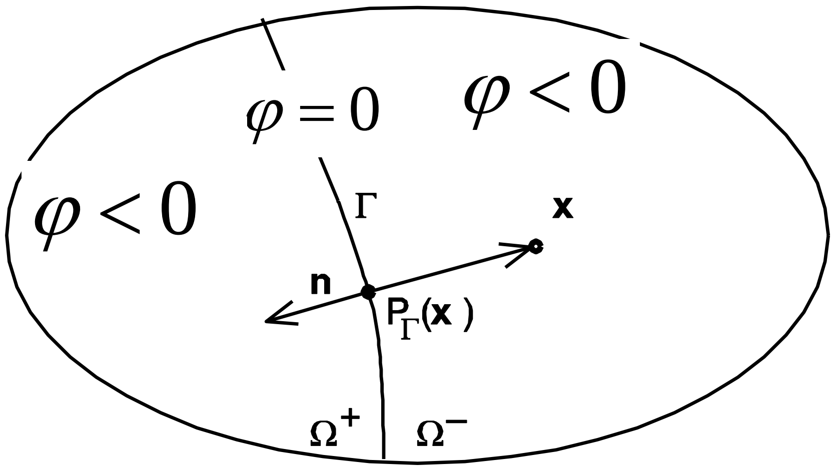


They are defined by their values at nodes around the crack and interpolated using the standard shape functions:

$$\varphi(\mathbf{x}) = \sum_I N_I(\mathbf{x})\,\varphi_I, \quad \psi(\mathbf{x}) = \sum_I N_I(\mathbf{x})\,\psi_I$$

XFEM – level set functions

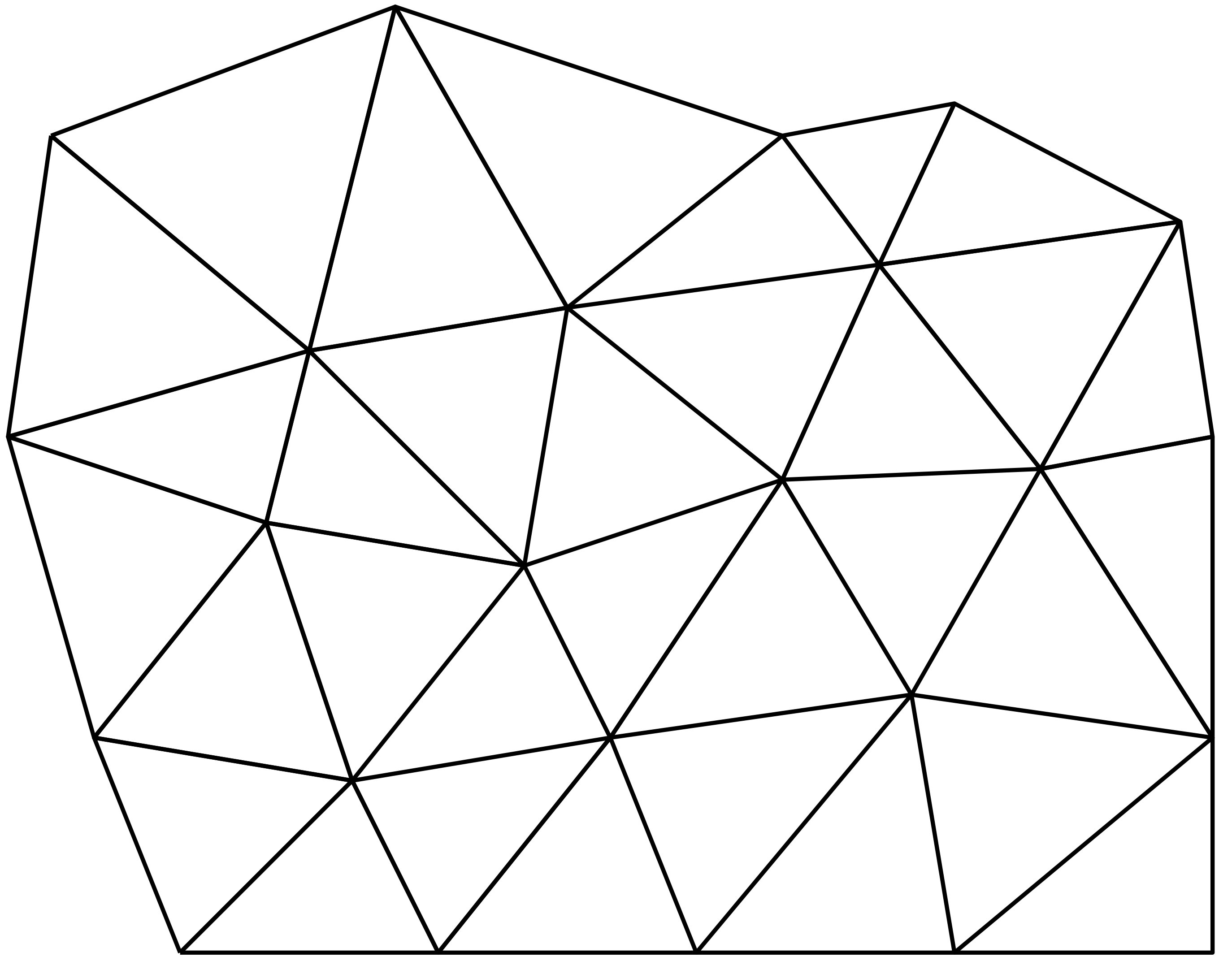
For an existing crack, function φ can be constructed as the signed distance function:



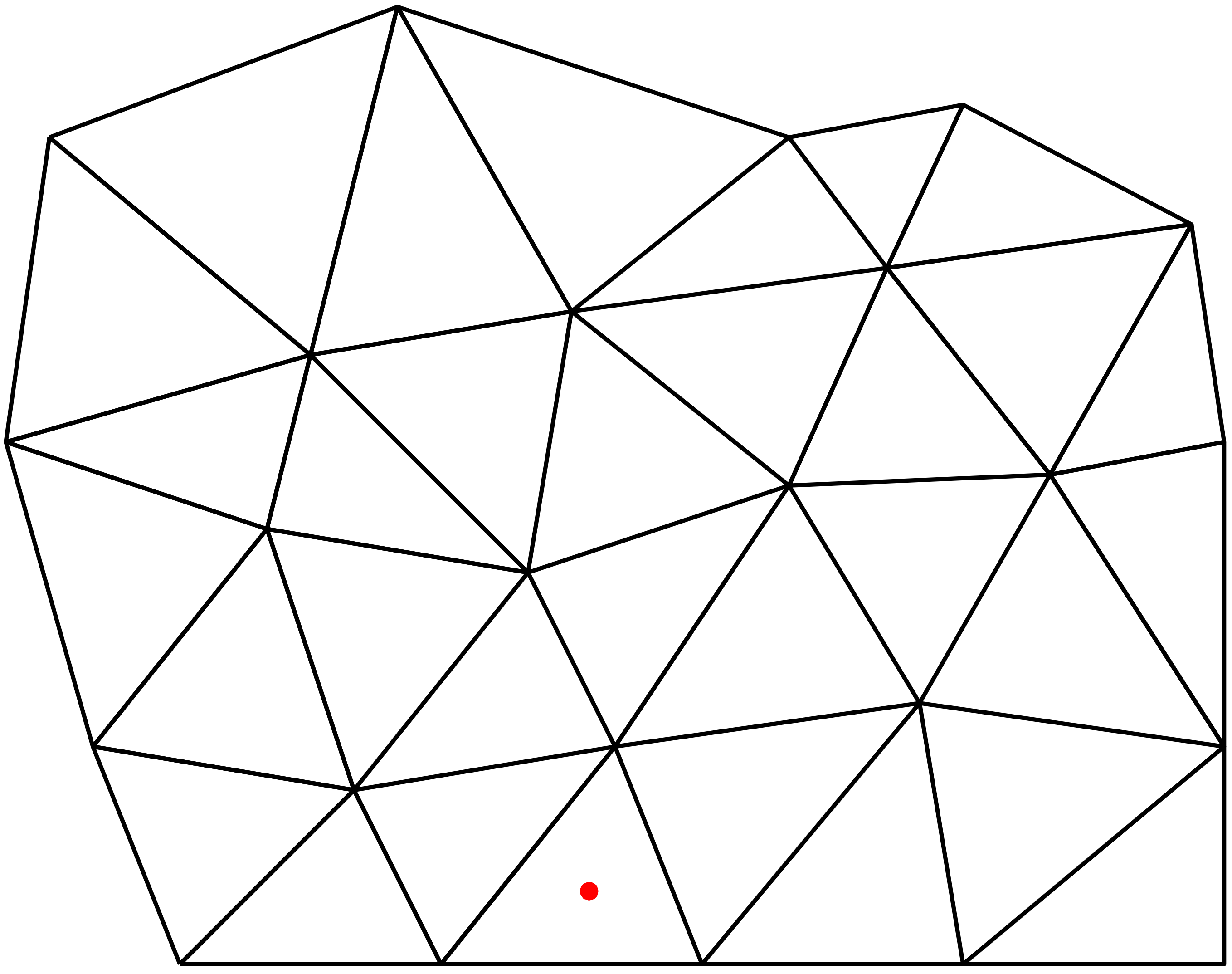
$$\varphi(\mathbf{x}) = \|\mathbf{x} - P_\Gamma(\mathbf{x})\| \operatorname{sgn}\left[(\mathbf{x} - P_\Gamma(\mathbf{x})) \cdot \mathbf{n}(P_\Gamma(\mathbf{x}))\right]$$

Criteria for Direction of Crack Propagation

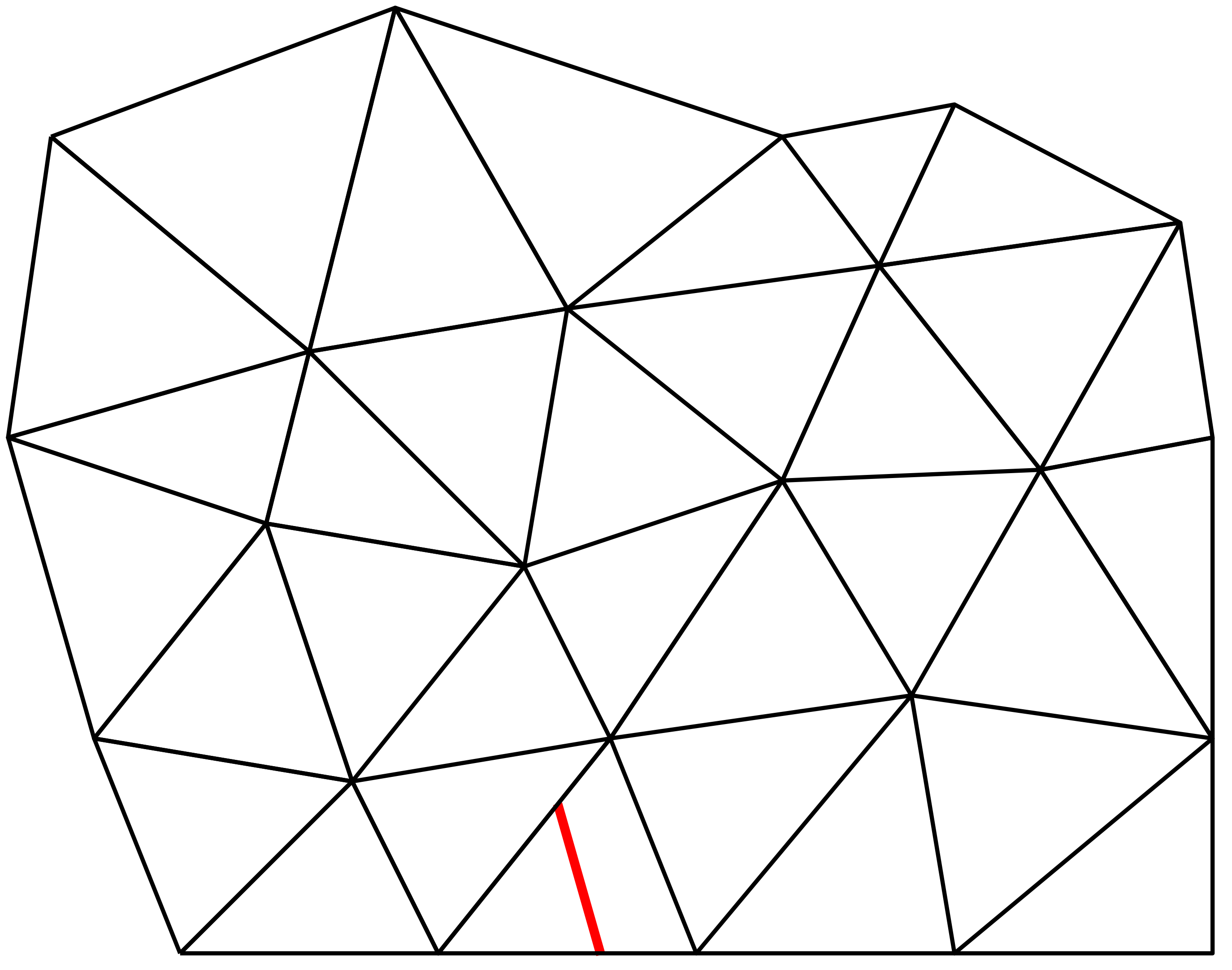
Tracking of a propagating crack



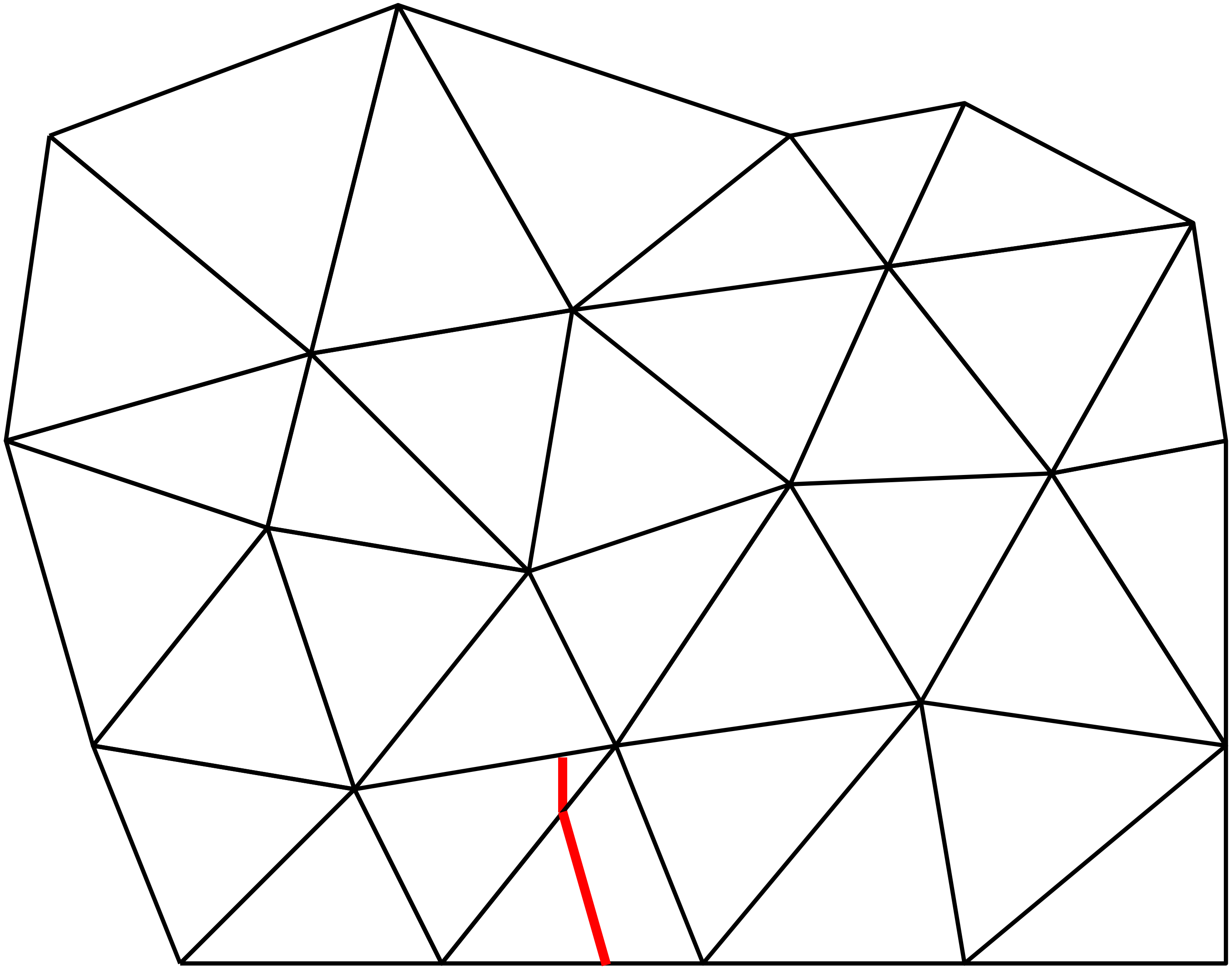
Tracking of a propagating crack



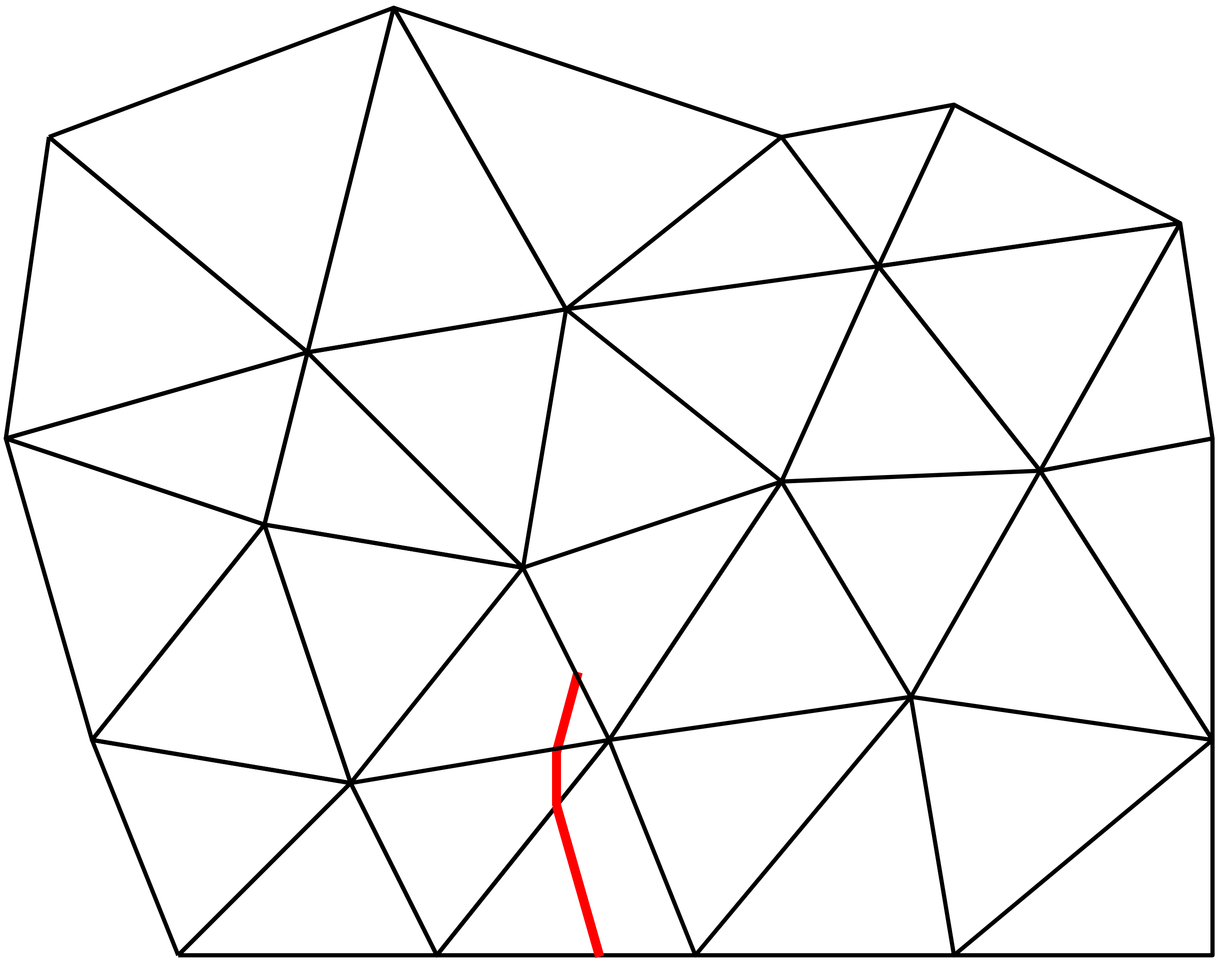
Tracking of a propagating crack



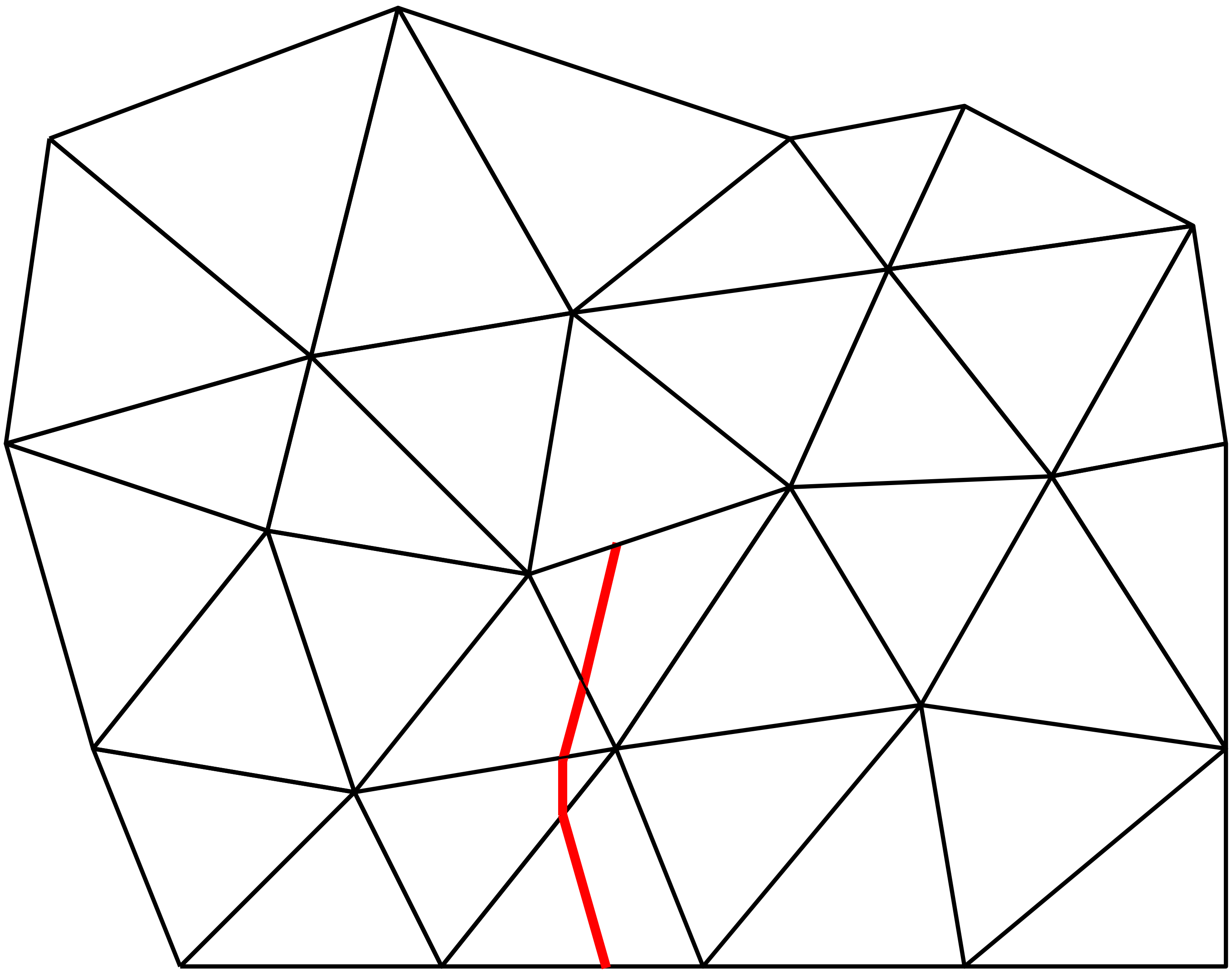
Tracking of a propagating crack



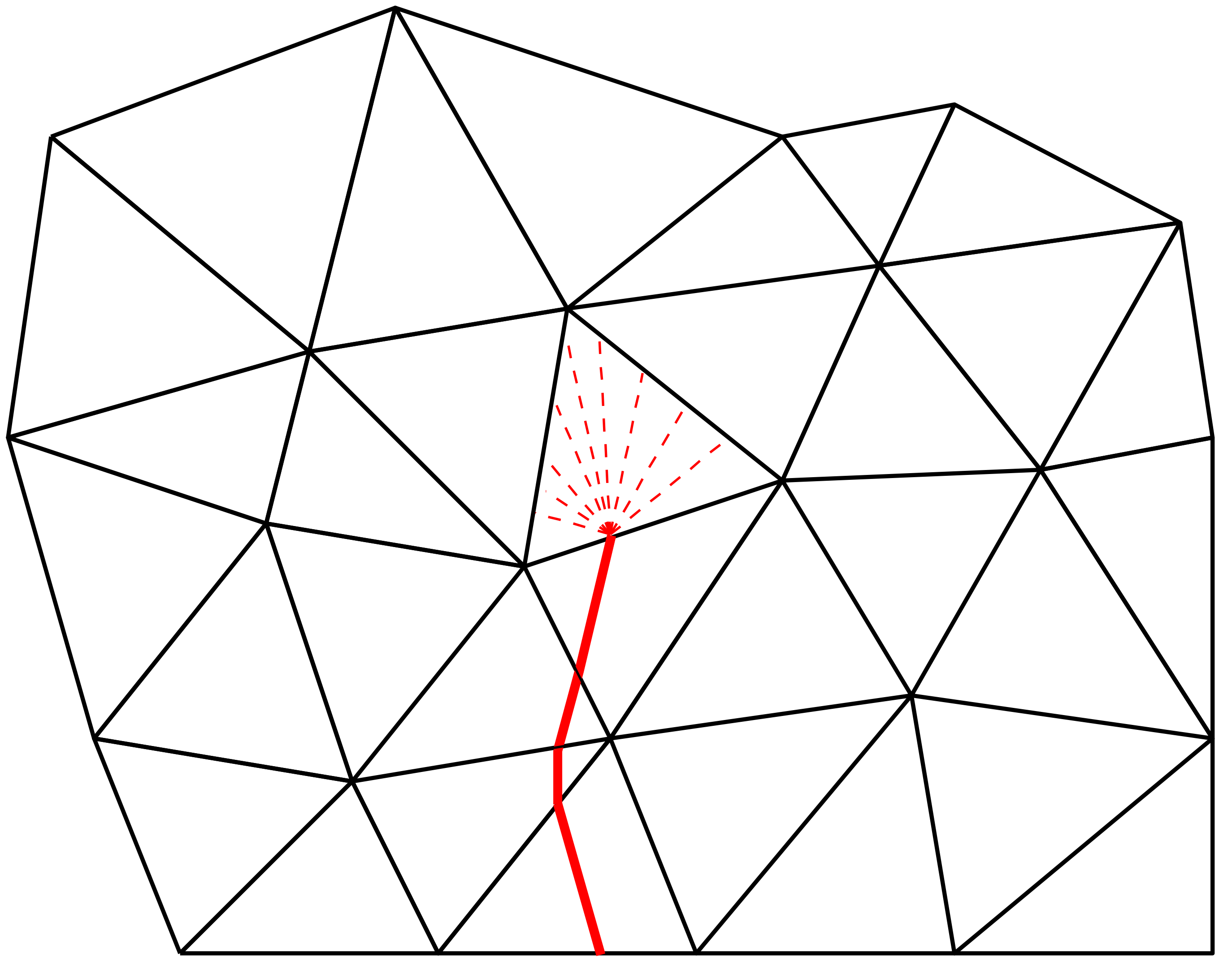
Tracking of a propagating crack



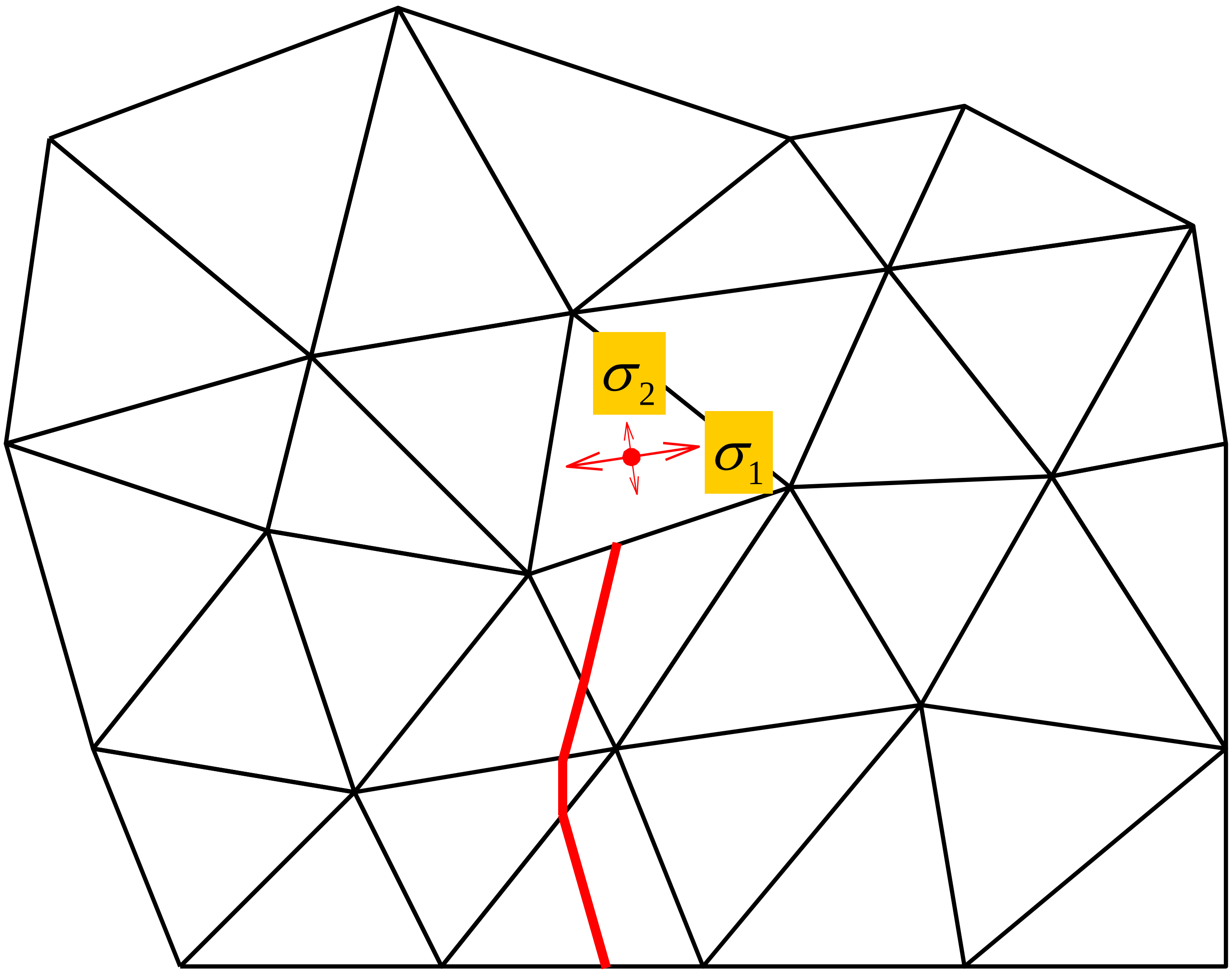
Tracking of a propagating crack



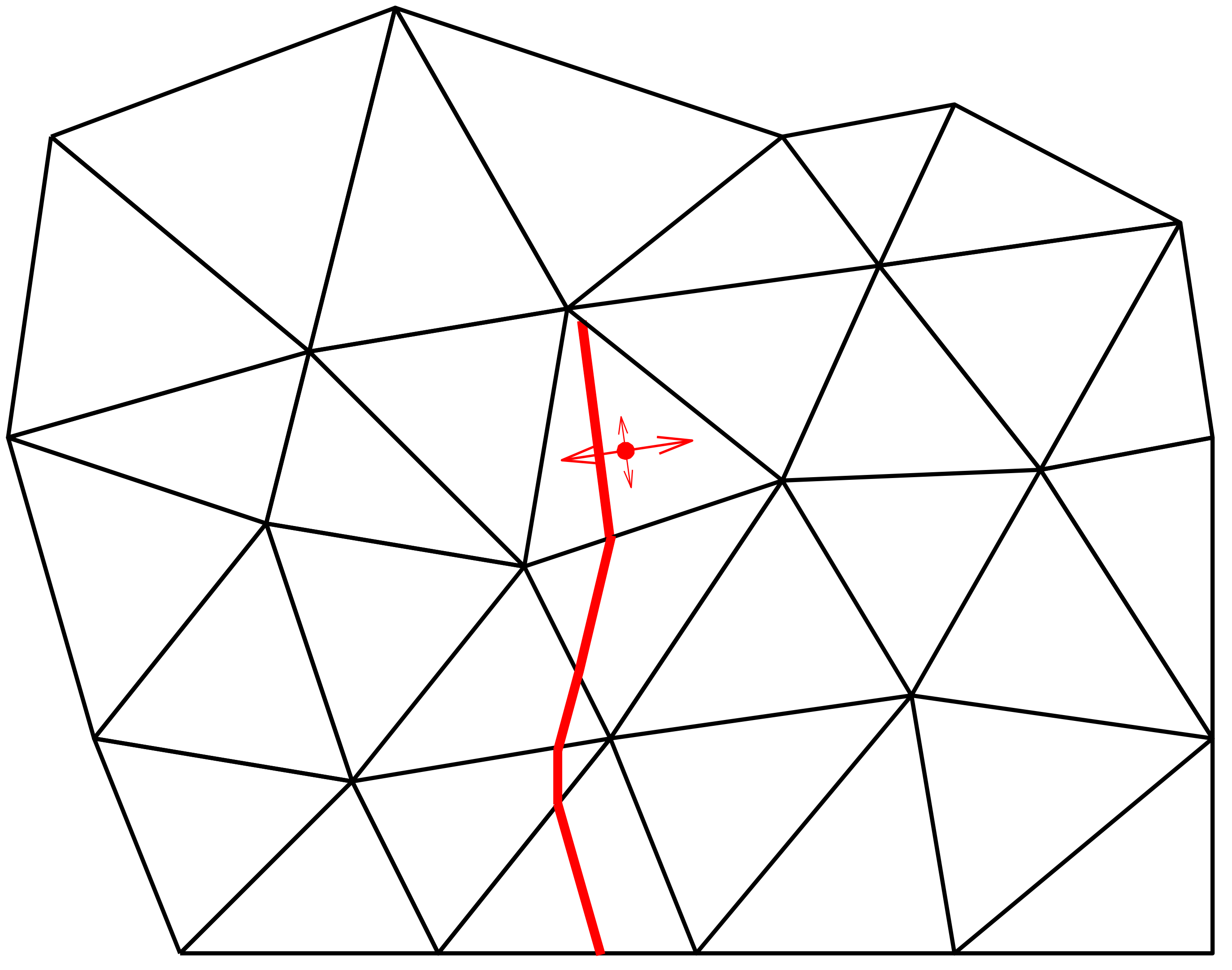
Tracking of a propagating crack



Tracking of a propagating crack

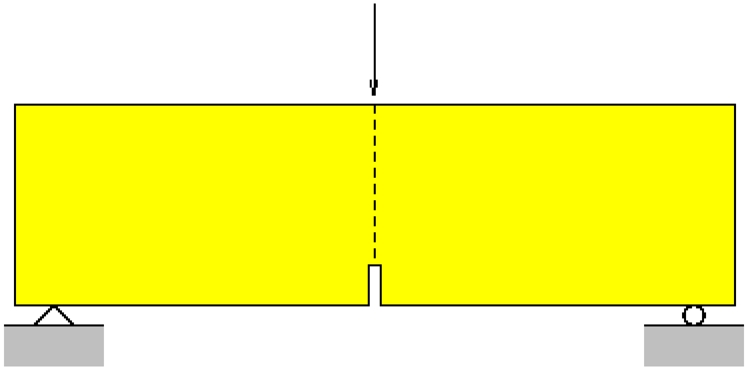


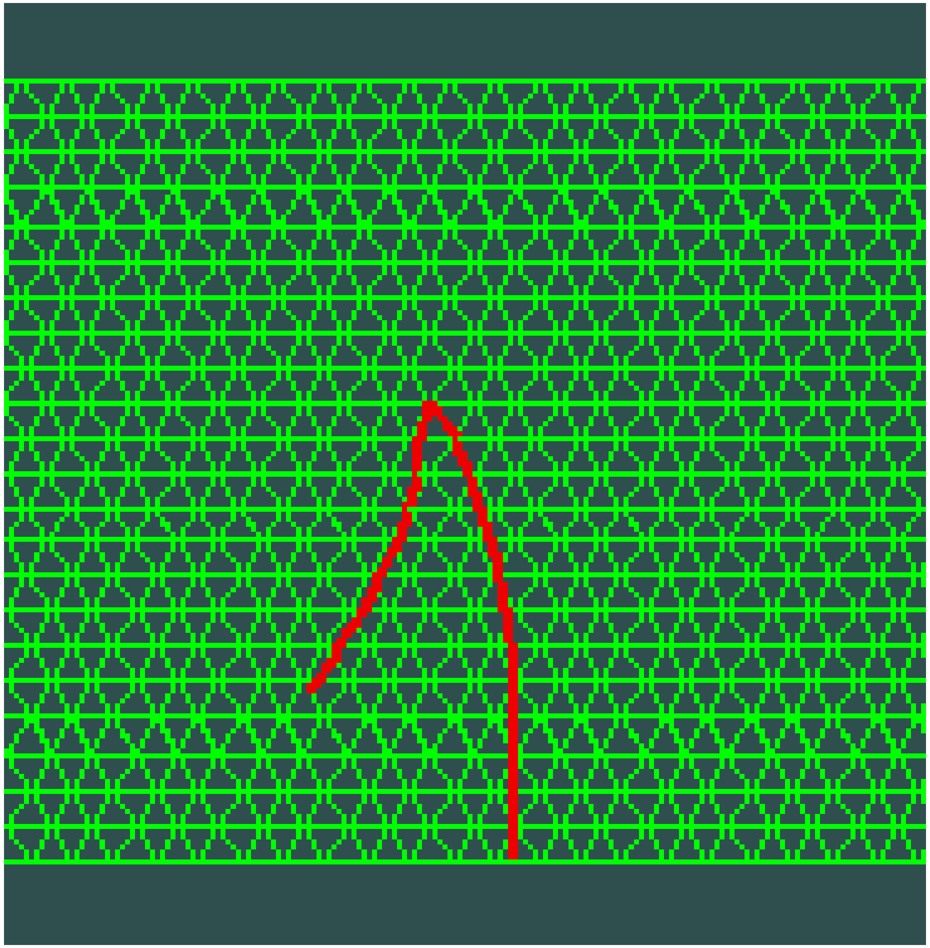
Tracking of a propagating crack



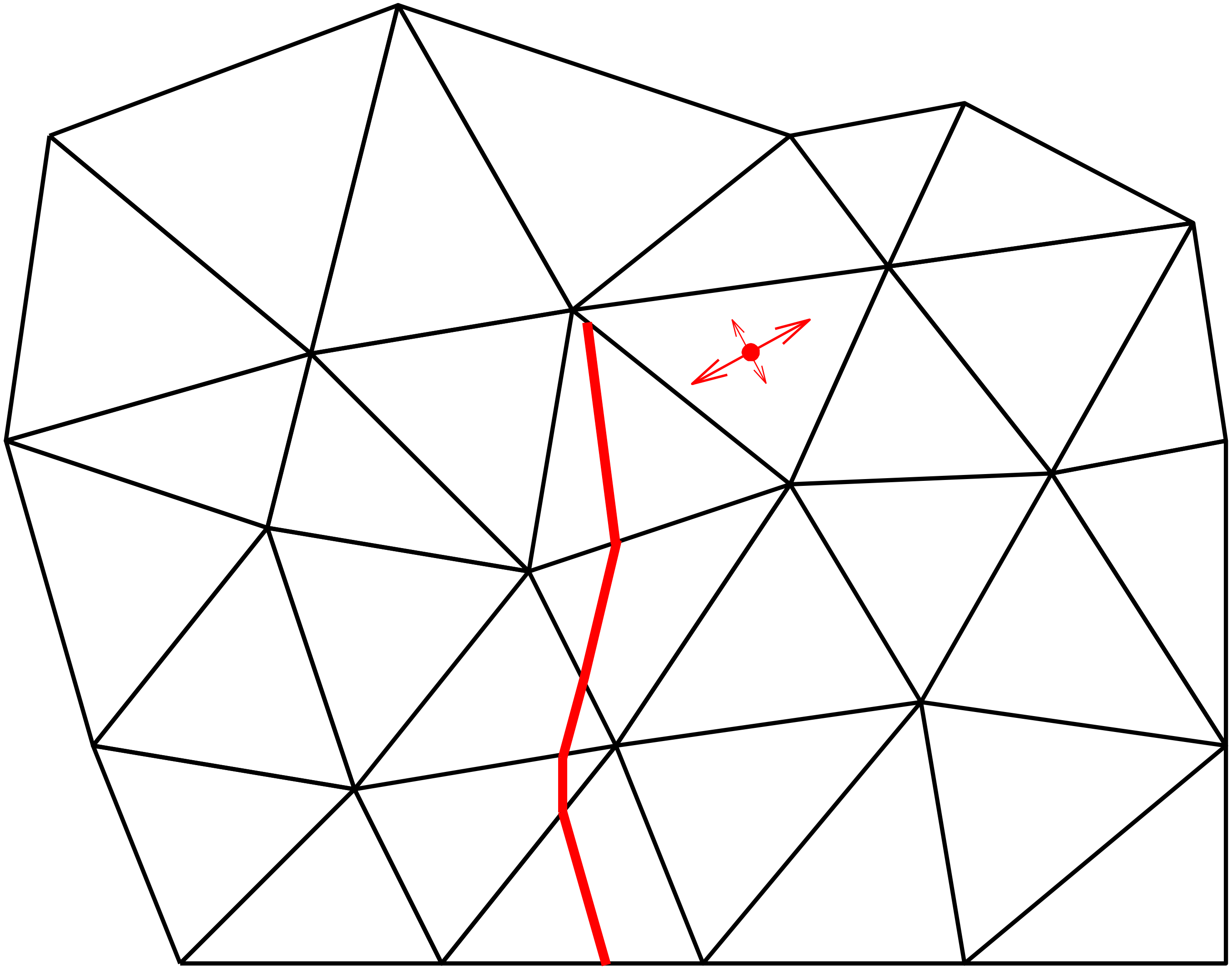
Tracking of a propagating crack

Crack direction = normal to the maximum principal stress direction

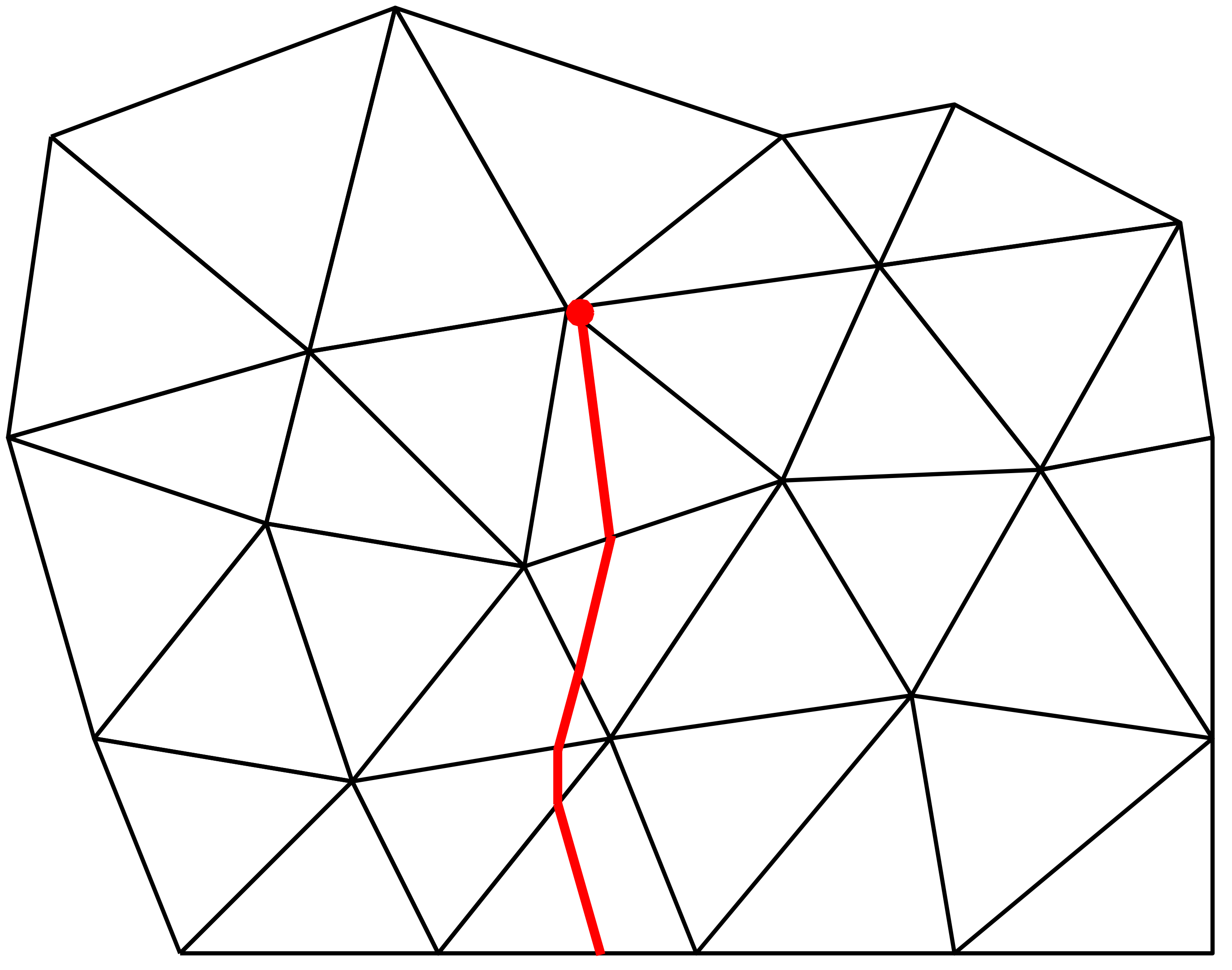




Tracking of a propagating crack

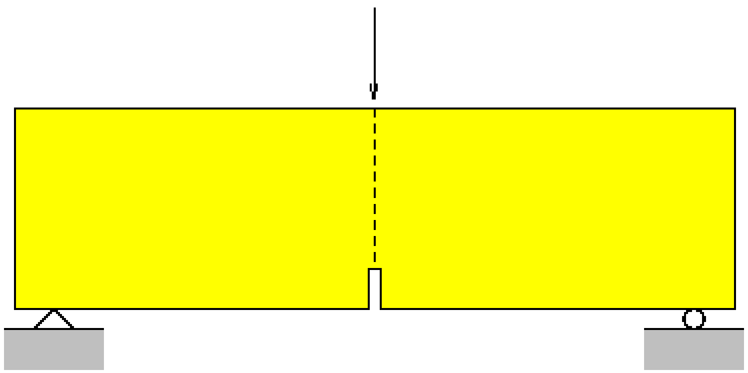


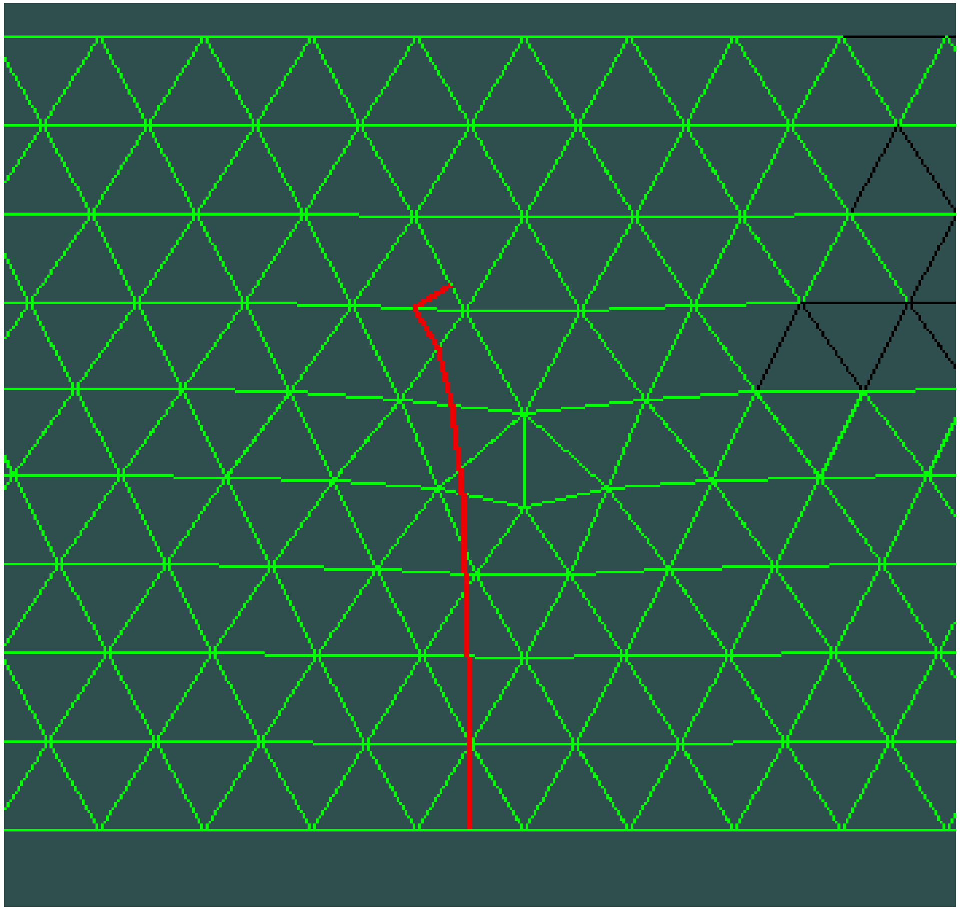
Tracking of a propagating crack



Tracking of a propagating crack

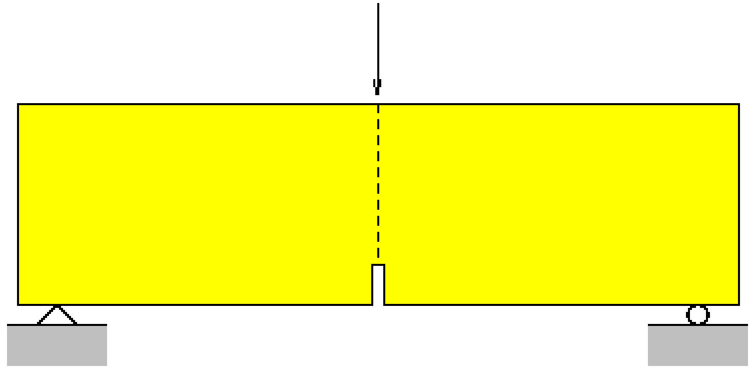
Crack direction = normal to the direction of
maximum principal **nonlocal** stress (or strain)



