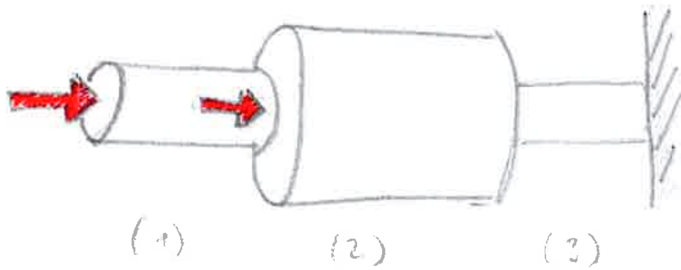
HANDOUT 1

Finite Element Method (FEM):

1. **SUBDIVISION** of a problem into finite elements



trusses ⊕ nodes whenever loads are applied
① nodes where section / material properties change

2. **ELEMENT FORMULATION**
   response of the structure to loading ⇒ stiffness matrix

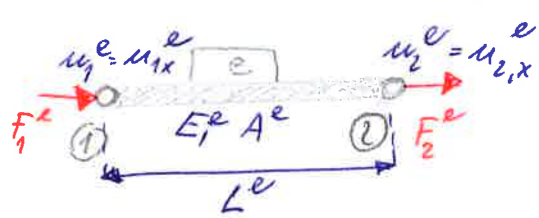
3. **ASSEMBLY**
   putting all element equations together to get global equations of the entire system

4. **SOLUTION**

5. **POST-PROCESSING**
   stresses, strains, visualization, ...

BAR ELEMENT

- slender (very thin) and straight
  → negligible resistance to — torsion, shear, bending
  → only axial internal forces
  (similar to springs)
- notation



superscript .... element number
supscript ... node number

GOVERNING EQUATIONS

1) EQUILIBRIUM $\quad F_1^e + F_2^e = 0$

2) STRESS-STRAIN (HOOKE'S) LAW, $\quad \sigma^e = E^e \cdot \varepsilon^e$

- linear elasticity
- ε - strain
- σ - stress

3) COMPATIBILITY. $\quad \varepsilon^e = \frac{\delta^e}{L^e} = \frac{u_2^e - u_1^e}{L^e}$

→ no gaps, overlaps
→ continuous structure

internal axial force

$$F_2^e = F^e = A^e \cdot \sigma^e = A^e \cdot E^e \cdot \varepsilon^e = A^e E^e \cdot \frac{u_2^e - u_1^e}{L^e} = -A^e E^e \frac{u_1^e - u_2^e}{L^e}$$

$$F_1^e = -F^e = A^e E^e \frac{u_1^e - u_2^e}{L^e}$$

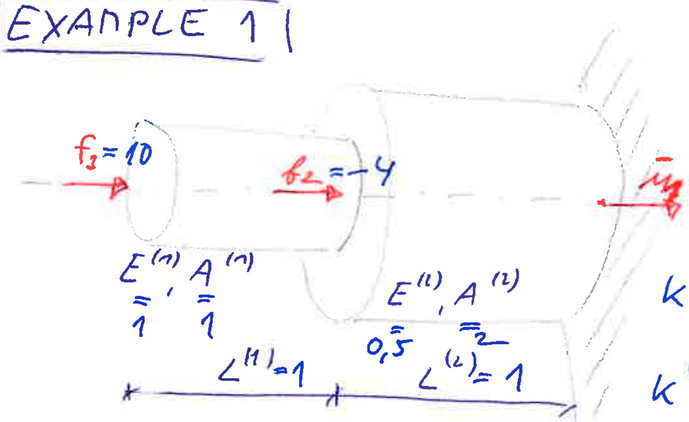
in matrix form:

$$\boxed{\begin{bmatrix} F_1^e \\ F_2^e \end{bmatrix} = \underbrace{\frac{A^e E^e}{L^e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}} \begin{bmatrix} u_1^e \\ u_2^e \end{bmatrix}}$$

