HANDOUT 2

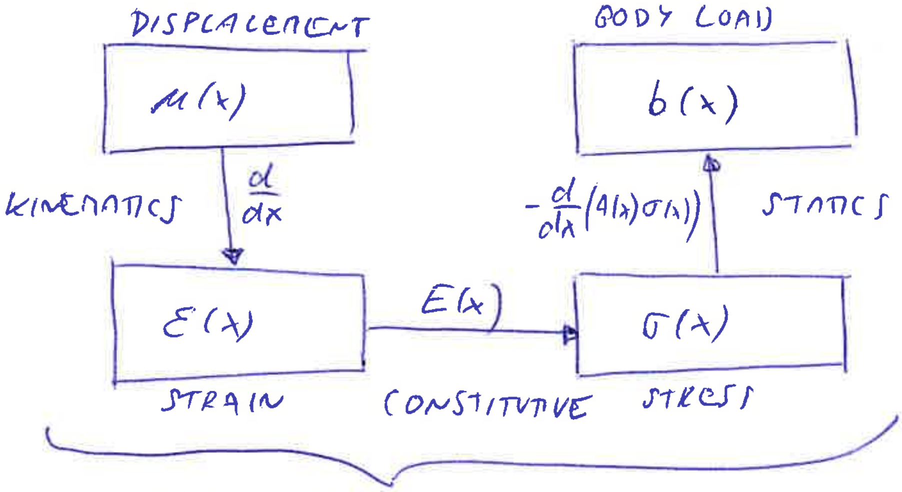
STRONG AND WEAK FORMS FOR 1D PROBLEMS

1) STRONG FORM
   → GOVERNING EQUATIONS AND BOUNDARY CONDITIONS FOR A PHYSICAL SYSTEM. PARTIAL DIFF. EQ. → ORDINARY DIFF. EQ. (1D)

2) WEAK FORM
   → INTEGRAL FORM OF THE STRONG FORM, NEEDED FOR FEM

3) APPROXIMATION FUNCTIONS
   → OF THE WEAK FORM

STRONG FORM – 1D ELASTICITY (LINEAR)

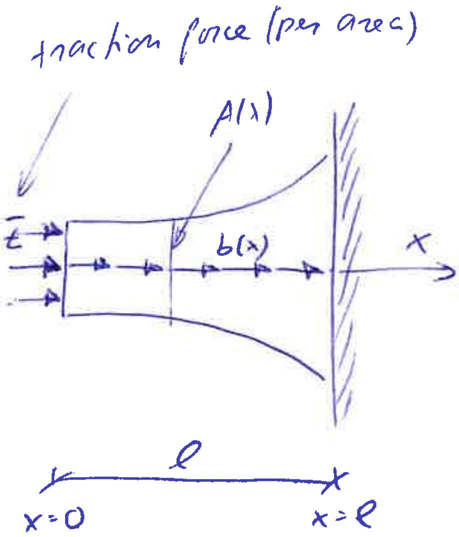


$$\frac{d}{dx}\left(AE\frac{du}{dx}\right) + b = 0 \qquad 0 < x < \ell$$

$$u(\ell) = \bar{u} = 0$$

$$\sigma\big|_{x=0} = \left[E\frac{du}{dx}\right]_{x=0} = -\bar{t}$$

– compression
+ tension



WEAK FORM – 1D LINEAR ELASTICITY

→ EQUIVALENT TO THE STRONG FORM IN INTEGRAL MANNER
→ PRINCIPLE OF VIRTUAL WORK

a) STATIC EQUILIBRIUM → VIRTUAL WORK

$$\int_0^\ell \left[\frac{d}{dx}\left(AE\frac{du}{dx}\right) + b\right]\delta u\, dx = \int_0^\ell \frac{d}{dx}\left(AE\frac{du}{dx}\right)\delta u\, dx + \int_0^\ell b\,\delta u\, dx =$$

$$= \Big[A\underbrace{E\frac{du}{dx}}_{\sigma}\delta u\Big]_0^\ell - \int_0^\ell AE\frac{du}{dx}\frac{d\delta u}{dx}dx + \int_0^\ell b\,\delta u\, dx$$

$$\Leftrightarrow A\sigma n\,\delta u\big|_{\Gamma_u} + A\bar{t}\,\delta u\big|_{\Gamma_t} - \int_\Omega AE\frac{du}{dx}\frac{d\delta u}{dx}dx + \int_\Omega b\,\delta u\, dx$$

b) TRACTION BOUNDARY CONDITION

$$\left[A(\bar{t} - \sigma n)\, \delta u \right]_{\Gamma_t}$$

$$\Rightarrow \int_\Omega AE \frac{du}{dx} \frac{\delta u}{dx}\, dx = AE\, \delta u \Big|_{\Gamma_t} + \int_\Omega b\, \delta u\, dx \qquad \forall w,\ w = 0 \text{ on } \Gamma_u$$