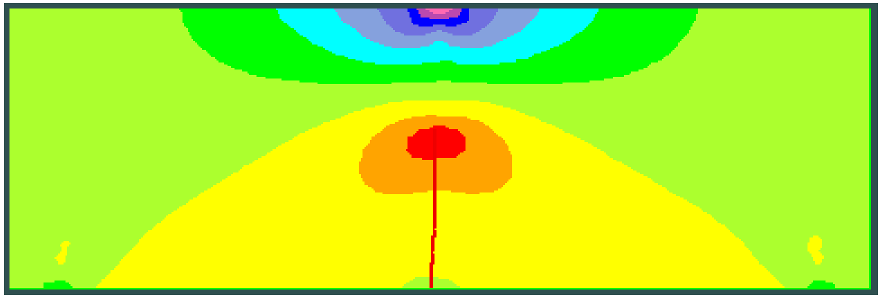


Tracking of a propagating crack

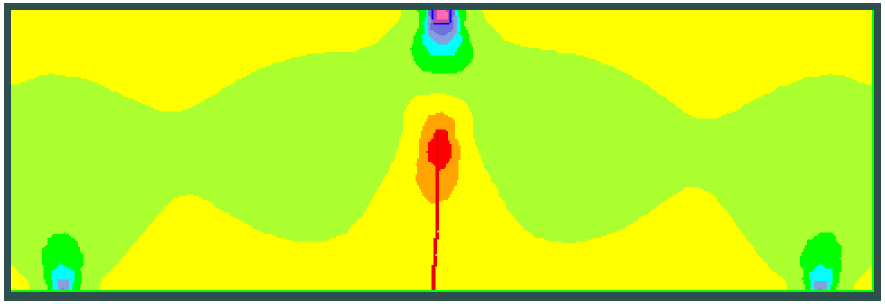
Stress state around the tip of a cohesive crack
is very close to equibiaxial tension



σ_x

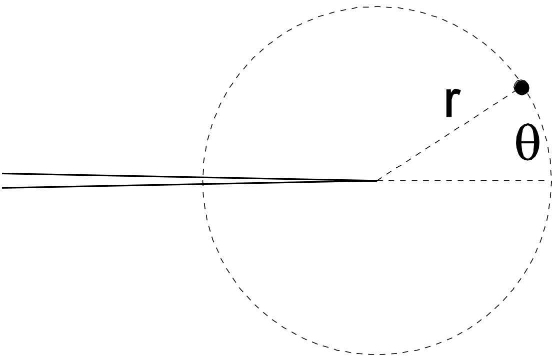


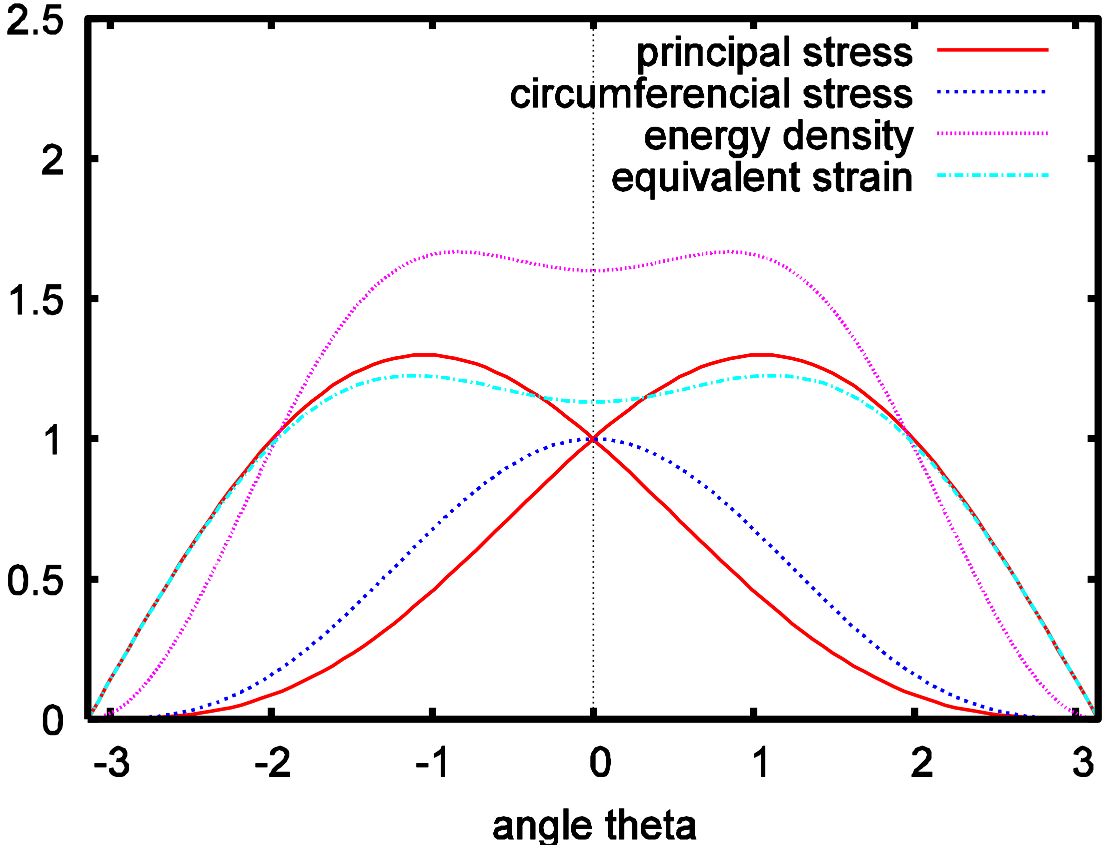
σ_y



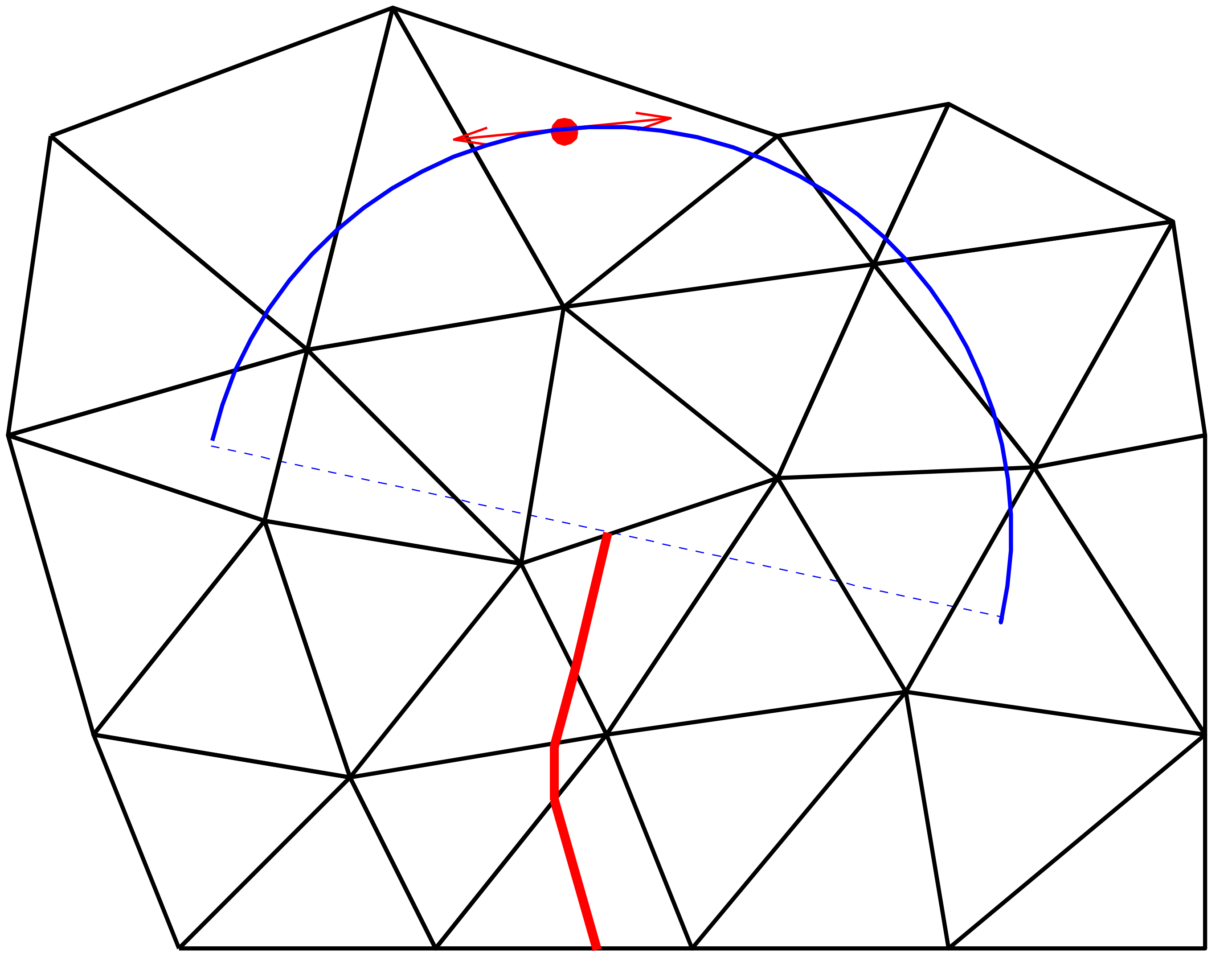
Tracking of a propagating crack

Stress distribution at constant distance from the tip of a stress-free crack

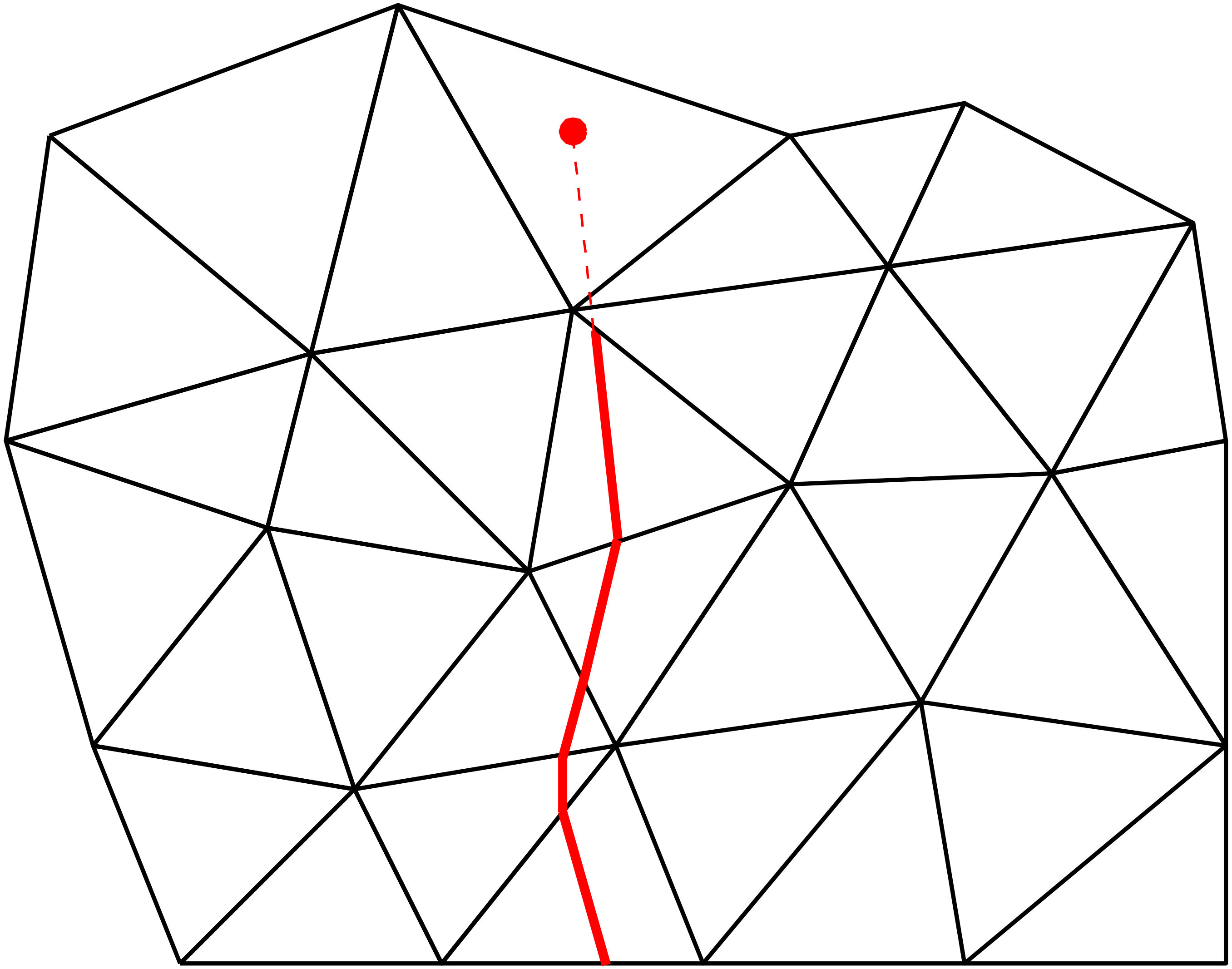




Tracking of a propagating crack

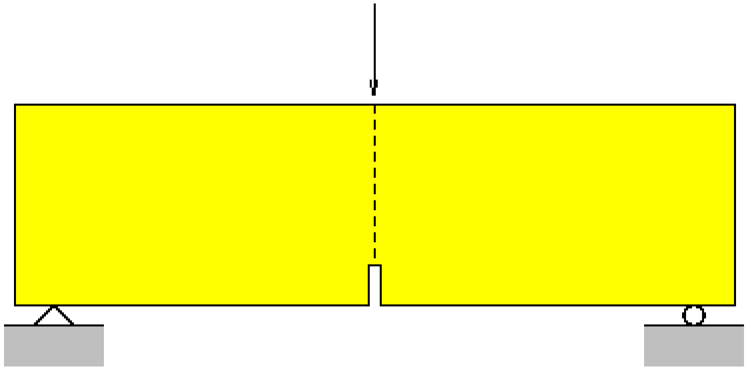


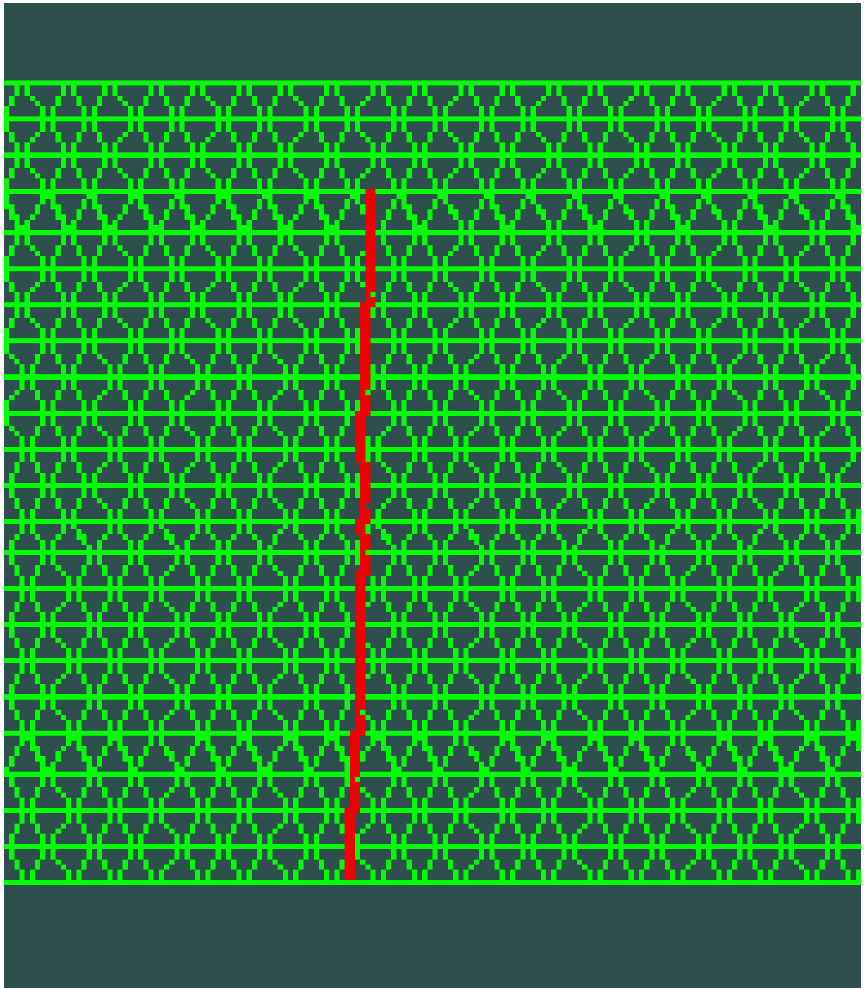
Tracking of a propagating crack



Tracking of a propagating crack

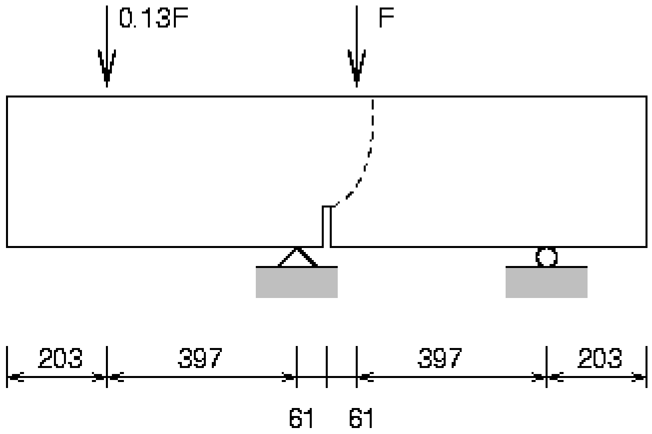
Crack direction = normal to the direction of maximum circumferential stress

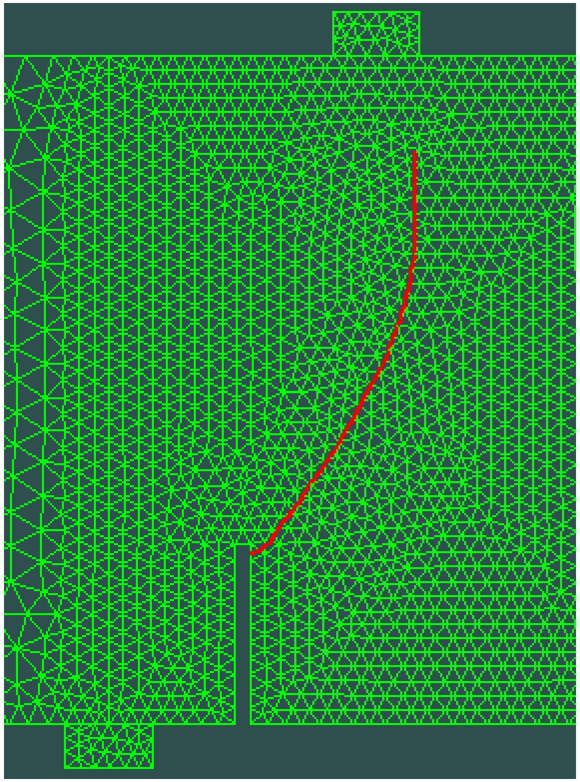




Tracking of a propagating crack

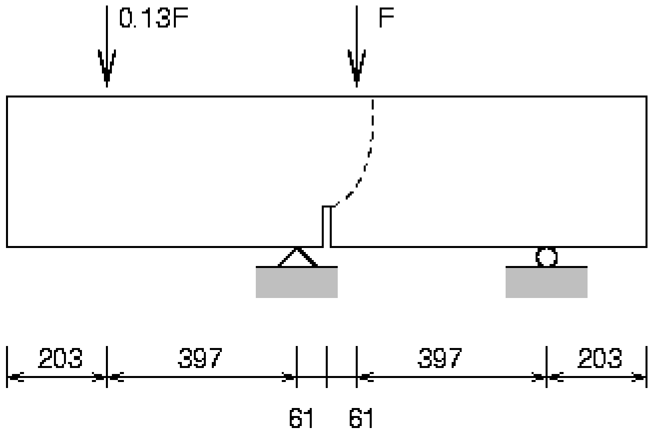
Crack direction = normal to the direction of
maximum circumferential stress

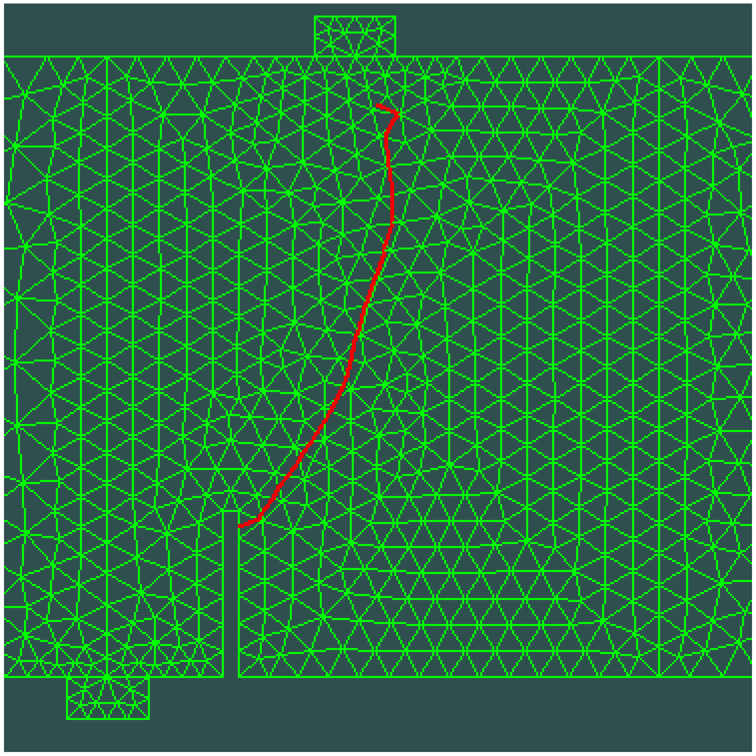




Tracking of a propagating crack

Crack direction = normal to the direction of
maximum circumferential stress





F.5

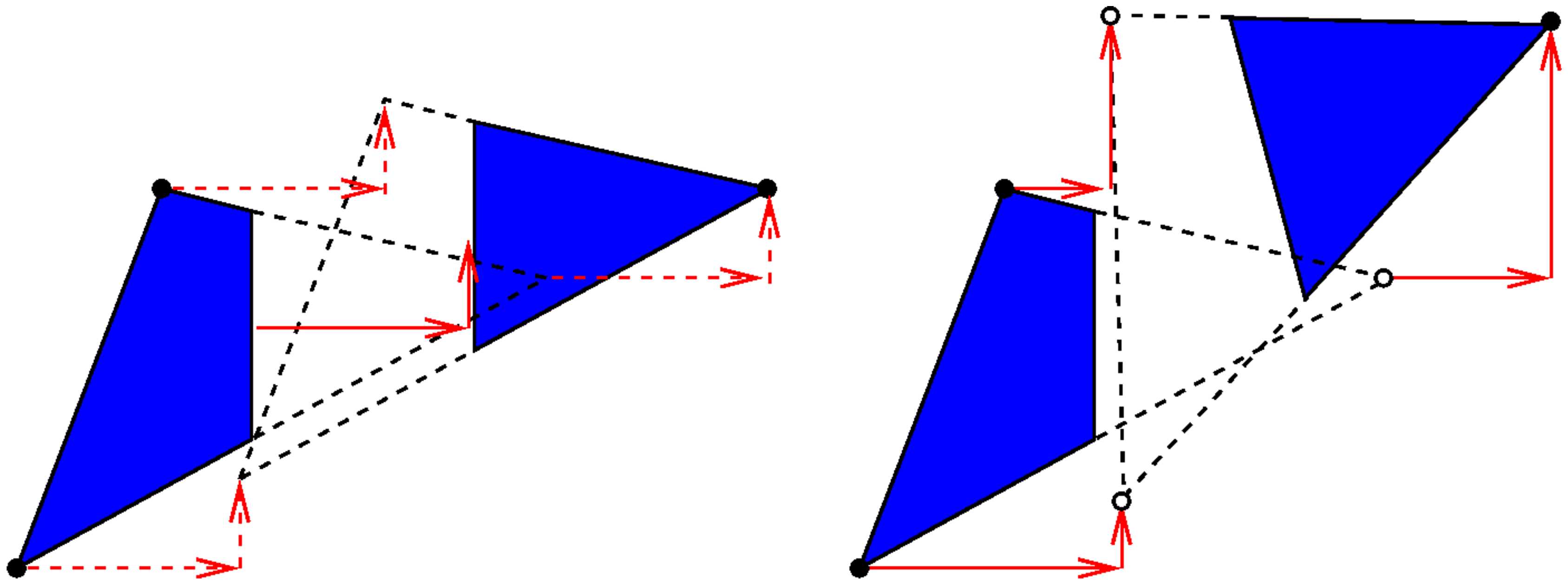
Comparison:

EED-EAS versus XFEM-PUM

Comparison of EED-EAS and XFEM-PUM

Embedded discontinuity

Extended finite elements



Comparison of EED-EAS and XFEM-PUM

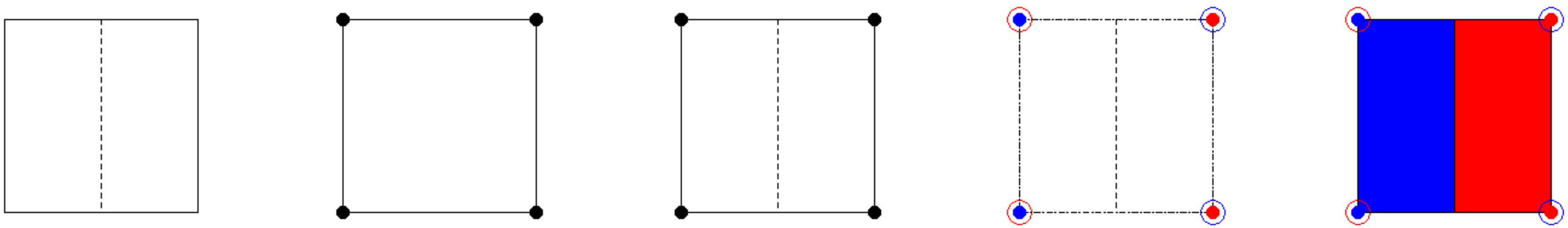
	Embedded discontinuity	Extended finite elements
DOF's added	locally	globally
and related to	elements	nodes

Comparison of EED-EAS and XFEM-PUM

	Embedded discontinuity	Extended finite elements
DOF‘s added	locally	globally
and related to	elements	nodes
Approximation of crack opening	discontinuous	continuous
Enrichment	incompatible	compatible

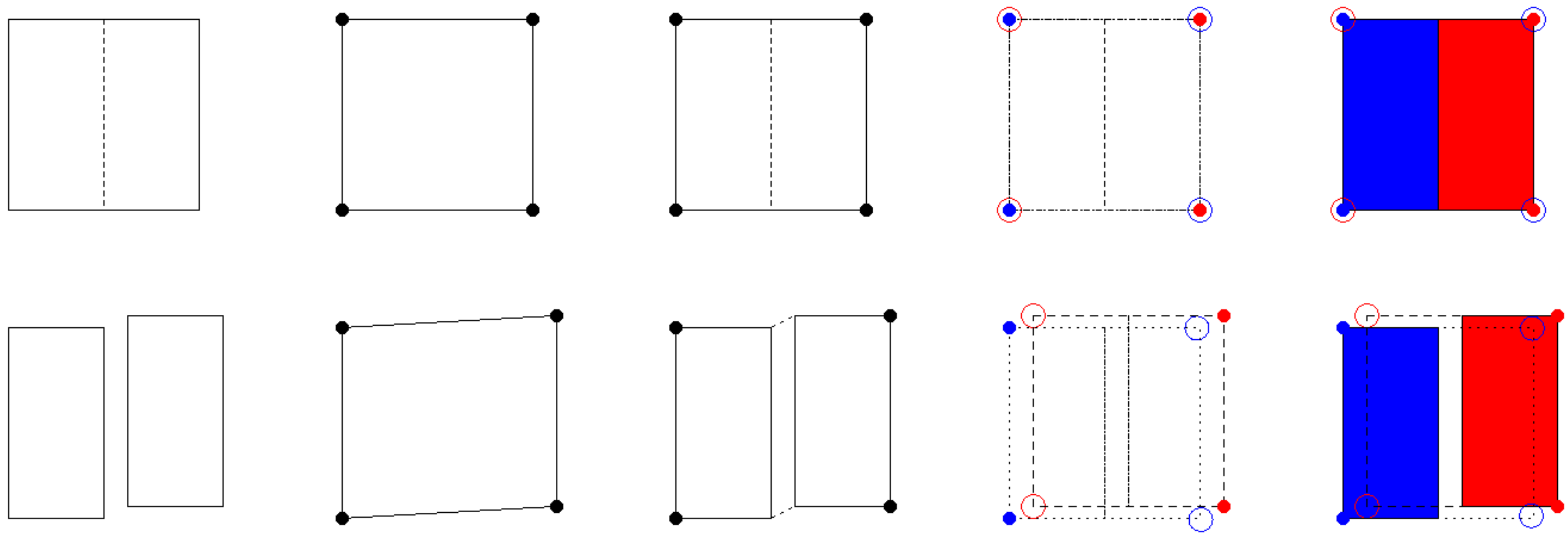
Separation test

physical smeared EED-EAS XFEM-PUM



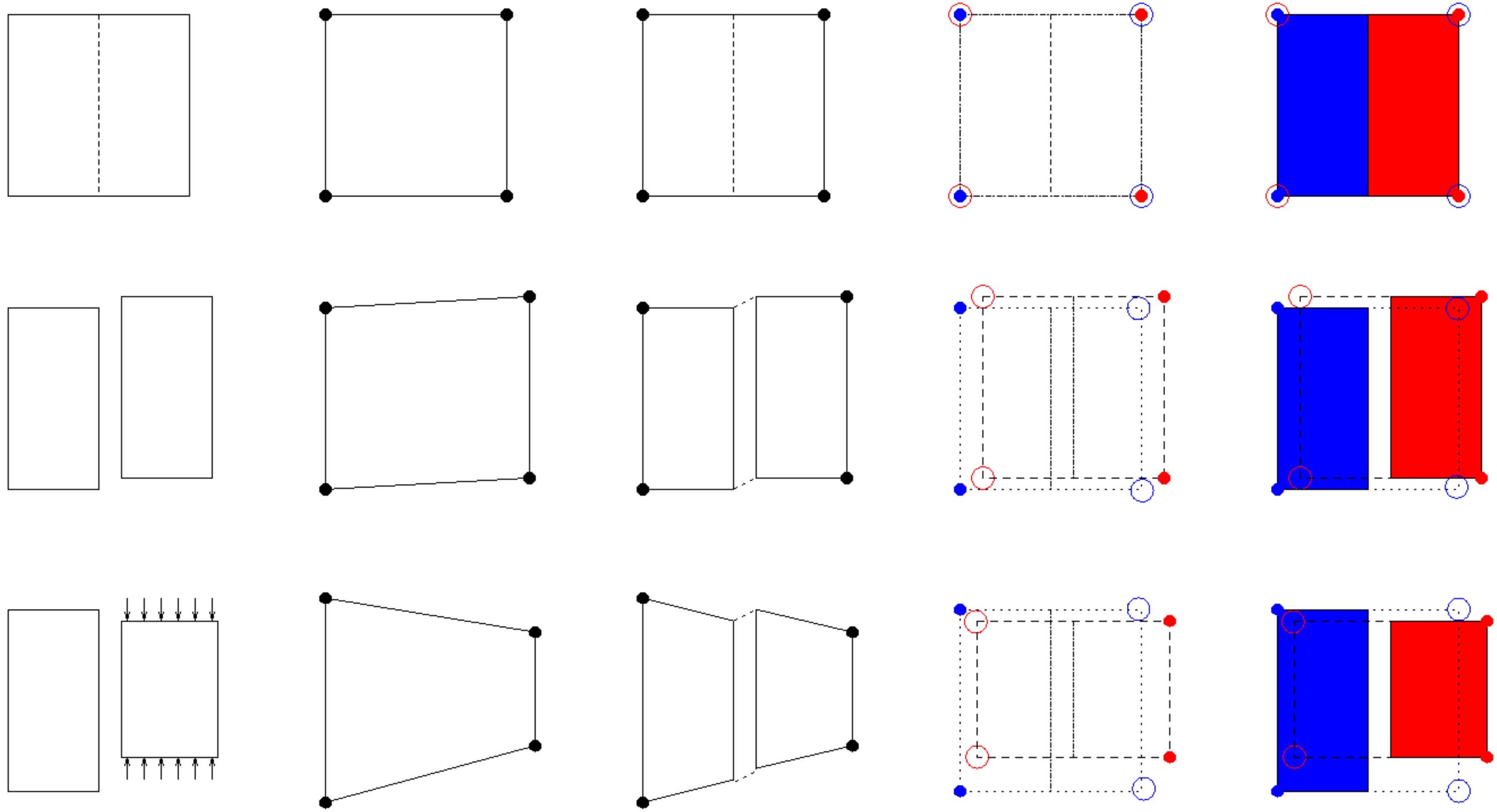
Separation test

physical smeared EED-EAS XFEM-PUM

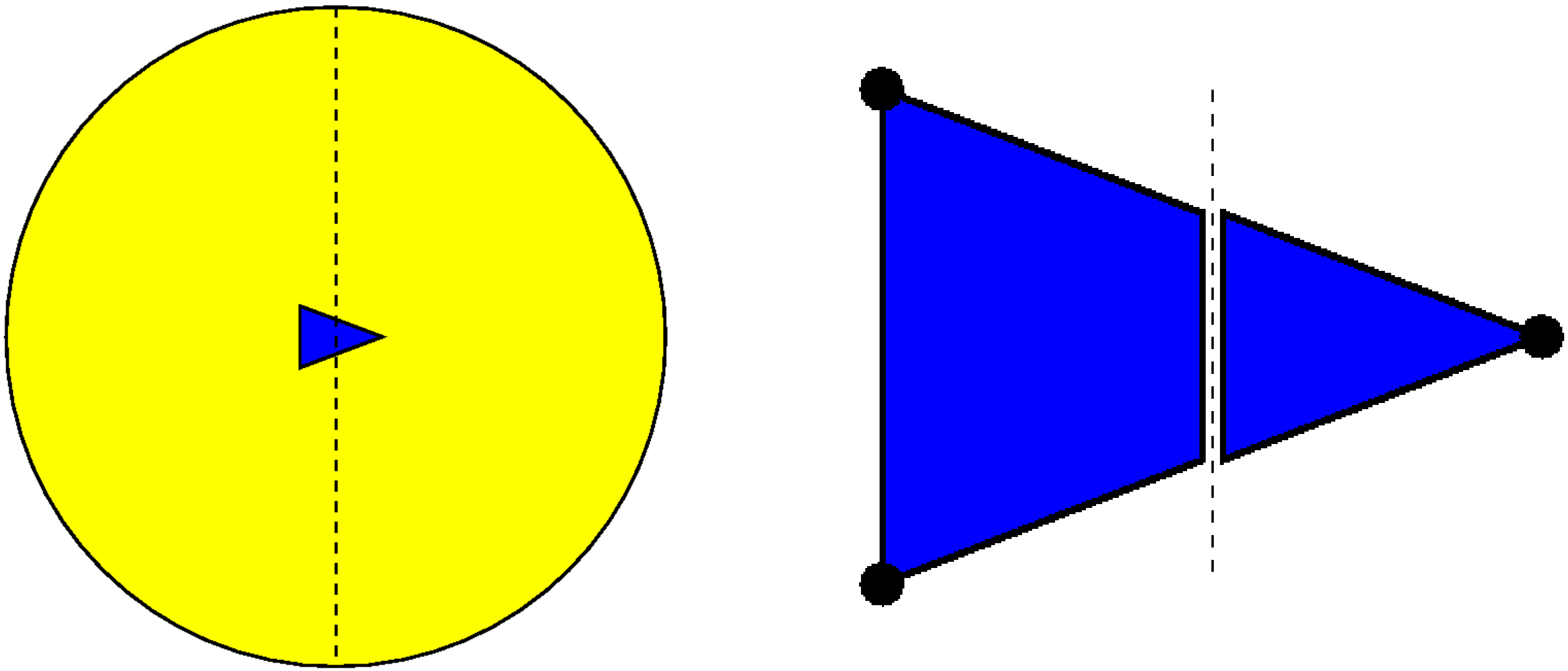


Separation test

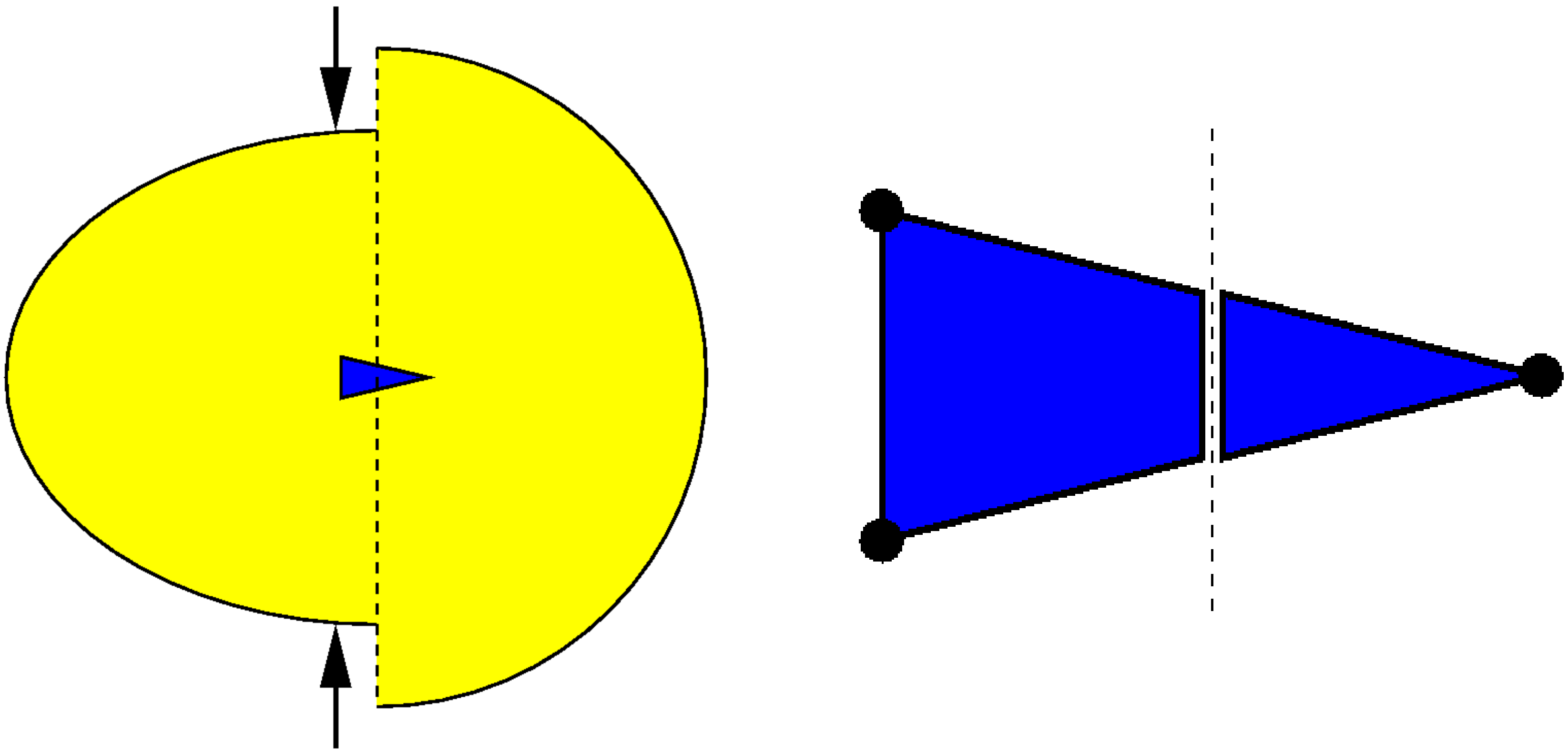
physical smeared EED-EAS XFEM-PUM



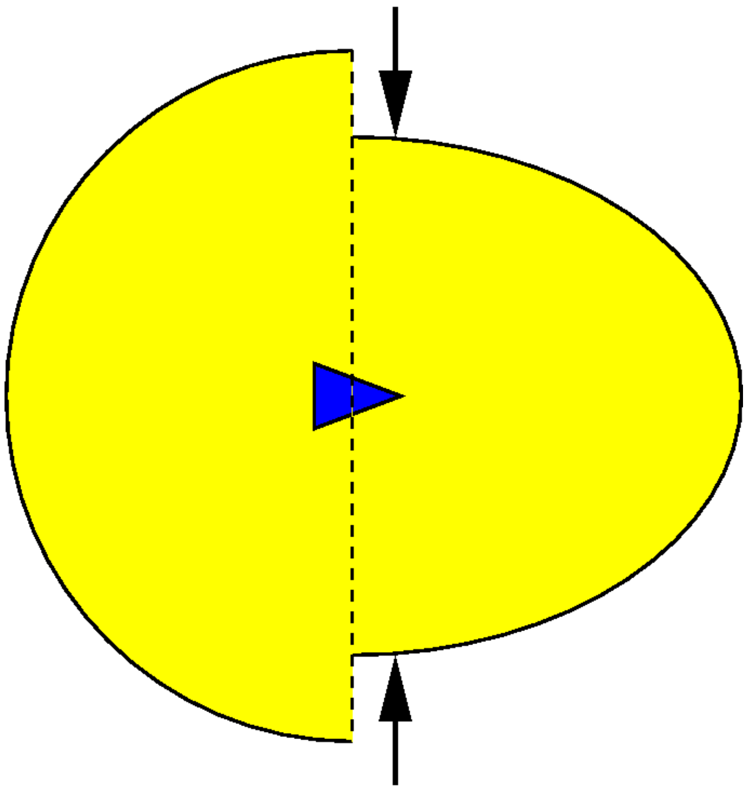
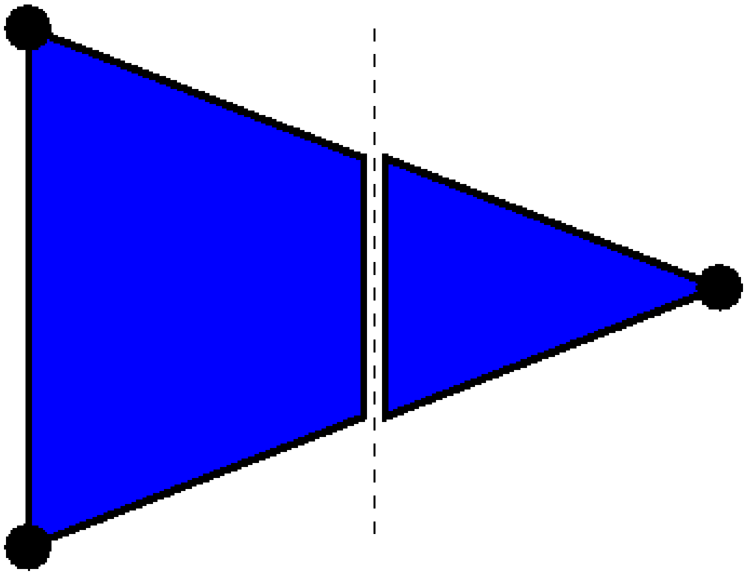
EED-EAS approach: partial coupling