$$\underline{F}^e = \underline{K}^e \quad \underline{d}^e$$

symmetric stiffness matrix

EXAMPLE 1

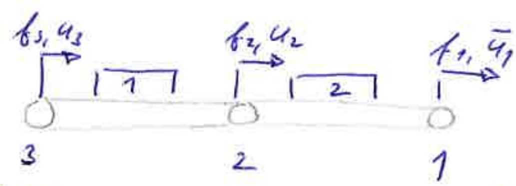


$$\bar{u}_1 = \frac{4}{K^{(2)}} = 4$$

$$k^{(1)} = \frac{E^{(1)} A^{(1)}}{L^{(1)}} = 1$$

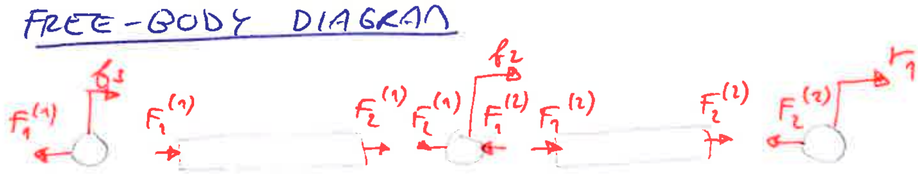
$$k^{(2)} = \frac{E^{(2)} A^{(2)}}{L^{(2)}} = 1$$

SCHEME



1) SUBDIVISION – forces, mat. properties
2) NUMBERING
3) UNKNOWNS – EITHER FORCE or displacement
→ relation through stiffness matrix

FREE-BODY DIAGRAM



4) ELEMENT STIFFNESS EQUATIONS OF ALL BARS

→ relate nodal forces to nodal displacements

5) ASSEMBLY

EQUILIBRIUM OF NODES

$$\begin{array}{c} \\ (1) \\ (2) \\ (3) \end{array} \overset{\boxed{1}}{\begin{bmatrix} 0 \\ F_2^{(1)} \\ F_1^{(1)} \end{bmatrix}} + \overset{\boxed{2}}{\begin{bmatrix} F_2^{(2)} \\ F_1^{(2)} \\ 0 \end{bmatrix}} = \begin{bmatrix} r_1 \\ f_2^{-4} \\ f_3^{10} \end{bmatrix} = \overbrace{\begin{bmatrix} 0 \\ f_2^{-4} \\ f_3^{10} \end{bmatrix}}^{\text{PRESCRIBED}} + \overbrace{\begin{bmatrix} r_1 \\ 0 \\ 0 \end{bmatrix}}^{\text{UNKNOWN REACTIONS}}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & k^{(1)} & -k^{(1)} \\ 0 & -k^{(1)} & k^{(1)} \end{bmatrix} \begin{bmatrix} u_1^{\frac{4}{k^{(2)}}} \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} k^{(2)} & -k^{(2)} & 0 \\ -k^{(2)} & k^{(2)} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1^{\frac{4}{k^{(2)}}} \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ f_2 \\ f_3 \end{bmatrix} + \begin{bmatrix} r_1 \\ 0 \\ 0 \end{bmatrix}$$

$$\underbrace{\underline{K}^{(1)} \quad \underline{d} \quad + \quad K^{(2)} \quad \underline{d}}_{\underline{K} = \sum_{e=1}^{n_e} \underline{K}^e} = \underline{f} + \underline{r}$$

$$\underline{K} \cdot \underline{d} = \underline{f} + \underline{r}$$

E-nodes

→ "essential", known displacements

F-nodes

→ "free", unknown displacements

$$\underline{d} = \begin{bmatrix} \bar{d}_E \\ d_F \end{bmatrix} \qquad \underline{f} = \begin{bmatrix} f_E^{\,\emptyset} \\ \bar{f}_F \end{bmatrix} \qquad \underline{r} = \begin{bmatrix} r_E \\ r_F^{\,\emptyset} \end{bmatrix}$$

$$\left[\begin{array}{c|cc} k^{(2)} & -k^{(2)} & 0 \\ \hline -k^{(2)} & k^{(1)}+k^{(2)} & -k^{(1)} \\ 0 & -k^{(1)} & k^{(1)} \end{array}\right] \left[\begin{array}{c} \bar{u}_1^{\frac{4}{k^{(2)}}} \\ \hline u_2 \\ u_3 \end{array}\right] = \left[\begin{array}{c} r_1 \\ \hline f_2^{-4} \\ f_3^{10} \end{array}\right] \iff \left[\begin{array}{c|c} \underline{K}_E & \underline{K}_{EF} \\ \hline \underline{K}_{EF}^T & \underline{K}_F \end{array}\right] \left[\begin{array}{c} \bar{d}_E \\ \hline d_F \end{array}\right] = \left[\begin{array}{c} r_E \\ \hline f_F \end{array}\right]$$