Find $u(x)$ among the smooth functions that satisfy $u|_\Gamma = \bar{u}$
such that

$$\int_0^l AE \frac{du}{dx} \frac{\delta u}{dx}\, dx = \left[A\bar{t}\, \delta u \right]_{\Gamma_t} + \int_0^l b\, \delta u\, dx \qquad \forall \delta u : \delta u|_\Gamma = 0$$

EXAMPLE 1

STRONG FORM:

$$\frac{d}{dx}\left(AE \frac{du}{dx}\right) + 2x = 0$$

$$\sigma(1) = \left(E \frac{du}{dx}\right)_{x=1} = 0.1$$

$$u(3) = 0.001$$

$$x \in \langle 1, 3 \rangle$$

WEAK FORM:

$$\int_1^3 AE \frac{du}{dx} \frac{\delta u}{dx}\, dx = \int_1^3 2x\, \delta u\, dx + \left[-0.1\, A\, \delta u \right]_{x=1}$$

$$\forall \delta u : \delta u(3) = 0$$

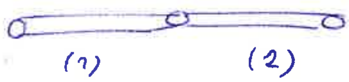
FINITE ELEMENT APPROXIMATION

- domain subdivided into elements (MESH)
- APPROXIMATION USING POLYNOMIAL

$$\theta^e = \alpha_0^e + \alpha_1^e x + \alpha_2^e x^2 + \dots$$

→ CONTINUITY AMONG ELEMENTS

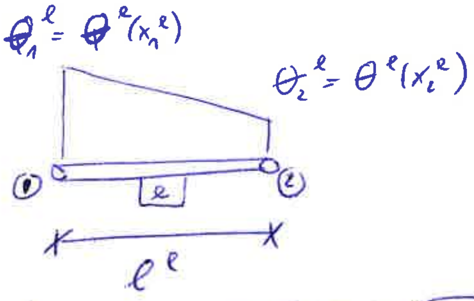
$$\theta^{(1)}(x_2^{(1)}) = \theta^{(2)}(x_1^{(2)})$$



→ CONVERGENCE WITH MESH REFINEMENT

→ INCOMPLETE APPROXIMATION WILL NOT CONVERGE $\quad \theta^e = \alpha_0^e + \alpha_1^e x^2$

TWO-NODE LINEAR ELEMENT



$$\theta^e(x) = \alpha_0^e + \alpha_1^e x.$$

$$\theta^e(x) = \begin{bmatrix} 1 & x \end{bmatrix} \begin{bmatrix} \alpha_0^e \\ \alpha_1^e \end{bmatrix} = \underbrace{\underline{p}(x)}_{\text{polynomial matrix}} \underbrace{\underline{\alpha}^e}_{\text{coefficients}}$$

at NODES 1 AND 2

$$\left.\begin{aligned} \theta^e(x_1^e) &= \alpha_0^e + \alpha_1^e x_1^e \\ \theta^e(x_2^e) &= \alpha_0^e + \alpha_1^e x_2^e \end{aligned}\right\} \quad \begin{bmatrix} \theta_1^e \\ \theta_2^e \end{bmatrix} = \begin{bmatrix} 1 & x_1^e \\ 1 & x_2^e \end{bmatrix} \begin{bmatrix} \alpha_0^e \\ \alpha_1^e \end{bmatrix}$$

$$\underbrace{\underline{d}^e}_{\text{nodal values}} = \underbrace{\underline{M}^e}_{\text{positions}} \underline{\alpha}^e$$