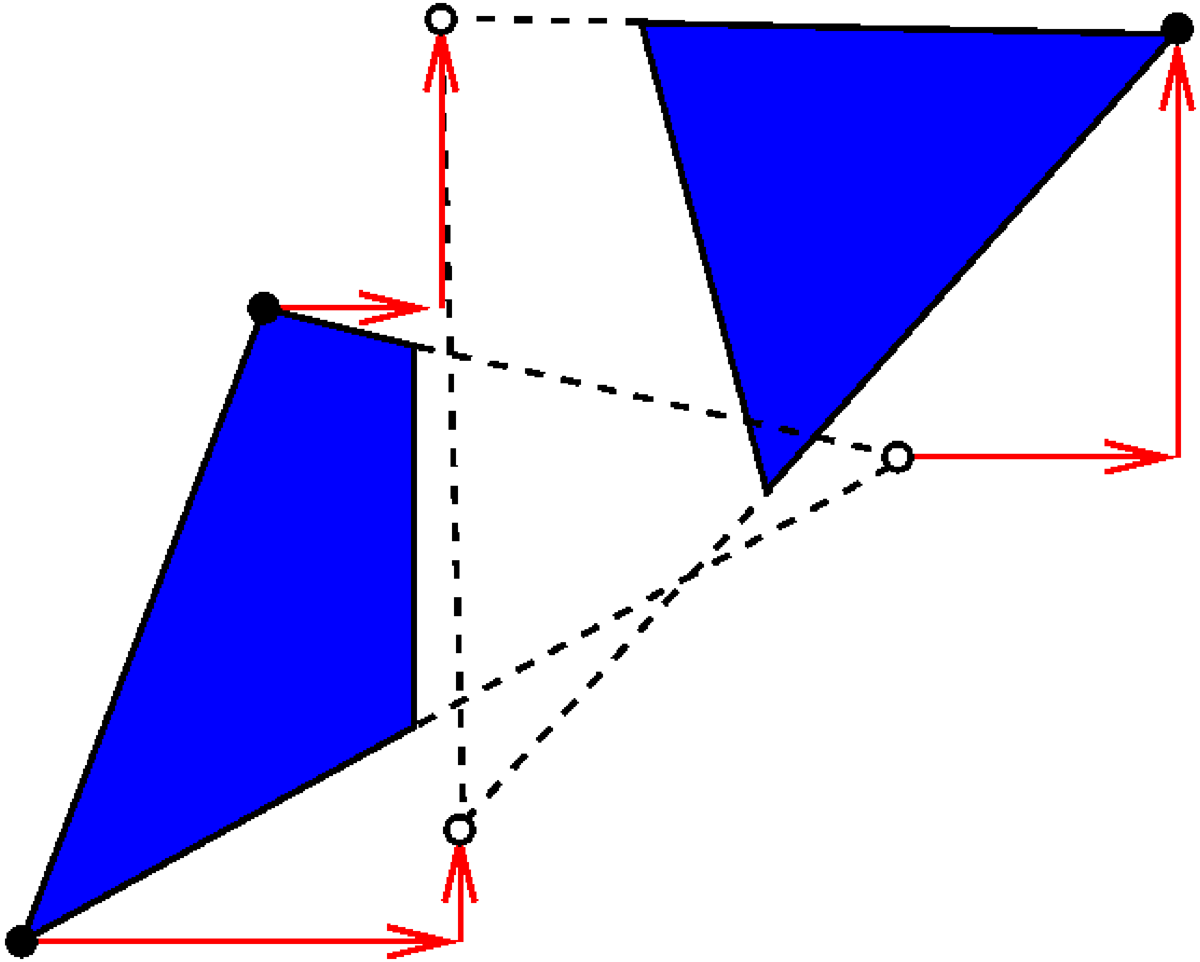
EED- EAS approach: partial coupling



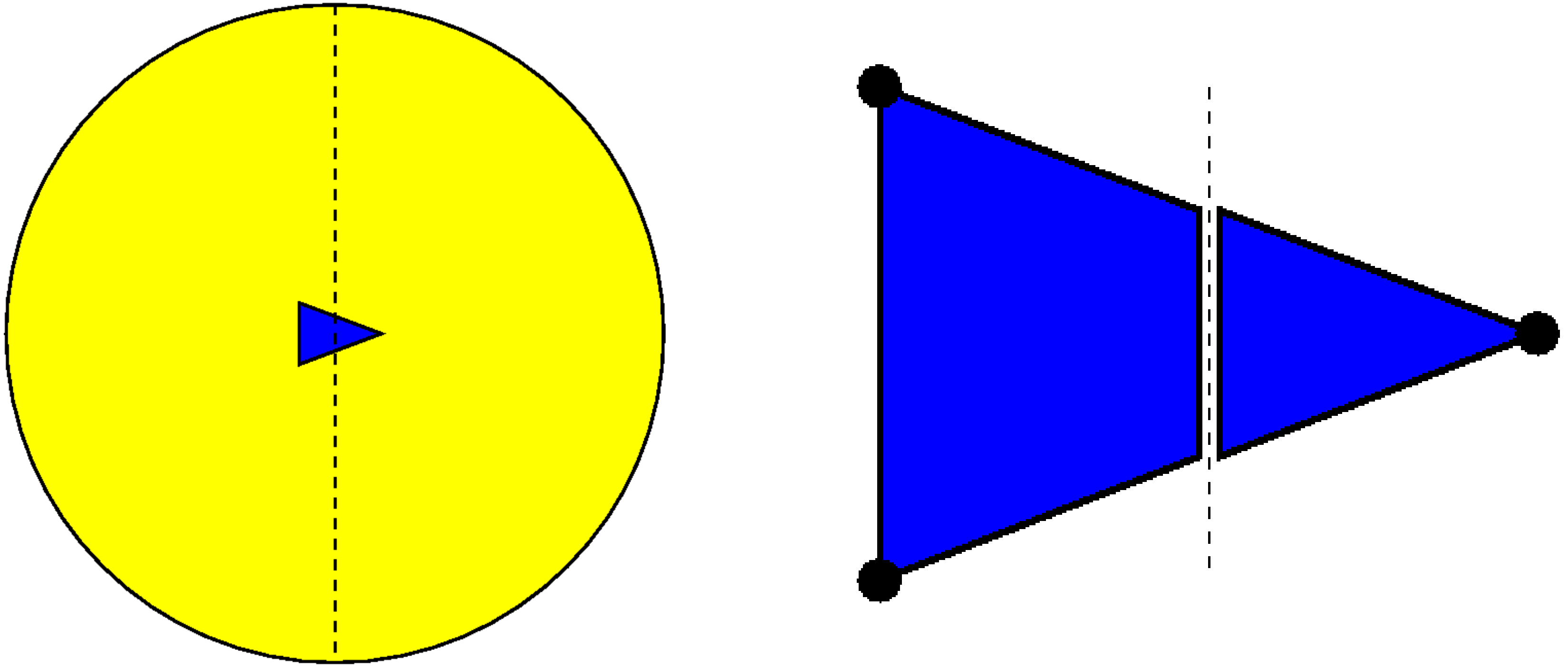
EED- EAS approach: partial coupling

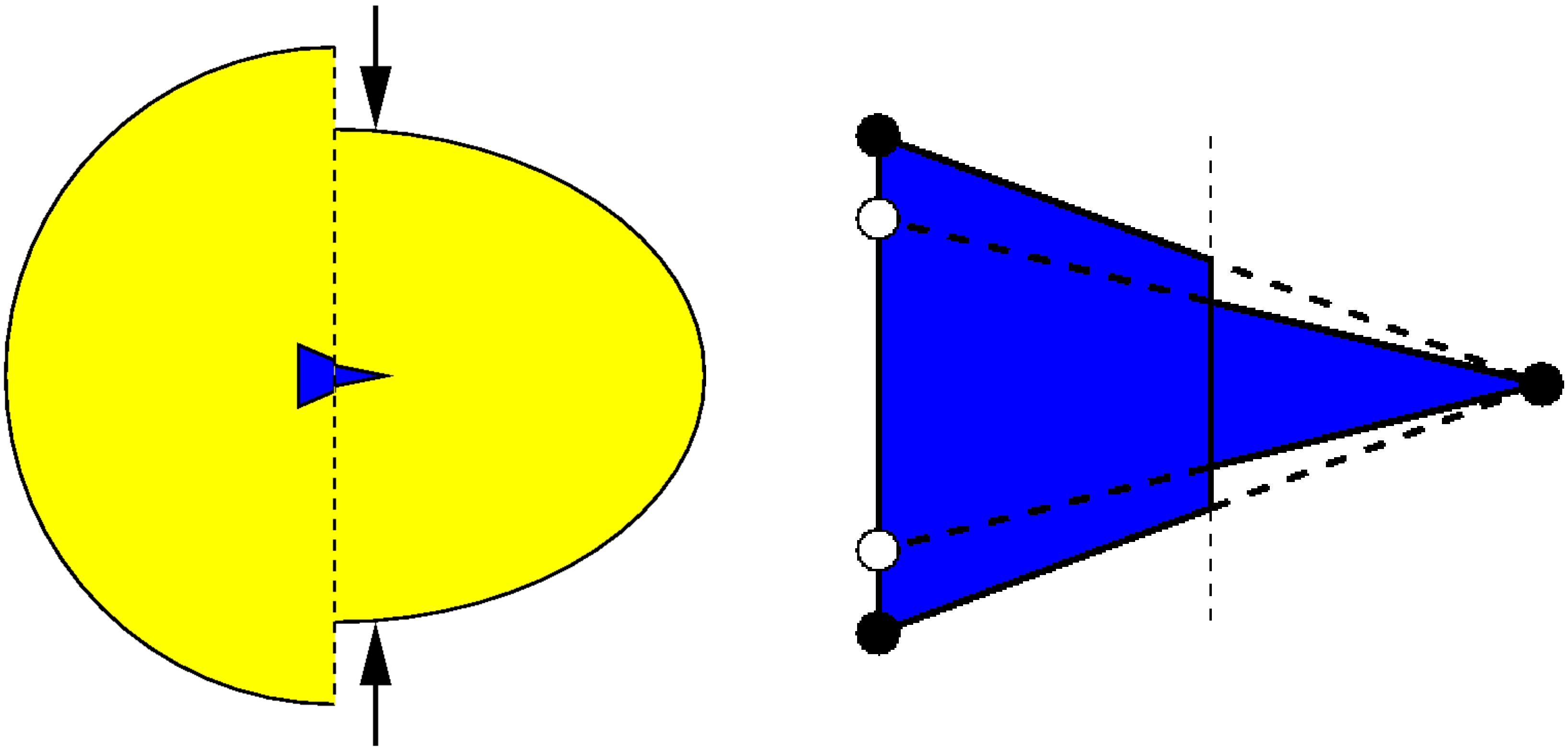
XFEM-PUM approach: complete decoupling



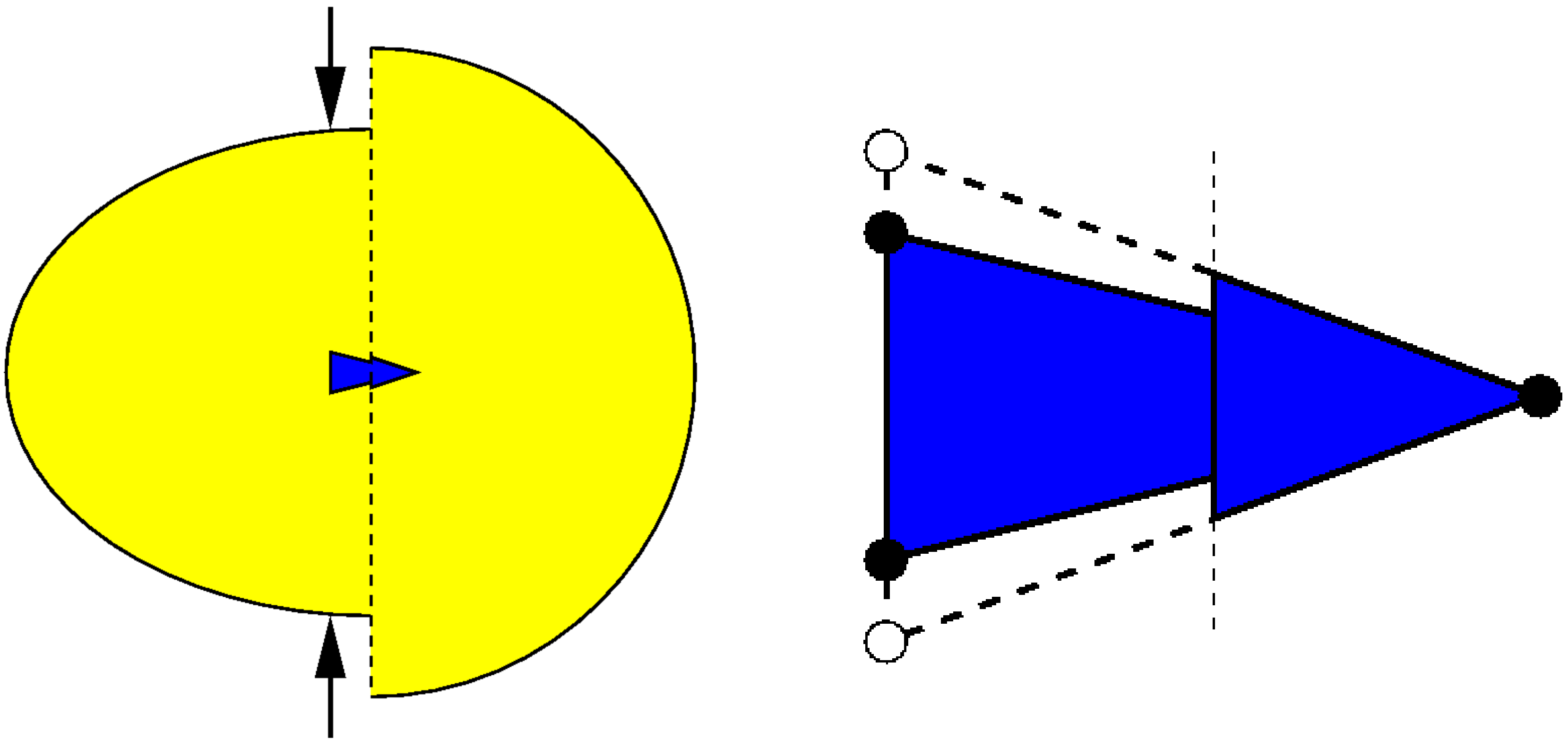
XFEM-PUM approach: complete decoupling



XFEM-PUM approach: complete decoupling



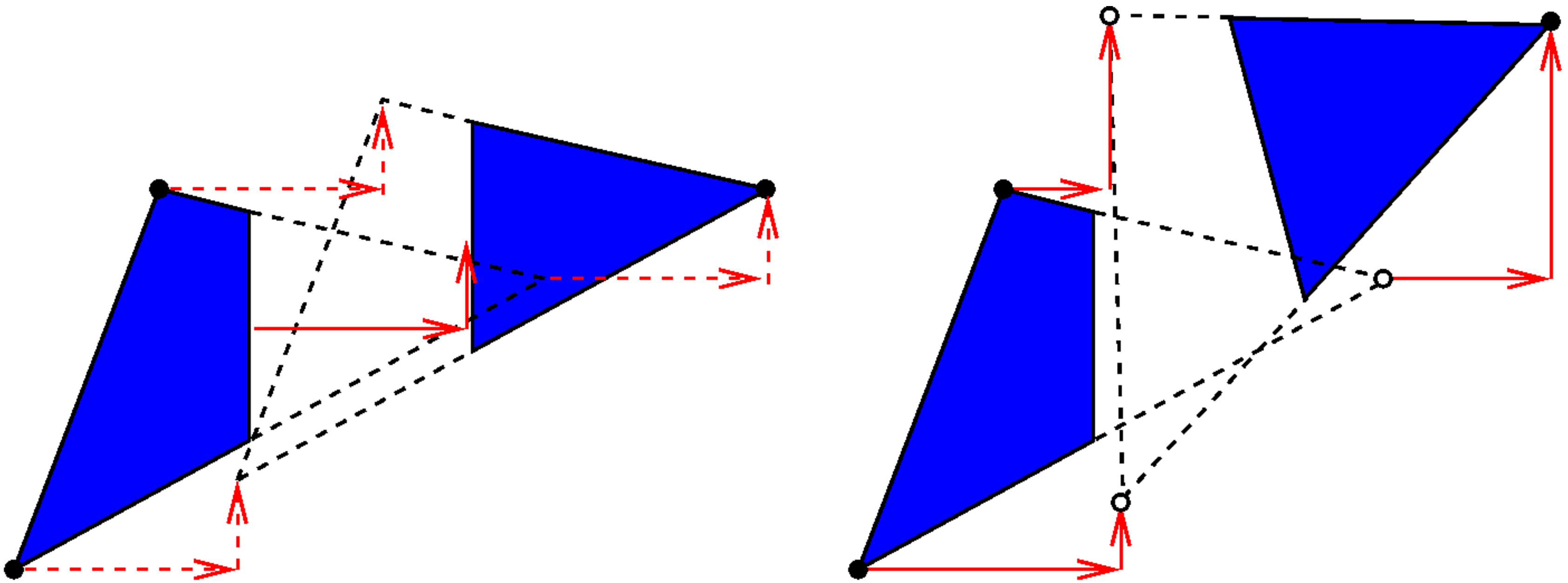
XFEM-PUM approach: complete decoupling



Comparison of EED-EAS and XFEM-PUM

Embedded discontinuity

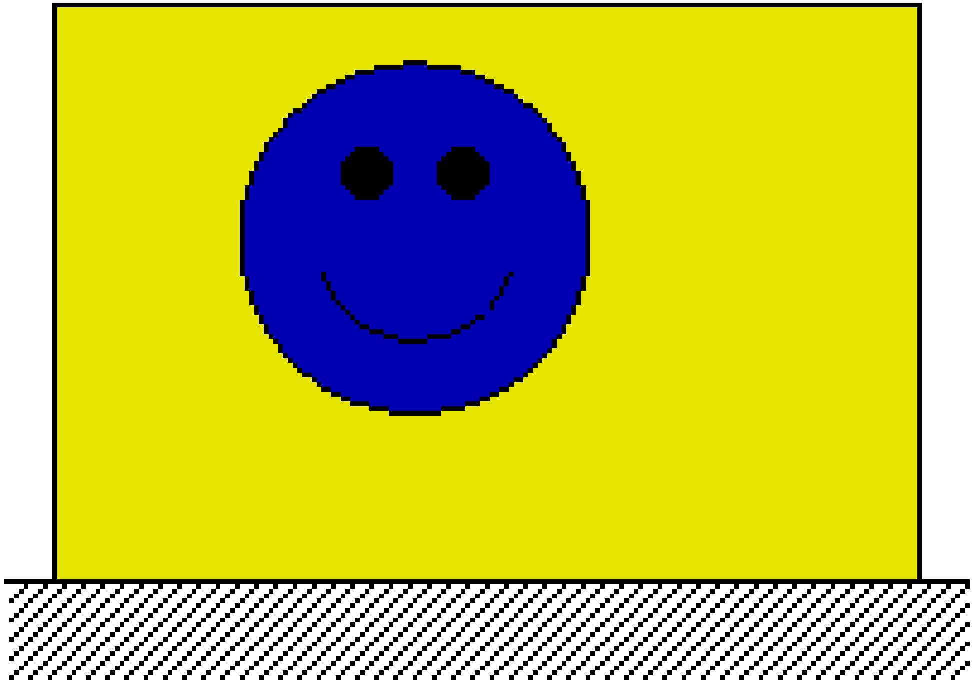
Extended finite elements



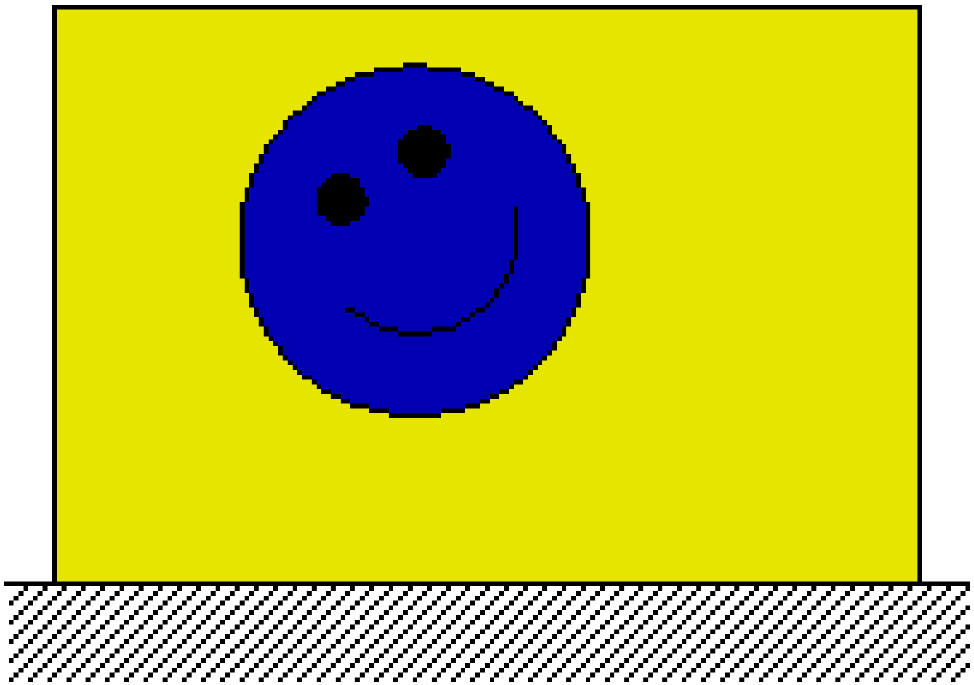
Comparison of EED-EAS and XFEM-PUM

	Embedded discontinuity	Extended finite elements
DOF‘s added	locally	globally
and related to	elements	nodes
Approximation of crack opening	discontinuous	continuous
Enrichment	incompatible	compatible
Separated parts	**partially coupled**	**fully decoupled**

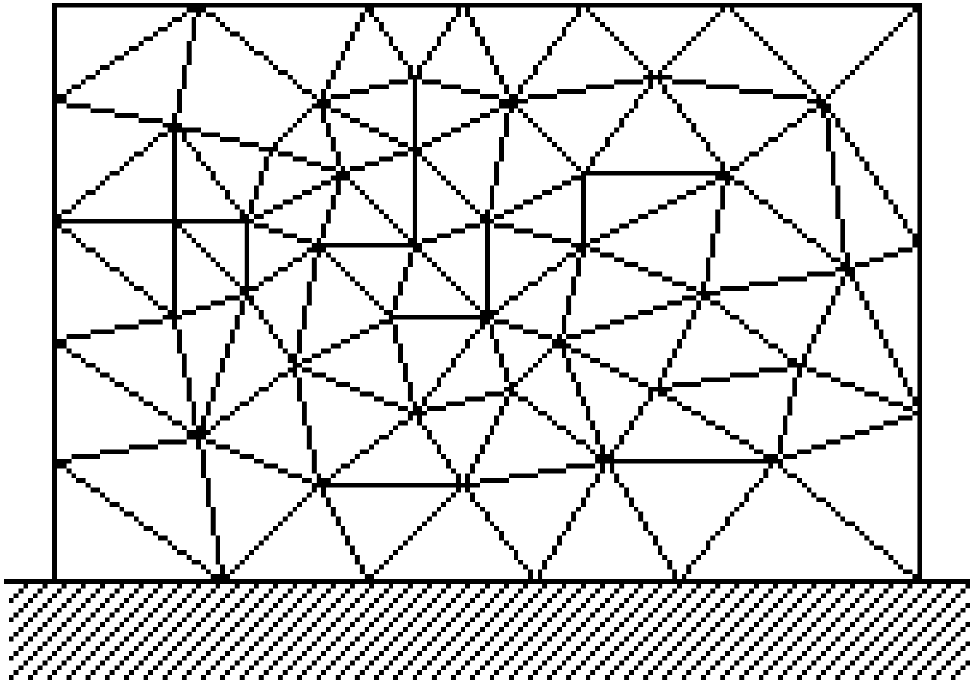
Journal bearing: Physical process



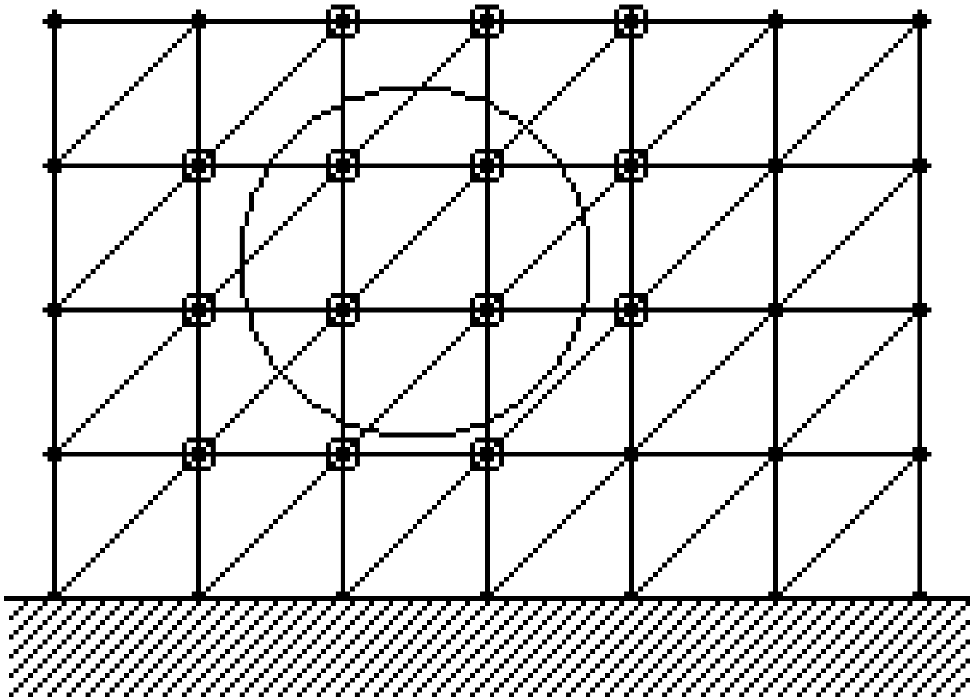
Journal bearing: Physical process



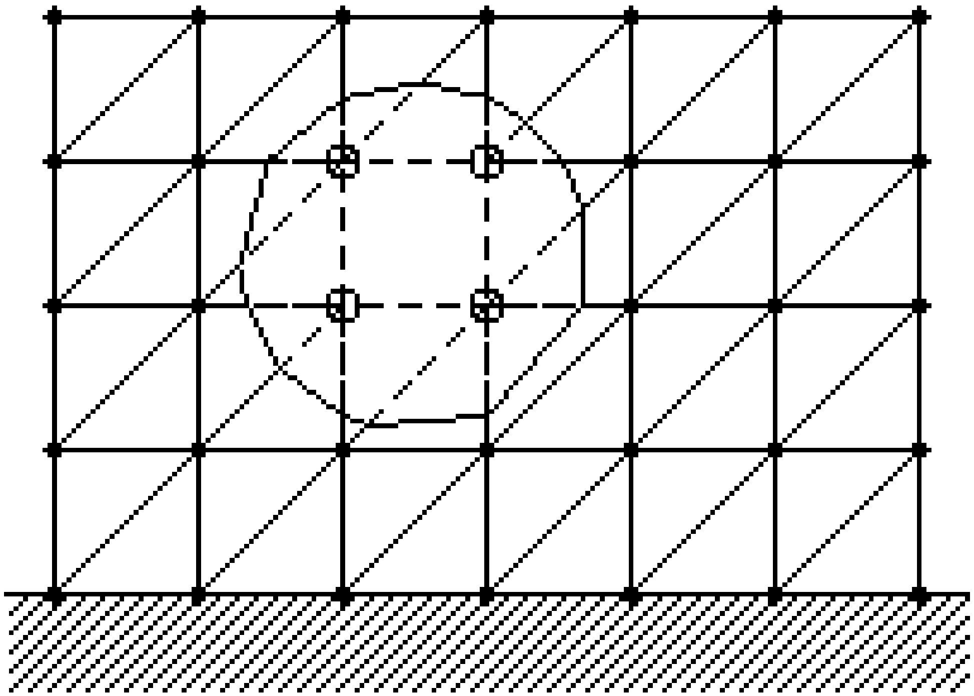
Journal bearing: Mesh respecting material boundaries



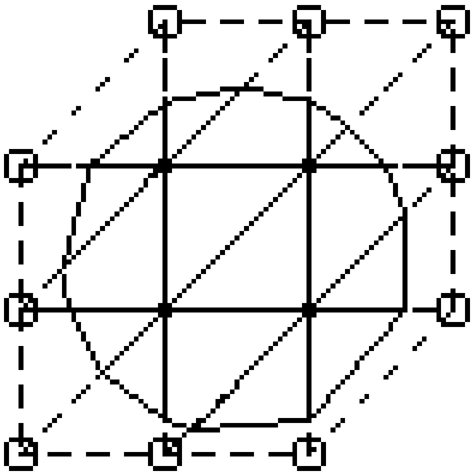
Journal bearing: Structured mesh with enrichment



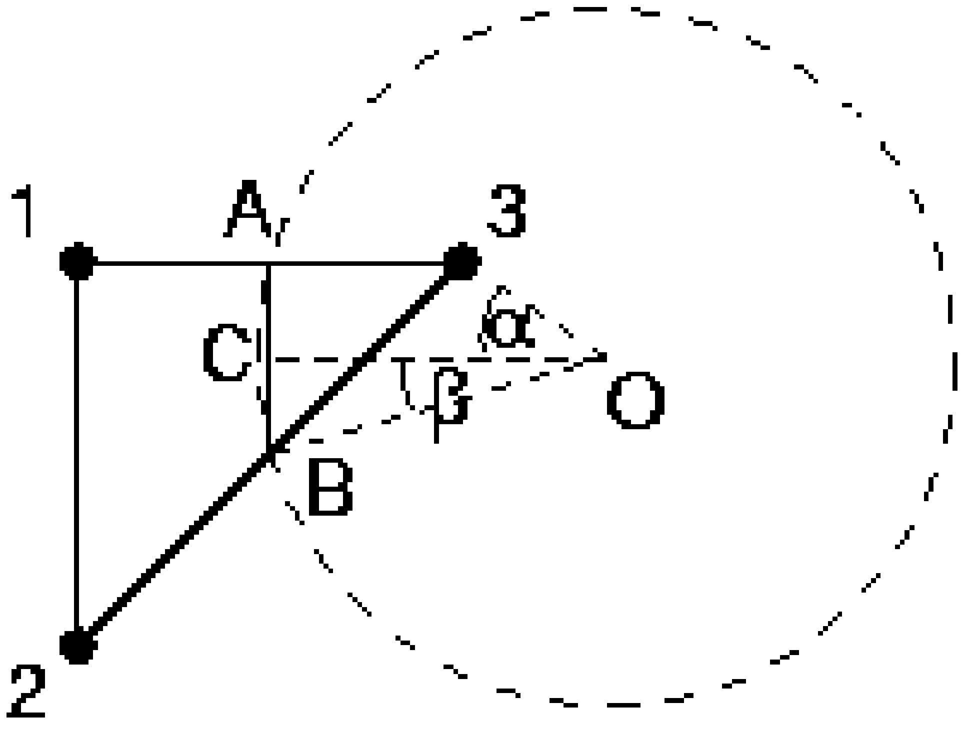
Journal bearing: Structured mesh with enrichment



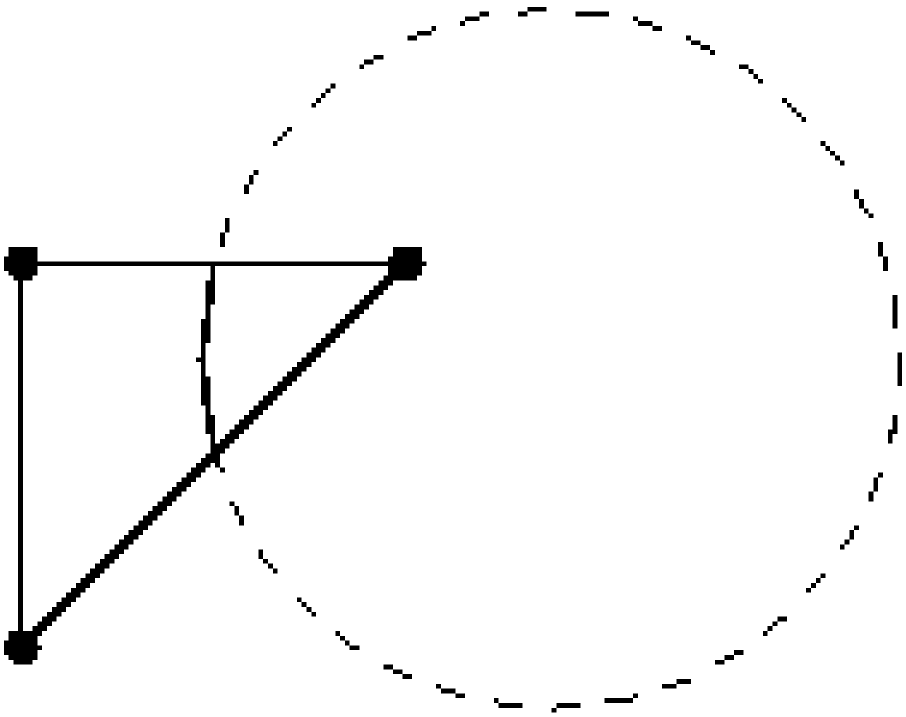
+



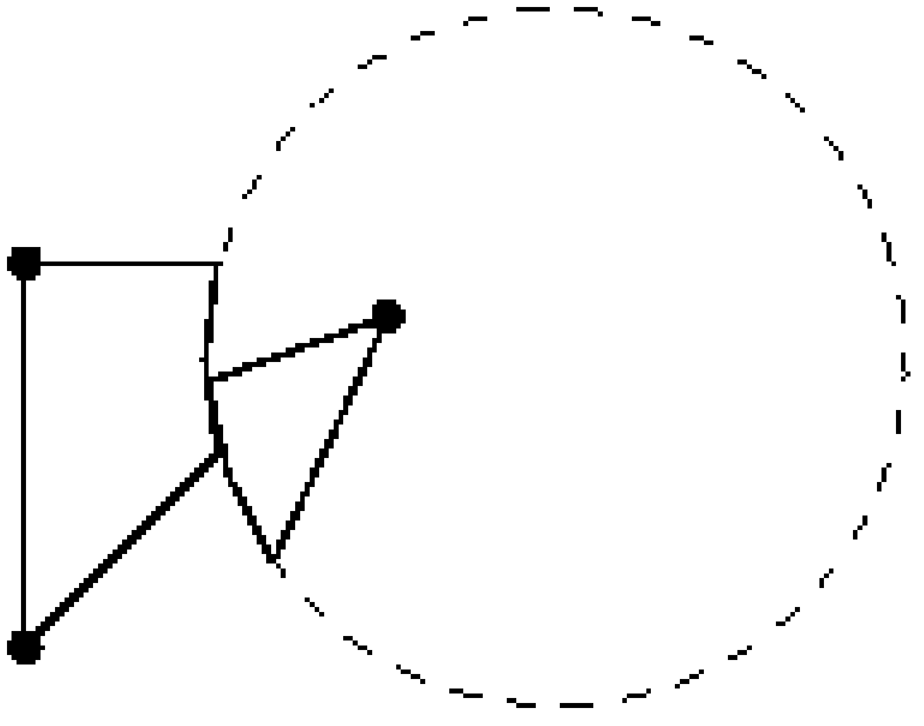
One element crossed by pre-existing discontinuity



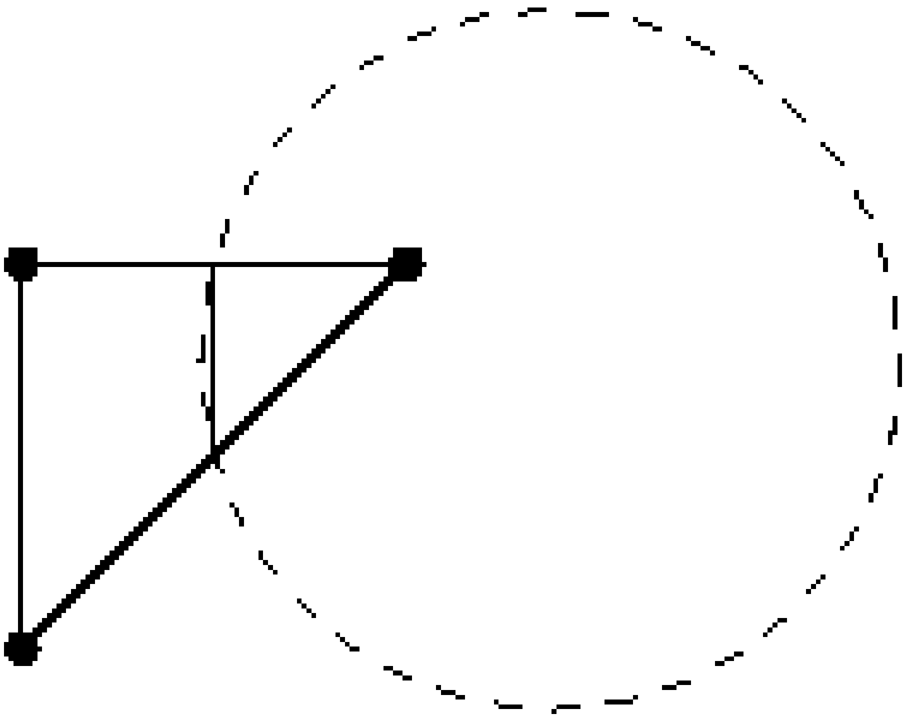
One element: Physical process



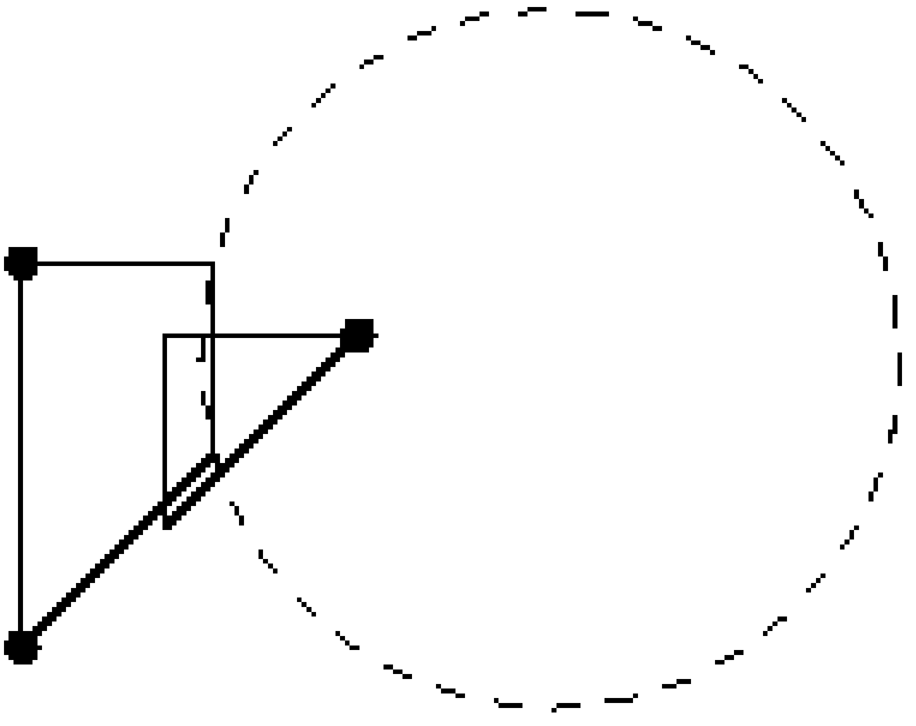
One element: Physical process



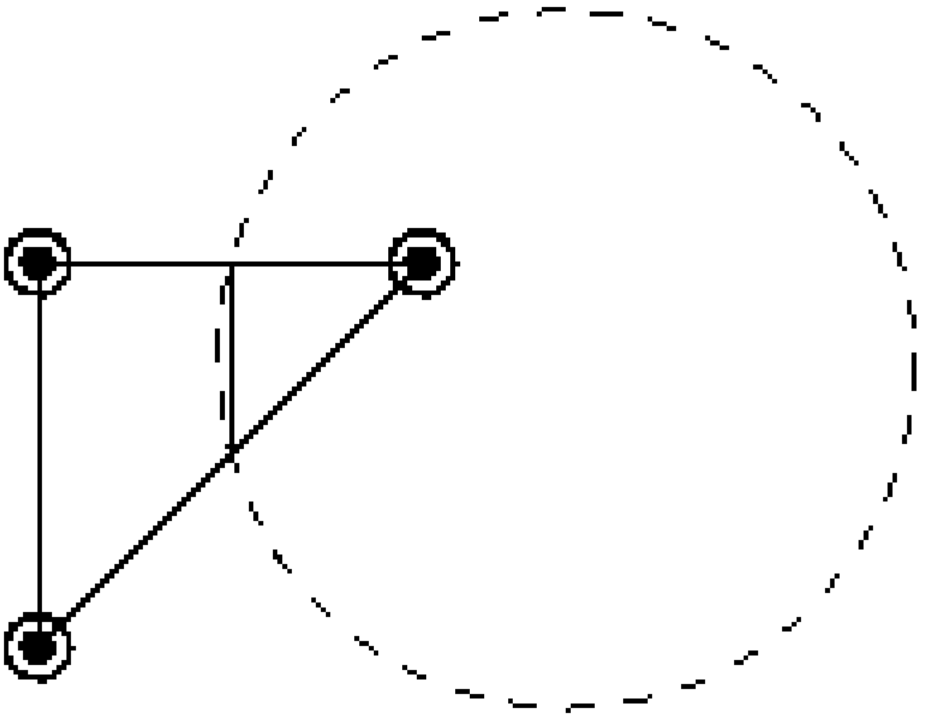
One element: EED-EAS



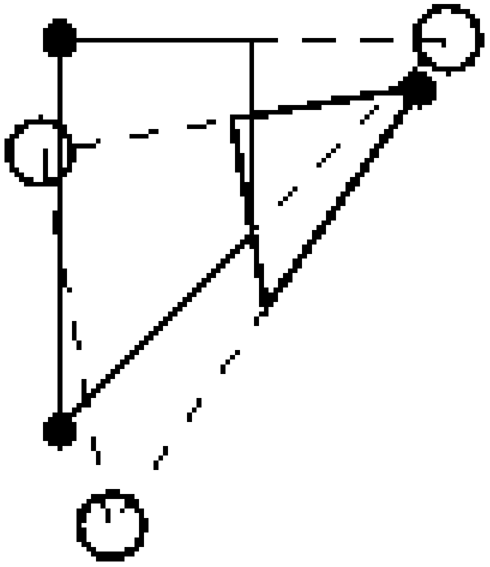
One element: EED-EAS



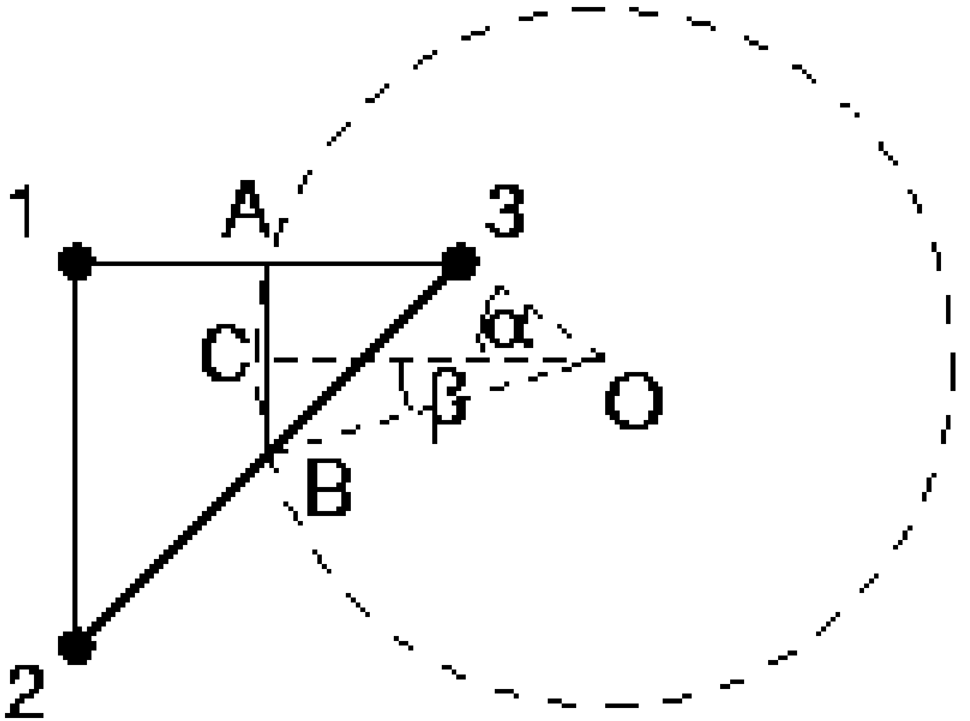
One element: XFEM-PUM



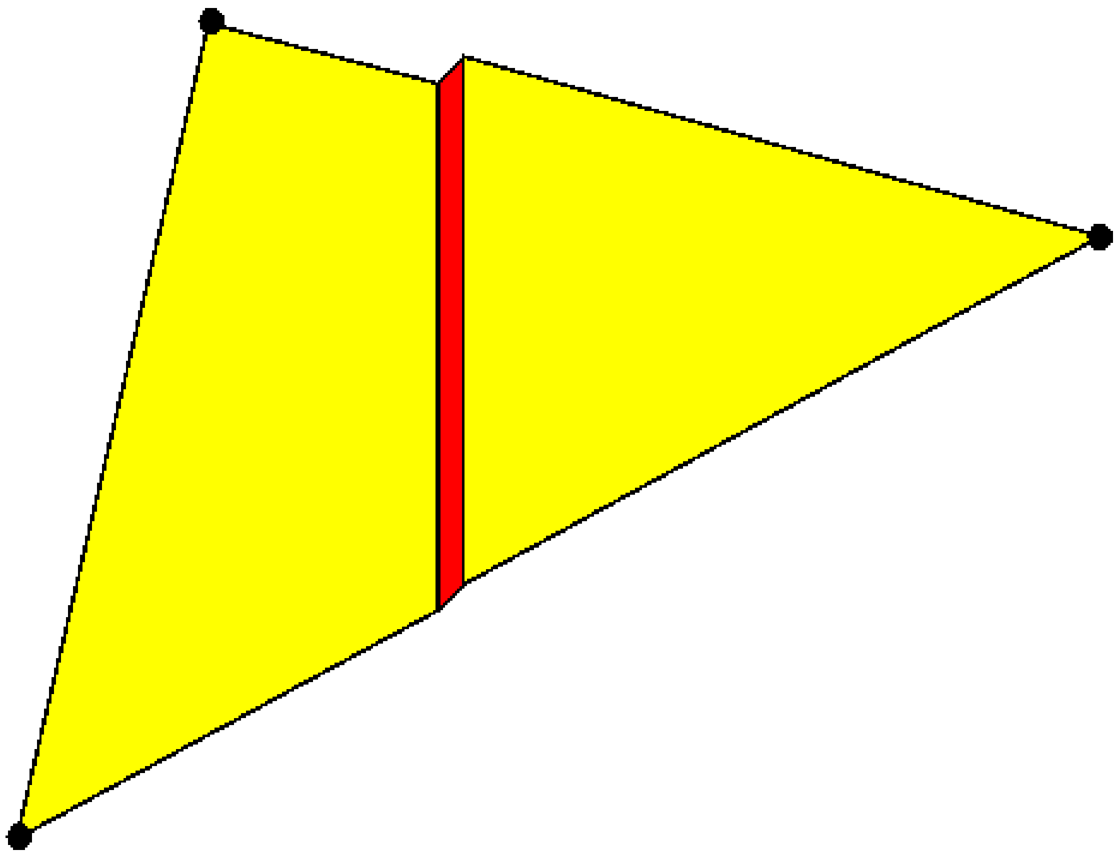
One element: XFEM-PUM



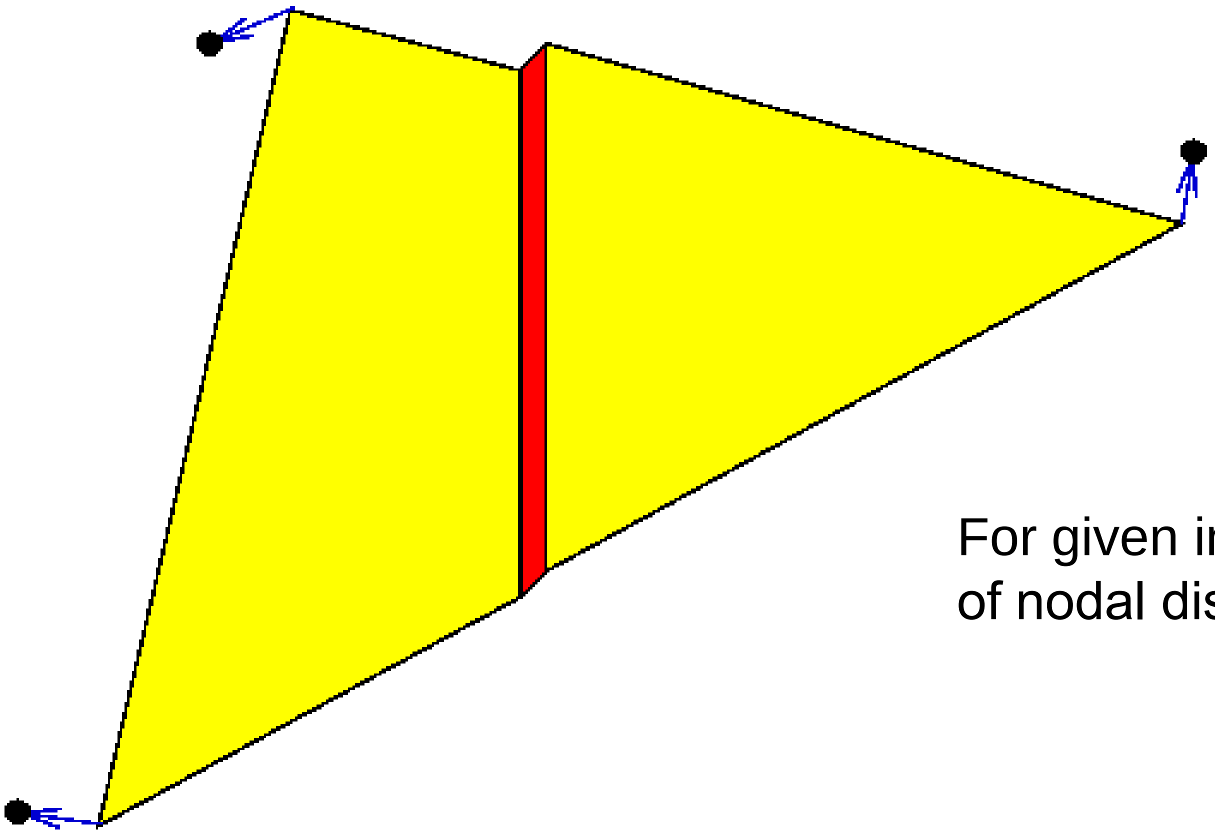
One element crossed by pre-existing discontinuity