$$\underline{K}_{EF}^T \underline{\bar{d}_E} + \underline{K}_F \underline{d}_F = \underline{f}_F \implies \underline{d}_F = \underline{K}_F^{-1} (\underline{f}_F - \underline{K}_{EF}^T \underline{\bar{d}_E})$$

$$\underline{r}_E = \underline{K}_E \underline{\bar{d}}_E + \underline{K}_{EF} \underline{d}_F$$

$$\underline{\delta}_F = \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} k^{(1)}+k^{(2)} & -k^{(1)} \\ -k^{(1)} & k^{(1)} \end{bmatrix}^{-1} \left\{ \begin{bmatrix} -4 \\ 10 \end{bmatrix} - \begin{bmatrix} -k^{(2)} \\ 0 \end{bmatrix} \left[\frac{4}{k^{(2)}} \right] \right\}$$

$$u_2 = \frac{10}{k^{(1)}} \qquad u_3 = 10\left(\frac{1}{k^{(1)}} + \frac{1}{k^{(2)}}\right)$$

$$r_1 = \underbrace{k^{(2)} \cdot \frac{4}{k^{(2)}}}_{4} + \underbrace{\begin{bmatrix} -k^{(2)} & 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{10}{k^{(1)}} \\ 10\cdot\left(\frac{1}{k^{(1)}} + \frac{1}{k^{(2)}}\right) \end{bmatrix}}_{-10} = -6$$

OR :

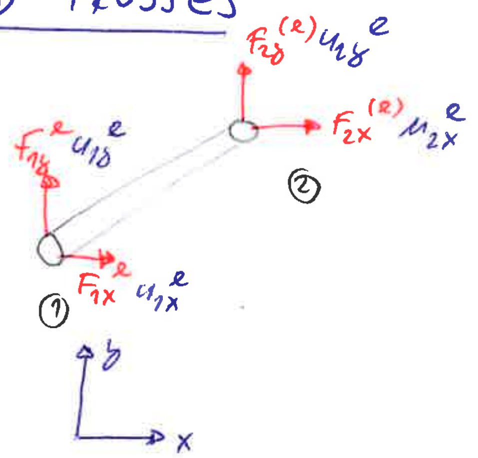
$$\begin{bmatrix} \overset{2}{k^{(1)}+k^{(2)}} & \overset{-1}{-k^{(1)}} \\ \overset{-1}{-k^{(1)}} & \overset{1}{k^{(1)}} \end{bmatrix} \cdot \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -4 \\ 10 \end{bmatrix} - \overset{-1}{\begin{bmatrix} -k^{(2)} \\ 0 \end{bmatrix}} \left[\overset{4}{\frac{4}{k^{(2)}}} \right]}_{\begin{bmatrix} 0 \\ 10 \end{bmatrix}} \Rightarrow \begin{matrix} u_2 = 10 \\ u_3 = 20 \end{matrix}$$

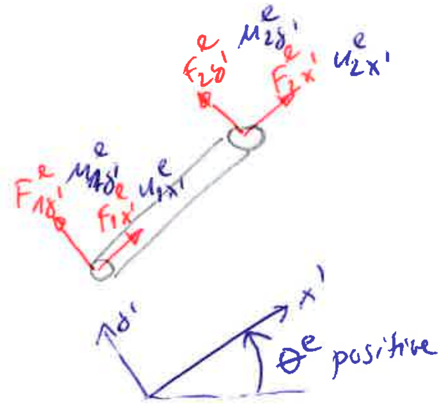
STRAINS, STRESSES

$\varepsilon^{(1)} = -10$ $\quad \sigma^{(1)} = -10$

$\varepsilon^{(2)} = -6$ $\quad \sigma^{(2)} = -3$ $\quad$ } compression

2D TRUSSES





$$\underline{F}^e = \begin{bmatrix} F_{1x}^e \\ F_{1y}^e \\ F_{2x}^e \\ F_{2y}^e \end{bmatrix} \qquad \underline{d}^e = \begin{bmatrix} d_{1x}^e \\ d_{1y}^e \\ d_{2x}^e \\ d_{2y}^e \end{bmatrix}$$