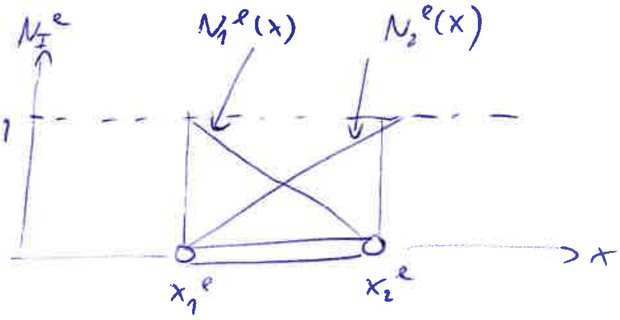
INVERSE $\underline{\alpha}^e = (\underline{M}^e)^{-1} \underline{d}^e = \cancel{\ldots}$

$$\theta^e(x) = \underline{p}(x)\underline{\alpha}^e = \underline{p}(x)(\underline{M}^e)^{-1}\underline{d}^e = \underline{N}^e(x)\,\underline{d}^e$$

$\underline{N}^e = \begin{bmatrix} N_1^e(x) & N_2^e(x) \end{bmatrix}$ ELEMENT SHAPE FUNCTION MATRIX

$$\underline{N}^e = \frac{1}{\ell^e}\begin{bmatrix} x_2^e - x & x - x_1^e \end{bmatrix}$$

→ nonzero only on $\boxed{e}$



$$\underbrace{\begin{matrix} N_1^e(x_1^e) = 1 & N_1^e(x_2^e) = 0 \\ N_2^e(x_1^e) = 0 & N_2^e(x_2^e) = 1 \end{matrix}}$$

$$N_i^e(x_j^e) = \delta_{ij} \qquad \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

KRONECKER DELTA

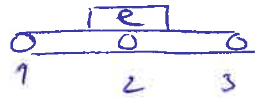
DERIVATIVES OF THE SHAPE FUNCTIONS

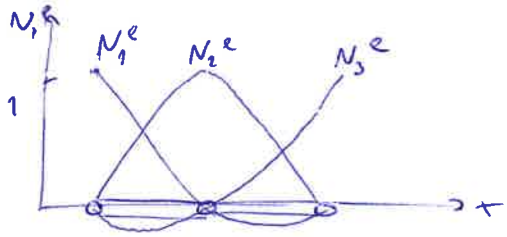
$$\frac{d\theta^e}{dx} = \frac{d}{dx}(\underline{N}^e \underline{d}^e) = \frac{d\underline{N}^e}{dx}\underline{d}^e = \begin{bmatrix} \frac{dN_1^e}{dx} & \frac{dN_2^e}{dx} \end{bmatrix} \begin{bmatrix} \theta_1^e \\ \theta_2^e \end{bmatrix} = \underline{B}^e \underline{d}^e$$

$$\underline{B}^e = \frac{1}{\ell^e}\begin{bmatrix} -1 & 1 \end{bmatrix}$$

QUADRATIC 1D ELEMENT

$$\theta^e(x) = \alpha_0^e + \alpha_1^e x + \alpha_2^e x^2 = \underbrace{[1 \quad x \quad x^2]}_{\underline{p}(x)} \underbrace{\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix}}_{\underline{\alpha}^e} = \underline{p}(x)\underline{\alpha}^e$$





$$\left.\begin{aligned} \theta_1^e &= \alpha_0^e + \alpha_1^e x_1^e + \alpha_2^e (x_1^e)^2 \\ \theta_2^e &= \alpha_0^e + \alpha_1^e x_2^e + \alpha_2^e (x_2^e)^2 \\ \theta_3^e &= \alpha_0^e + \alpha_1^e x_3^e + \alpha_2^e (x_3^e)^2 \end{aligned}\right\} \quad \underline{d}^e = \underbrace{\begin{bmatrix} 1 & x_1^e & (x_1^e)^2 \\ 1 & x_2^e & (x_2^e)^2 \\ 1 & x_3^e & (x_3^e)^2 \end{bmatrix}}_{\underline{M}^e} \underbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}}_{\underline{\alpha}^e}$$

$$\underline{N}^e = \underline{p}(x)\,(\underline{M}^e)^{-1} = \frac{2}{(l^e)^2}\left[(x-x_2^e)(x-x_3^e) \quad -2(x-x_1^e)(x-x_3^e) \quad (x-x_1^e)(x-x_2^e)\right]$$