Uniqueness of the element response (EED-EAS)

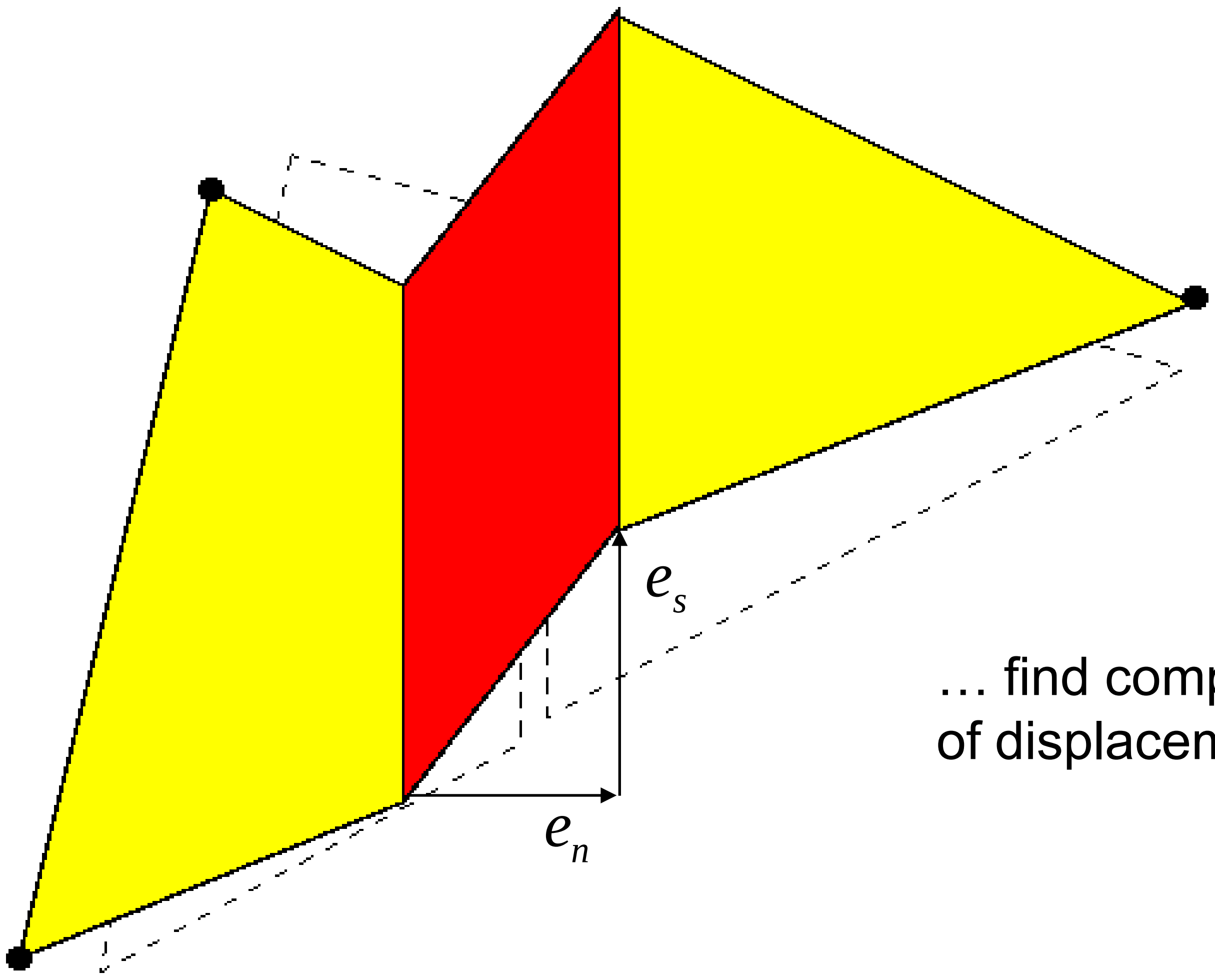


Uniqueness of the element response



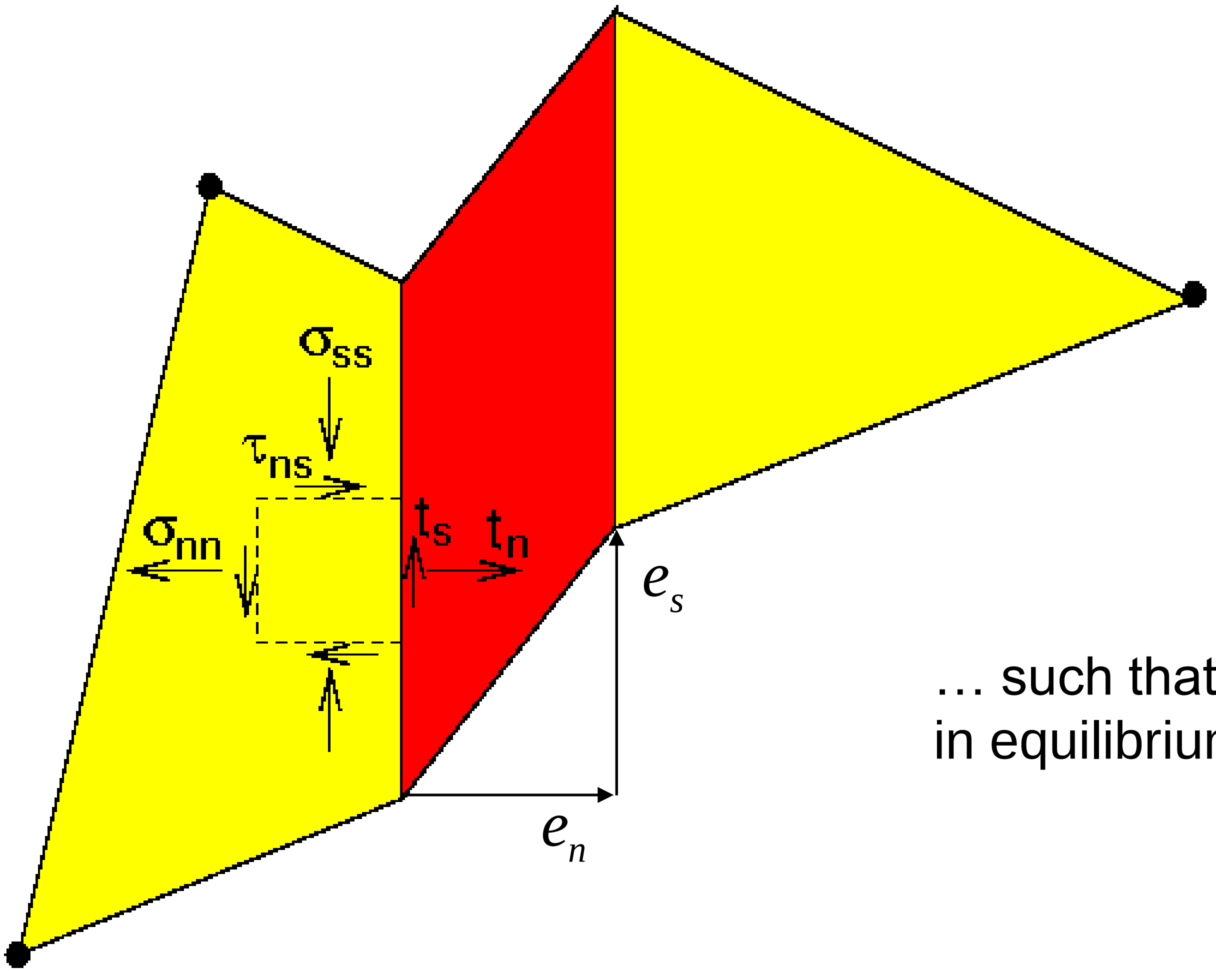
For given increments
of nodal displacements …

Uniqueness of the element response



… find components
of displacement jump …

Uniqueness of the element response



… such that tractions are in equilibrium with stresses.

Uniqueness of the element response

The solution is unique for infinitesimal displacement increments of an arbitrary direction if

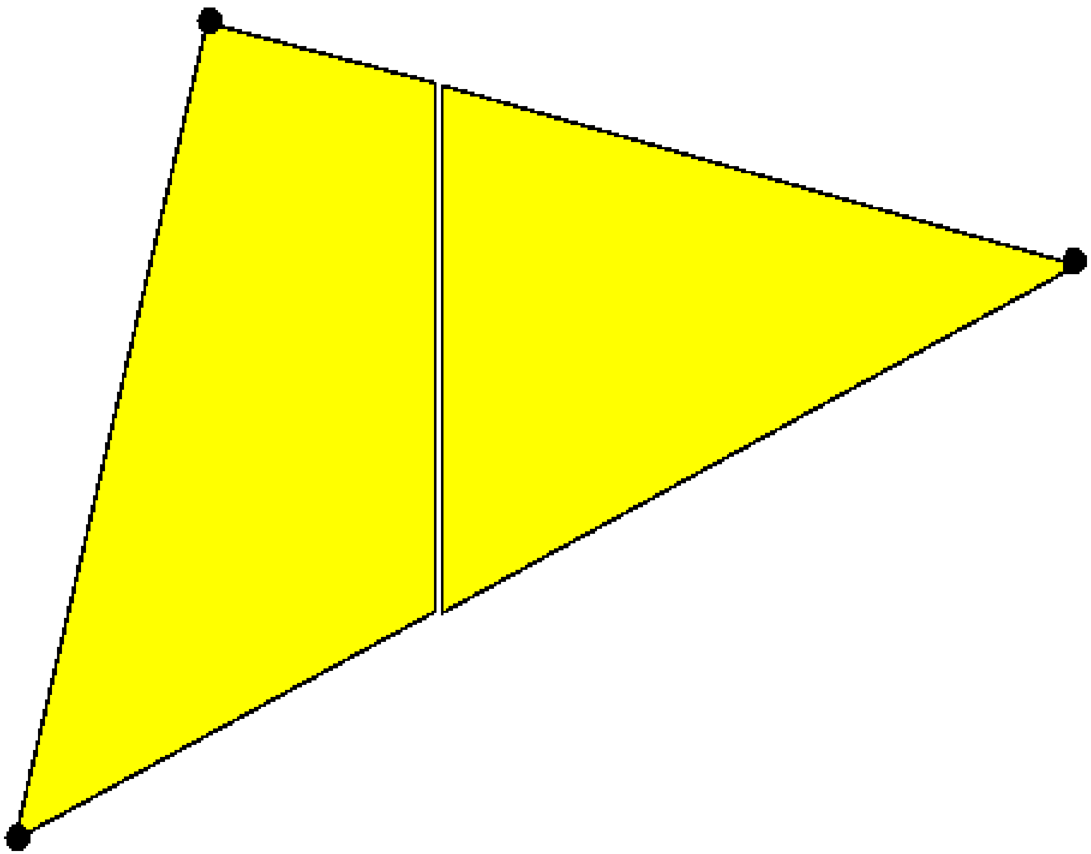
$$\lambda_{\min}\left(\mathbf{Q}_{sym}\right) + H > 0$$

where $\mathbf{Q}_{sym}$ is the symmetric part of $\mathbf{Q} = \mathbf{P}^T \mathbf{D}_e \mathbf{B} \mathbf{H}$

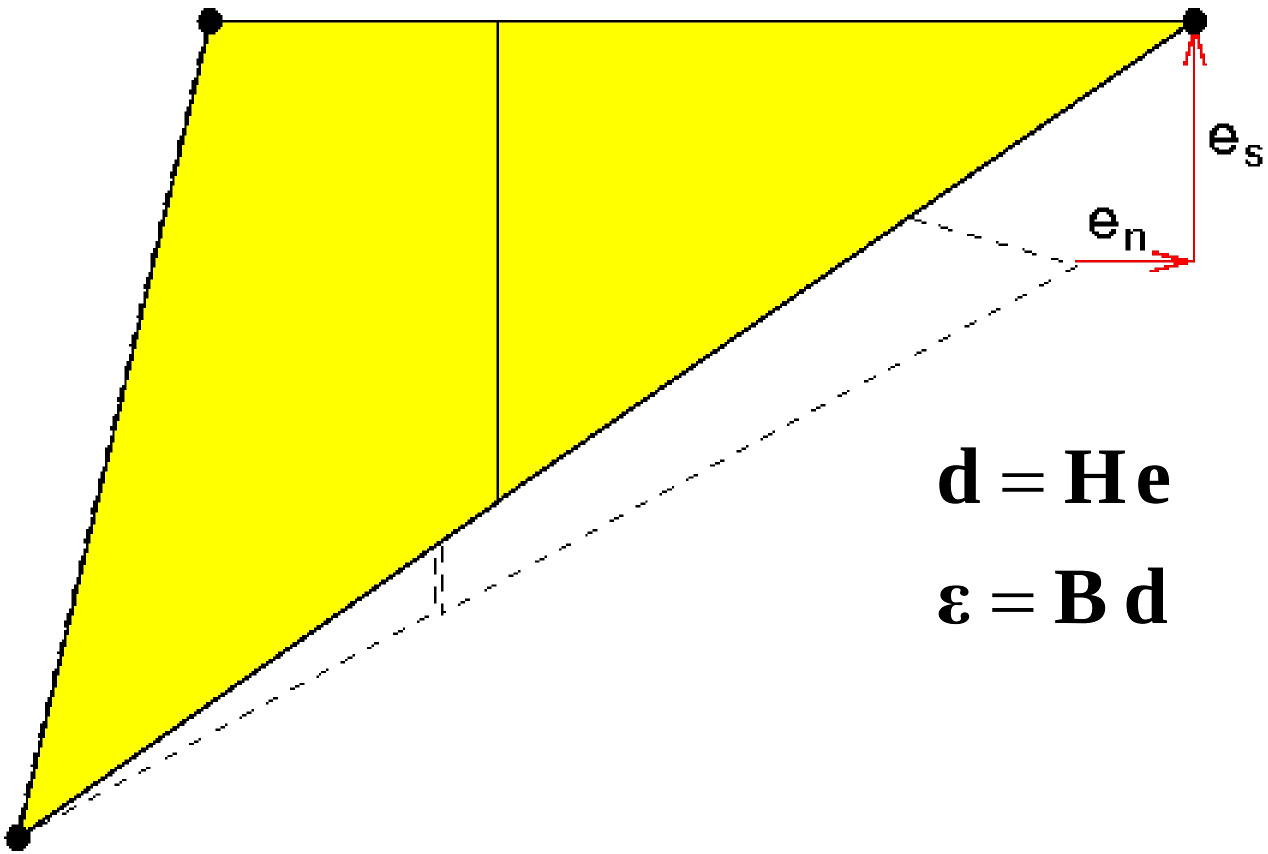
and $H < 0$ is the discrete softening modulus.

Physical meaning of $\mathbf{Q}$ …

Uniqueness of the element response



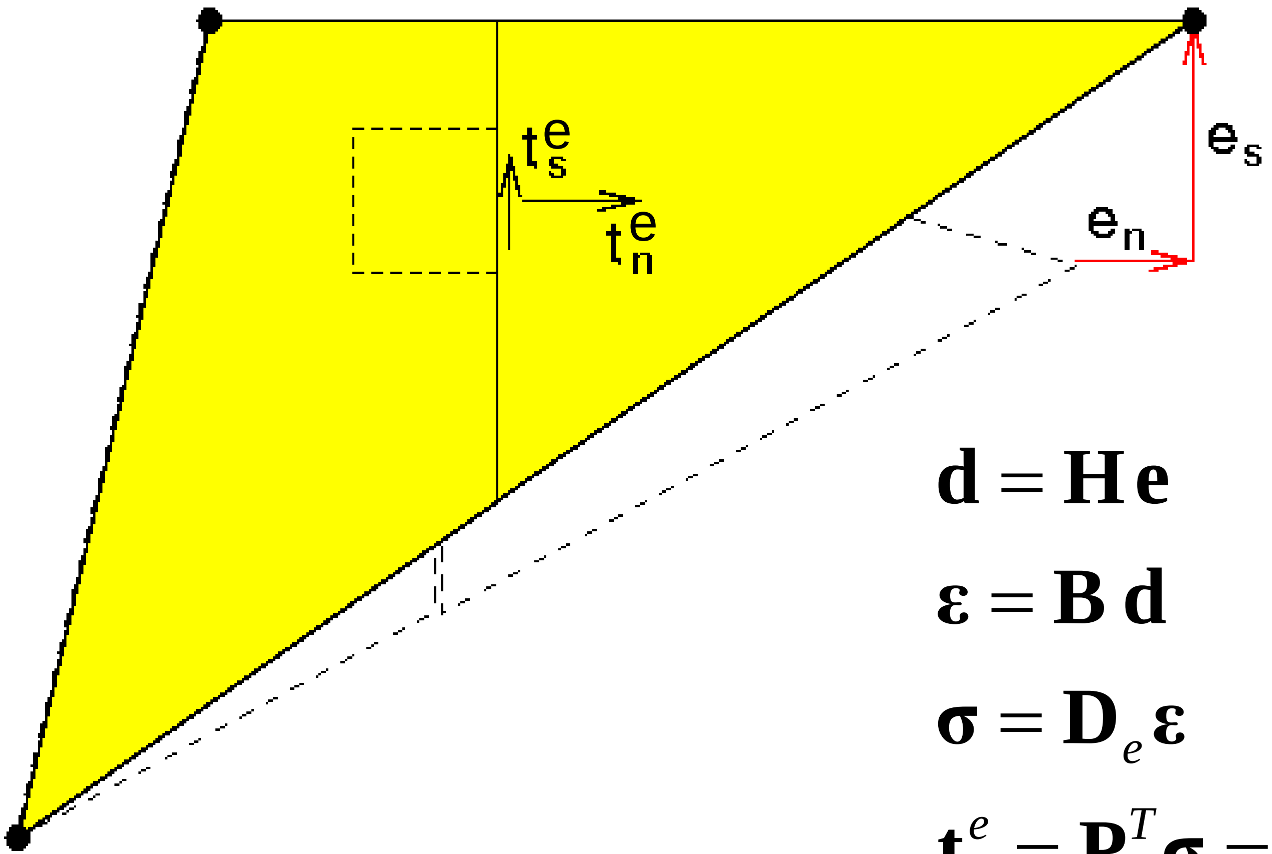
Uniqueness of the element response



$$\mathbf{d} = \mathbf{H}\,\mathbf{e}$$

$$\boldsymbol{\varepsilon} = \mathbf{B}\,\mathbf{d}$$

Uniqueness of the element response



$$\mathbf{d} = \mathbf{H}\,\mathbf{e}$$

$$\boldsymbol{\varepsilon} = \mathbf{B}\,\mathbf{d}$$

$$\boldsymbol{\sigma} = \mathbf{D}_e\,\boldsymbol{\varepsilon}$$

$$\mathbf{t}^e = \mathbf{P}^T\boldsymbol{\sigma} = \mathbf{P}^T\mathbf{D}_e\,\mathbf{B}\,\mathbf{H}\,\mathbf{e}$$

Uniqueness of the element response

$$\lambda_{\min}\left(\mathbf{Q}_{sym}\right) > -H_{\min}$$

$$\mathbf{Q} = \mathbf{P}^T \mathbf{D}_e \mathbf{B} \mathbf{H}$$

is proportional to the elastic modulus
and inversely proportional to the element size

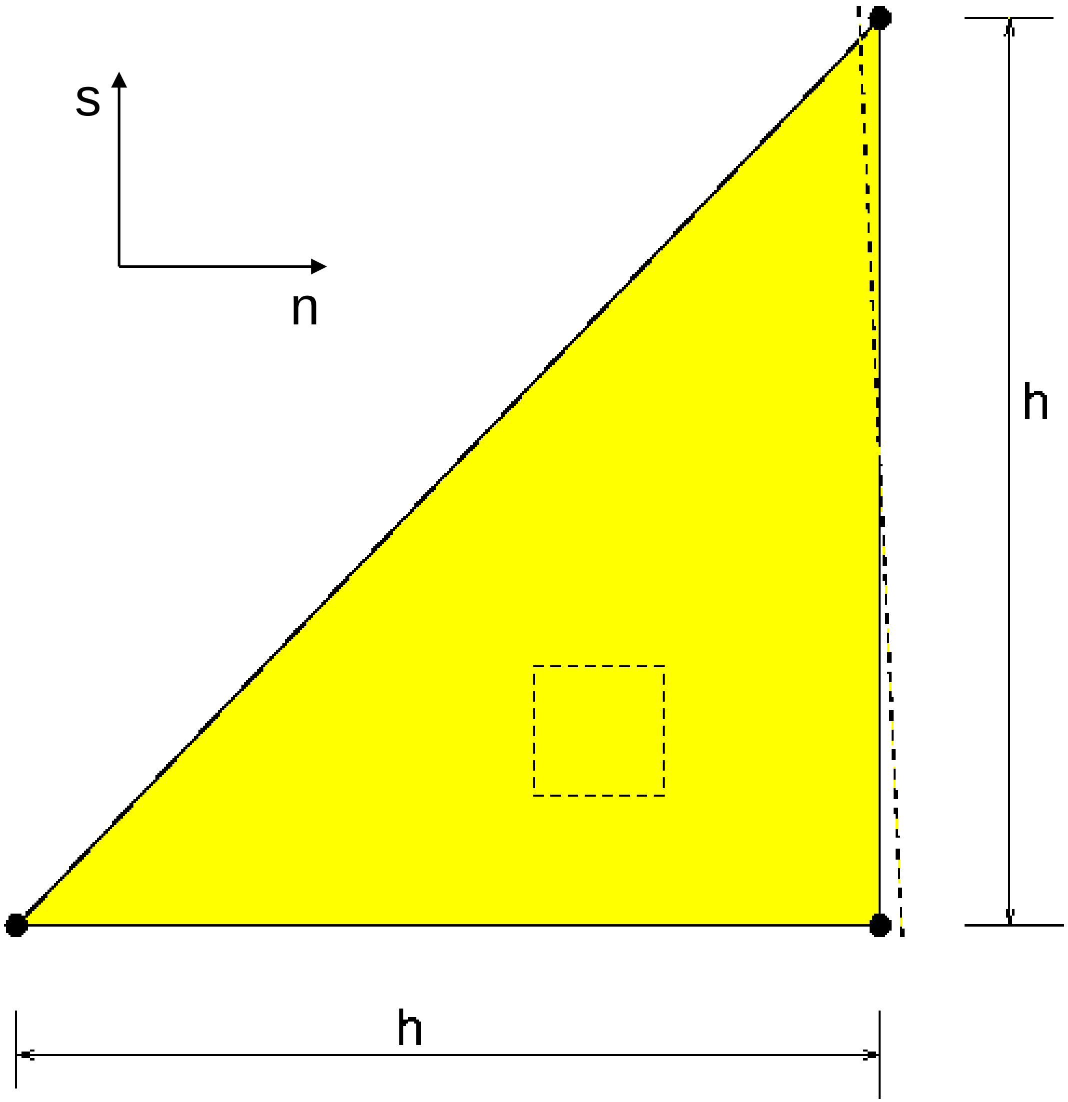
Uniqueness of the element response

$$\lambda_{\min}\left(\mathbf{Q}_{sym}\right) > -H_{\min}$$

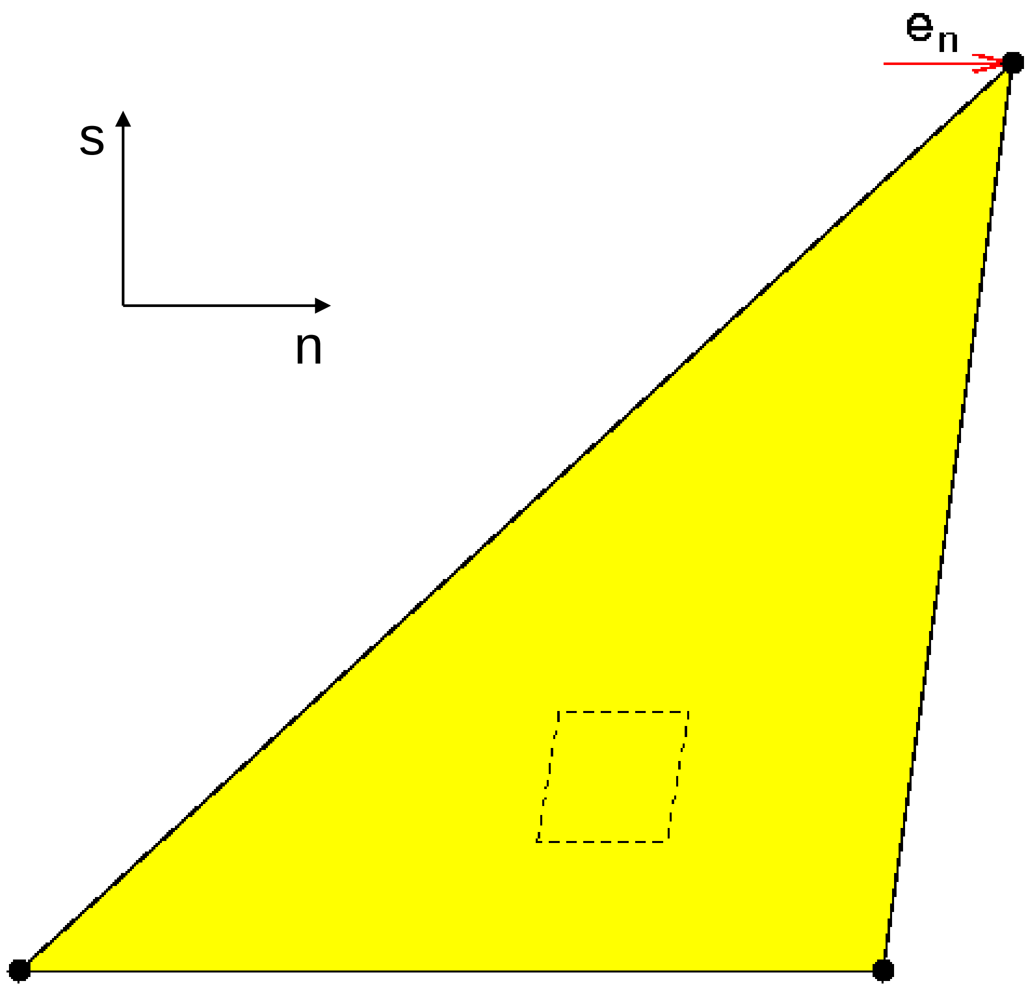
$\mathbf{Q} = \mathbf{P}^T \mathbf{D}_e \mathbf{B}\mathbf{H}$ is proportional to the elastic modulus and inversely proportional to the element size

$\mathbf{e}^T \mathbf{Q}_{sym} \mathbf{e} = \mathbf{e}^T \mathbf{Q} \mathbf{e} = \mathbf{e}^T \mathbf{t}^e < 0$ can happen

Uniqueness of the element response



Uniqueness of the element response



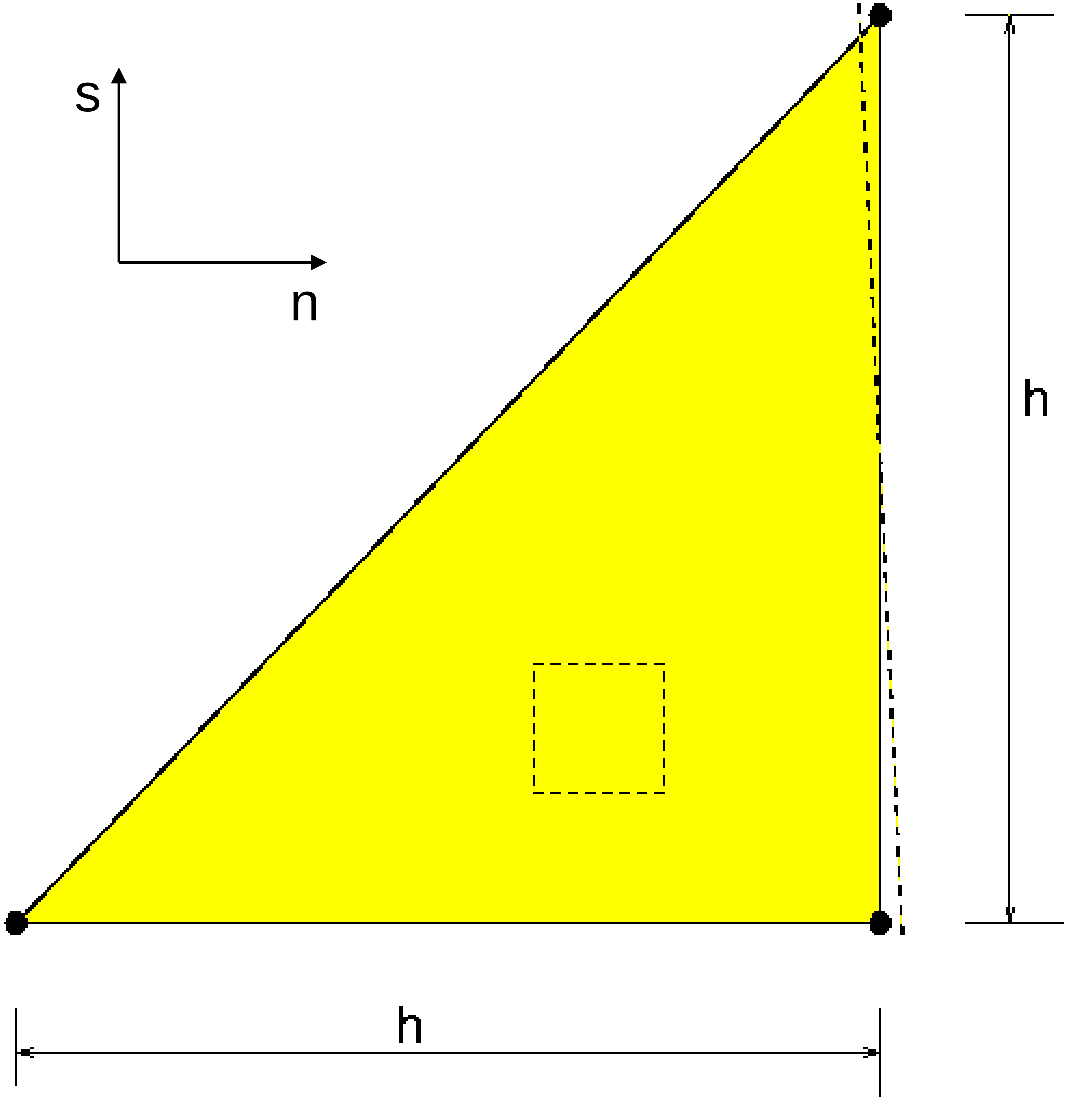
$$\gamma = e_n / h$$

$$\tau = G\gamma$$

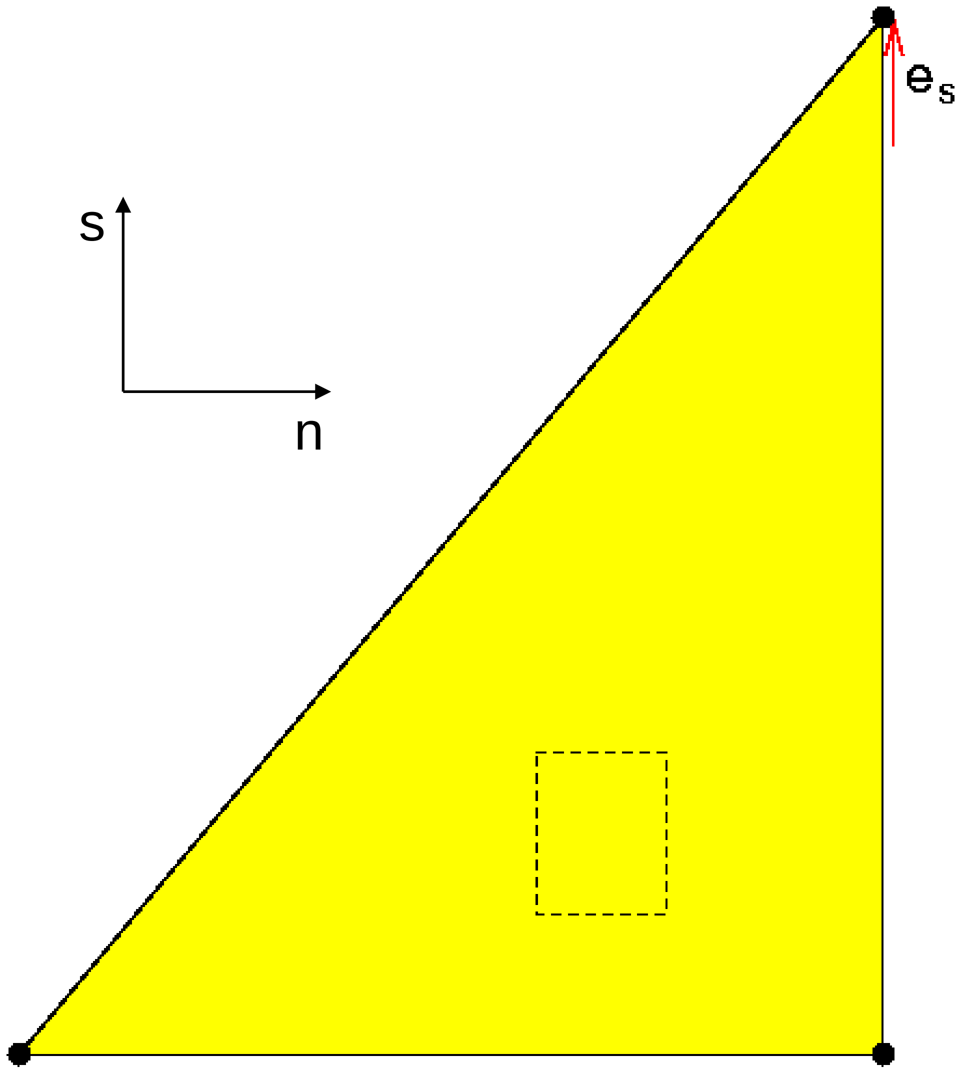
$$t_n = 0$$

$$t_s = \tau = G e_n / h$$

Uniqueness of the element response



Uniqueness of the element response



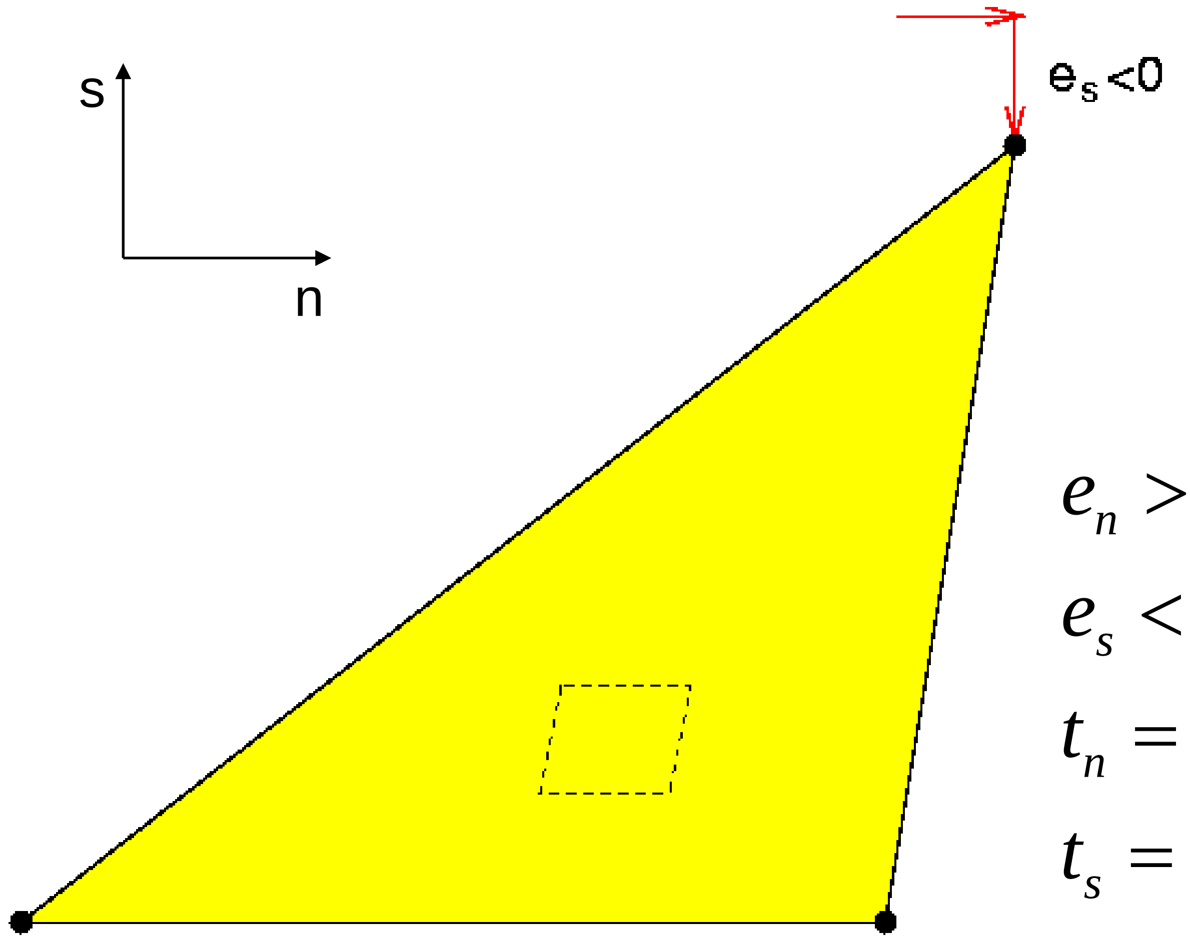
$$\varepsilon_{ss} = e_s / h$$

$$\sigma_{nn} = D_{12}\varepsilon_{ss}$$

$$t_n = \sigma_{nn} = D_{12} e_s / h$$

$$t_s = 0$$

Uniqueness of the element response



$$e_n > 0$$

$$e_s < 0$$

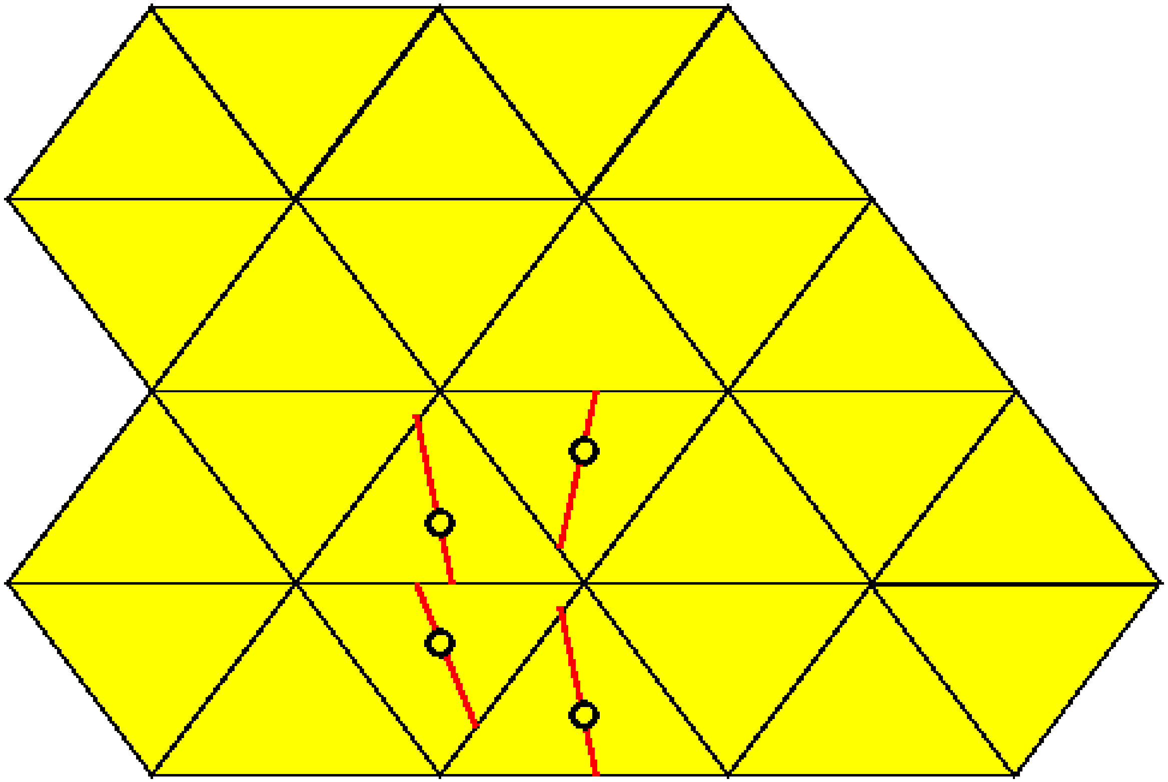
$$t_n = D_{12} e_s / h < 0$$

$$t_s = G e_n / h > 0$$

$$\mathbf{e}^T \mathbf{t}^e < 0$$

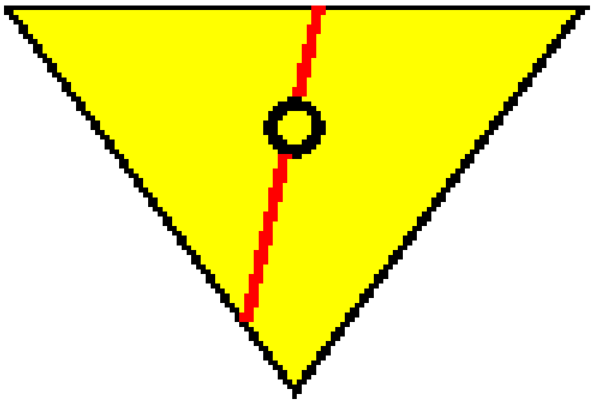
Uniqueness of the element response

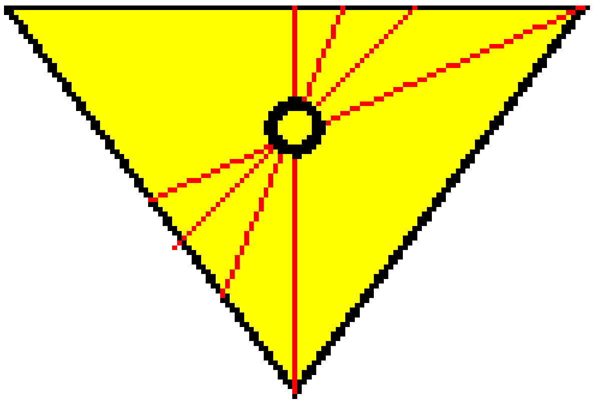
discontinuity segments placed at element centers