2 unknowns per node

a) ~~FORCES~~ TRANSFORMATIONS $\qquad c^e = \cos\theta^e \quad s^e = \sin\theta^e$

$$\left.\begin{aligned} u_{1x'}^e &= u_{1x}^e \cdot \cos\theta^e + u_{1y}^e \cdot \sin\theta^e \\ u_{1y'}^e &= -u_{1x}^e \cdot \sin\theta^e + u_{1y}^e \cdot \cos\theta^e \\ u_{2x'}^e &= \ldots \\ u_{2y'}^e &= \ldots \end{aligned}\right\} \begin{bmatrix} u_{1x'}^e \\ u_{1y'}^e \\ u_{2x'}^e \\ u_{2y'}^e \end{bmatrix} = \underbrace{\begin{bmatrix} c^e & s^e & 0 & 0 \\ -s^e & c^e & 0 & 0 \\ 0 & 0 & c^e & s^e \\ 0 & 0 & -s^e & c^e \end{bmatrix}}_{\underline{T}^e \text{ - rotation matrix}} \begin{bmatrix} u_{1x}^e \\ u_{1y}^e \\ u_{2x}^e \\ u_{2y}^e \end{bmatrix}$$

$$c^e = \frac{x_2^e - x_1^e}{l^e} \qquad s^e = \frac{y_2^e - y_1^e}{l^e}$$

orthogonal : $(T^e)^{-1} = (T^e)^T$

$T^e (T^e)^T = I$

b) Expanded 1D stiffness equation

$$\begin{bmatrix} k^e & -k^e \\ -k^e & k^e \end{bmatrix} \cdot \begin{bmatrix} u^e_{1x'} \\ u^e_{2x'} \end{bmatrix} = \begin{bmatrix} F^e_{1x'} \\ F^e_{2x'} \end{bmatrix}$$

$$\underbrace{k^e \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\underline{k}'^e} \underbrace{\begin{bmatrix} u^e_{1x'} \\ u^e_{1y'} \\ u^e_{2x'} \\ u^e_{2y'} \end{bmatrix}}_{\underline{d}'^e} = \underbrace{\begin{bmatrix} F^e_{1x'} \\ F^e_{1y'} \\ F^e_{2x'} \\ F^e_{2y'} \end{bmatrix}}_{\underline{f}'^e} \quad F^e_{1y'} \to 0,\; F^e_{2y'} \to 0$$

shear forces negligible

$$\underline{k}'^e \cdot \underline{d}'^e = \underline{f}'^e$$

$$\underline{k}'^e \cdot \underline{T}^e \cdot \underline{d}^e = \underline{T}^e \cdot \underline{f}^e \qquad / \cdot (\underline{T})^{-1}$$

$$(\underline{T}^e)^{-1} \underline{k}'^e \underline{T}^e \underline{d}^e = \underline{I}\, \underline{F}^e \qquad / (\underline{T}^e)^{-1} = (\underline{T}^e)^T$$

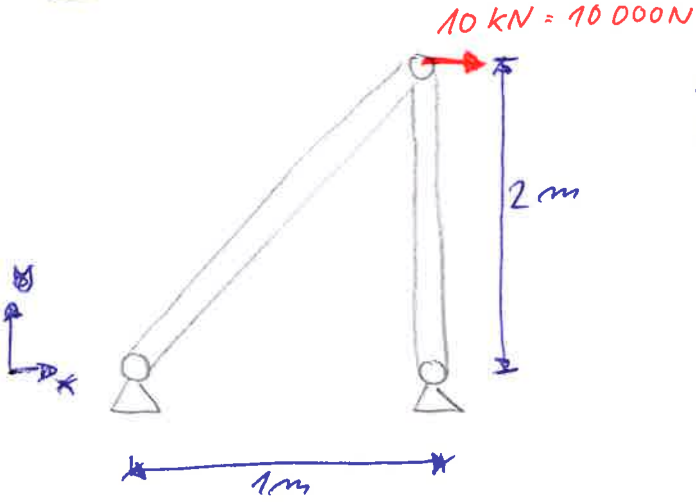
$$\underbrace{(\underline{T})^T \underline{k}'^e \underline{T}^e}_{\underline{k}^e} \underline{d}^e = \underline{F}^e$$

$$\underline{K}^e = (\underline{T})^T \underline{k}'^e \underline{T}^e = \begin{bmatrix} c^e & -s^e & 0 & 0 \\ s^e & c^e & 0 & 0 \\ 0 & 0 & c^e & -s^e \\ 0 & 0 & s^e & c^e \end{bmatrix} \frac{E^e A^e}{L^e} \begin{bmatrix} c^e & s^e & 0 & 0 \\ -s^e & c^e & 0 & 0 \\ 0 & 0 & c^e & s^e \\ 0 & 0 & -s^e & c^e \end{bmatrix} =$$