$$\underline{B}^e = \frac{d\underline{N}^e}{dx}$$

DIRECT CONSTRUCTION OF SHAPE FUNCTIONS

→ KRONECKER DELTA PROPERTY – SHAPE FUNCTION I MUST VANISH IN ALL NODES EXCEPT FOR I, WHERE IT MUST EQUAL 1

→ QUADRATIC $N_1^e(x)$

$$N_1^e(x) = \frac{(x-x_2^e)(x-x_3^e)}{(x_1^e-x_2^e)(x_1^e-x_3^e)}$$

← VANISH AT x_2^e AND x_3^e

← DENOMINATOR EQUALS NUMERATOR AT $x = x_1^e$

APPROXIMATION OF THE WEIGHT FUNCTIONS

→ GALERKIN FEM – <u>SAME</u> APPROXIMATION FOR WEIGHT FUNCTIONS δu TRIAL FUNCTIONS u

$$\delta u^e(x) = \underline{N}^e \delta \underline{d}^e \qquad \frac{\delta u^e(x)}{dx} = \underline{B}^e \delta \underline{d}^e$$

$$u^e(x) = \underline{N}^e \underline{d}^e \qquad \frac{du^e(x)}{dx} = \underline{B}^e \underline{d}^e$$

GLOBAL APPROXIMATION

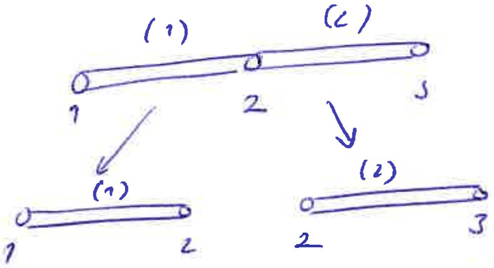
→ GATHERING CONTRIBUTIONS FROM INDIVIDUAL ELEMENTS

$\underline{L}^e$ - gather matrix

$$\underline{u} = \sum_{e=1}^{n_{el}} \underline{N}^e \underline{d}^e = \left(\sum_{e=1}^{n_{el}} \underline{N}^e \underline{L}^e\right)\underline{d}$$

$$\underline{\delta u} = \sum_{e=1}^{n_{el}} \underline{N}^e \underline{d}^e = \left(\sum_{e=1}^{n_{el}} \underline{N}^e \underline{L}^e\right)\underline{d}$$

$$(\underline{N})^T = \sum_{e=1}^{n_{el}} (\underline{N}^e \underline{L}^e)^T = \sum_{e=1}^{n_{el}} (\underline{L}^e)^T (\underline{N}^e)^T$$

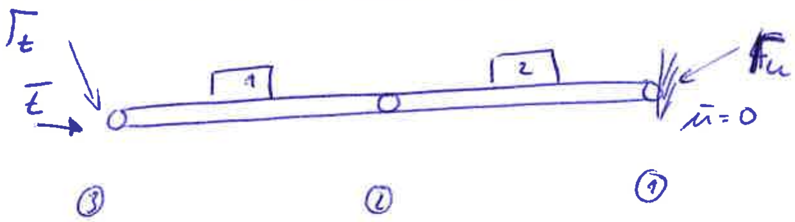


$$\underline{d}^{(1)} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \underline{L}^{(1)} \underline{d}$$

$$\underline{d}^{(2)} = \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \underline{L}^{(2)} \underline{d}$$

$$\underline{N} = \underline{N}^{(1)} \underline{L}^{(1)} + \underline{N}^{(2)} \underline{L}^{(2)} = [\underbrace{N_1^{(1)}}_{N_1} \quad \underbrace{N_2^{(1)} + N_1^{(2)}}_{N_2} \quad \underbrace{N_2^{(2)}}_{N_3}]$$

FE FOR 1D PROBLEMS



weak solution:

$$\int_0^\ell \left(\frac{\delta u}{dx}\right)^T EA \left(\frac{du}{dx}\right) dx - \int_0^\ell (\delta u)^T b \, dx - \left[(\delta u)^T \bar{t} A\right]_{x=0} = 0$$

$$\forall \delta u(x), \quad \delta u(\ell) = 0$$

Approximation:

$\delta u(x) \simeq \underline{N}(x) \, \delta \underline{d}$ weight (test) function