Uniqueness of the element response

discontinuity segments placed at element centers

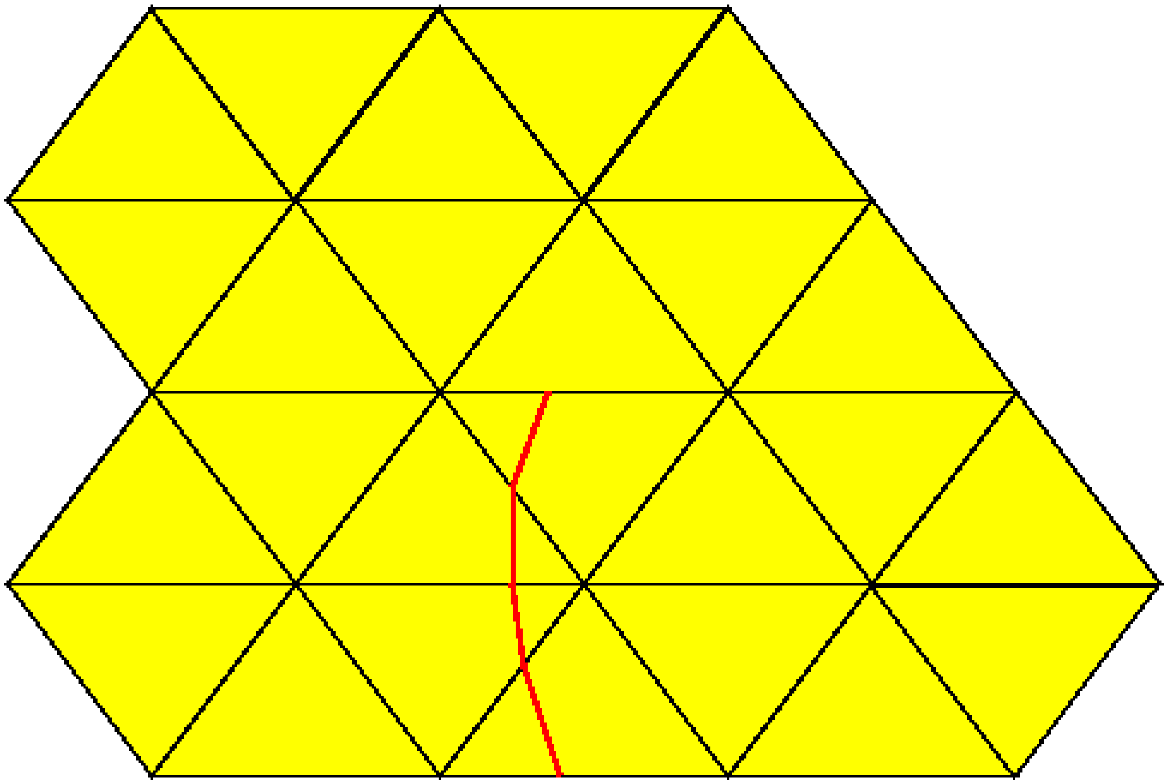




maximum deviation α between element side and discontinuity is limited (e.g., 30 degrees for an equilateral triangle)

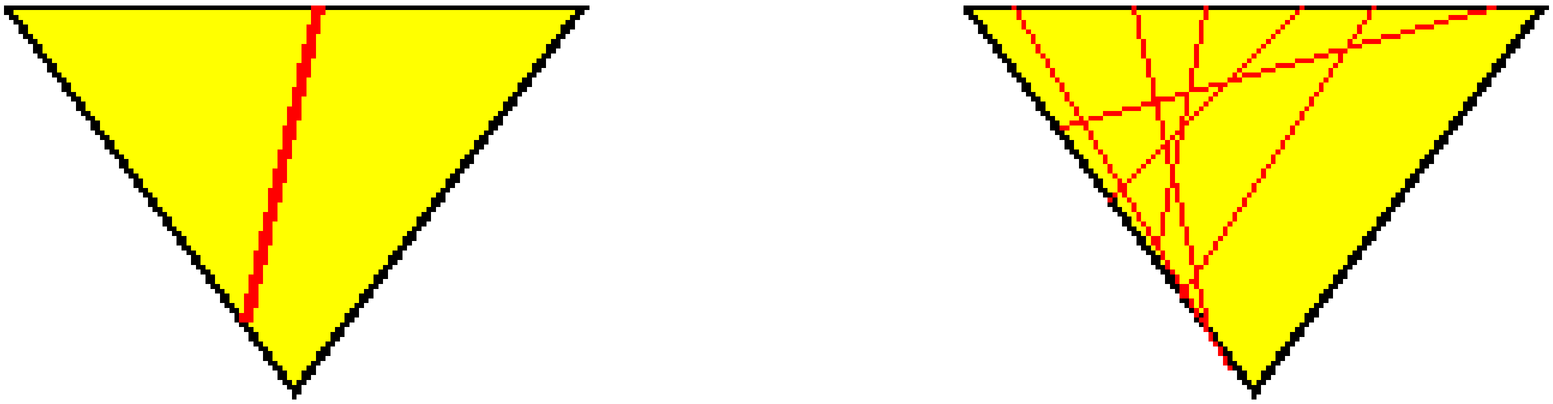
Uniqueness of the element response

discontinuity segments form a continuous path



Uniqueness of the element response

discontinuity segments form a continuous path



maximum deviation α between element side and discontinuity
is given by the largest angle of the triangle
(e.g., 60 degrees for an equilateral triangle)

Uniqueness of the element response

Condition under which uniqueness can be guaranteed if the element is sufficiently small:

plane stress … $\cos\alpha > \dfrac{1+\nu}{3-\nu}$

true only if $\nu < 1/3$ and the element is close to equilateral

Uniqueness of the element response

Condition under which uniqueness can be guaranteed if the element is sufficiently small:

plane stress … $$\cos\alpha > \frac{1+\nu}{3-\nu}$$

true only if $\nu < 1/3$ and the element is close to equilateral

plane strain … $$\cos\alpha > \frac{1}{3-4\nu}$$

true only if $\nu < 1/4$ and the element is close to equilateral

Uniqueness of the element response

Condition under which uniqueness can be guaranteed if the element is sufficiently small:

plane stress … $\cos\alpha > \dfrac{1+\nu}{3-\nu}$

true only if $\nu < 1/3$ and the element is close to equilateral

plane strain … $\cos\alpha > \dfrac{1}{3-4\nu}$

true only if $\nu < 1/4$ and the element is close to equilateral

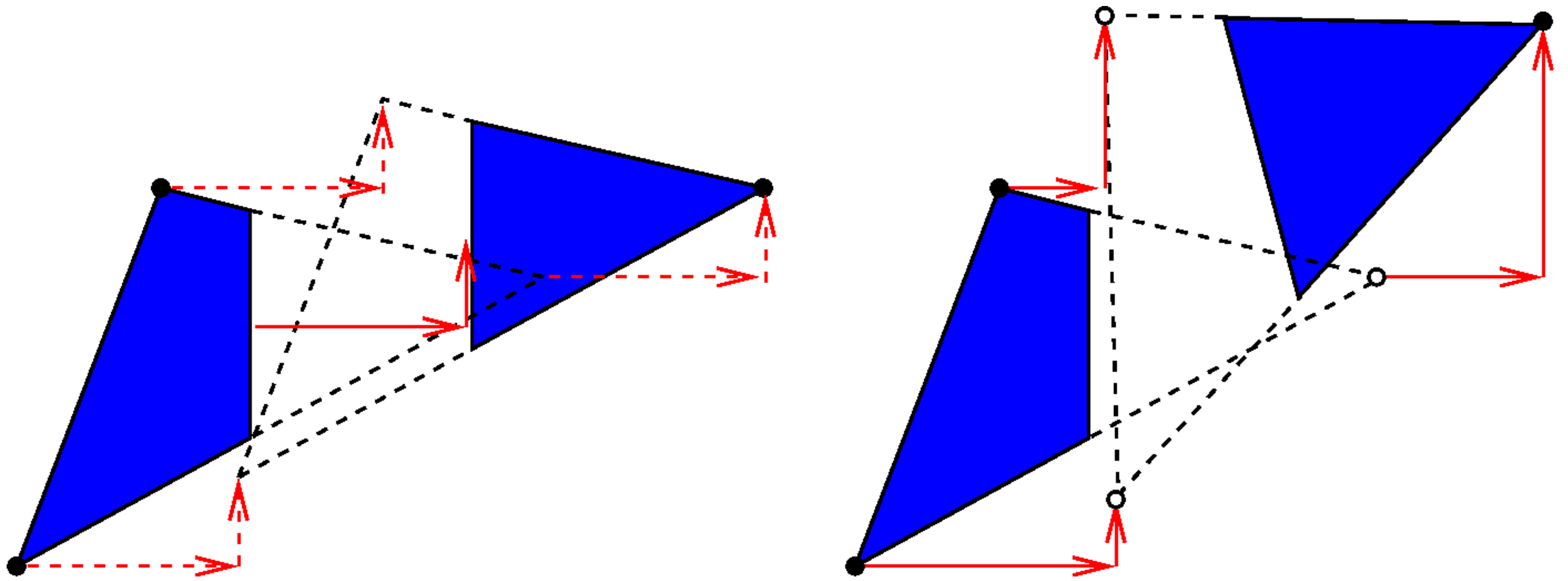
three dimensions … $\cos\alpha > \dfrac{1}{3-4\nu}$

violated even if the tetrahedral element is regular

Comparison of EED-EAS and XFEM-PUM

Embedded discontinuity

Extended finite elements



Comparison of EED-EAS and XFEM-PUM

	Embedded discontinuity	Extended finite elements
DOF‘s added	locally	globally
and related to	elements	nodes
Approximation of crack opening	discontinuous	continuous
Enrichment	incompatible	compatible
Separated parts	partially interacting	independent
Numerical behavior	**rather fragile**	**more robust**

Comparison of EED-EAS and XFEM-PUM

	Embedded discontinuity	Extended finite elements
Stiffness matrix	always nonsymmetric	can be symmetric
Integration scheme for continuous part	remains standard	must be modified
Global degrees of freedom	do not change	added during simulation
Implementation effort	smaller	larger

Comparison of EED-EAS and XFEM-PUM

	Embedded discontinuity	Extended finite elements
Stiffness matrix	always nonsymmetric	can be symmetric
Integration scheme for continuous part	remains standard	must be modified
Global degrees of freedom	do not change	added during simulation
Implementation effort	smaller	larger
		... but it pays off

THE END

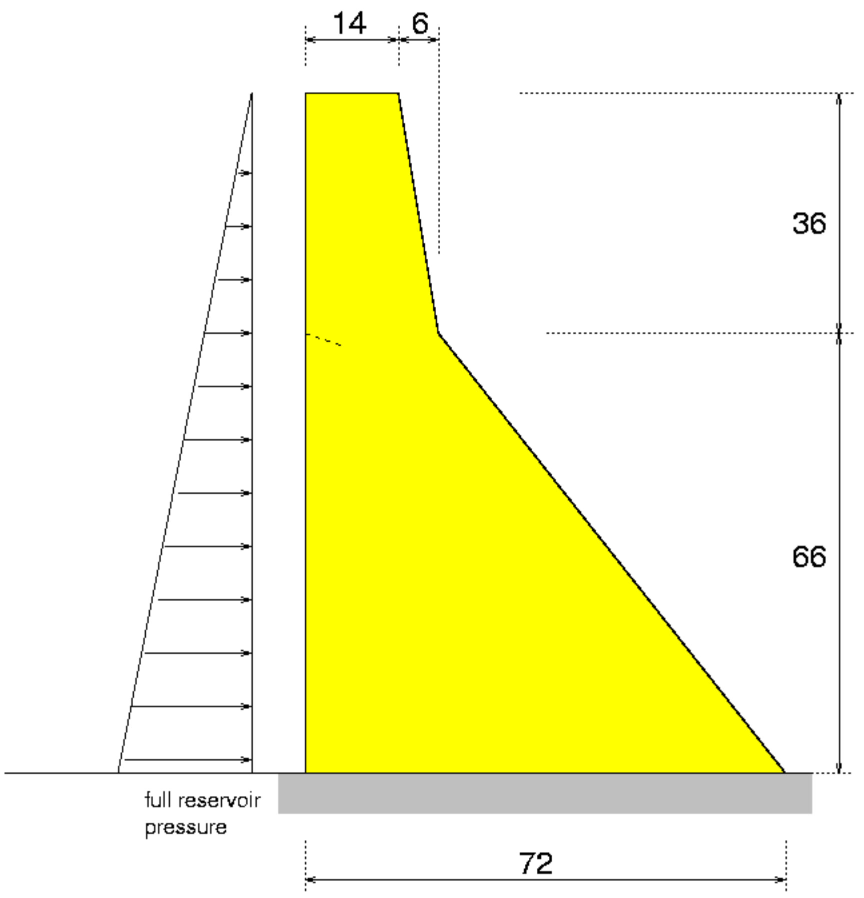
F.6

Regularized Continua

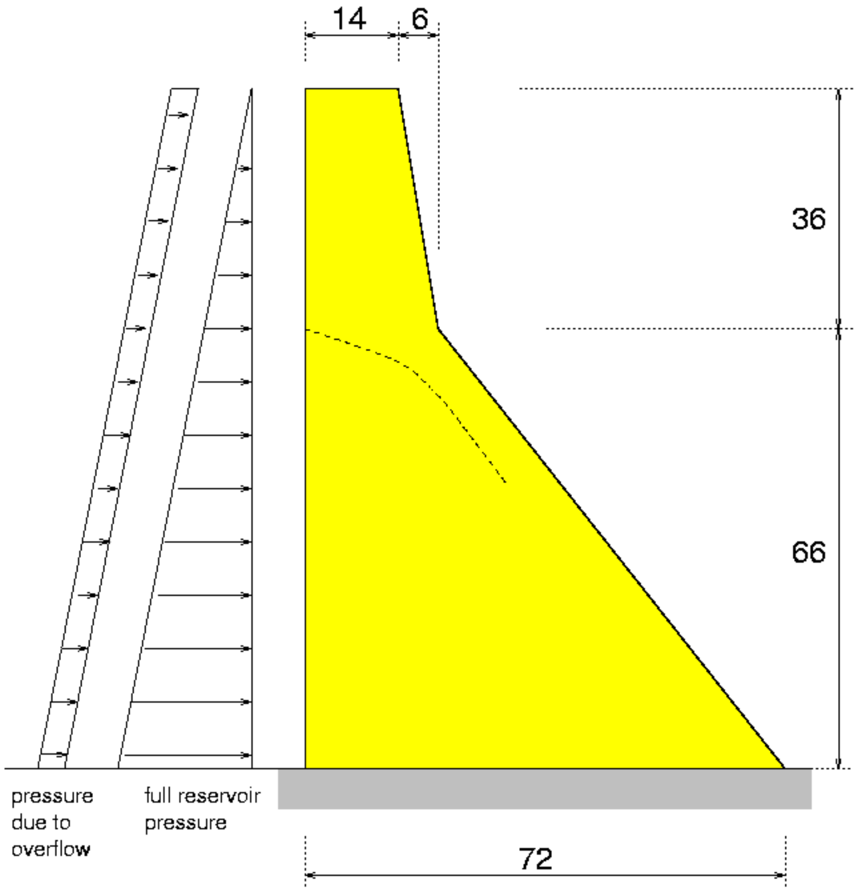
with Strong Discontinuities

F.6.1
Strong Discontinuities versus Regularized Continuum Models

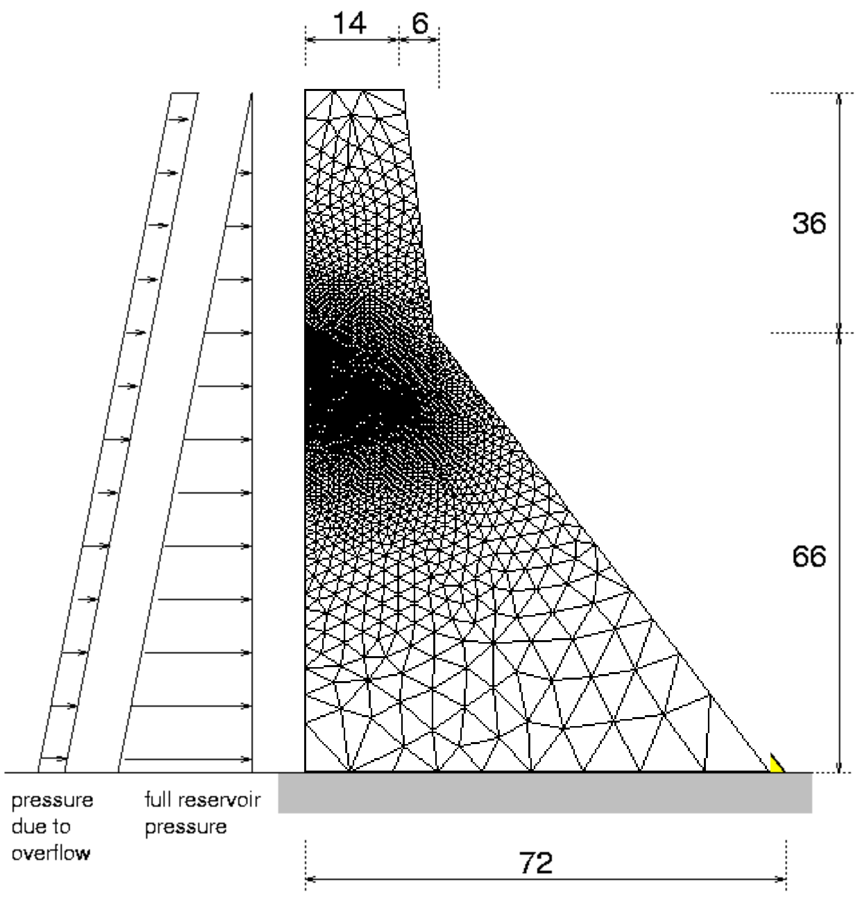
Crack propagation in a gravity dam



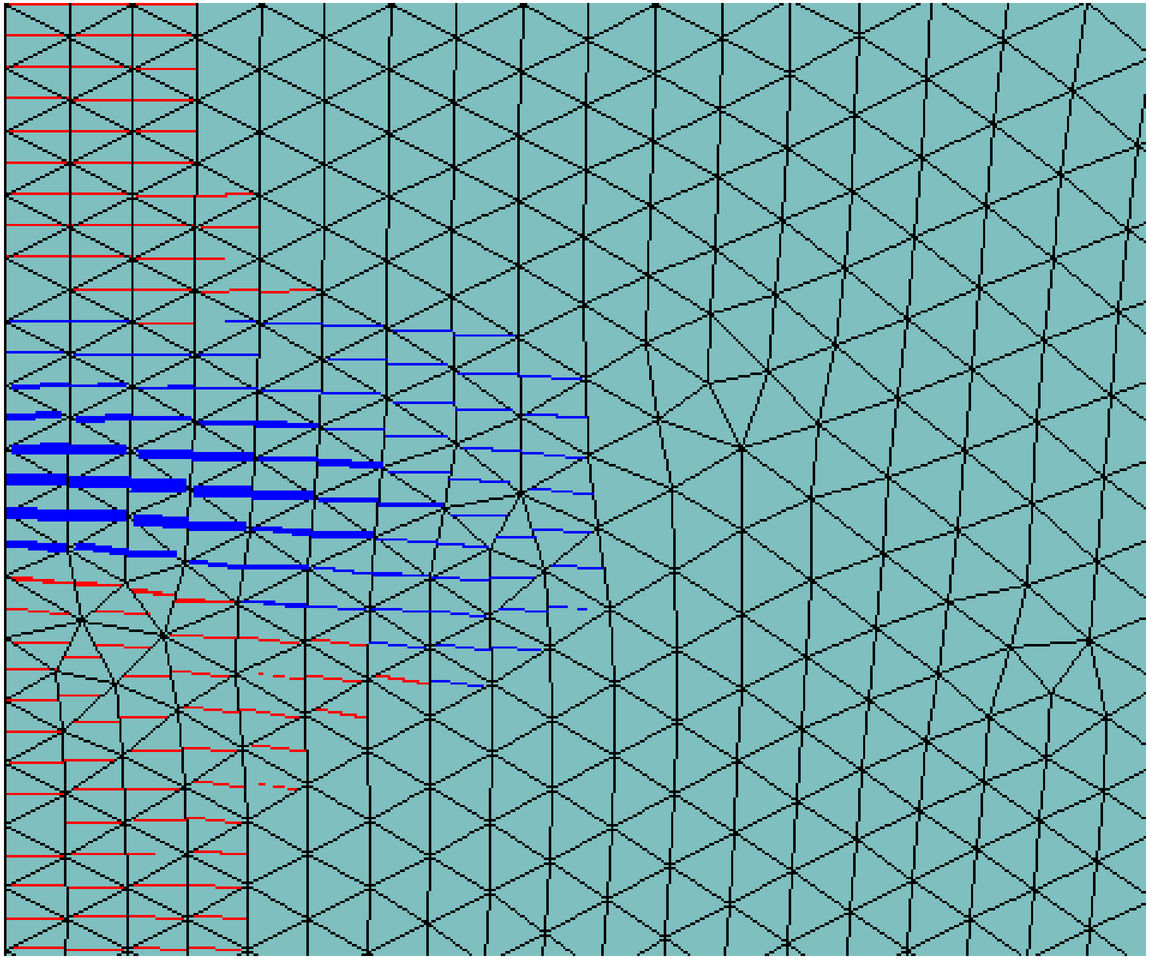
Crack propagation in a gravity dam



Crack propagation in a gravity dam

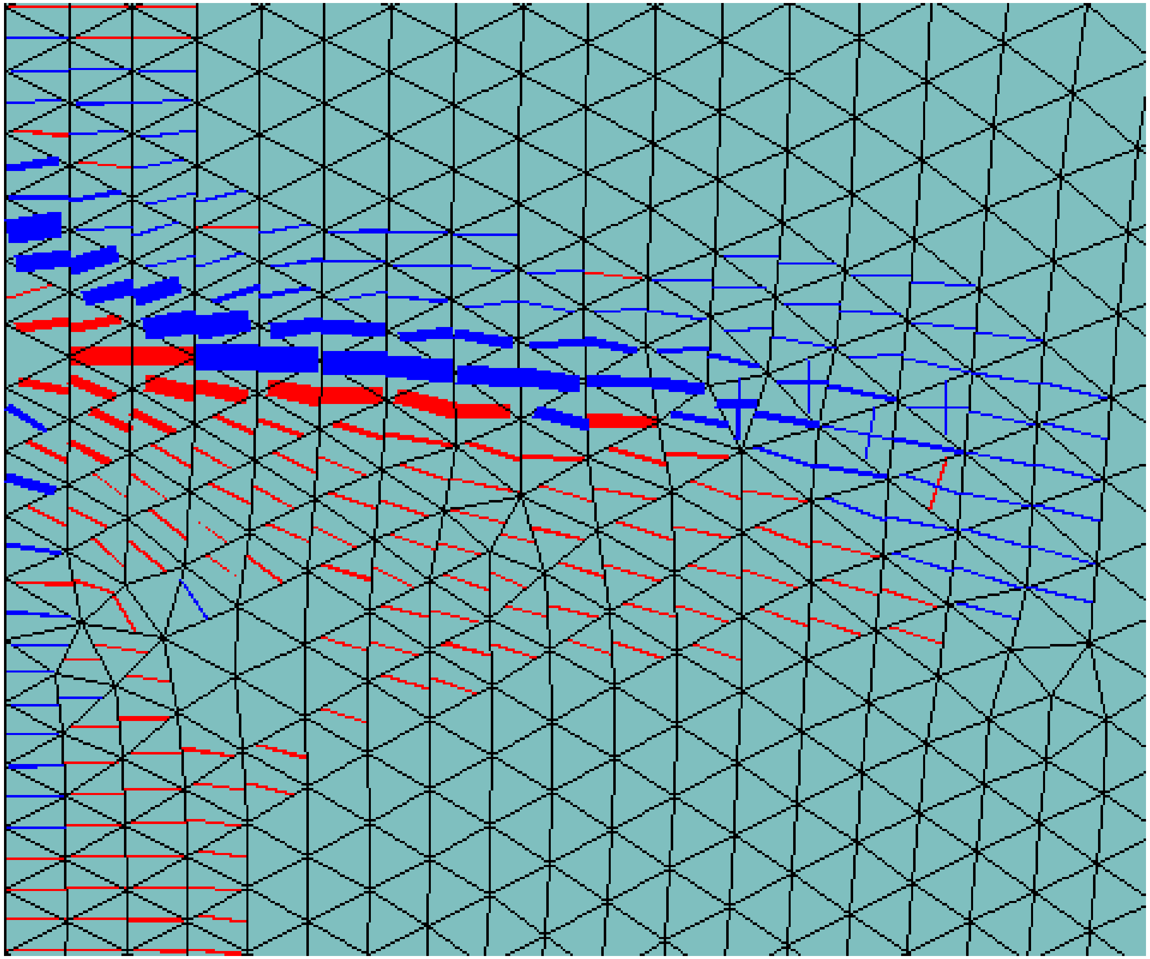


Crack propagation in a gravity dam



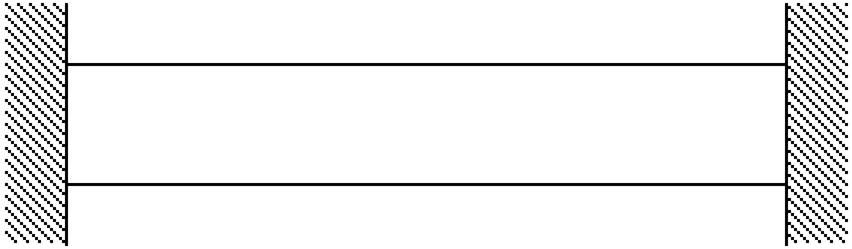
nonlocal damage model

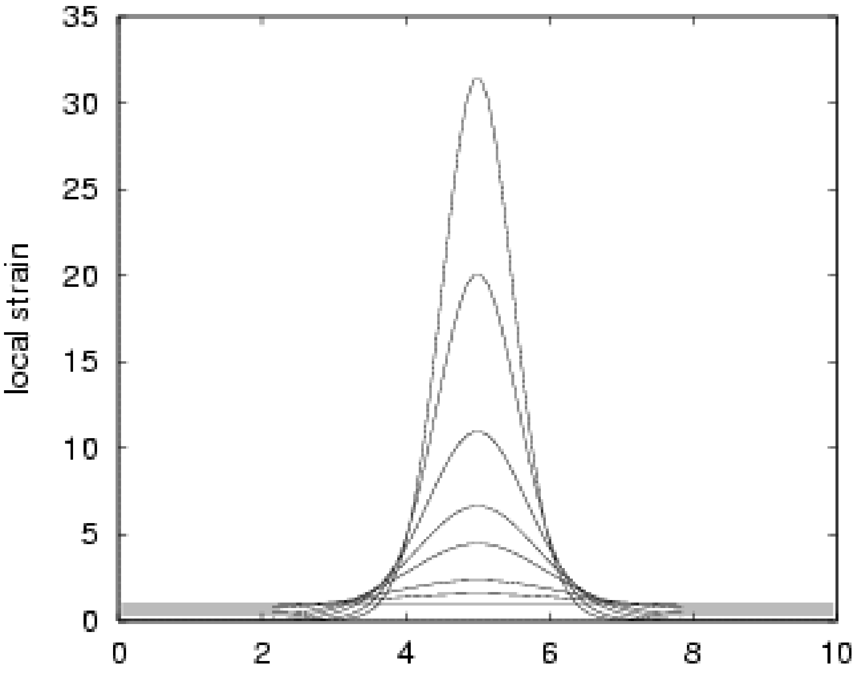
Crack propagation in a gravity dam



nonlocal damage model

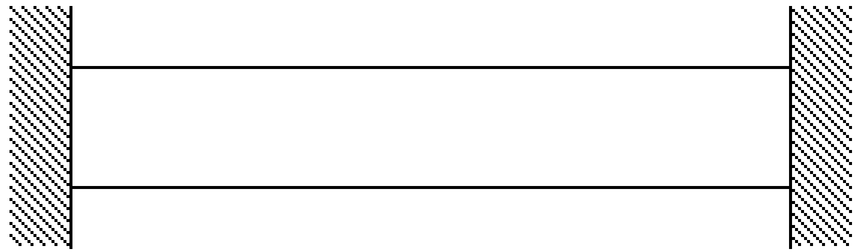
One-dimensional localization test

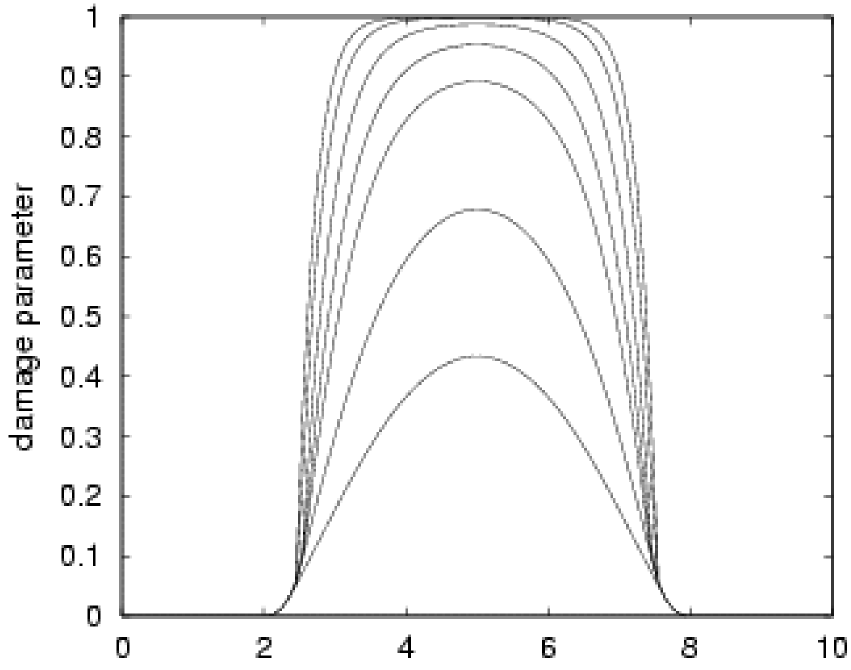




evolution of strain profile

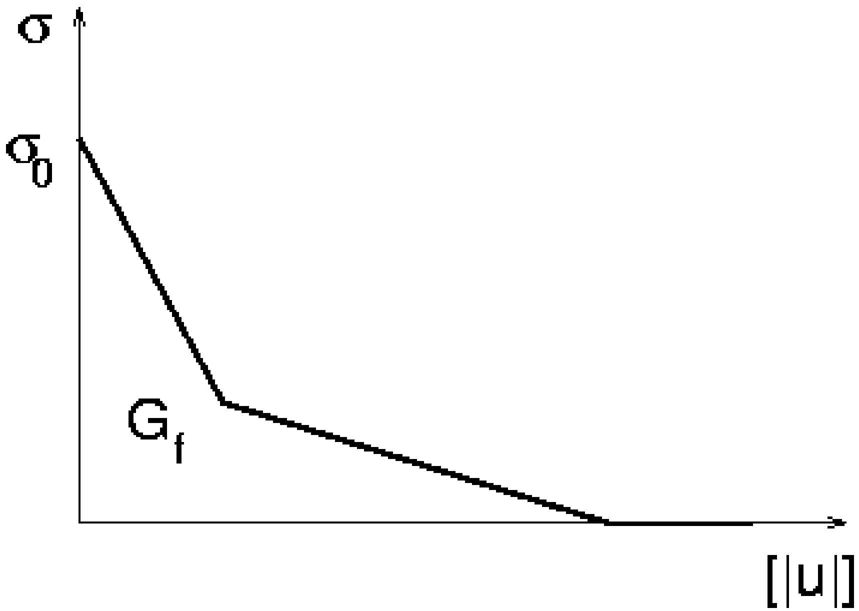
One-dimensional localization test





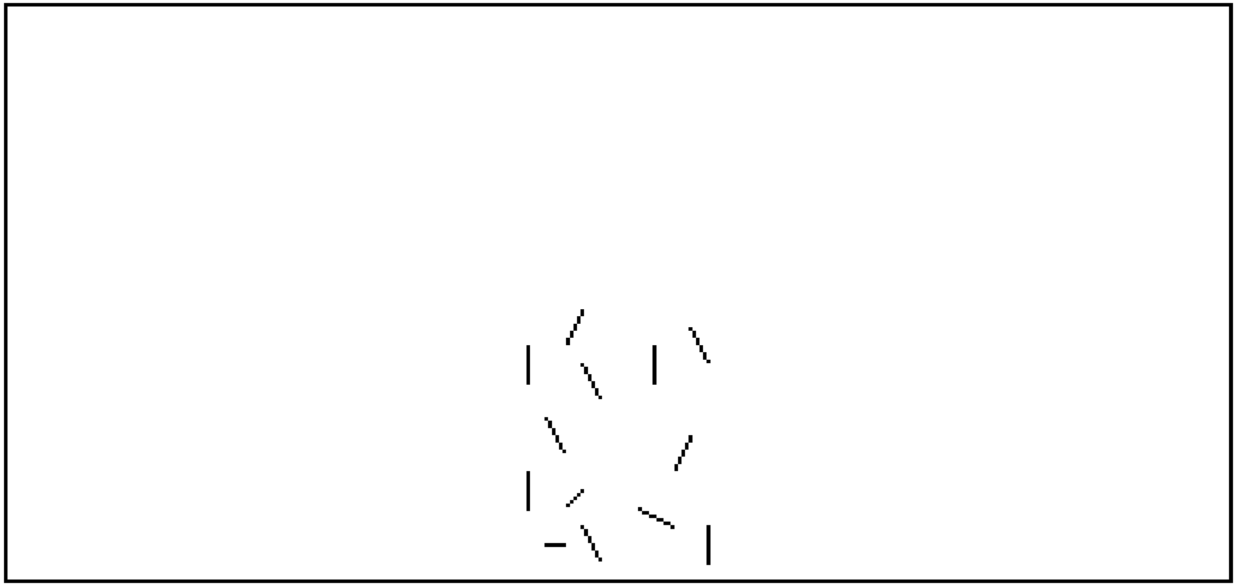
evolution of damage profile

Problem with definition of fracture energy

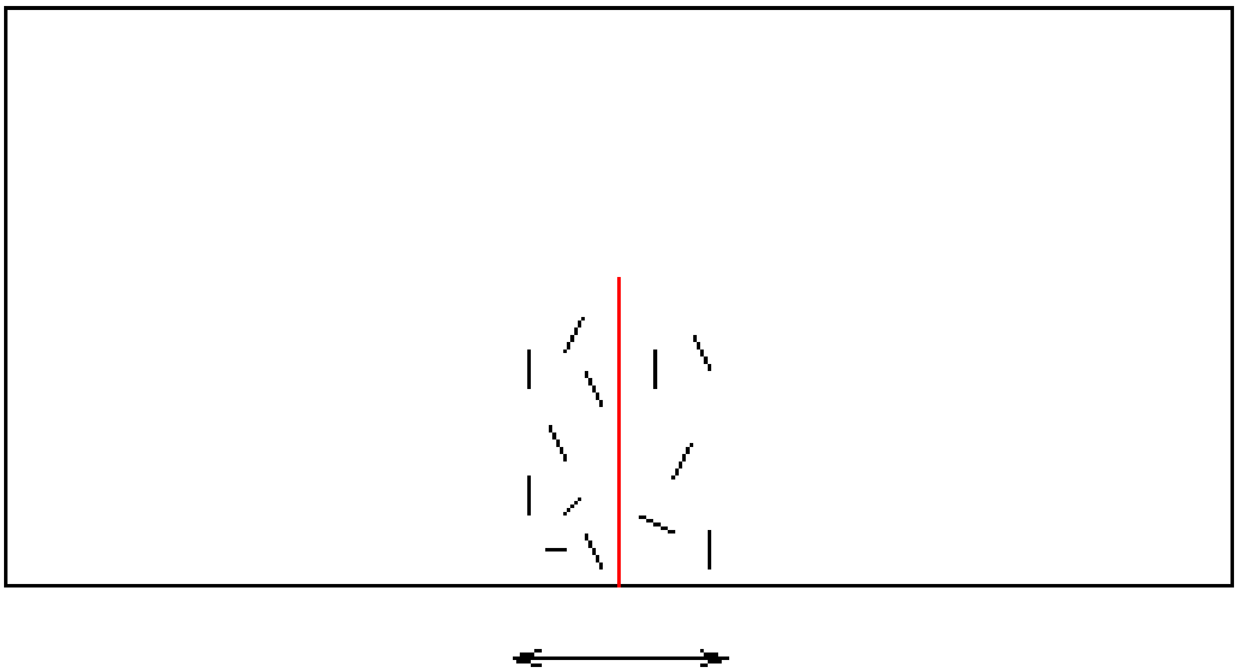


traction-separation law

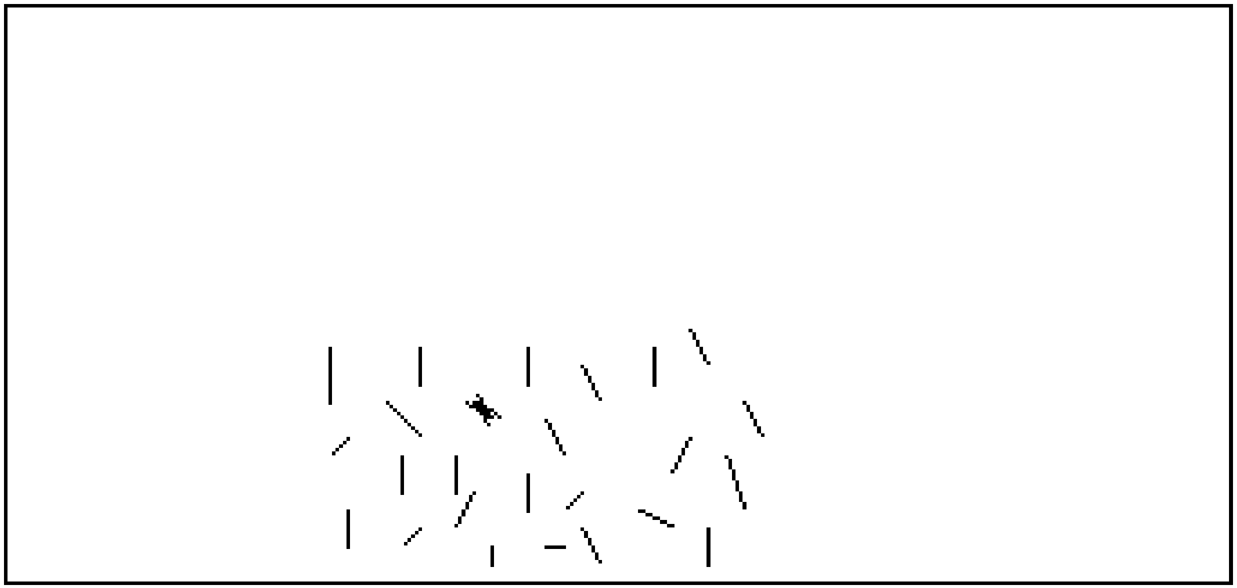
Process zone replaced by cohesive crack



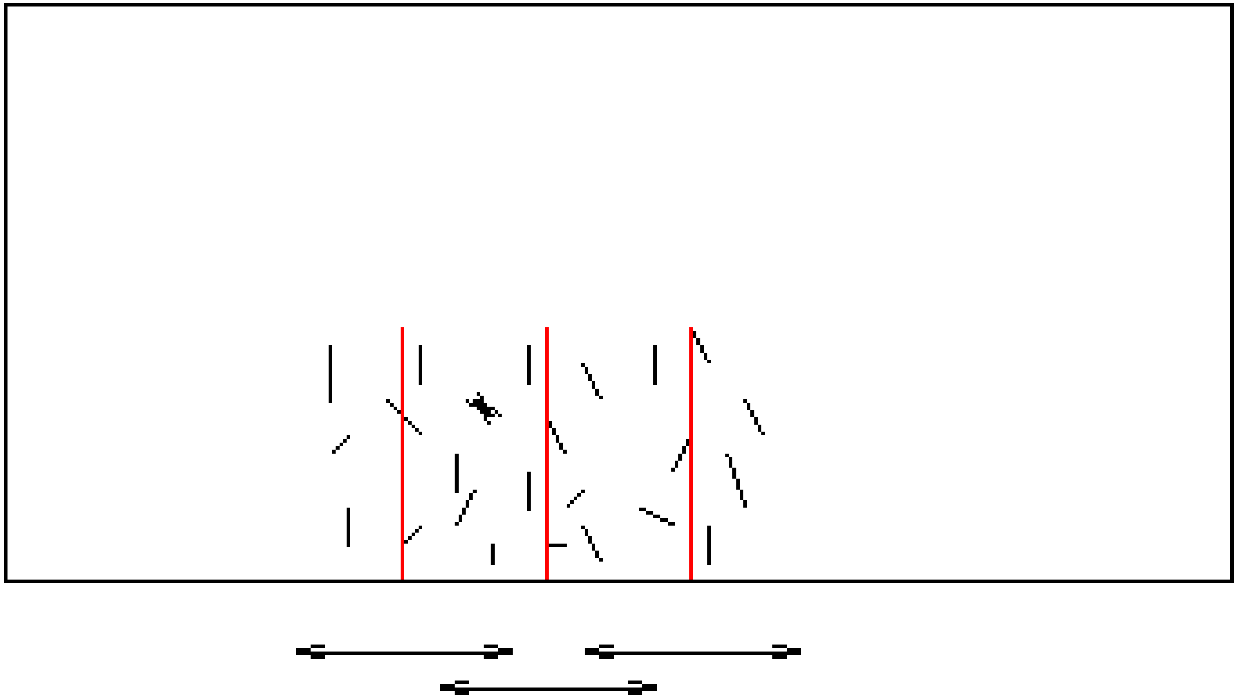
Process zone replaced by cohesive crack



Diffuse damage zone replaced by cohesive cracks

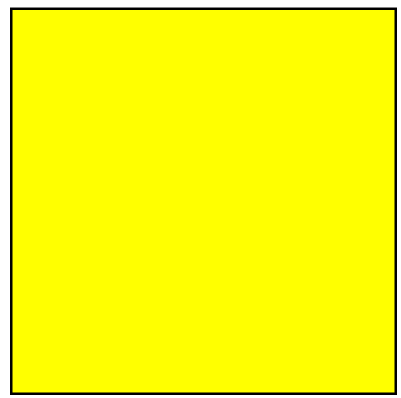


Diffuse damage zone replaced by cohesive cracks

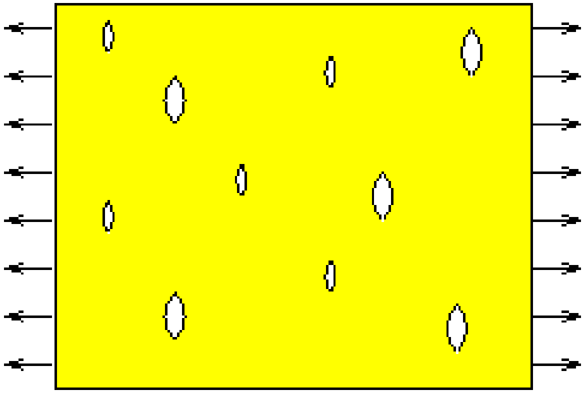


F.6.2
Nonlocal Model with Transition to Strong Discontinuities

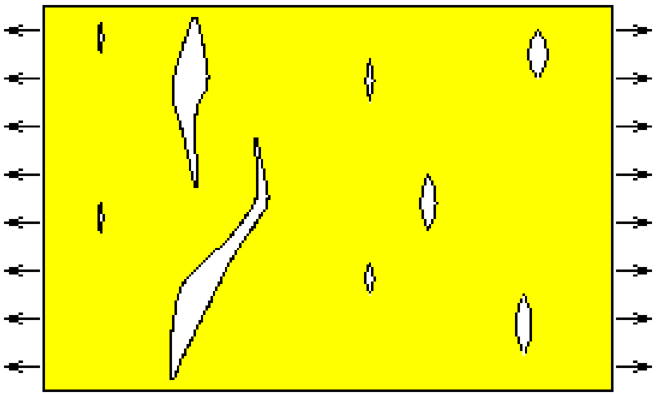
From diffuse damage to discrete cracking