$$= \underbrace{\boxed{\frac{E^e A^e}{L^e} \begin{bmatrix} c_e^2 & c_e s_e & -c_e^2 & -c_e s_e \\ c_e s_e & s^2 & -c_e s_e & -s_e^2 \\ -c_e^2 & -c_e s_e & c_e^2 & c_e s_e^2 \\ -c_e s_e & -s_e^2 & c_e s_e^2 & s_e^2 \end{bmatrix}}}_{\underline{k}^e}$$

k^e

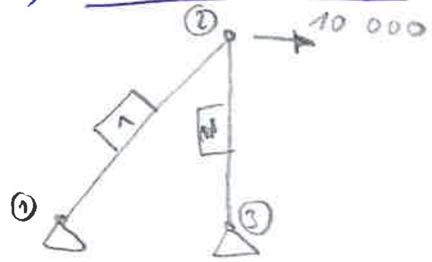
EXAMPLE 2



$E = 200\ GPa = 200 \cdot 10^{9}\ Pa = 2 \cdot 10^{11}\ Pa$

$A = 500\ mm^2 = 500 \cdot 10^{-6}\ m^2 = 5 \cdot 10^{-4}\ Pa$

a) DISCRETIZATION



b) ELEMENT BEHAVIOR (LOCAL STIFFNESS MATRICES)

$\boxed{1}$ 1 - 2

$$\left.\begin{aligned} x_1^{(1)} &= 0\,m \\ x_2^{(1)} &= 1\,m \\ y_1^{(1)} &= 0\,m \\ y_2^{(1)} &= 2\,m \end{aligned}\right\}$$

$$L^{(1)} = \sqrt{\left(x_2^{(1)} - x_1^{(1)}\right)^2 + \left(y_2^{(1)} - y_1^{(1)}\right)^2} = \sqrt{1^2 + 2^2} = 2{,}236\ m$$

$$c^{(1)} = \frac{x_2^{(1)} - x_1^{(1)}}{L_1^{(1)}} = \frac{1}{2{,}236} = 0{,}447$$

$$s^{(1)} = \frac{y_2^{(1)} - y_1^{(1)}}{L_1^{(1)}} = \frac{2}{2{,}236} = 0{,}894$$

$$\underline{K}^{(1)} = \frac{200 \cdot 10^{9} \cdot 500 \cdot 10^{-6}}{2{,}236} \cdot \begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix} =$$

$$= 0{,}894 \cdot 10^{7} \cdot \begin{bmatrix} 1 & 2 & -1 & -2 \\ 2 & 4 & -2 & -4 \\ -1 & -2 & 1 & 2 \\ -2 & -4 & 2 & 4 \end{bmatrix}$$

[N/m]

$\boxed{2}$ 2 – 3

$$\left.\begin{aligned} x_1^{(2)} &= 1\,m \\ x_2^{(2)} &= 1\,m \\ y_1^{(2)} &= 2\,m \\ y_2^{(2)} &= 0\,m \end{aligned}\right\} \quad \begin{aligned} L^{(2)} &= 2\,m \\ c^{(2)} &= 0 \\ s^{(2)} &= -1 \end{aligned}$$

$$K^{(2)} = 5 \cdot 10^{7} \cdot \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} [N/m]$$

c) ASSEMBLY

$$\underline{K} = \begin{array}{c} \begin{bmatrix} 0{,}894 & 1{,}788 & -0{,}894 & -1{,}788 & 0 & 0 \\ 1{,}788 & 3{,}576 & -1{,}788 & -3{,}576 & 0 & 0 \\ -0{,}894 & -1{,}788 & 0{,}894 & 1{,}788 & 0 & 0 \\ -1{,}788 & -3{,}576 & 1{,}788 & 8{,}576 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 & 5 \end{bmatrix} \begin{array}{l} \left.\begin{matrix}\\ \\\end{matrix}\right\} 1 \\ \left.\begin{matrix}\\ \\\end{matrix}\right\} 2 \\ \left.\begin{matrix}\\ \\\end{matrix}\right\} 3 \end{array} \\ \underbrace{\qquad\quad}_{1}\ \underbrace{\qquad\quad}_{2}\ \underbrace{\qquad\quad}_{3} \end{array} \cdot 10^{7}$$

$$\underline{d} = \begin{bmatrix} 0 \\ 0 \\ u_{2x} \\ u_{2y} \\ 0 \\ 0 \end{bmatrix} \qquad \underline{f} = \begin{bmatrix} 0 \\ 0 \\ 10\,000 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