$u(x) \simeq \underline{N}(x) \, \underline{d}$ trial function

↳ trial function need to satisfy E-boundary condition $u_1 = \bar{u} = 0$

SUM OF THE INTEGRALS OVER ELEMENT DOMAINS

$$\sum_{e=1}^{n_{el}} \left[\int_{x_1^e}^{x_2^e} \left(\frac{\delta u^e}{dx}\right)^T E^e A^e \left(\frac{du^e}{dx}\right) dx - \int_{x_1^e}^{x_2^e} (\delta u^e)^T b \, dx - \left[(\delta u^e)^T A^e \bar{t}\right]_{x=0} \right] = 0$$

$$(\delta u^e)^T = (\underline{N}^e \delta \underline{d}^e)^T = (\delta \underline{d}^e)^T (\underline{N}^e)^T \qquad u^e = \underline{N}^e \underline{d}^e$$

$$\left(\frac{\delta u^e}{dx}\right)^T = (\delta \underline{d}^e)^T (\underline{B}^e)^T \qquad \frac{du^e}{dx} = \underline{B}^e \underline{d}^e$$

$$\sum_{e=1}^{n_{el}} (\delta \underline{d}^e)^T \Big[\underbrace{\int_{x_1^e}^{x_2^e} (\underline{B}^e)^T E^e A^e \underline{B}^e \, dx \; \underline{d}^e}_{\underline{K}^e} - \underbrace{\int_{x_1^e}^{x_2^e} (\underline{N}^e)^T b \, dx}_{\underline{f}_{\Omega}^e} - \underbrace{\big[(\underline{N}^e)^T A^e \bar{t}\big]_{x=0}}_{\underline{f}_{\Gamma}^e} \Big] = 0$$

ARBITRARY

ELEMENT STIFFNESS MATRIX

$$\underline{K}^e = \int_{x_1^e}^{x_2^e} (\underline{B}^e)^T E^e A^e \underline{B}^e \, dx = \boxed{\int_{\Omega^e} (\underline{B}^e)^T E^e A^e \underline{B}^e \, dx}$$

EXTERNAL FORCE MATRIX

$$\underline{f}^e = \int_{x_1^e}^{x_2^e} (\underline{N}^e)^T b \, dx + \big[(\underline{N}^e)^T A^e \bar{t}\big]_{x=0} = \int_{\Omega} (\underline{N}^e)^T b \, dx + \big[(\underline{N}^e)^T A^e \bar{t}\big]_{\Gamma_t^e}$$

USING GATHERING MATRICES

$$(\delta \underline{d})^T \Big[\underbrace{\sum_{e=1}^{n_{el}} (\underline{L}^e)^T \underline{K}^e \underline{L}^e \underline{d}} - \underbrace{\sum_{e=1}^{n_{el}} (\underline{L}^e)^T \underline{f}^e} \Big] = 0$$

GLOBAL STIFFNESS MATRIX

$$\underline{K} = \sum_{e=1}^{n_{el}} (\underline{L}^e)^T \underline{K}^e \underline{L}^e$$

GLOBAL FORCE MATRIX

$$\underline{f} = \sum_{e=1}^{n_{el}} (\underline{L}^e)^T \underline{f}^e$$

$$(\delta \underline{d})^T (\underline{K}\,\underline{d} - \underline{f}) = 0 \qquad \forall \, \delta \underline{d}, \; \delta d_1 = 0$$

RESIDUAL

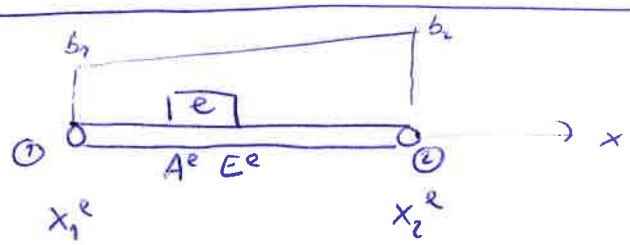
$$\underline{r} = \underline{K}\,\underline{d} - \underline{F} \qquad \Rightarrow (\delta \underline{d})^T \underline{r} = 0$$

$$\rightarrow \delta d_2 r_2 + \delta d_3 r_3 = 0 \qquad (\delta d_1 = 0)$$

UNBALANCED FORCE

$$\underline{r} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \underline{K} \cdot \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} - \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$$

ELEMENT MATRICES FOR 2-NODE ELEMENT



$$\underline{N}^e = \left[\frac{x_2^e - x}{L^e} \quad \frac{x - x_1^e}{L^e} \right] \rightarrow \underline{B}^e = \frac{d}{dx}\underline{N}^e = \left[-\frac{1}{L^e} \quad \frac{1}{L^e} \right]$$

$$\underline{K}^e = \int_{x_1^e}^{x_2^e} (\underline{B}^e)^T A^e E^e \underline{B}^e \, dx = \int_{x_1^e}^{x_2^e} \begin{bmatrix} -\frac{1}{L^e} \\ \frac{1}{L^e} \end{bmatrix} A^e E^e \left[-\frac{1}{L^e} \quad \frac{1}{L^e} \right] dx =$$