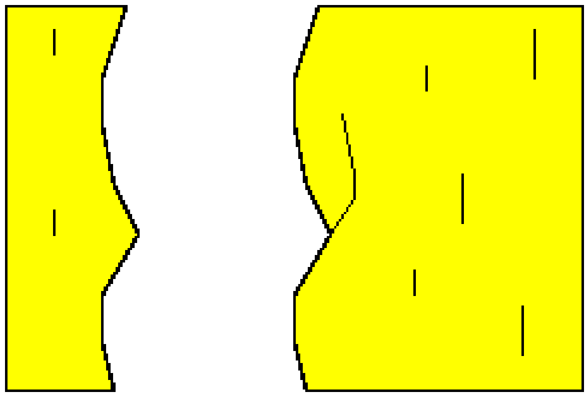
From diffuse damage to discrete cracking



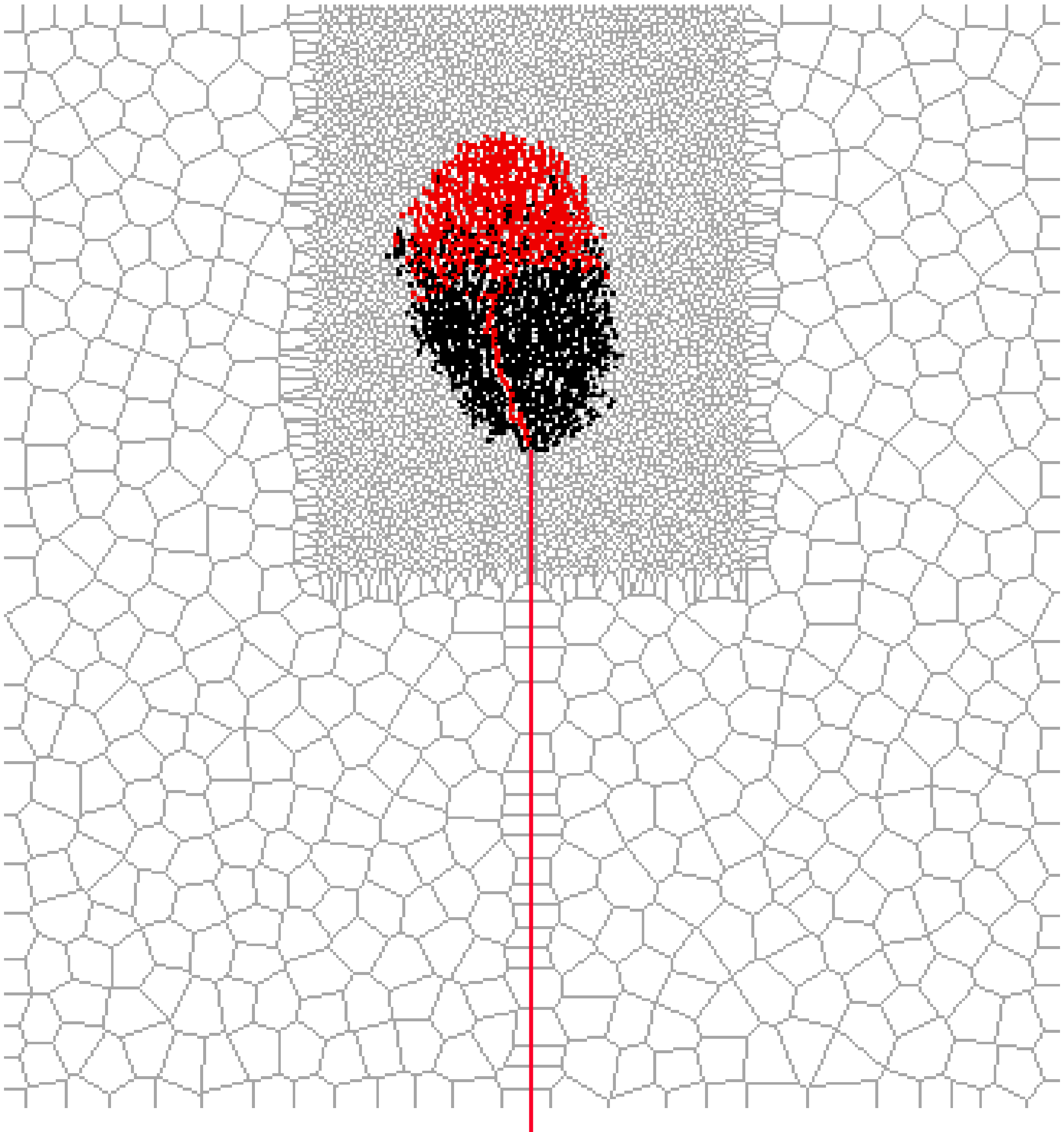
From diffuse damage to discrete cracking

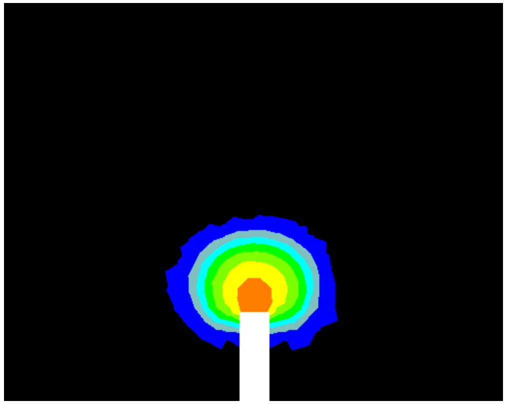


From diffuse damage to discrete cracking

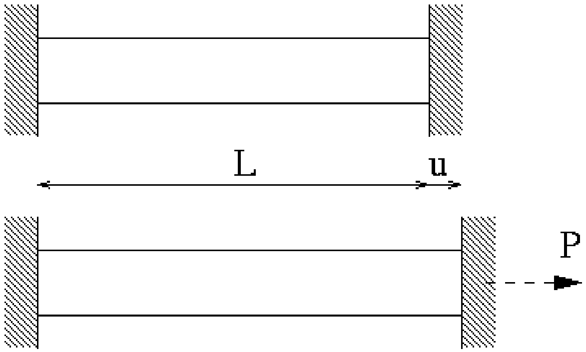


Transition from diffuse to localized failure pattern

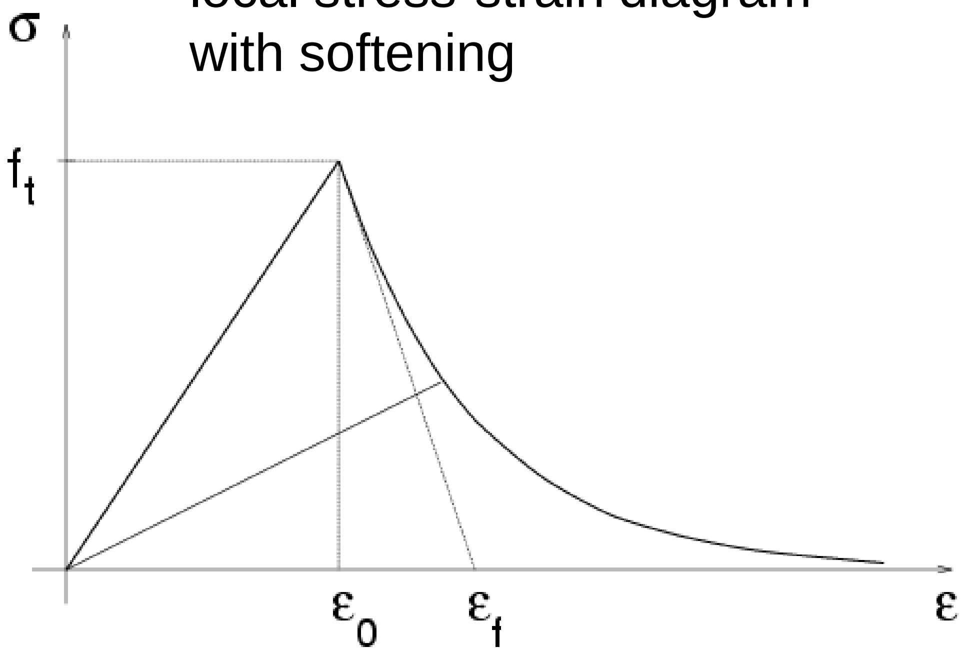




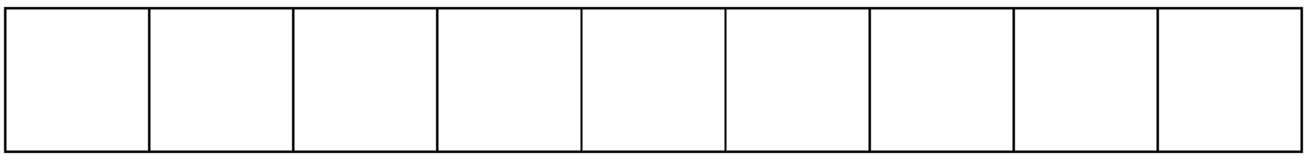
One-dimensional localization test



local stress-strain diagram
with softening



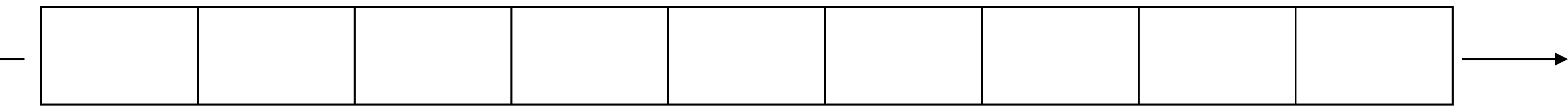
Continuum damage combined with a discontinuity



Continuum damage combined with a discontinuity

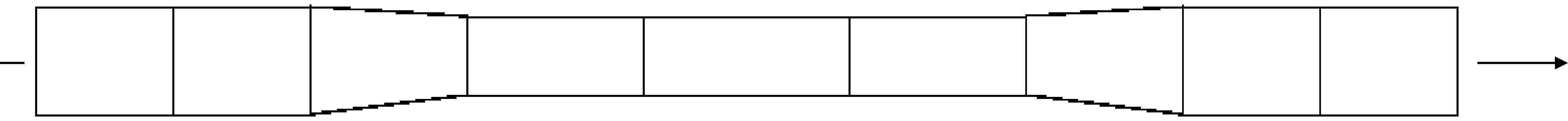
uniform strain distribution

Continuum damage combined with a discontinuity



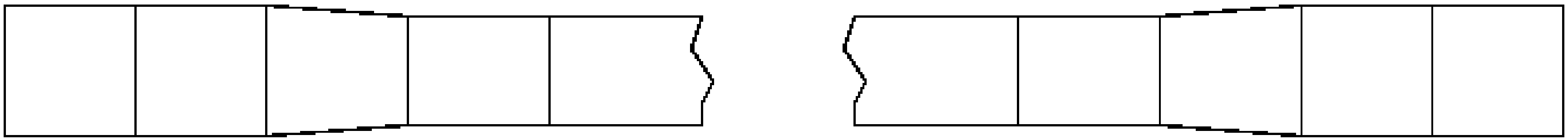
uniform strain distribution

Continuum damage combined with a discontinuity



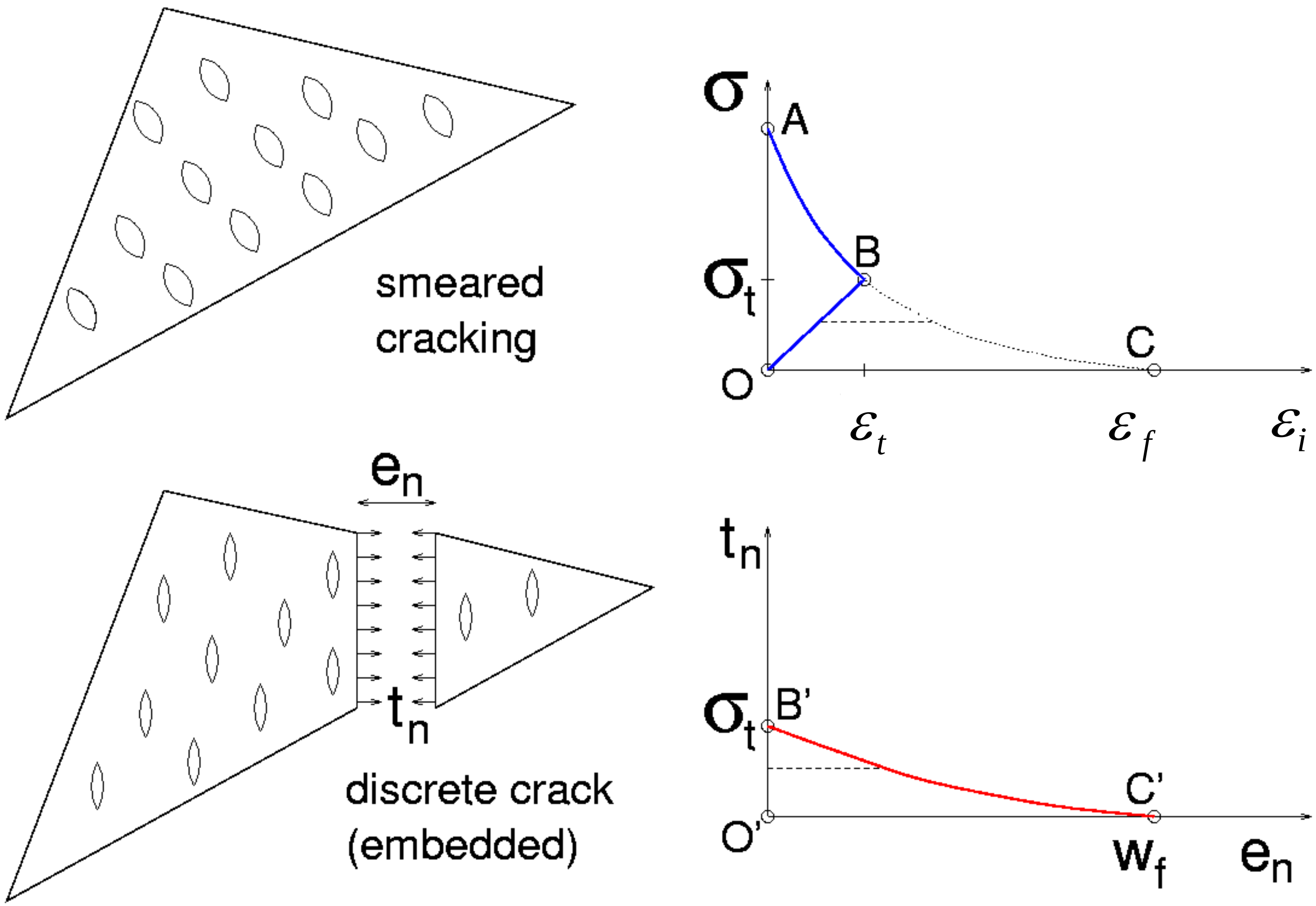
localized strain distribution, continuous

Continuum damage combined with a discontinuity

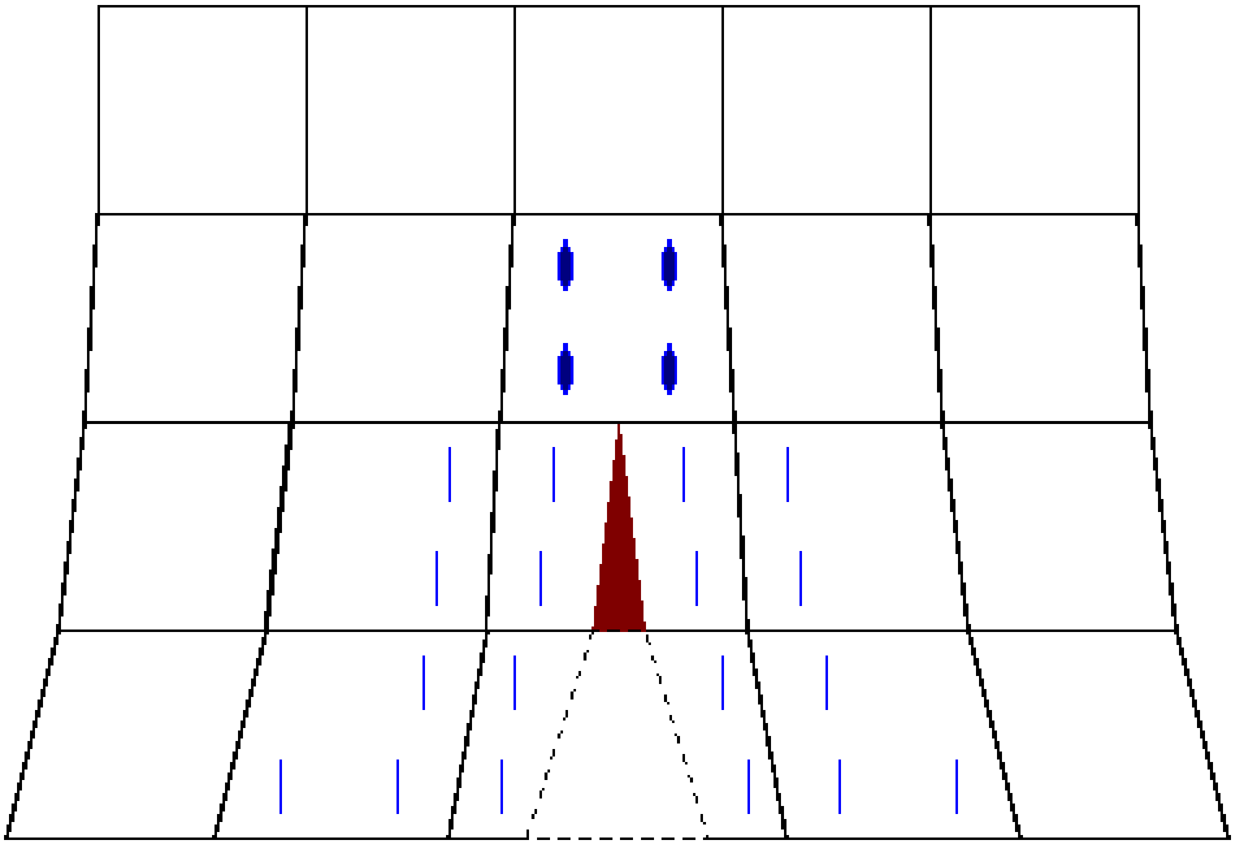


localized strain distribution, discontinuous

Continuum damage combined with a discontinuity



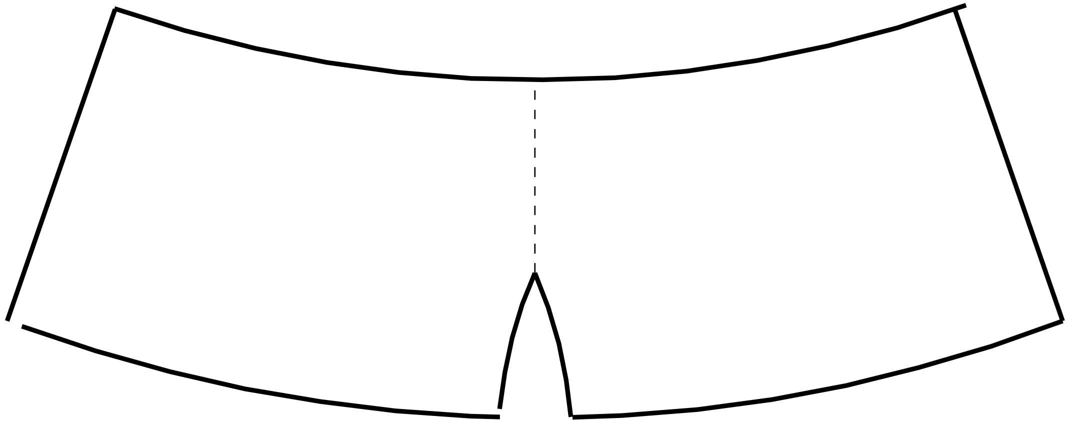
Continuum damage combined with a discontinuity



F.6.4
Influence of Crack on Nonlocal Strain

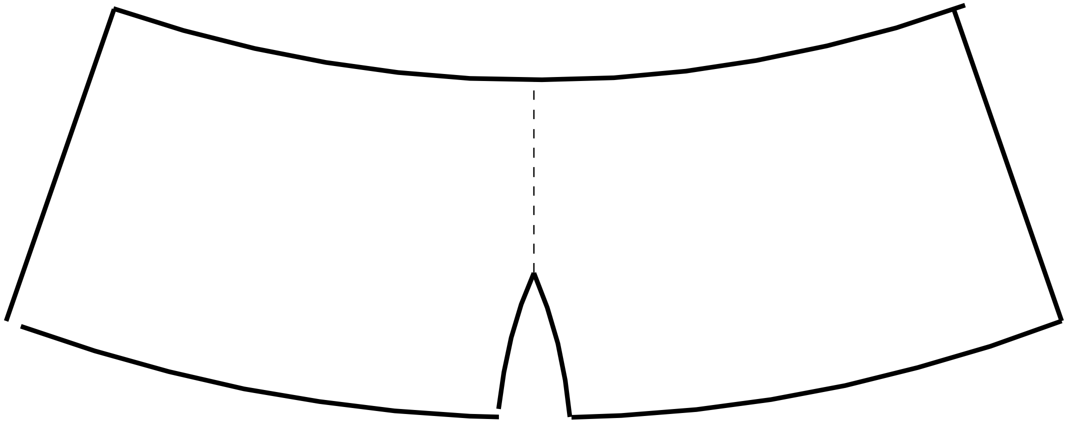
Influence of crack on nonlocal strain

Observation (Simone et al.):
maximum value of nonlocal strain **is not** attained at the crack tip

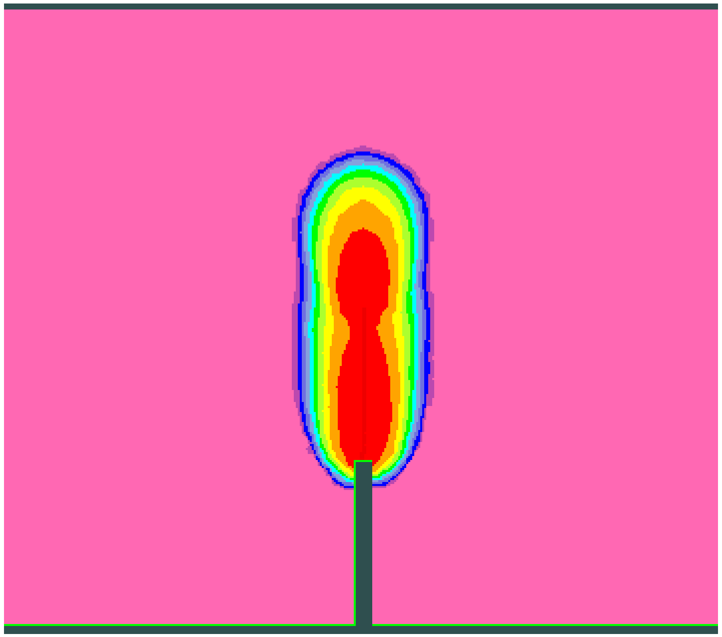


Influence of crack on nonlocal strain

Observation (Simone et al.): with standard averaging,
maximum value of nonlocal strain **is not** attained at the crack tip

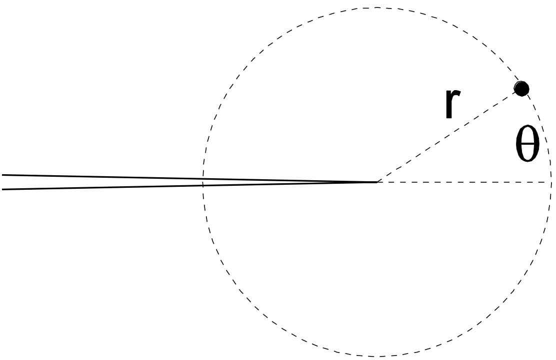


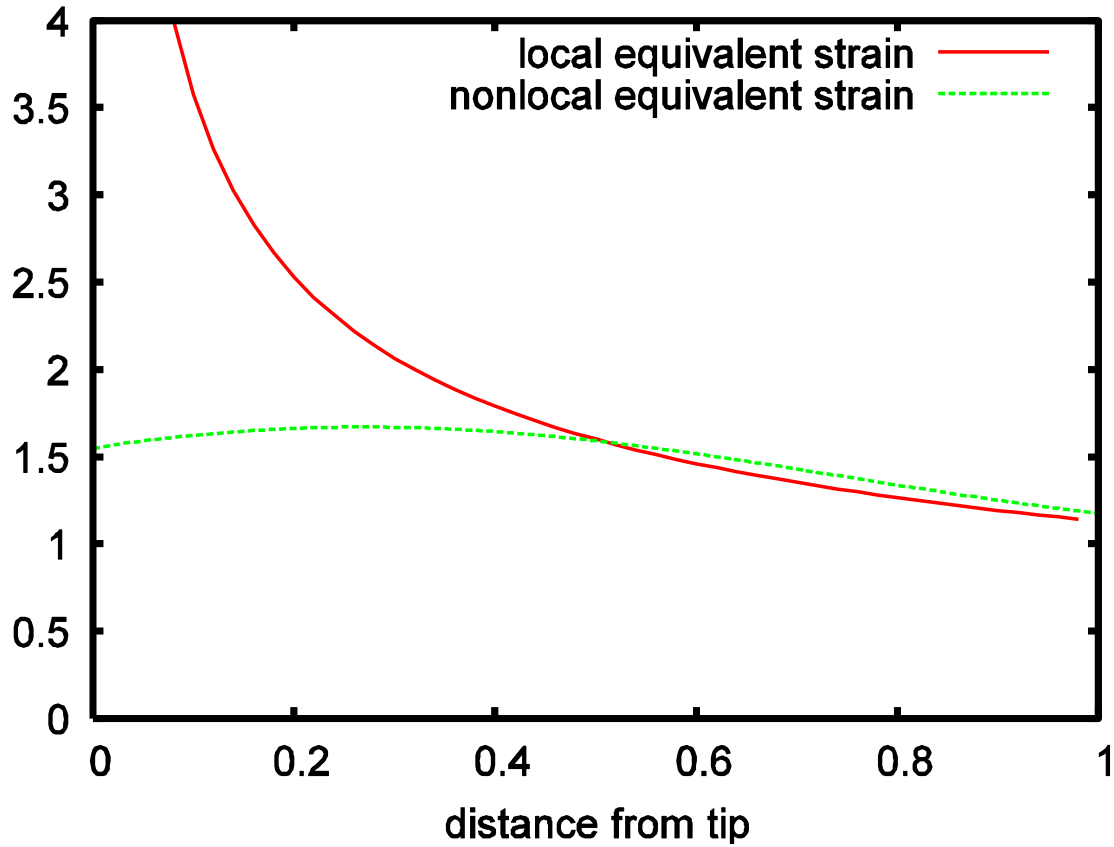
Possible consequence:
crack propagation in jumps



Influence of crack on nonlocal strain

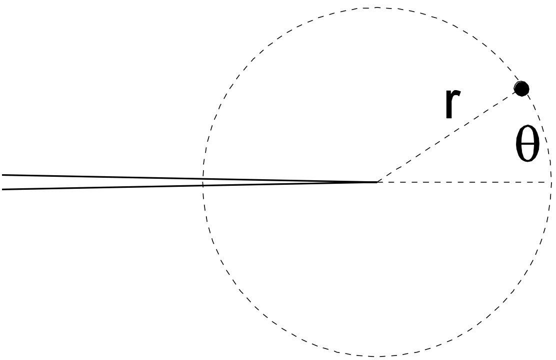
maximum value of nonlocal strain **is not** attained at the crack tip ??

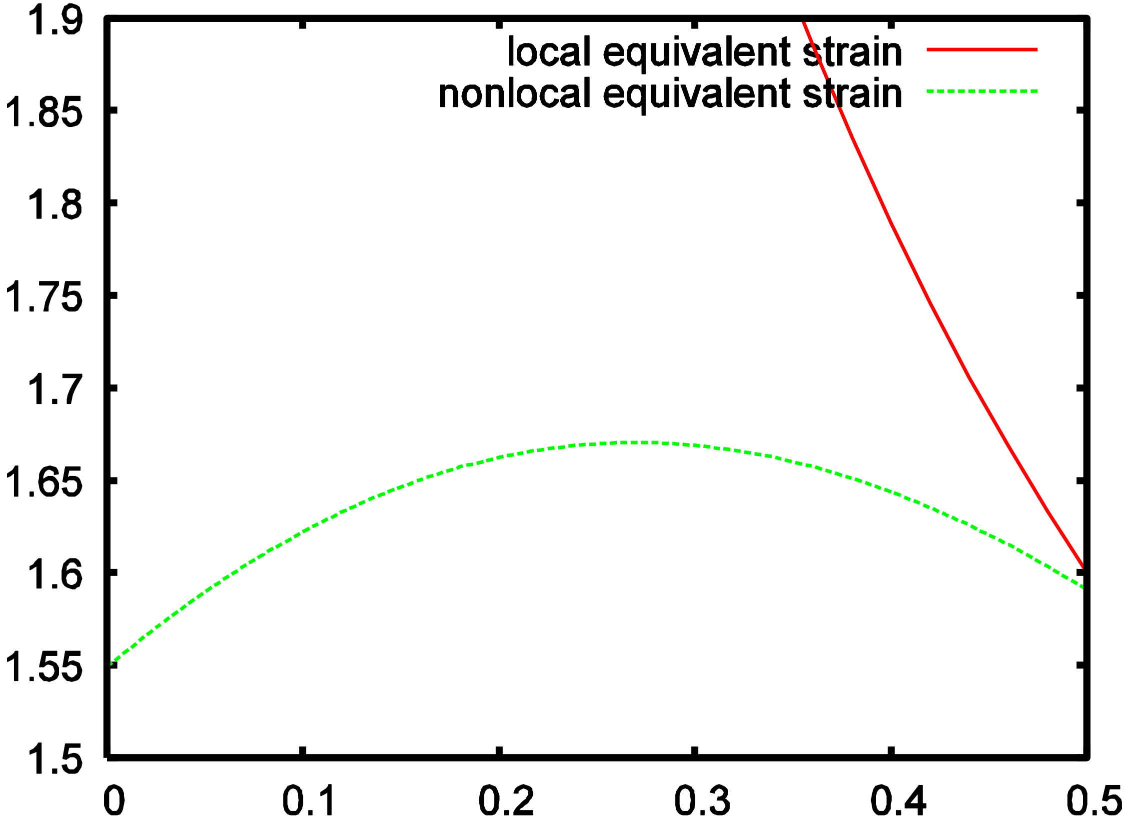




Influence of crack on nonlocal strain

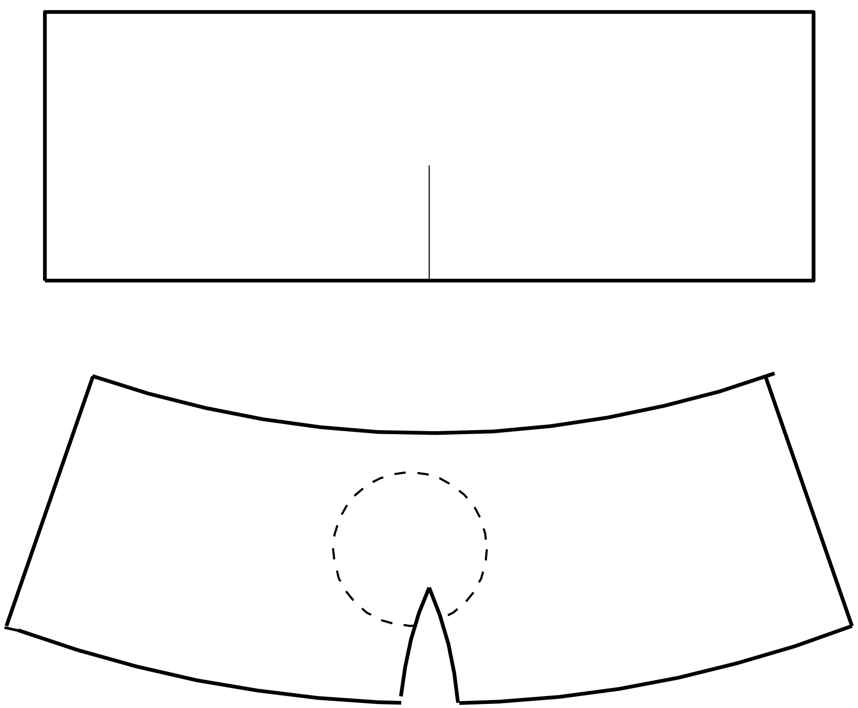
maximum value of nonlocal strain **is not** attained at the crack tip !!



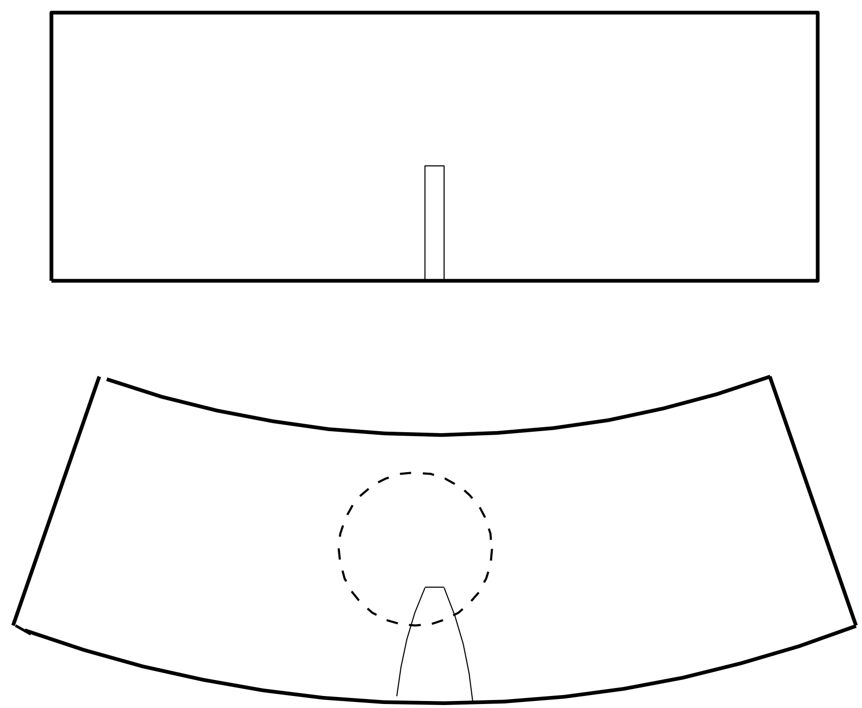


Influence of crack on nonlocal strain

line crack

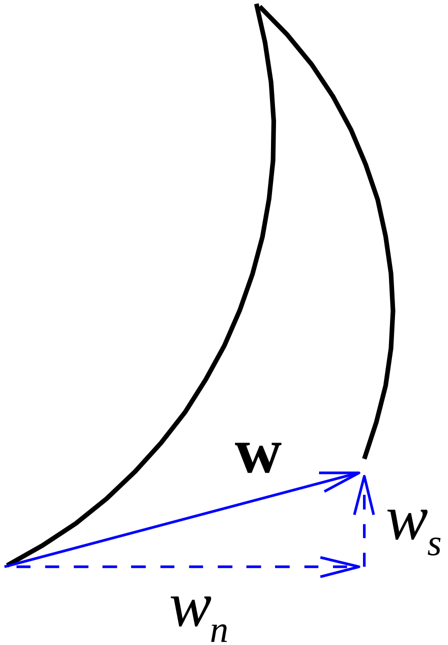


thin layer of damaged material



Influence of crack on nonlocal strain

Contribution of crack opening to nonlocal equivalent strain



$$w_n \quad \rightarrow \quad \varepsilon_n = \frac{w_n}{h}$$

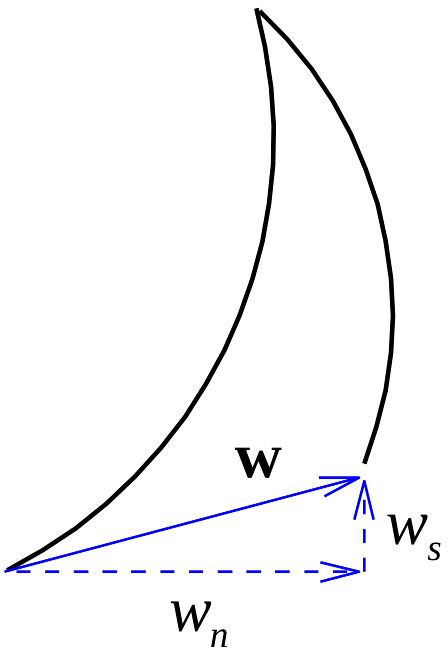
$$w_s \quad \rightarrow \quad \gamma_{ns} = \frac{w_s}{h}$$

$$\varepsilon_{1,2} = \frac{1}{2}\left(\varepsilon_n \pm \sqrt{\varepsilon_n^2 + \gamma_{ns}^2}\right)$$

$$\tilde{\varepsilon} = \sqrt{\langle \varepsilon_1 \rangle^2 + \langle \varepsilon_2 \rangle^2} = \varepsilon_1 = \frac{1}{2h}\left(w_n + \sqrt{w_n^2 + w_s^2}\right)$$

Influence of crack on nonlocal strain

Contribution of crack opening to nonlocal equivalent strain



$$w_n \quad \rightarrow \quad \varepsilon_n = \frac{w_n}{h}$$

$$w_s \quad \rightarrow \quad \gamma_{ns} = \frac{w_s}{h}$$

$$\varepsilon_{1,2} = \frac{1}{2}\left(\varepsilon_n \pm \sqrt{\varepsilon_n^2 + \gamma_{ns}^2}\right)$$

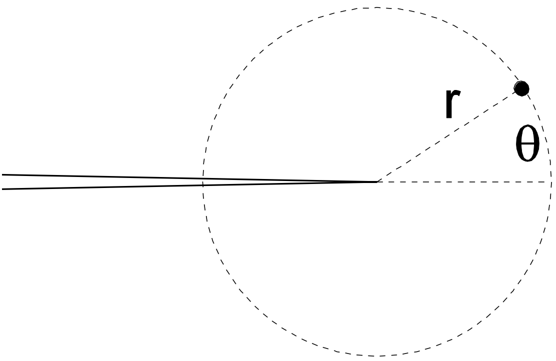
$$\tilde{\varepsilon} = \sqrt{\langle \varepsilon_1 \rangle^2 + \langle \varepsilon_2 \rangle^2} = \varepsilon_1 = \frac{1}{2h}\left(w_n + \sqrt{w_n^2 + w_s^2}\right)$$

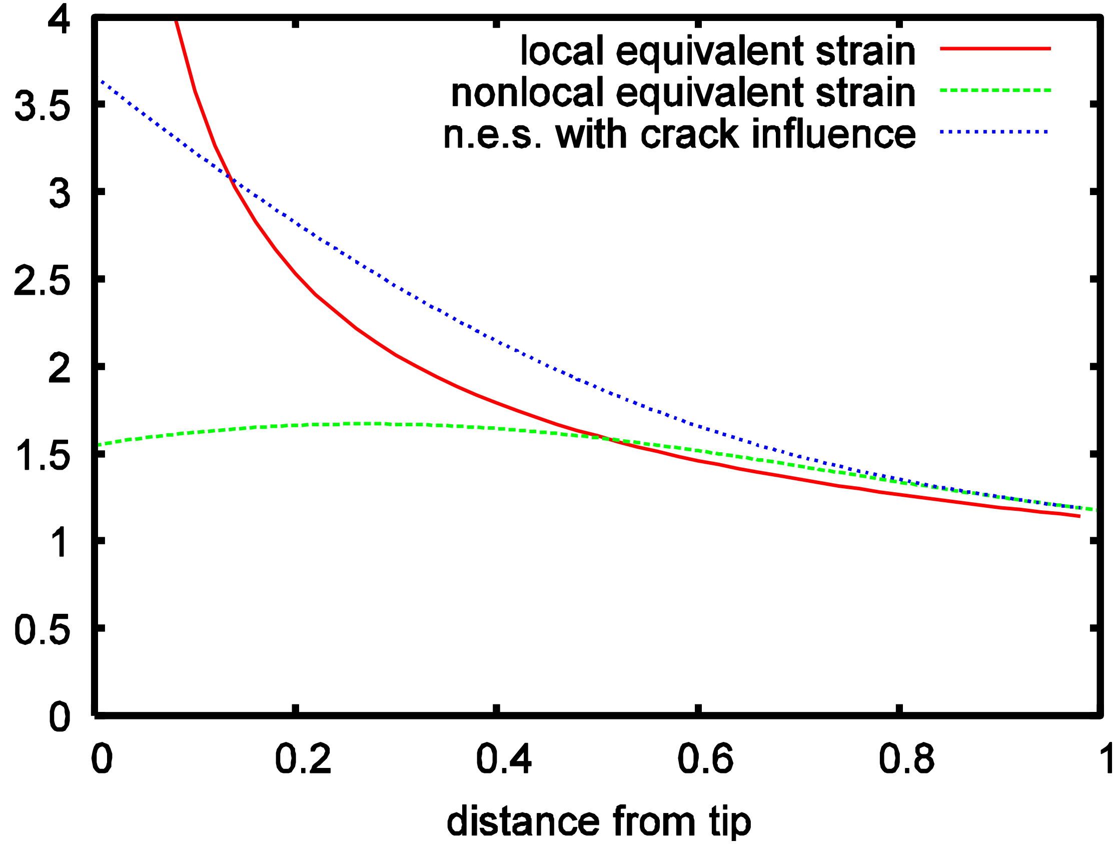
$$\int_{V_d} \alpha(\mathbf{x}, \xi)\tilde{\varepsilon}(\xi)\mathrm{dV}(\xi) = \int_{\Gamma_d} h\alpha(\mathbf{x}, \xi)\tilde{\varepsilon}(\xi)\mathrm{d}\Gamma(\xi) =$$

$$= \frac{1}{2}\int_{\Gamma_d} \alpha(\mathbf{x}, \xi)\left(w_n(\xi) + \sqrt{w_n^2(\xi) + w_s^2(\xi)}\right)\mathrm{d}\Gamma(\xi)$$

Influence of crack on nonlocal strain

after correction, maximum value of nonlocal strain **is** attained at the crack tip

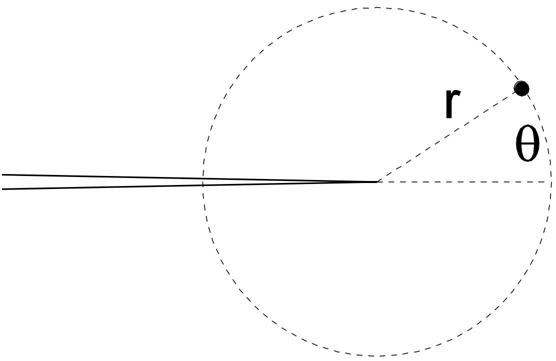


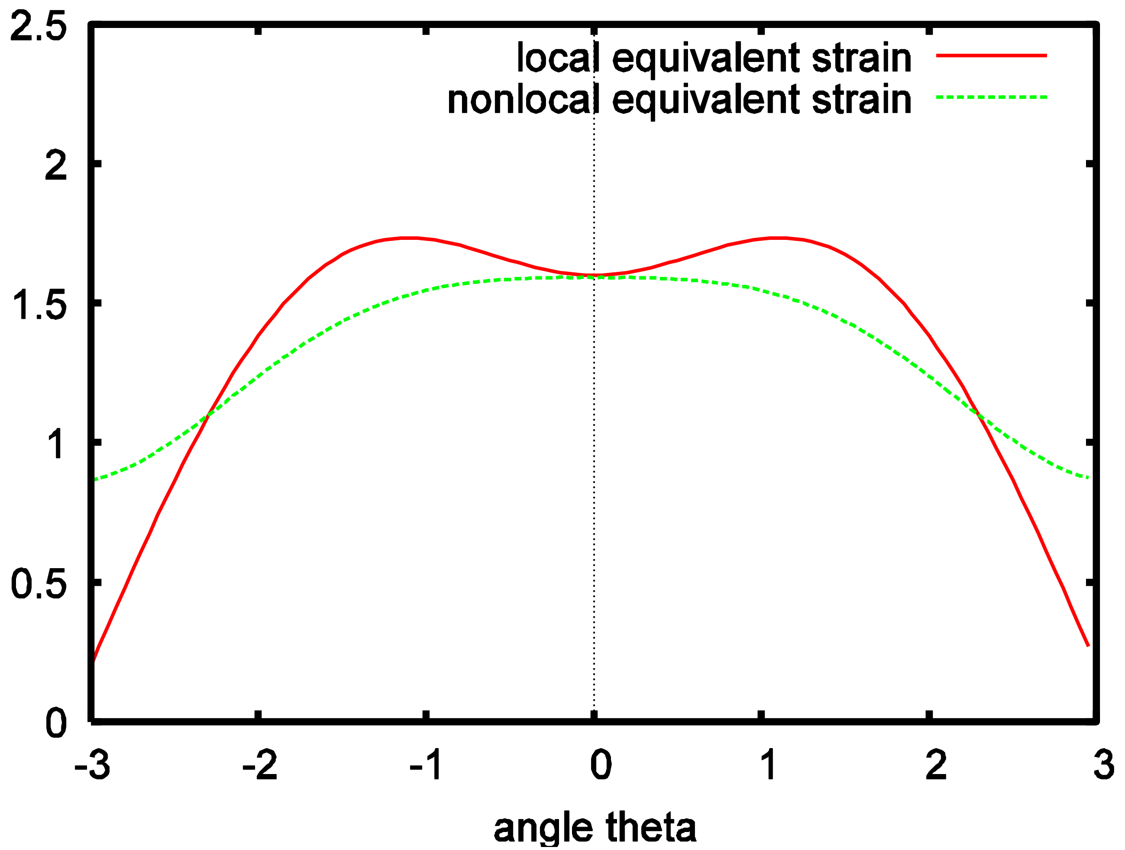


$$\bar{\varepsilon}(\mathbf{x}) = \int_V \alpha(\mathbf{x},\xi)\,\tilde{\varepsilon}(\xi)\,\mathrm{d}V(\xi) + \int_{\Gamma_d} \alpha(\mathbf{x},\xi)\,\tilde{w}(\xi)\,\mathrm{d}\Gamma(\xi)$$