→ free DOFs 3 and 4

$$\underline{K}_F = 10^{7} \cdot \begin{bmatrix} 0{,}894 & 1{,}788 \\ 1{,}788 & 8{,}576 \end{bmatrix} \quad \frac{N}{m} \qquad \underline{K}_E = 10^{7} \cdot \begin{bmatrix} 0{,}894 & 1{,}788 & 0 & 0 \\ 1{,}788 & 3{,}576 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

$$\underline{d}_F = \begin{bmatrix} u_{2x} \\ u_{2y} \end{bmatrix} \quad m \qquad \underline{d}_E = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \underline{r}_E = \begin{bmatrix} r_{1x} \\ r_{1y} \\ r_{2x} \\ r_{2y} \end{bmatrix}$$

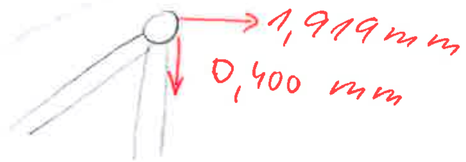
$$\underline{f}_F = \begin{bmatrix} 10\,000 \\ 0 \end{bmatrix} \quad N$$

$$\underline{K}_{EF} = \begin{bmatrix} -0{,}894 & -1{,}788 \\ -1{,}788 & -3{,}576 \\ 0 & 0 \\ 0 & -5 \end{bmatrix} \cdot 10^7 \ \frac{N}{m}$$

D) SOLUTION

$$\underline{K}_F \ \underline{d}_F = \underline{f}_F - \cancel{\underline{K}_{EF}^T \ \underline{d}_E} = 0$$

$$10^7 \cdot \begin{bmatrix} 0{,}894 & 1{,}788 \\ 1{,}788 & 8{,}576 \end{bmatrix} \cdot \begin{bmatrix} u_{2x} \\ u_{2y} \end{bmatrix} = \begin{bmatrix} 10\,000 \\ 0 \end{bmatrix}$$



$$u_{2x} = 0{,}001\,919 \ m$$

$$u_{2y} = -0{,}000\,4 \ m$$

~~E) POST PROCESSING~~

$$\underline{r}_E = \underline{K}_E \ \underline{d}_E + \underline{K}_{EF} \ \underline{d}_F = \begin{bmatrix} -0{,}894 & -1{,}788 \\ -1{,}788 & -3{,}576 \\ 0 & 0 \\ 0 & -5 \end{bmatrix} \cdot \begin{bmatrix} 0{,}001\,919 \\ -0{,}000\,4 \end{bmatrix} = \begin{bmatrix} -10\,000{,}00 \\ -20\,000{,}00 \\ 0 \\ 20\,000 \end{bmatrix}$$

$\Sigma F_x = 0 \qquad \Sigma F_y = 0 \qquad \boxed{\checkmark}$

E) POST - PROCESSING

$$\sigma^e = E^e \cdot \varepsilon^e = E^e \cdot \frac{u^e_{2x'} - u^e_{1x'}}{L^e} = \frac{E^e}{L^e} \cdot [-1 \ \ 0 \ \ 1 \ \ 0] \begin{bmatrix} u^e_{1x'} \\ u^e_{1y'} \\ u^e_{2x'} \\ u^e_{2y'} \end{bmatrix} =$$

$$= \frac{E^e}{L^e} \cdot [-1 \ \ 0 \ \ 1 \ \ 0] \ \underline{T}^e \ \underline{d}^e = \frac{E^e}{L^e} [-c^e \ \ -s^e \ \ c^e \ \ s^e] \ \underline{d}^e$$

$$\sigma^{(1)} = \frac{200 \cdot 10^9}{2{,}236} \cdot [-0{,}447 \ \ -0{,}894 \ \ 0{,}447 \ \ 0{,}894] \begin{bmatrix} 0 \\ 0 \\ 0{,}001\,919 \\ -0{,}000\,4 \end{bmatrix} =$$

$$= \underline{44{,}76 \cdot 10^6 \ Pa}$$

$$\sigma^{(2)} = \frac{200 \cdot 10^9}{2} \cdot [0 \ \ 1 \ \ 0 \ \ -1] \cdot \begin{bmatrix} 0{,}001\,919 \\ -0{,}000\,4 \\ 0 \\ 0 \end{bmatrix} = \underline{-35{,}78 \cdot 10^6 \ Pa}$$