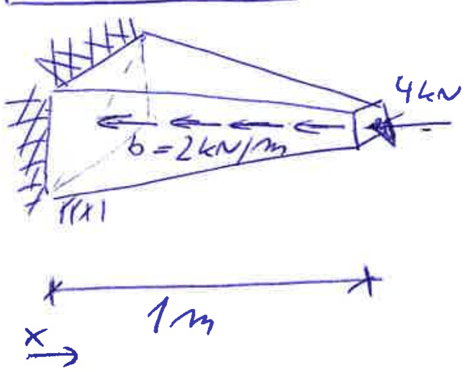
$$= \frac{A^e E^e}{(L^e)^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \int_{x_1^e}^{x_2^e} 1 \, dx = \frac{A^e E^e}{L^e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\underline{F}_\Omega^e = \int_{x_1^e}^{x_2^e} (\underline{N}^e)^T b(x) \, dx = \int_{x_1^e}^{x_2^e} (\underline{N}^e)^T \underline{N}^e \underline{b} \, dx = \frac{1}{(L^e)^2} \int_{x_1^e}^{x_2^e} \begin{bmatrix} (x_2^e - x)^2 & (x_2^e - x)(x - x_1^e) \\ (x_2^e - x)(x - x_1^e) & (x - x_1^e)^2 \end{bmatrix} dx \, b$$

$$b(x) = \underline{N}^e b = \underline{N}^e \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$= \frac{L^e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

EXAMPLE 1



$A(x) = 0.4 - 0.3x \quad [m^2]$

$E = 20 \ GPa$

DISCRETIZATION WITH 1 ELEMENT:

LINEAR BASIS FUNCTION:

$$\underline{N}(x)^e = \frac{1}{L^e} \left[x_2^e - x \quad x - x_1^e \right]$$

$$\underline{B}(x)^e = \frac{d\underline{N}(x)^e}{dx} = \frac{1}{L^e} \left[-1 \quad 1 \right]$$

a) ELEMENT STIFFNESS MATRIX

$$\underline{K}^e = \int_{x_1^e}^{x_2^e} (\underline{B}^e)^T E^e A^e \underline{B}^e \, dx = \int_0^1 \begin{bmatrix} -\frac{1}{1} \\ \frac{1}{1} \end{bmatrix} \cdot 20\times10^9 \cdot (0.4 - 0.3x) \begin{bmatrix} -1 & 1 \end{bmatrix} dx =$$

$$= 20\times10^9 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \int_0^1 (0.4 - 0.3x)\, dx =$$

$$= 20\times10^9 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \left[0.4x - \frac{0.3x^2}{2} \right]_0^1 = 20\times10^9 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \overbrace{(0.4 - 0.15)}^{0.25} =$$

$$= 5\times10^9 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad [N/m]$$

b) EXTERNAL FORCE MATRIX

$$\underline{f}_\Gamma^e = \begin{bmatrix} R_1 \\ -4000 \end{bmatrix} \quad [N]$$

c) BODY FORCE MATRIX

$$\underline{f}_\Omega^e = \frac{L^e}{6} \cdot \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \qquad b_1 = b_2 = b$$

$$= \frac{L^e b}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1 \cdot (-2000)}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1000 \\ -1000 \end{bmatrix} \quad [N]$$

d) DISPLACEMENT MATRIX

$$\underline{d}^e = \begin{bmatrix} 0 \\ d_2 \end{bmatrix} \quad [m]$$

ESSENTIAL BOUNDARY CONDITIONS: 1

UNKNOWN DOFS (F) 2

e) SOLUTION

$$\underline{K}_{FF}\, \underline{d}(F) = \underline{f}_F - \cancel{\underline{K}_{FE}\, \underline{d}_E} \quad 0$$