Influence of crack on nonlocal strain

variation of nonlocal strain at constant distance from the crack tip

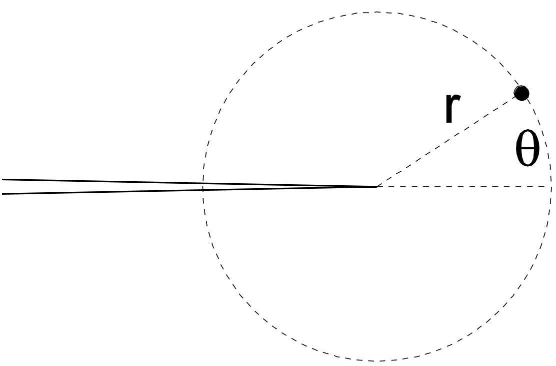


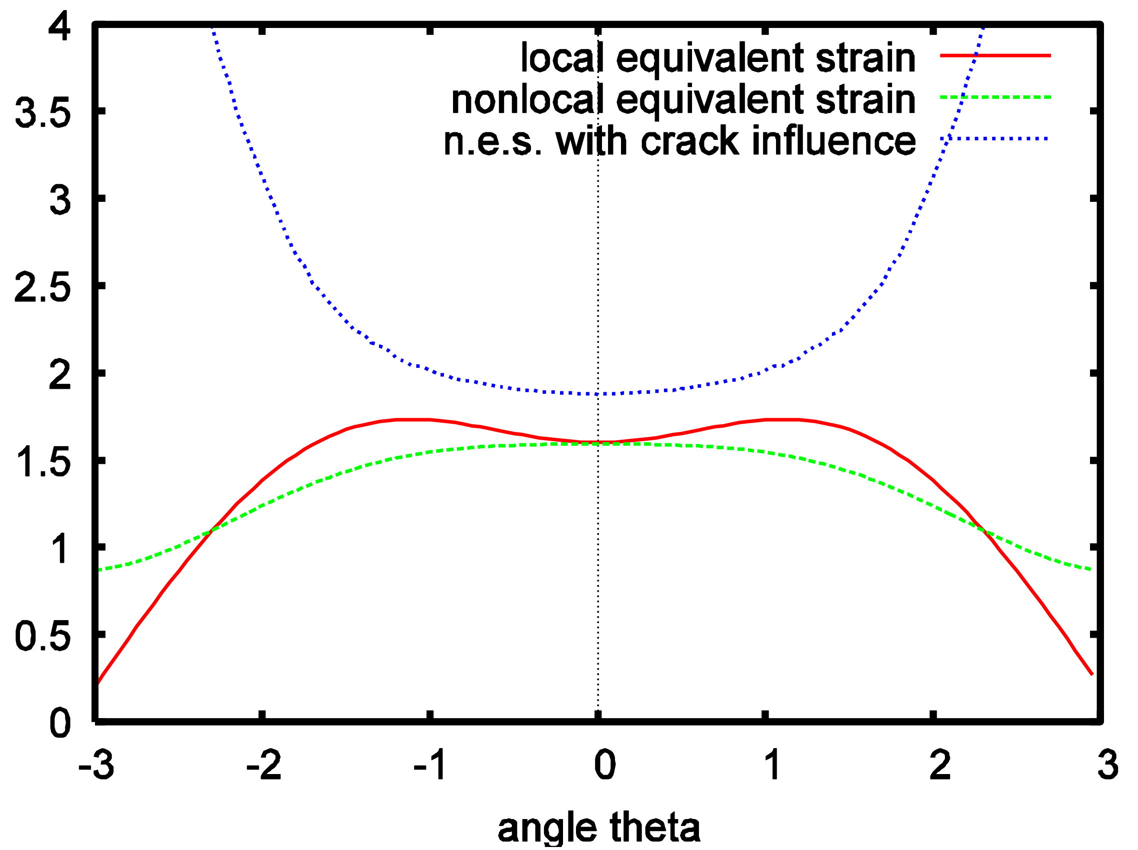


$$\bar{\varepsilon}(\mathbf{x}) = \int_V \alpha(\mathbf{x}, \xi)\, \tilde{\varepsilon}(\xi)\, \mathrm{d}\xi$$

Influence of crack on nonlocal strain

variation of nonlocal strain at constant distance from the crack tip

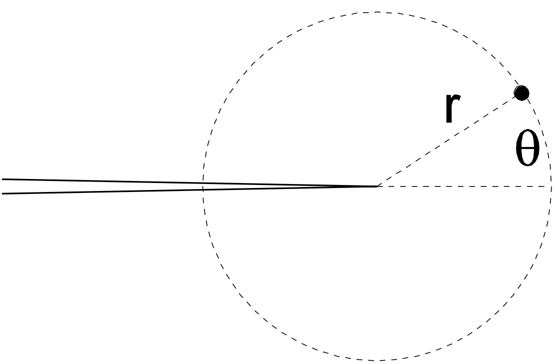


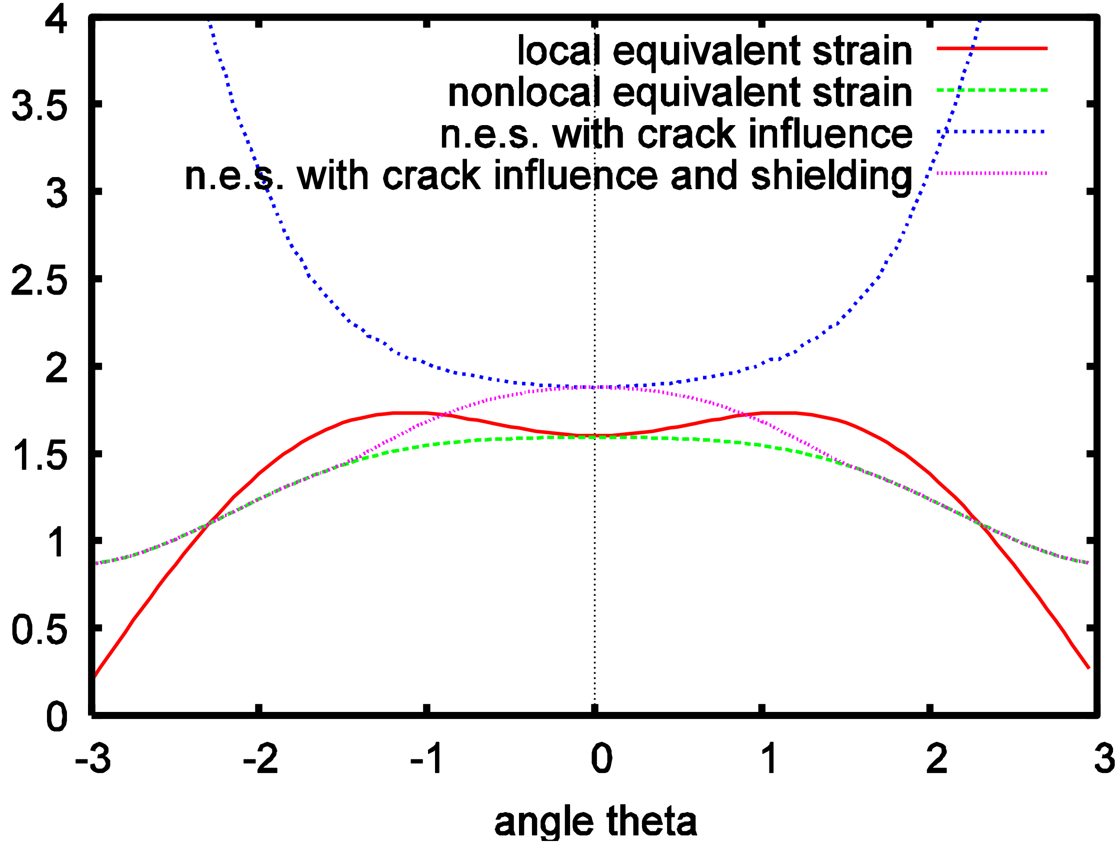


$$\bar{\varepsilon}(\mathbf{x}) = \int_V \alpha(\mathbf{x},\xi)\,\tilde{\varepsilon}(\xi)\,\mathrm{d}V(\xi) + \int_{\Gamma_d} \alpha(\mathbf{x},\xi)\,\tilde{w}(\xi)\,\mathrm{d}\Gamma(\xi)$$

Influence of crack on nonlocal strain

variation of nonlocal strain at constant distance from the crack tip

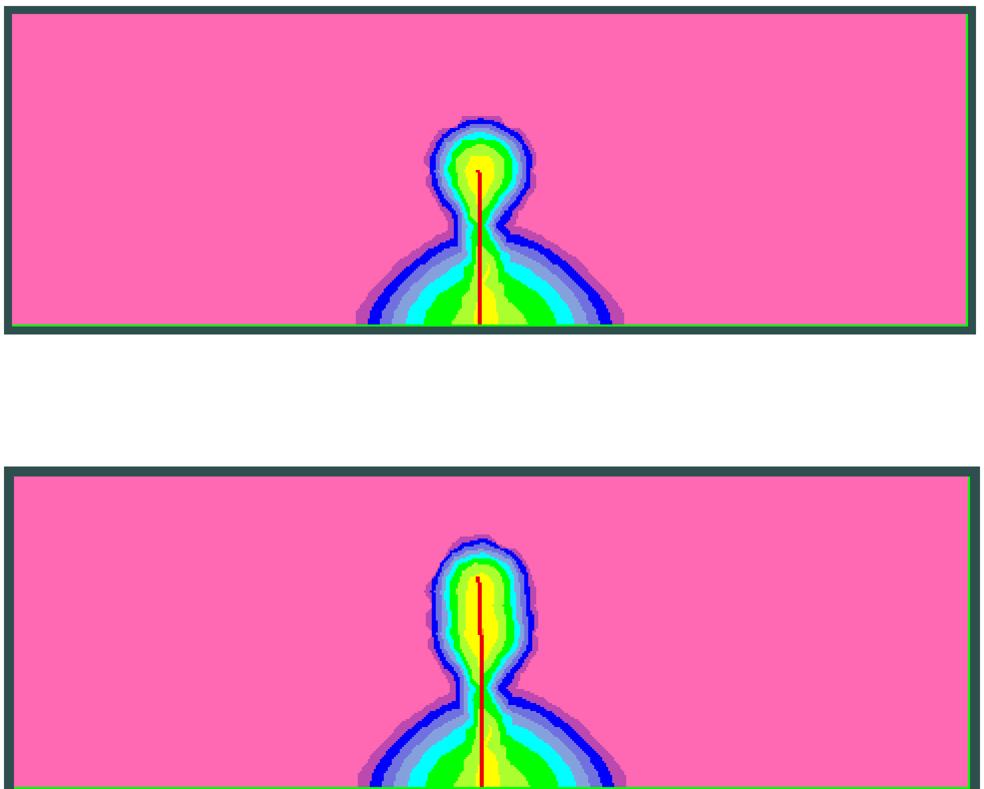




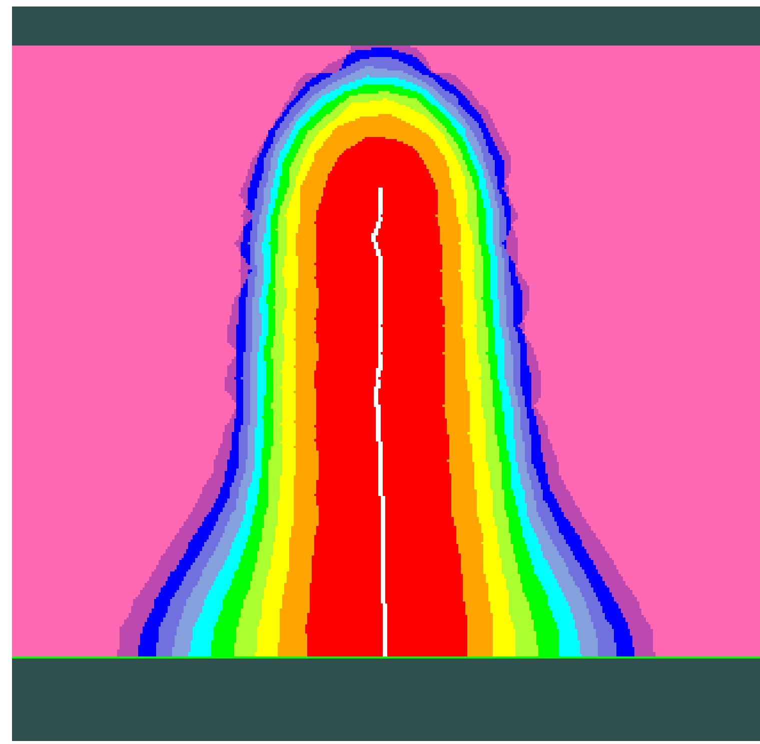
$$\bar{\varepsilon}(\mathbf{x}) = \int_V \alpha(\mathbf{x},\xi)\, \tilde{\varepsilon}(\xi)\, \mathrm{d}V(\xi) + \int_{\Gamma_d} \alpha(\mathbf{x},\xi)\, \tilde{w}(\xi)\, \mathrm{d}\Gamma(\xi) \langle \cos\theta \rangle^2$$

Influence of crack on nonlocal strain

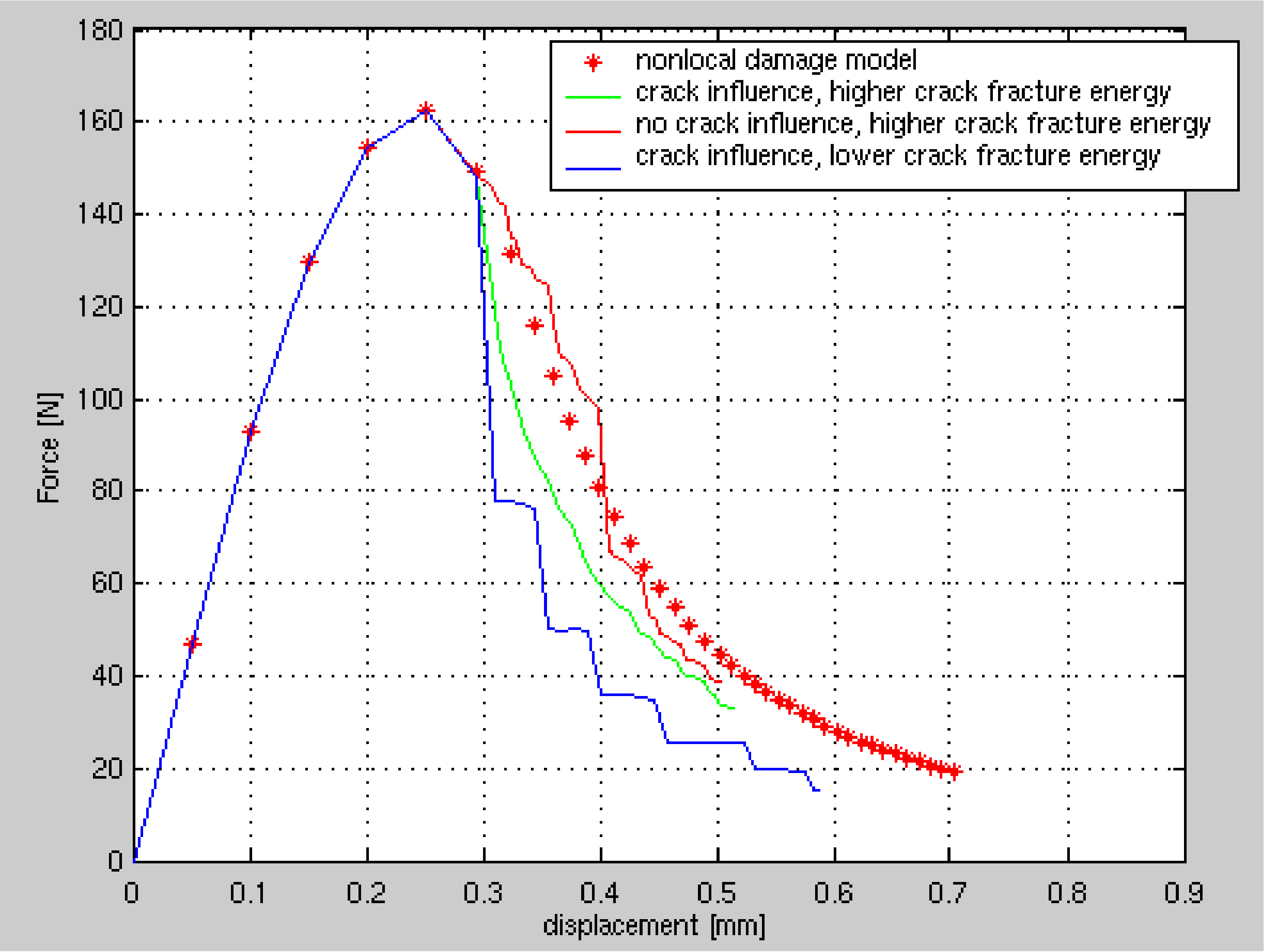
without crack influence
and improper energy balance



with crack influence
and proper energy balance

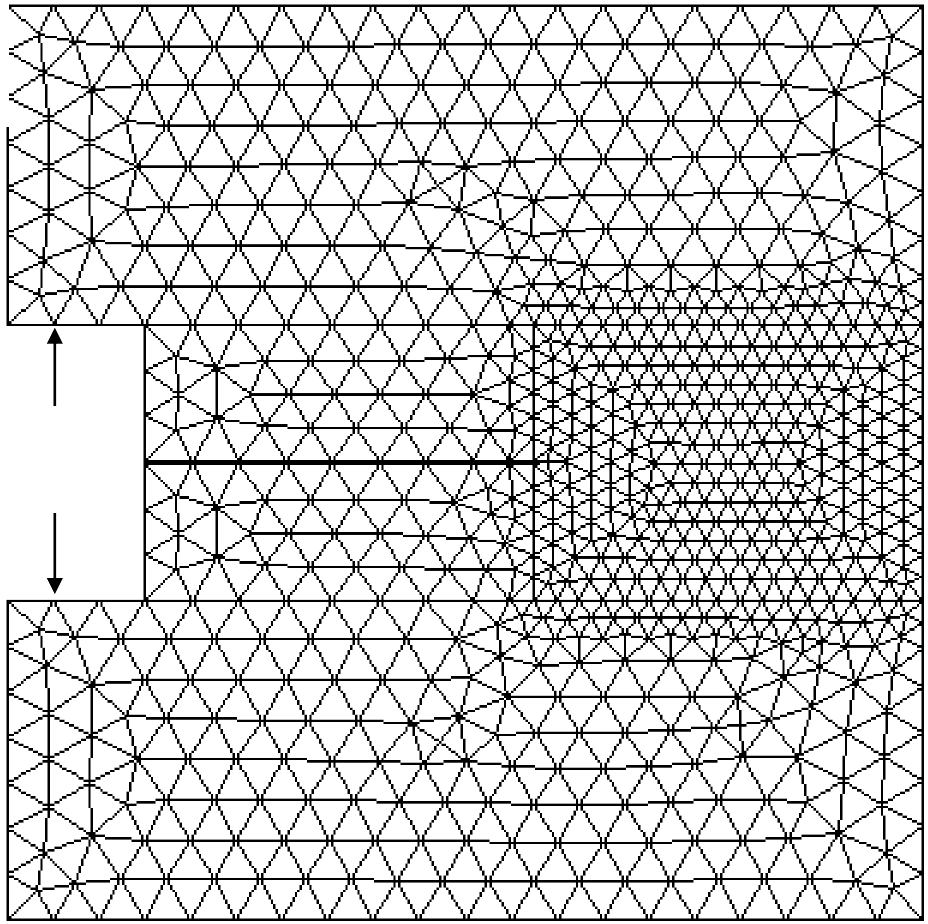


Influence of crack on nonlocal strain

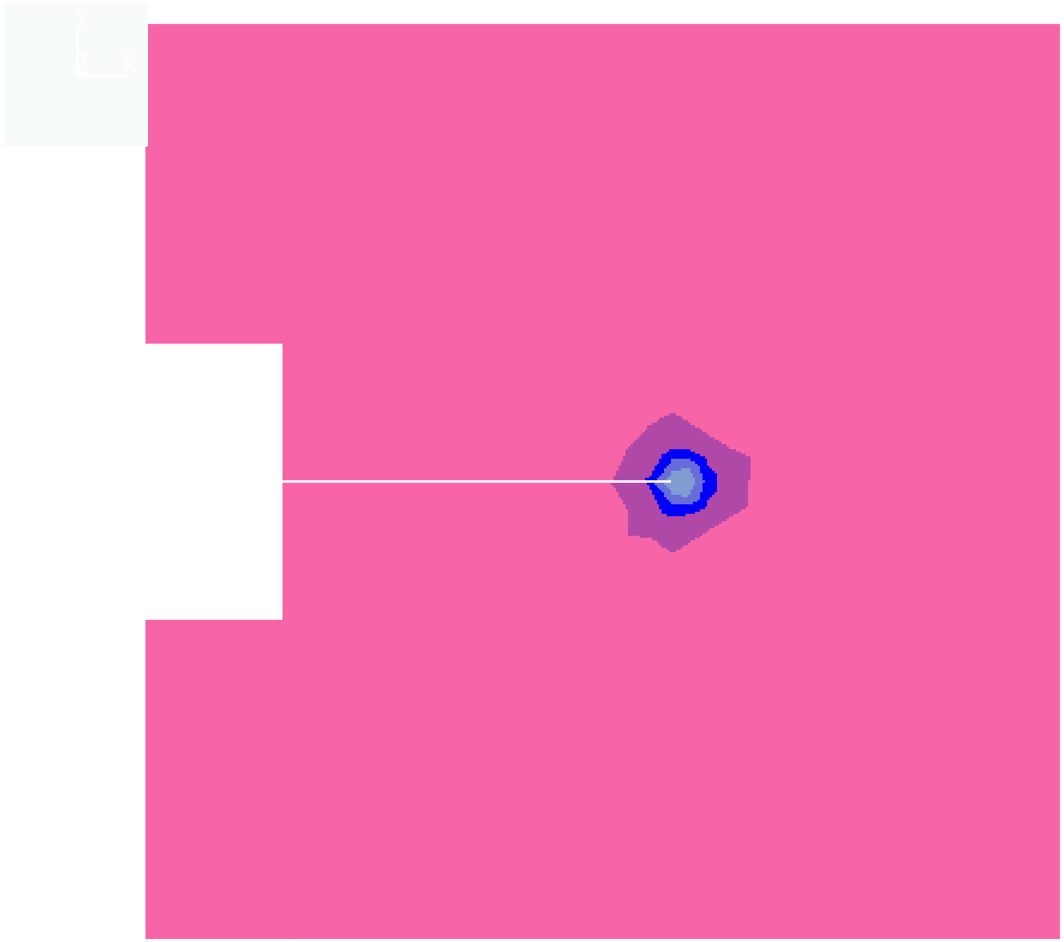


F.6.5
Examples

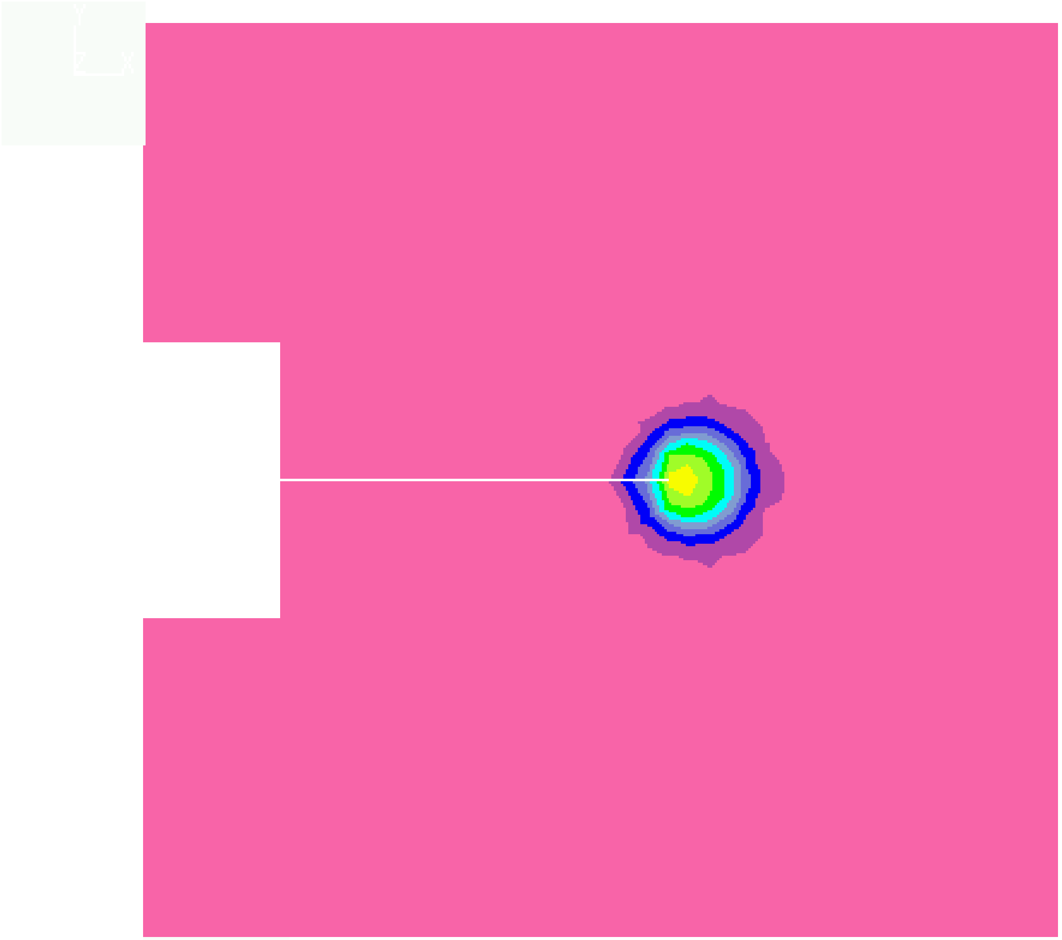
Compact tension test



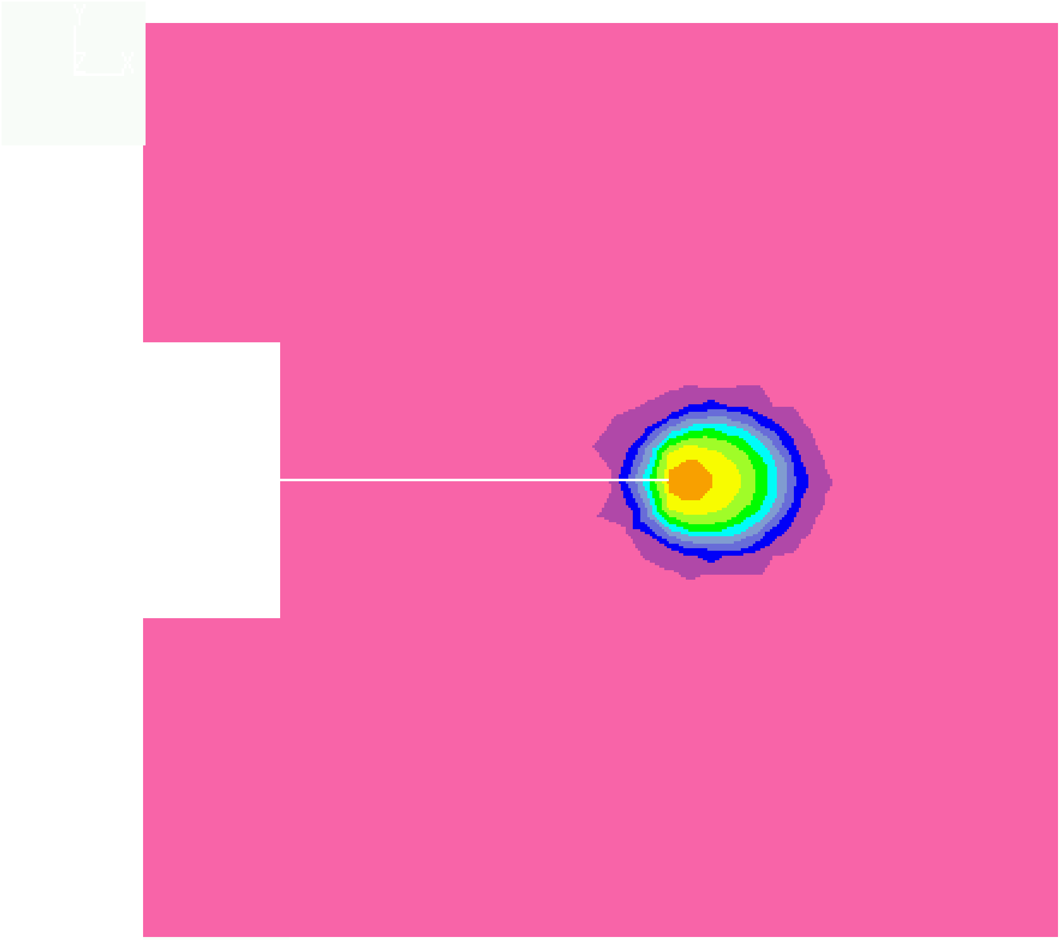
Compact tension test



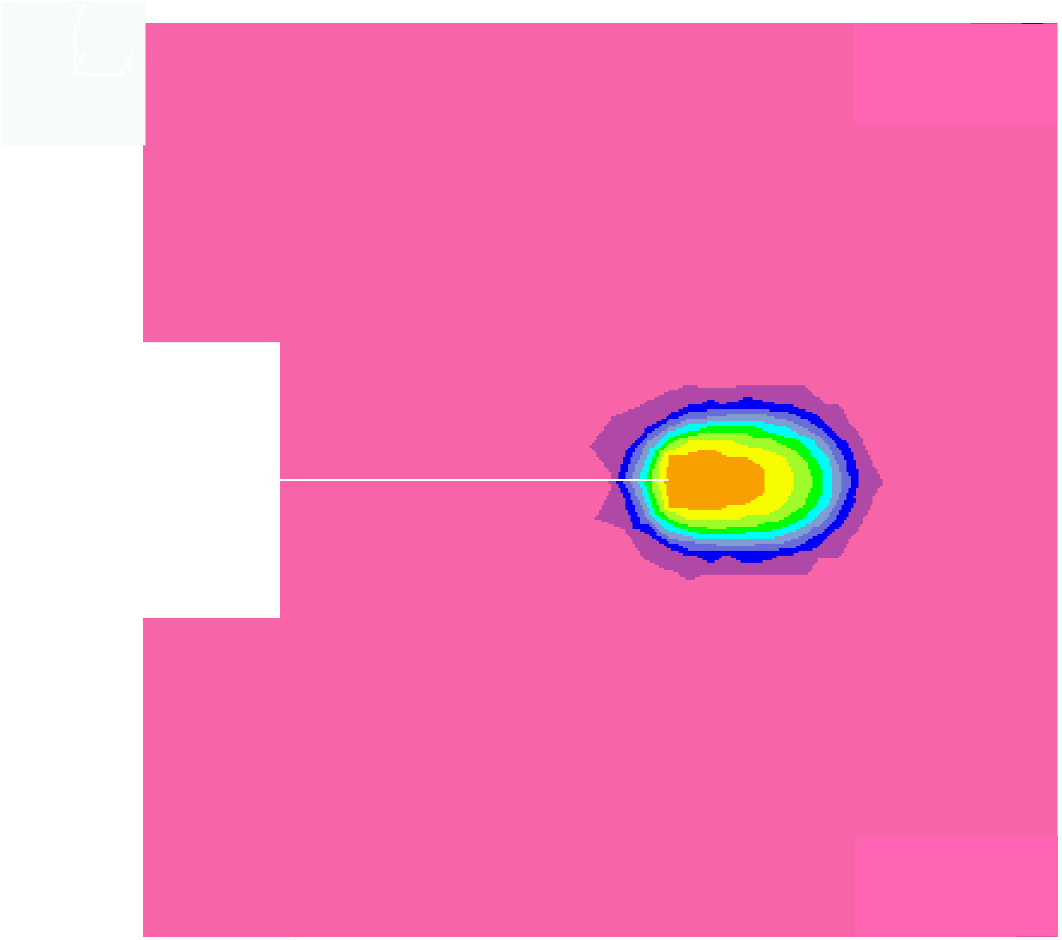
Compact tension test



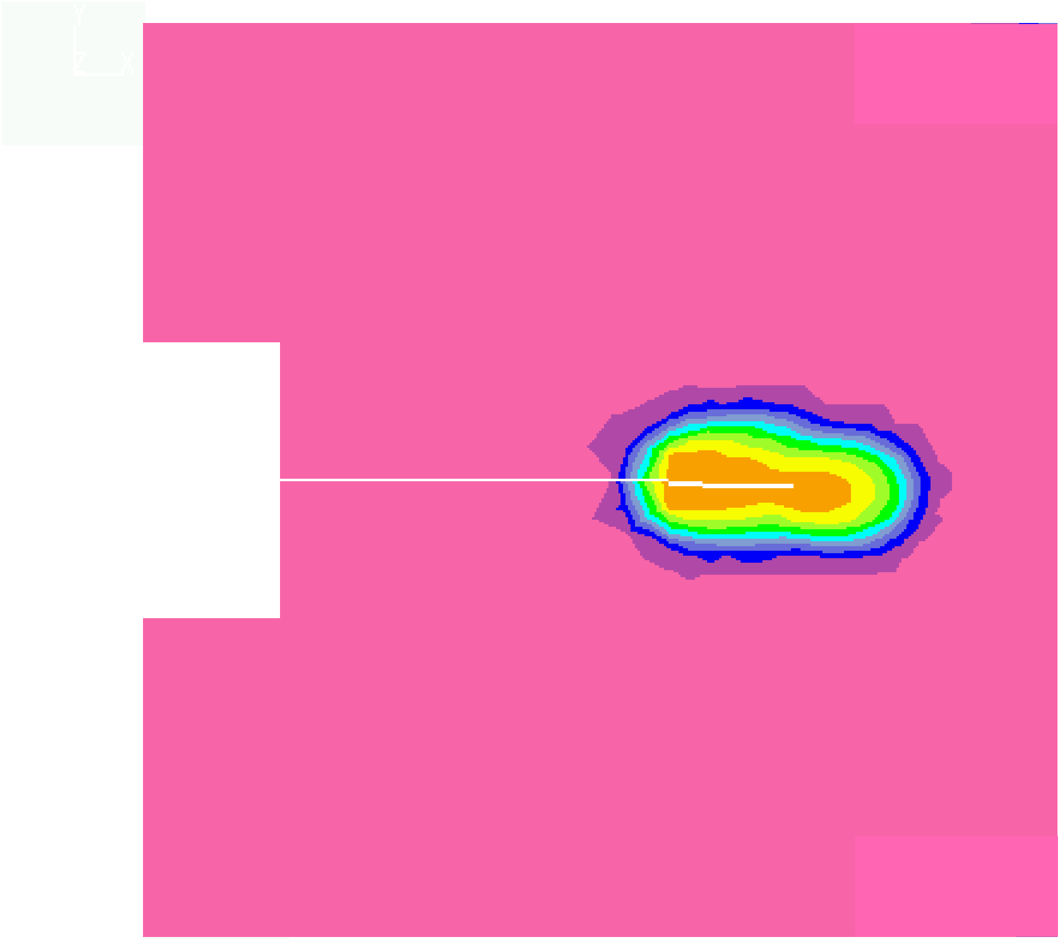
Compact tension test



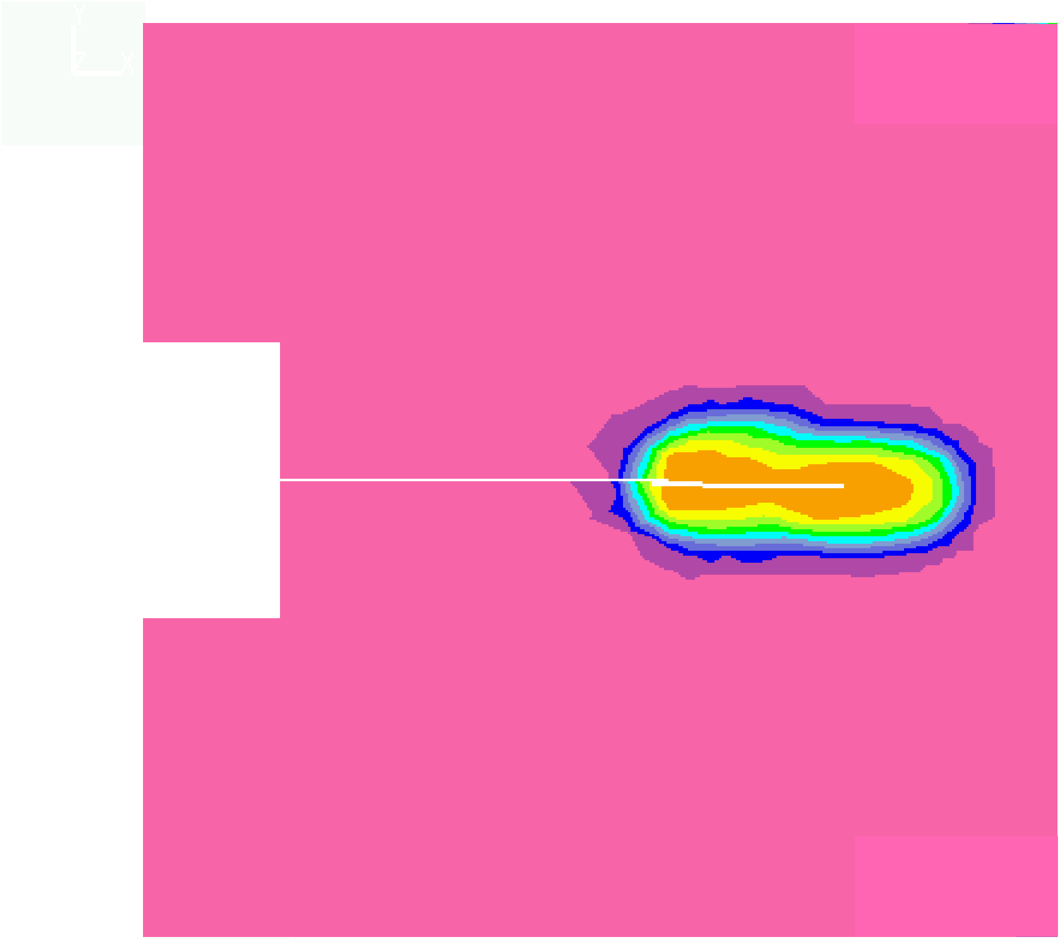
Compact tension test



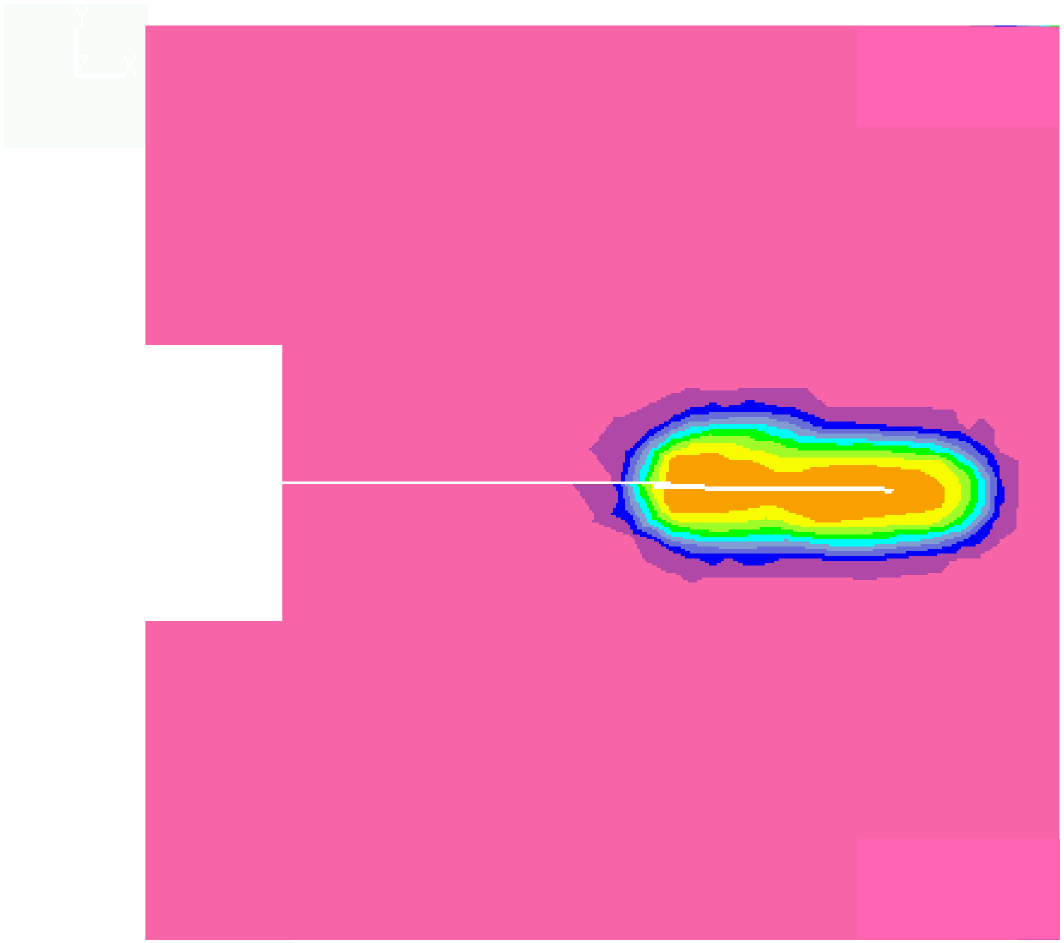
Compact tension test



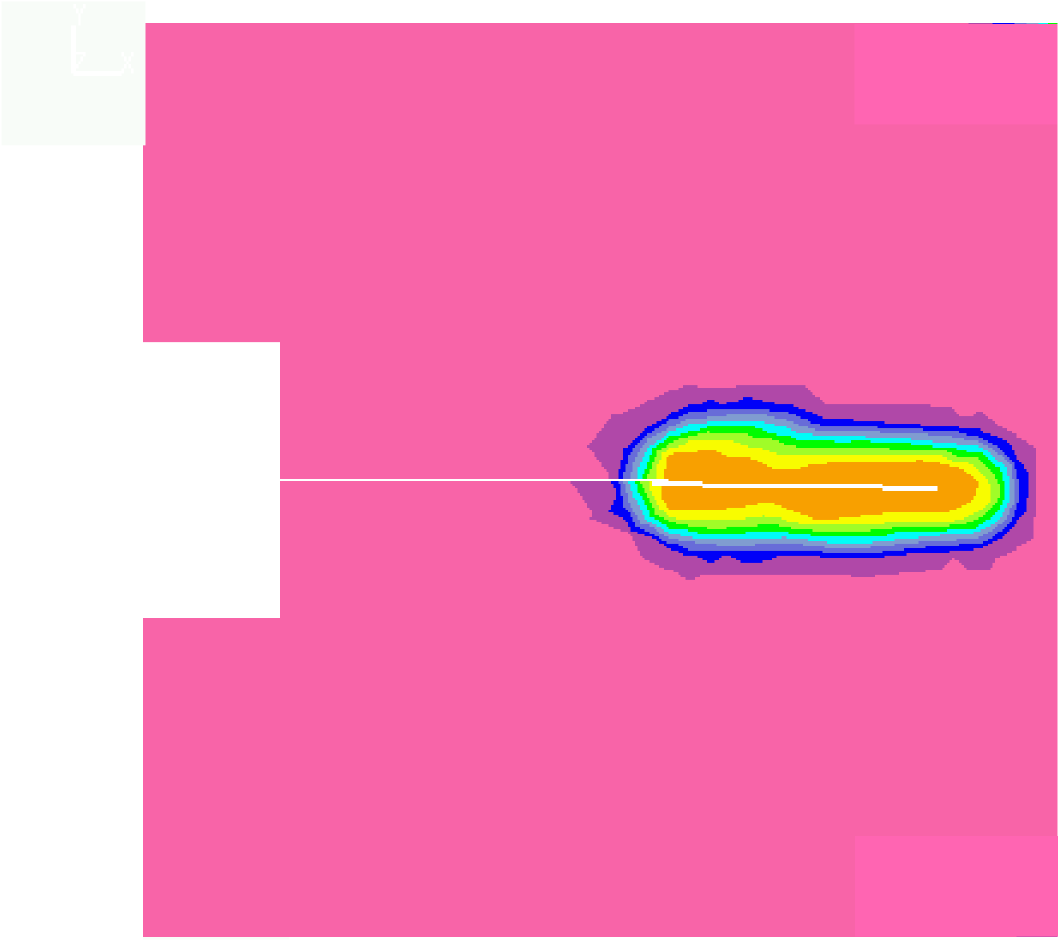
Compact tension test



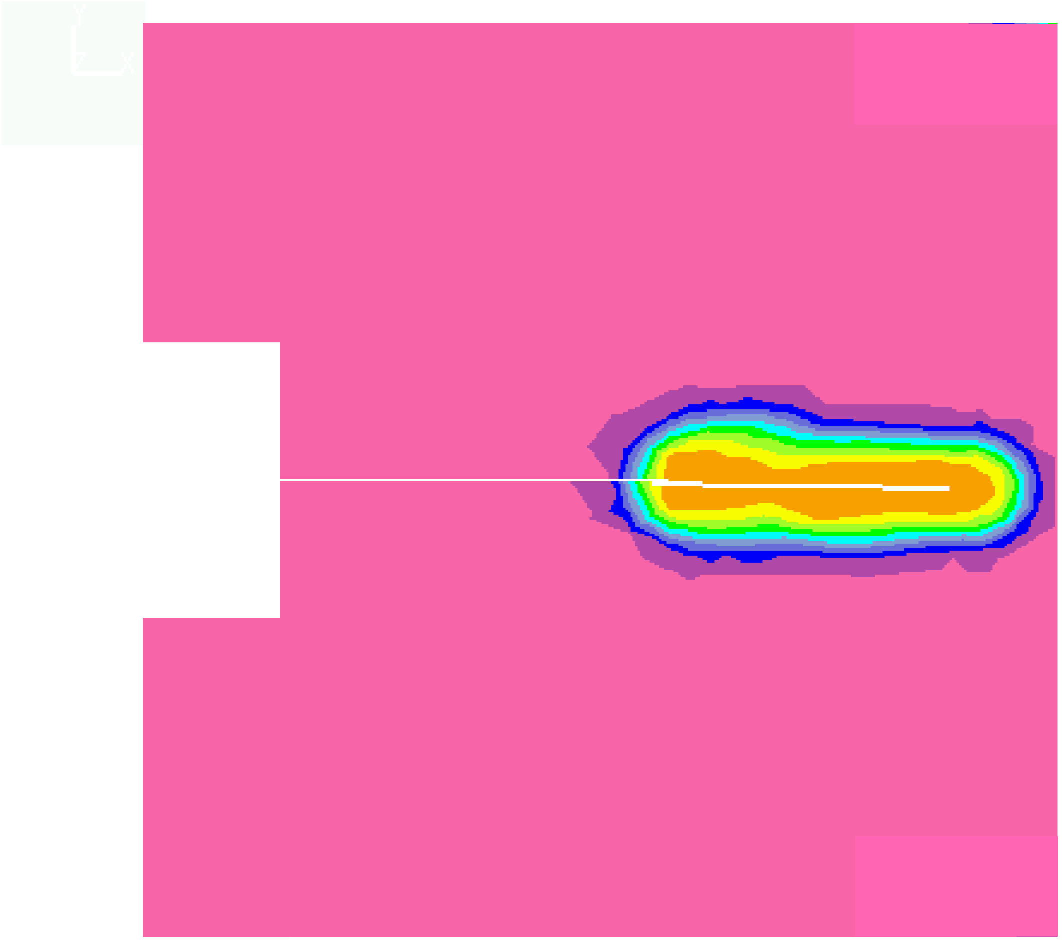
Compact tension test



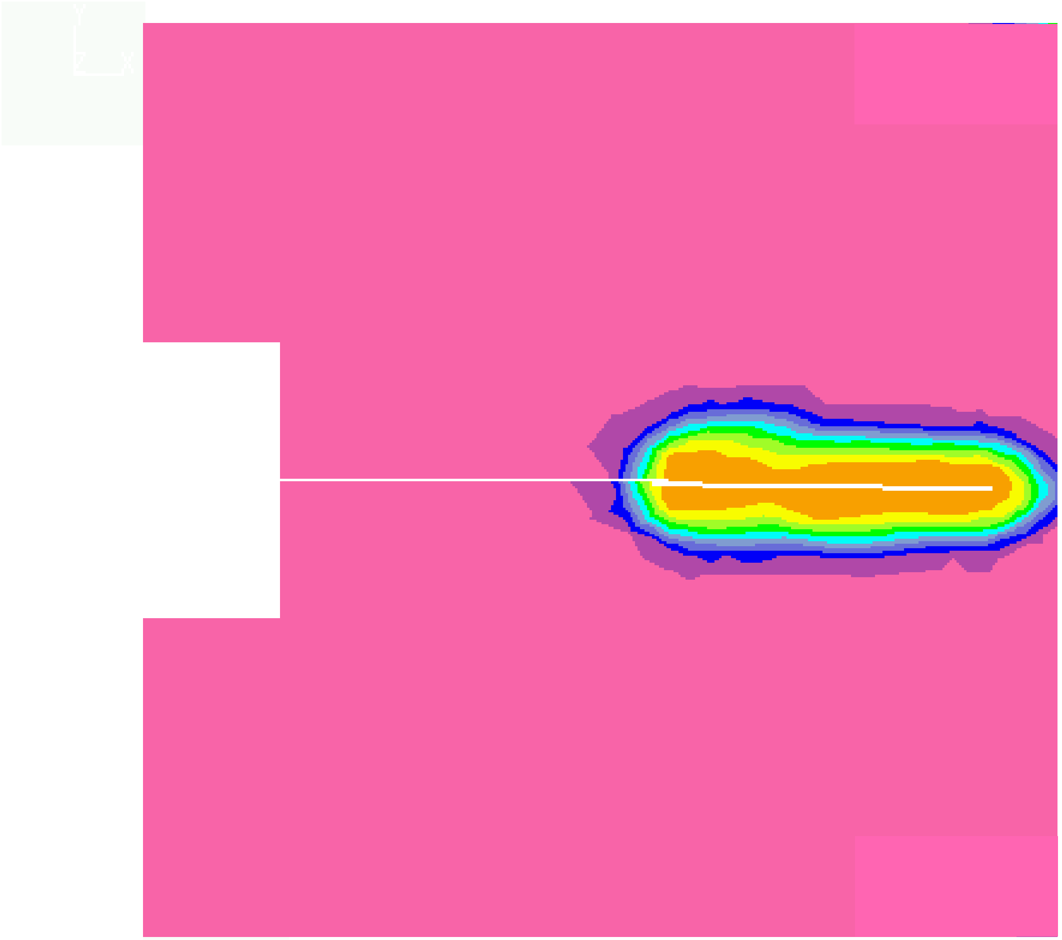
Compact tension test



Compact tension test



Compact tension test



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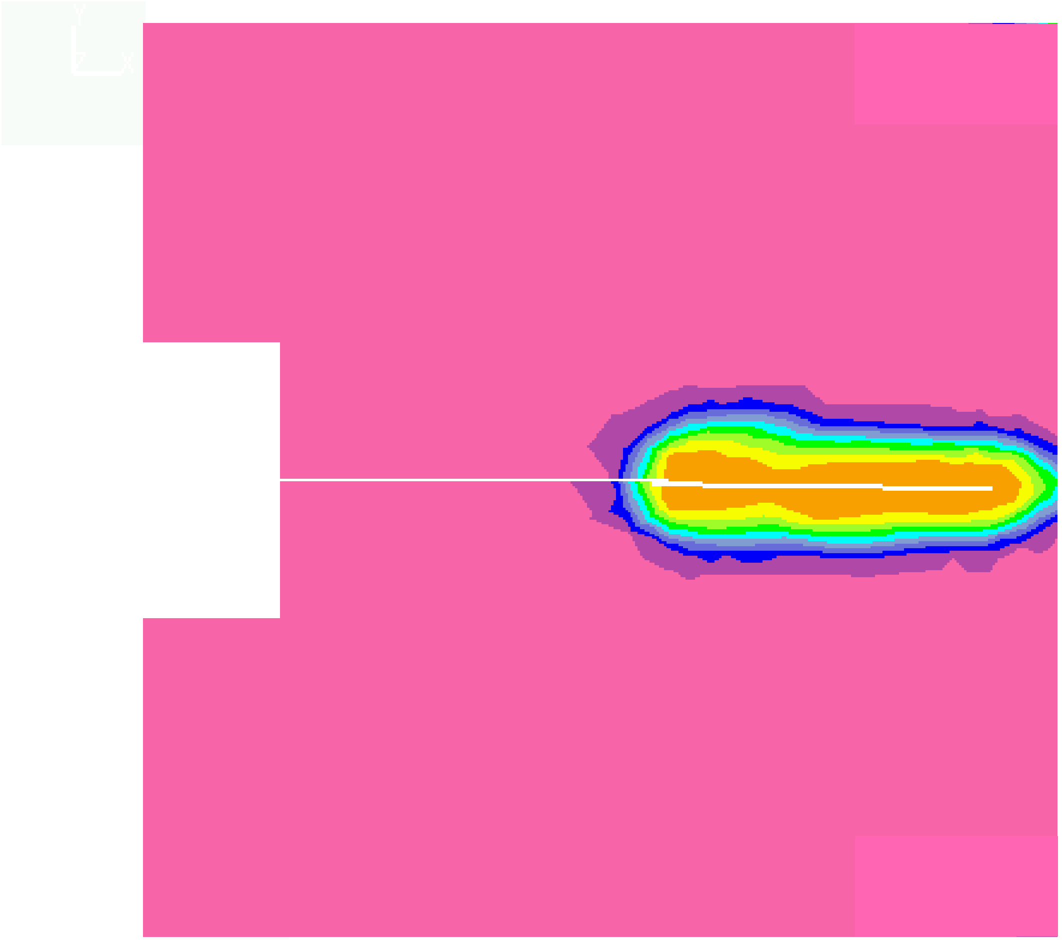

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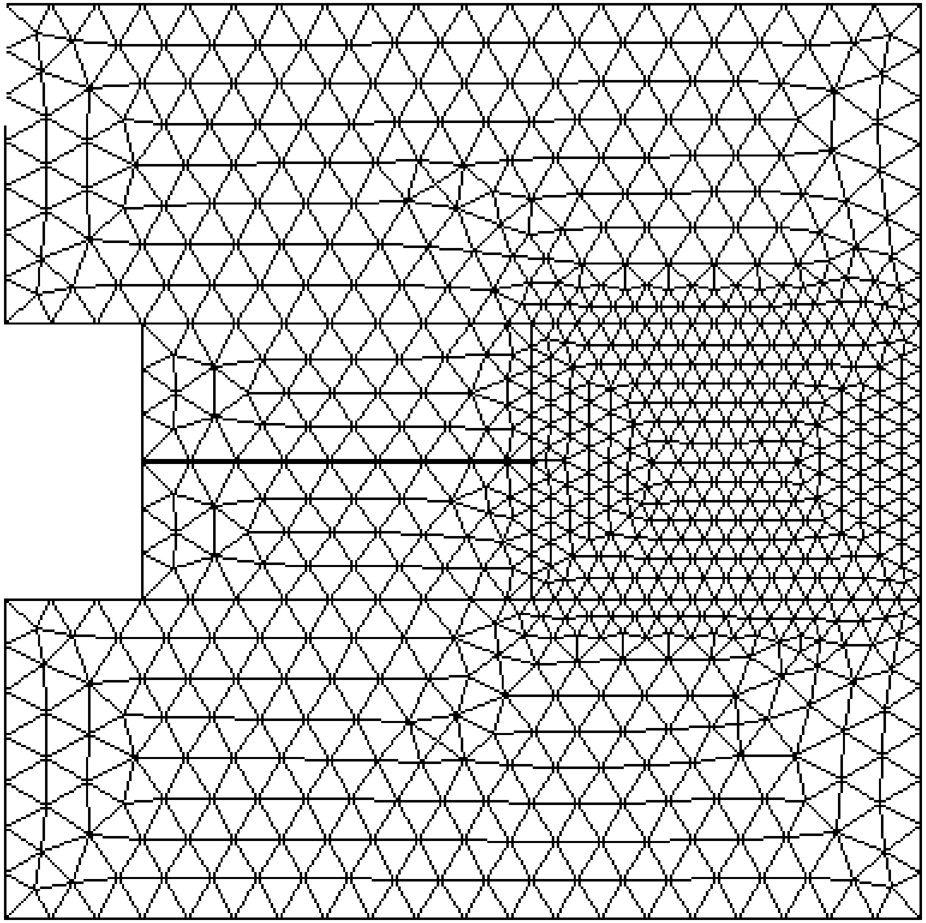

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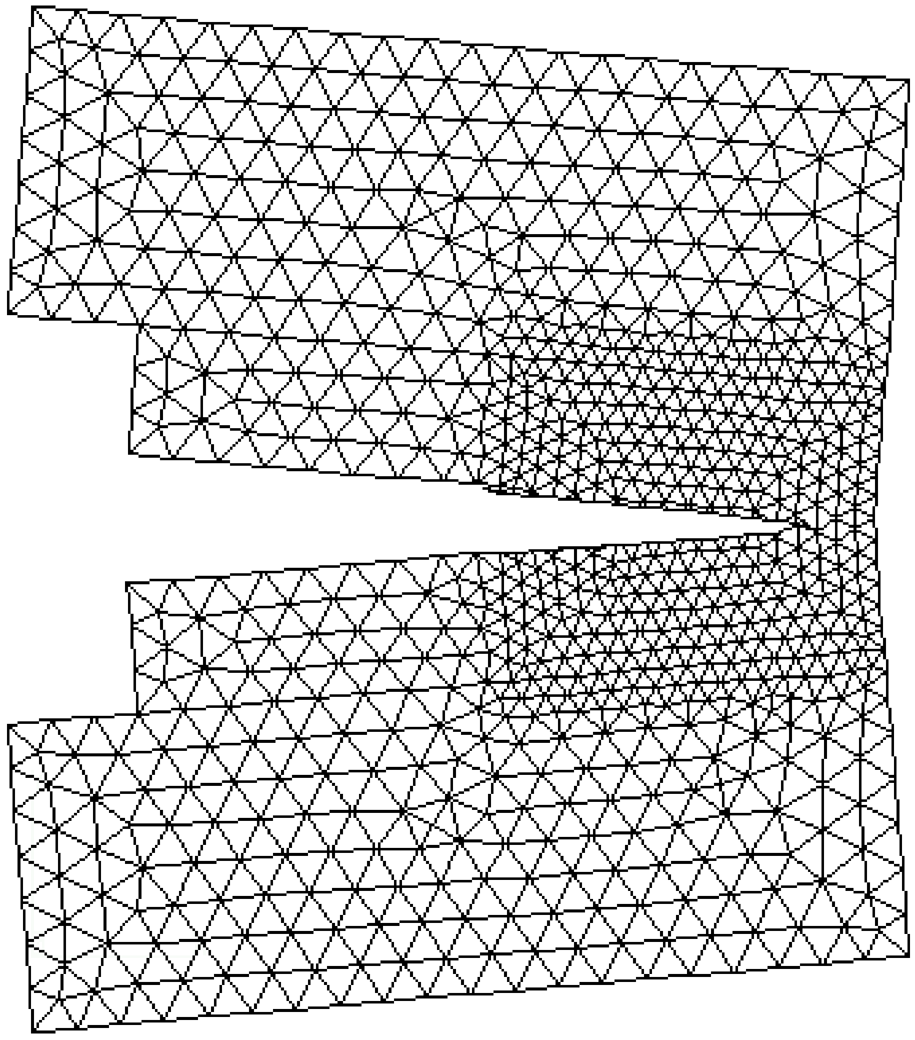

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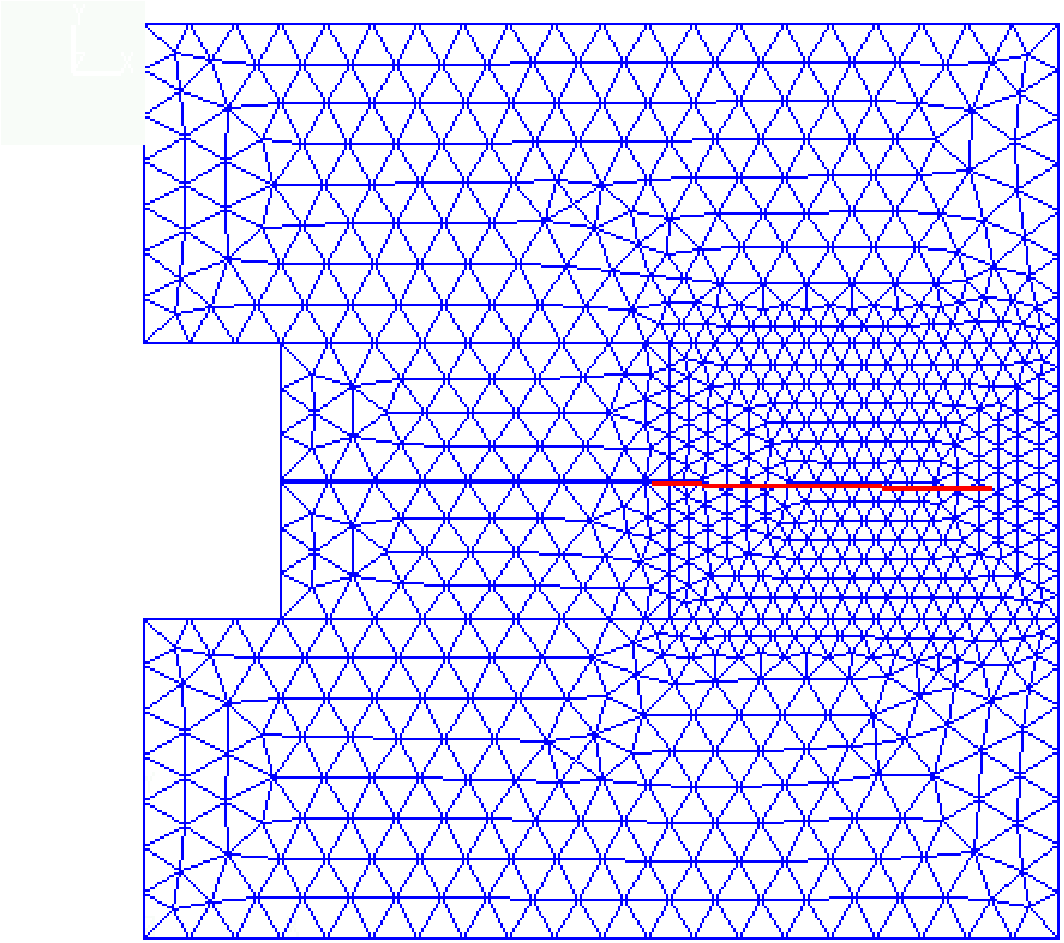

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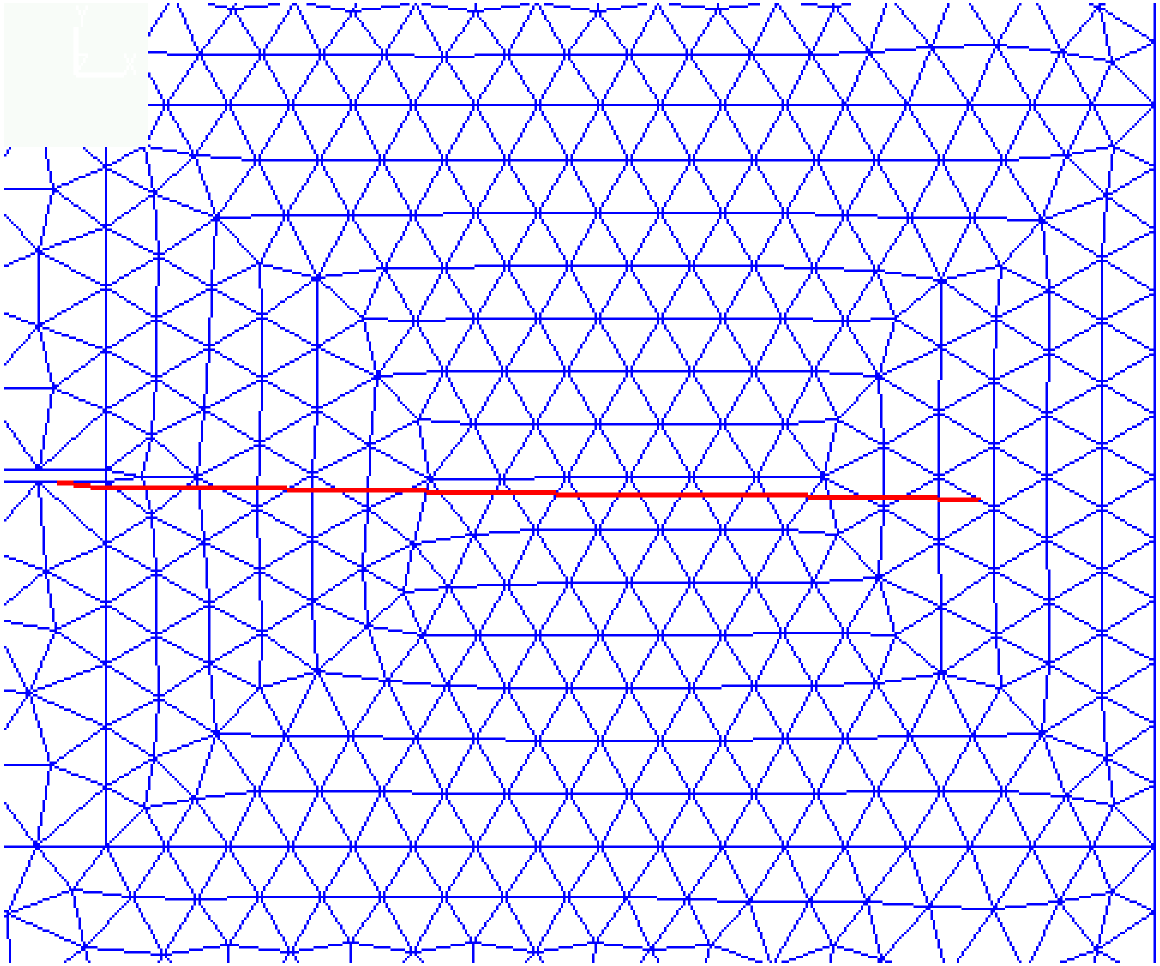

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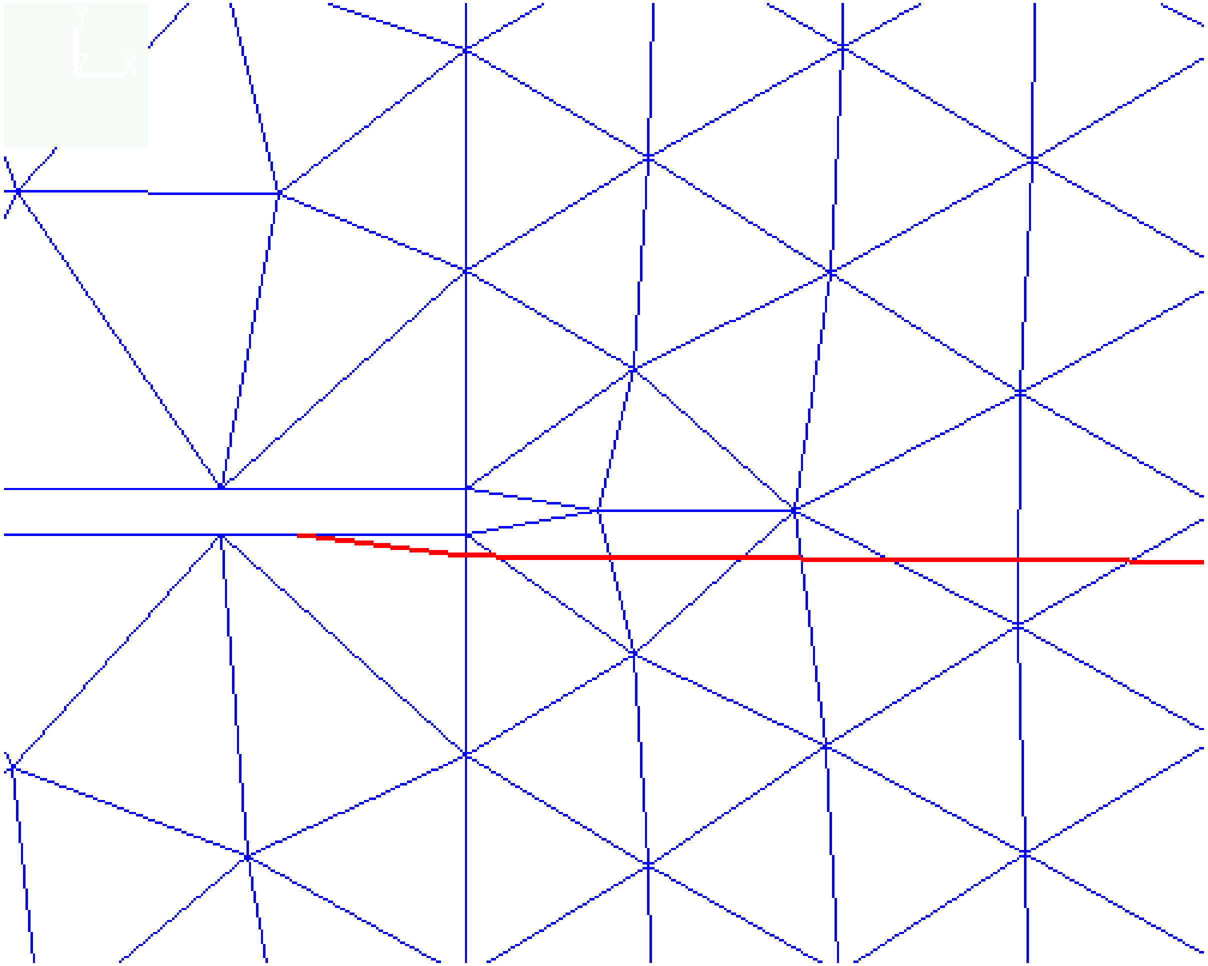
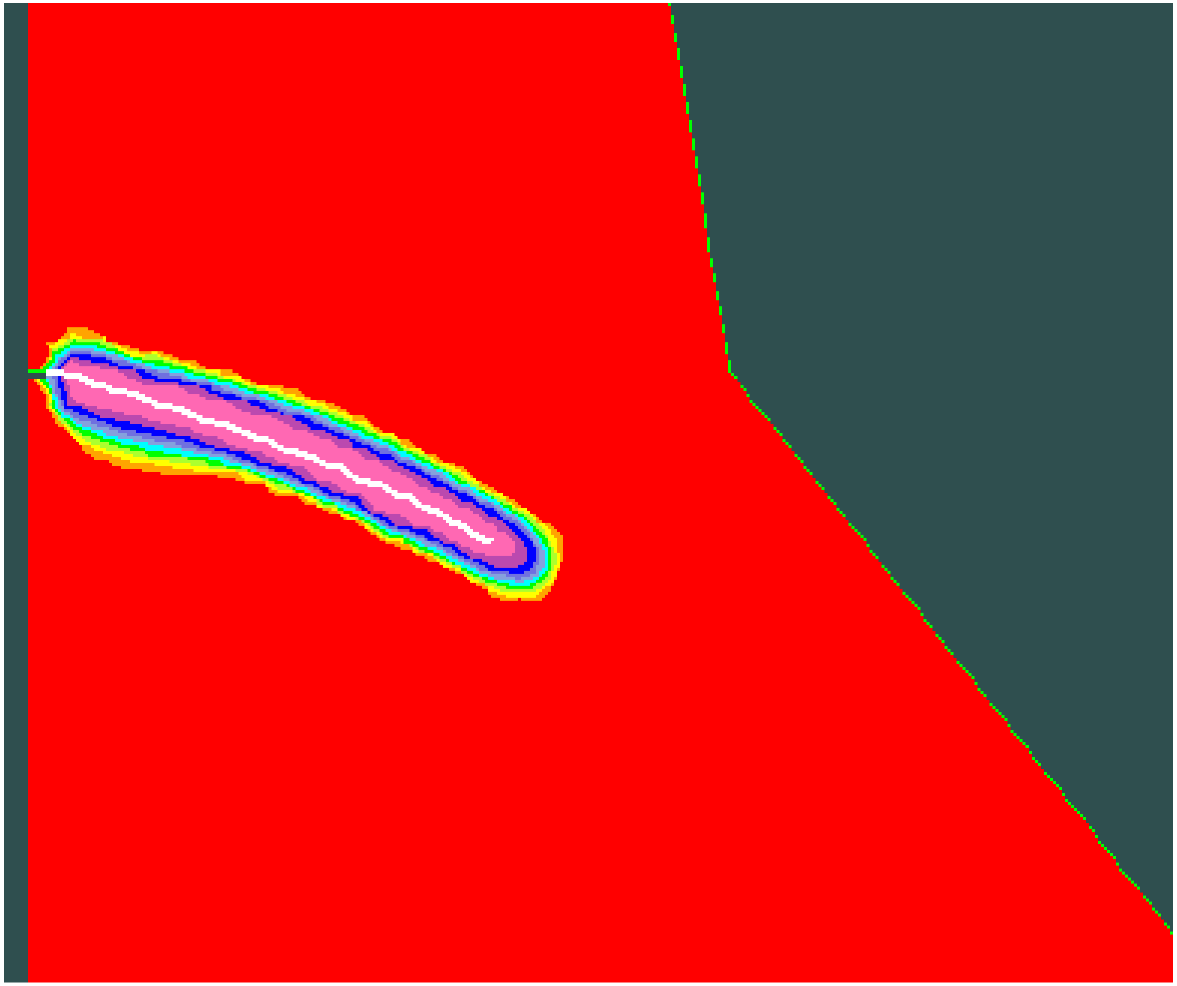
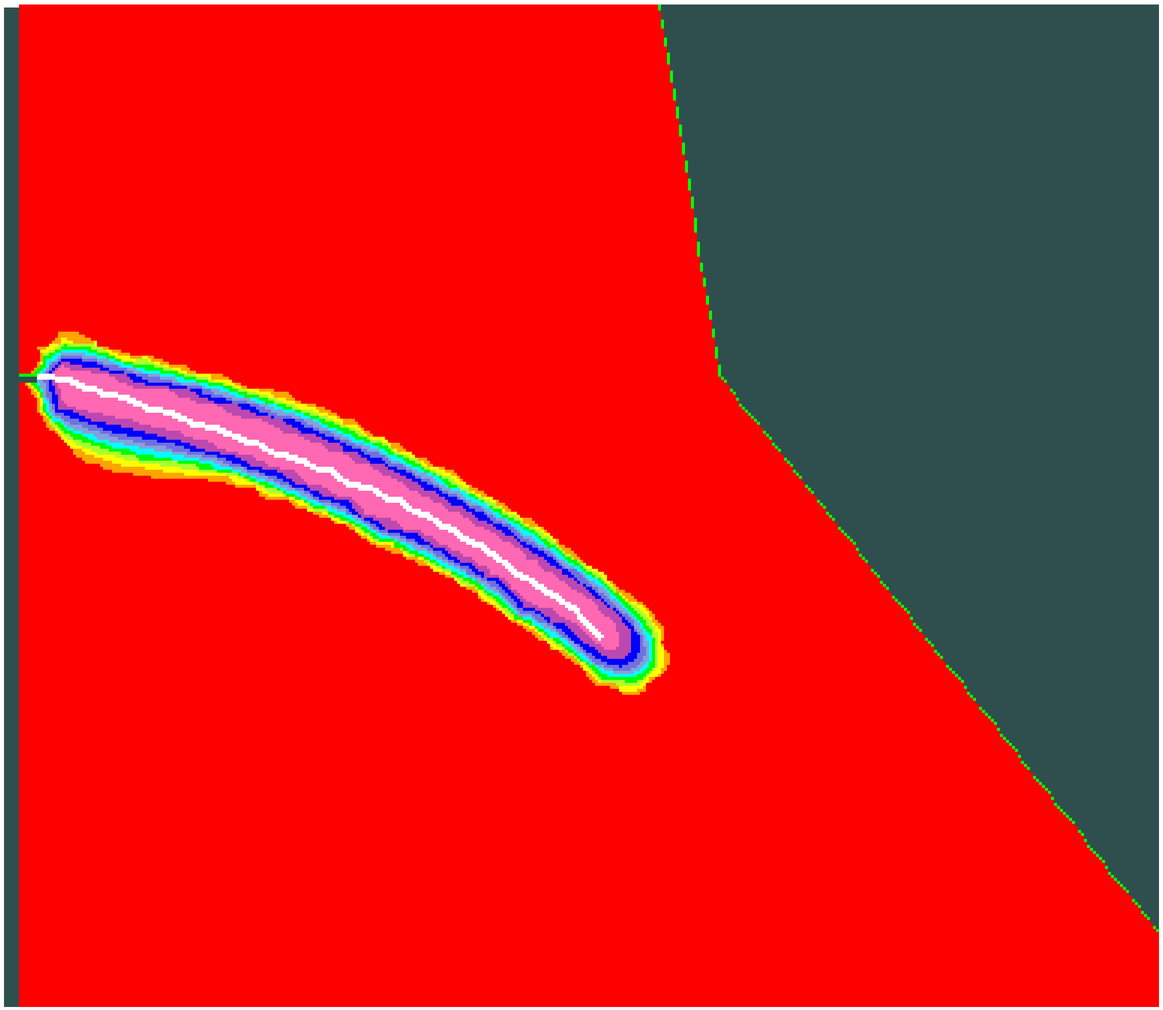


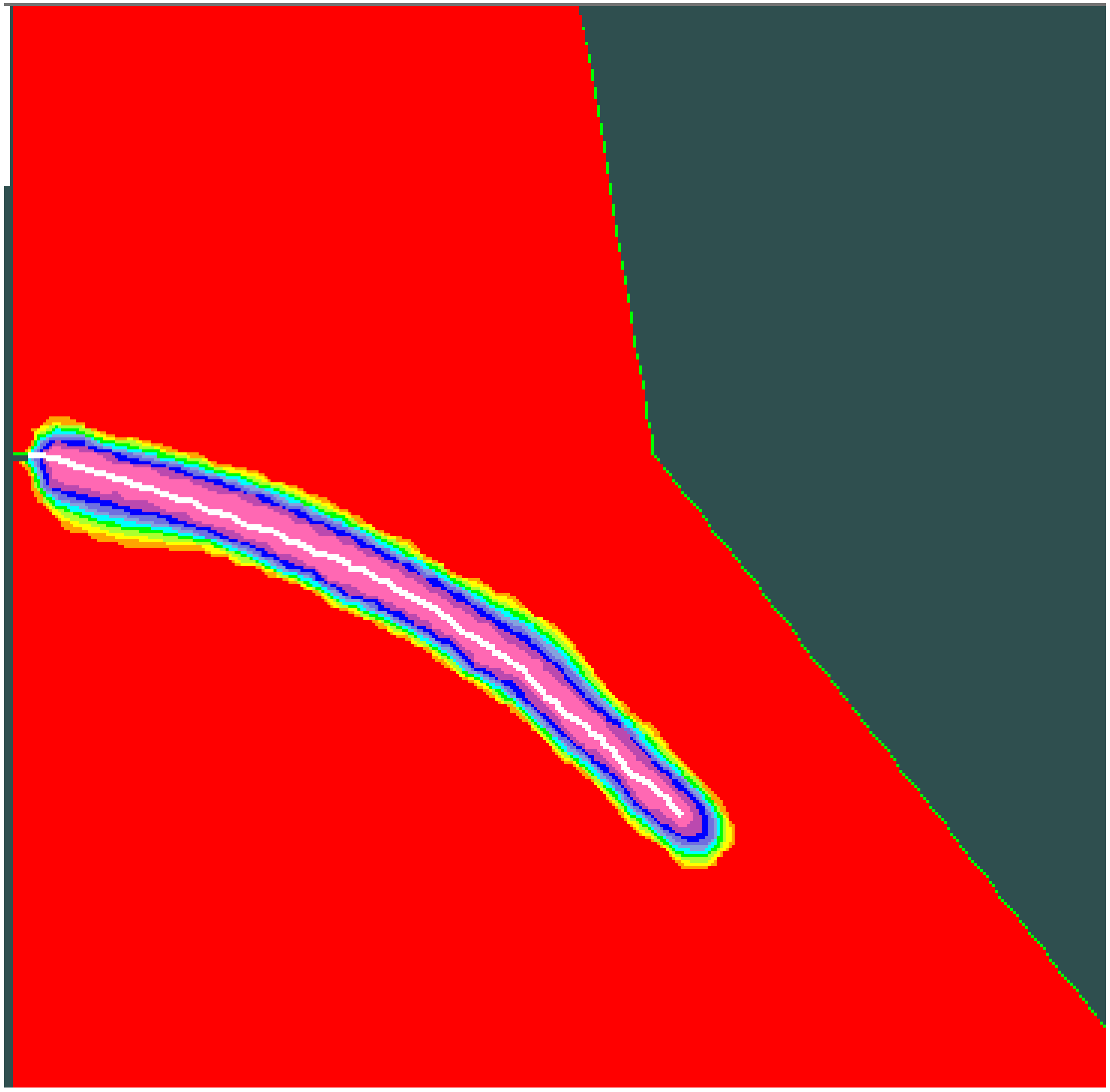
Crack propagation in gravity dam



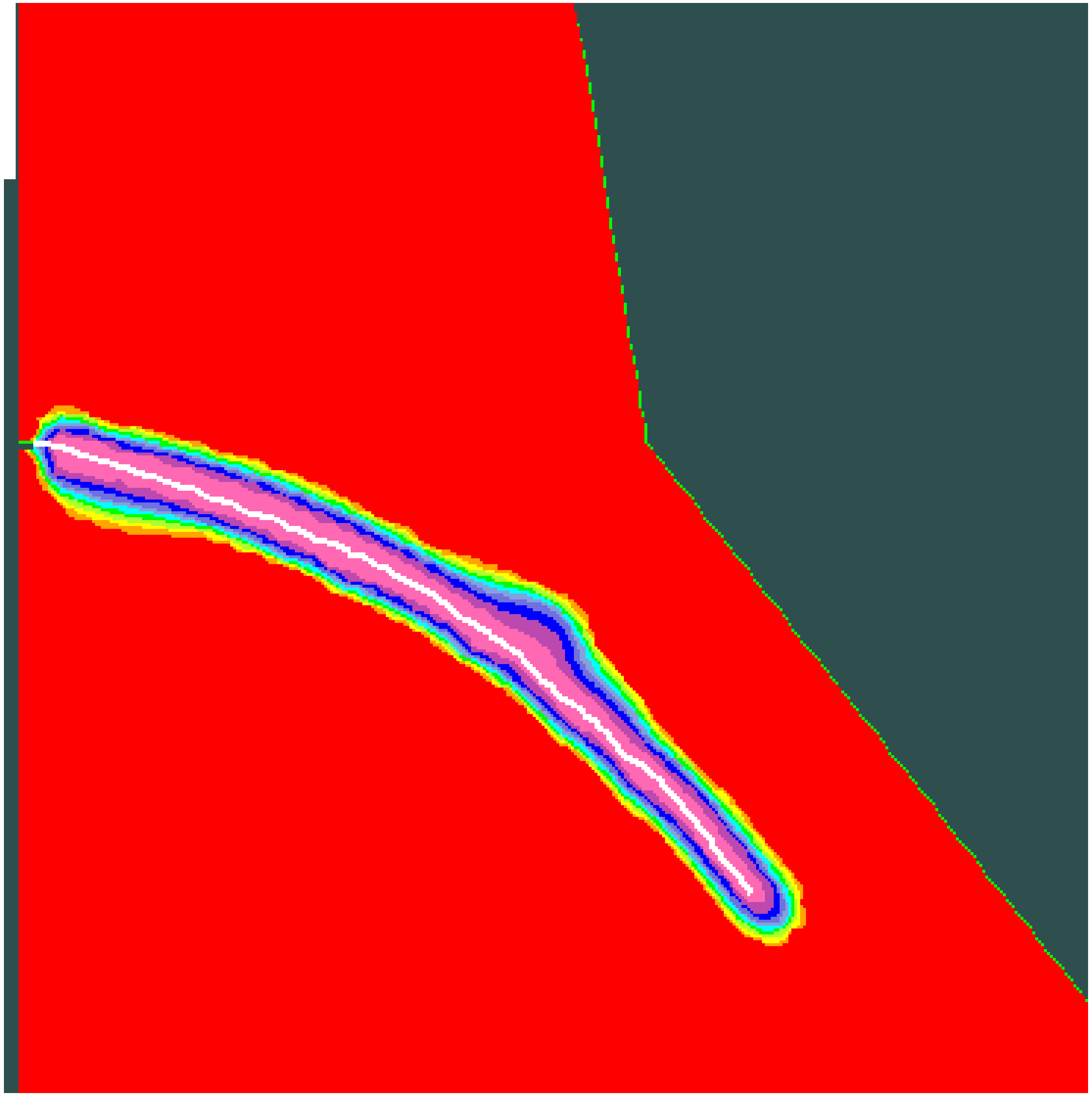
Crack propagation in gravity dam



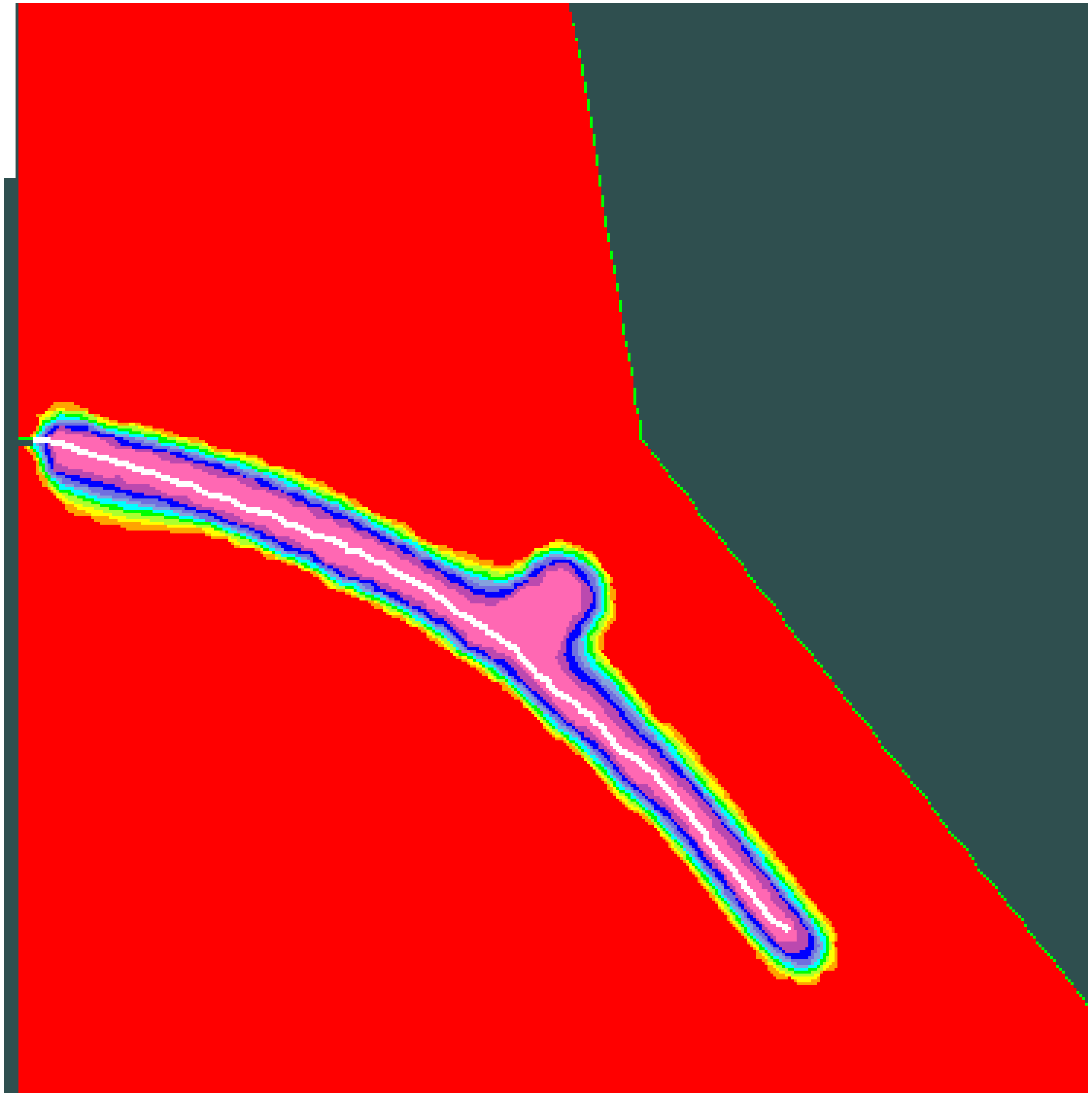
Crack propagation in gravity dam



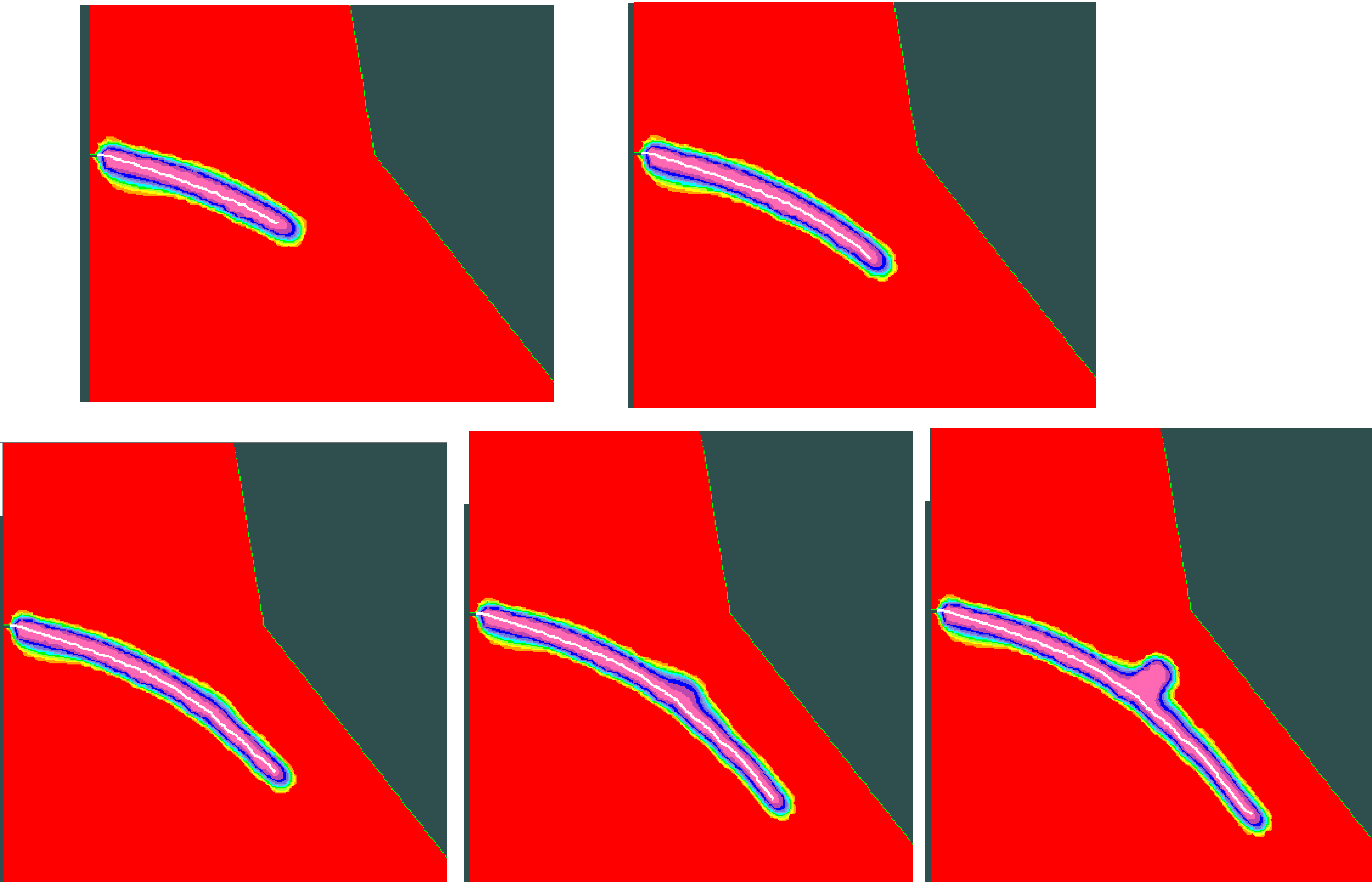
Crack propagation in gravity dam



Crack propagation in gravity dam



Crack propagation in gravity dam



Notched three-point bending test