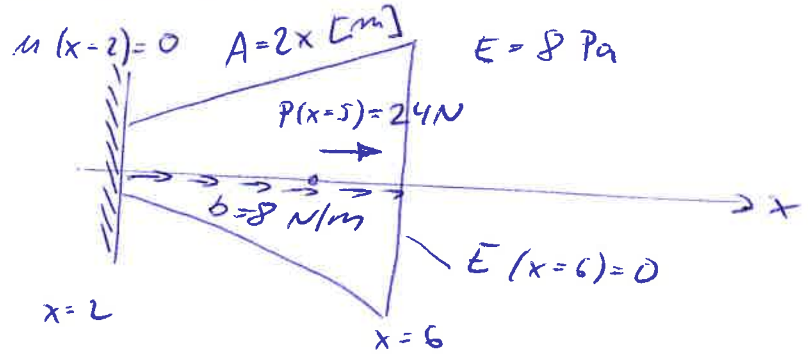
$$\underline{d}_F = (\underline{K}_{FF})^{-1} \underline{f}_F = \frac{\overset{-4000-1000}{-5000}}{5\times10^9} = \underline{-1\times10^{-6}\ m}$$

$$\underline{f}_E = \underline{K}_{EF}\, \underline{d}_F + \cancel{\underline{K}_{EE}\, \underline{d}_E} \Rightarrow \cancel{25\times10^9 (-6.4\times10^{-6}) = 4000\ N}$$

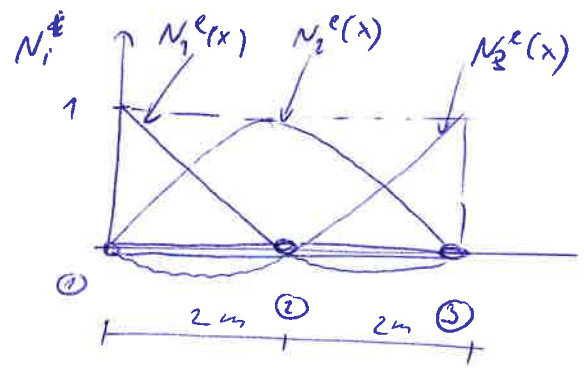
$$\Rightarrow R_1 - 1000 = -5\times10^9 \cdot (-1\times10^{-6})$$

$$\boxed{R_1 = 6000\ N}$$

EXAMPLE 2 TAPERED ELASTIC BAR



SINGLE THREE-NODE ELEMENT



$$N_1^{(1)}(x) = \frac{(x - x_2^{(1)})(x - x_3^{(1)})}{(x_1^{(1)} - x_2^{(1)})(x_1^{(1)} - x_3^{(1)})} = \frac{(x-4)(x-6)}{(2-4)(2-6)} = \frac{1}{8}(x-4)(x-6)$$

$$N_2^{(1)}(x) = \frac{(x - x_1^{(1)})(x - x_3^{(1)})}{(x_2^{(1)} - x_1^{(1)})(x_2^{(1)} - x_3^{(1)})} = \frac{(x-2)(x-6)}{(4-2)(4-6)} = -\frac{1}{4}(x-2)(x-6)$$

$$N_3^{(1)}(x) = \frac{(x - x_1^{(1)})(x - x_2^{(1)})}{(x_3^{(1)} - x_1^{(1)})(x_3^{(1)} - x_2^{(1)})} = \frac{(x-2)(x-4)}{(6-2)(6-4)} = \frac{1}{8}(x-2)(x-4)$$

$(f \cdot g)' = f'g + fg'$ PRODUCT RULE

$$B_1^{(1)}(x) = \frac{dN_1^{(1)}(x)}{dx} = \frac{1}{8}\cdot(x-6) + \frac{1}{8}(x-4) = \frac{1}{8}(2x-10) = \frac{1}{4}(x-5)$$

$$B_2^{(1)}(x) = \frac{dN_2^{(1)}(x)}{dx} = -\frac{1}{4}(x-6) - \frac{1}{4}(x-2) = -\frac{1}{4}(2x-8) = -\frac{1}{2}(x-4)$$

$$B_3^{(1)}(x) = \frac{dN_3^{(1)}(x)}{dx} = \frac{1}{8}(x-4) + \frac{1}{8}(x-2) = \frac{1}{8}(2x-6) = \frac{1}{4}(x-3)$$

STIFFNESS MATRIX

$$\underline{k}^{(1)} = \underline{K} = \int_{x_1}^{x_3} (\underline{B}^{(1)})^T A^{(1)} E^{(1)} \underline{B}^{(1)} \, dx =$$

$$= \int_2^6 \frac{1}{4} \begin{bmatrix} x-5 \\ 8-2x \\ x-3 \end{bmatrix} \cdot (2x) \cdot 8 \cdot \frac{1}{4} \begin{bmatrix} (x-5) & (8-2x) & (x-3) \end{bmatrix} dx =$$

$$= \int_2^6 \begin{bmatrix} x\cdot(x-5)^2 & x\cdot(x-5)(8-2x) & x(x-5)(x-3) \\ x\cdot(8-2x)(x-5) & x\cdot(8-2x)^2 & x\cdot(8-2x)(x-3) \\ x\cdot(x-3)(x-5) & x(x-3)(8-2x) & x\cdot(x-3)^2 \end{bmatrix} dx =$$

$$= \begin{bmatrix} 26.67 & -32 & 5.33 \\ -32 & 85.33 & -53.33 \\ 5.33 & -53.33 & 48 \end{bmatrix} = \underline{K} \qquad [N/m]$$

BODY FORCE MATRIX

→ distributed loading b

→ point force P

$$\underline{f}_\Omega = \underline{f}_\Omega^{(1)} = \int_{x_1}^{x_3} (\underline{N}^e)^T b \, dx + \left[(\underline{N}^e)^T P \right]_{x=5}$$

$$\underline{f}_\Omega = \underbrace{\int_2^6 \begin{bmatrix} \frac{1}{8}(x-4)(x-6) \\ -\frac{1}{4}(x-2)(x-6) \\ \frac{1}{8}(x-2)(x-4) \end{bmatrix} \times 8 \, dx}_{\text{sum } 8\times 4 = 32} + \underbrace{\left[\begin{bmatrix} \frac{1}{8}(x-4)(x-6) \\ -\frac{1}{4}(x-2)(x-6) \\ \frac{1}{8}(x-2)(x-4) \end{bmatrix} \times 24 \right]_{x=5}}_{\text{sum } 24} =$$

$$= \begin{bmatrix} 5.33 \\ 21.33 \\ 5.33 \end{bmatrix} + \begin{bmatrix} -3 \\ 18 \\ 9 \end{bmatrix} = \begin{bmatrix} 2.33 \\ 39.33 \\ 14.33 \end{bmatrix} [N]$$

$$\underline{f}_\Gamma = \begin{bmatrix} R_1 \\ 0 \\ 0 \end{bmatrix} \rightarrow \underline{f} = \underline{f}_\Omega + \underline{f}_\Gamma = \begin{bmatrix} 2.33 + R_1 \\ 39.33 \\ 14.33 \end{bmatrix} [N]$$

SOLUTION

$$\begin{bmatrix} 26.67 & -32 & 5.33 \\ & 85.33 & -53.33 \\ \text{sym.} & & 48 \end{bmatrix} \begin{bmatrix} \overset{\text{E-BC}}{0} \\ u_2 \\ \underset{\text{F-BC}}{u_3} \end{bmatrix} = \begin{bmatrix} r_1 + 2.33 \\ 39.33 \\ 14.33 \end{bmatrix}$$

$$\underline{K}_F \, \underline{d}_F = \underline{f}_F - \cancel{\underline{K}_{FE} \, \underline{d}_E}$$

$$\begin{bmatrix} 85.33 & -53.33 \\ -53.33 & 48 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 39.33 \\ 14.33 \end{bmatrix} \rightarrow \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 2.1193 \\ 2.6534 \end{bmatrix}$$

POST-PROCESSING

$$u = N_1^{(1)} u_1 + N_2^{(1)} u_2 + N_3^{(1)} u_3 \qquad \Rightarrow \quad \underline{d} = d^{(1)} = \begin{bmatrix} 0 \\ 2.1193 \\ 2.6534 \end{bmatrix}$$

DISPLACEMENT FIELD

$$u(x) = \frac{1}{8}(x-4)(x-6) \times 0 + -\frac{1}{4}(x-2)(x-6) \times 2.1193 +$$

$$+ \frac{1}{8}(x-2)(x-4) \times 2.6534 = 0.19815 x^2 + 2.24855 x - 3.7045$$

STRESS FIELD

$$\sigma(x) = E \frac{du}{dx} = E \frac{d}{dx}\left(N^{(1)} d^{(1)}\right) = E B^{(1)} d^{(1)} =$$

$$= 8 \cdot \frac{1}{4} \left[(x-5) \quad (8-2x) \quad (x-3) \right] \begin{bmatrix} 0 \\ 2.1193 \\ 2.6534 \end{bmatrix} = -3.17x + 17.99$$

DOES NOT CAMPTURE JUMP BELOW P!