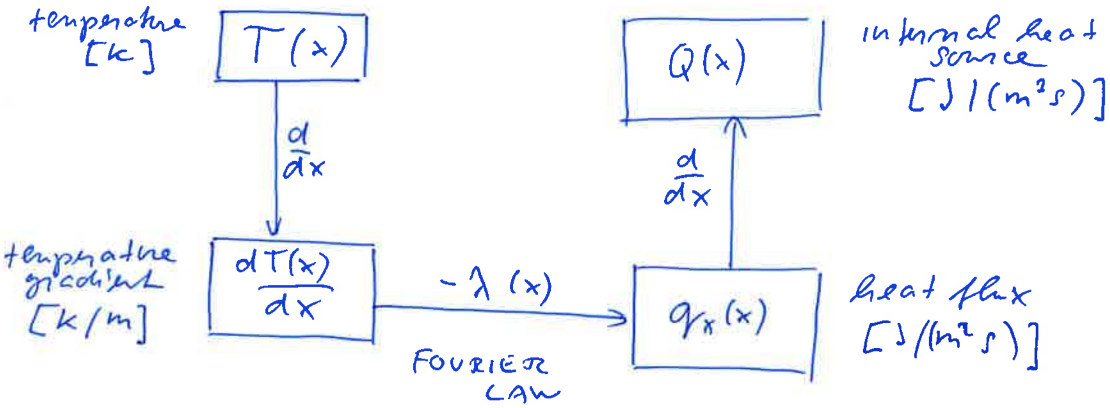
HANDOUT 3

- heat flow – temperature difference within a body, and ~~between~~ a body and surrounding medium
  - conduction → through wall of heated wall in winter
  - convection → temperature difference between body surface and medium
  - radiation
- energy balance : the rate of heat energy generated in the control volume must equal the heat energy leaving the control volume
- time does not matter in a steady-state problem



λ ... coefficient of thermal conductivity

$\ominus$ ... heat goes against the gradient

GOVERNING EQUATION :

$$\frac{d}{dx}\left(-\lambda \frac{dT}{dx}\right) = Q \quad , \quad 0 < x < \ell$$

boundary conditions : a) temperature at surface $T|_{\Gamma_t} = \bar{T}$ (Diricelet)

b) prescribed heat flux $q(0)$, or $q(\ell)$

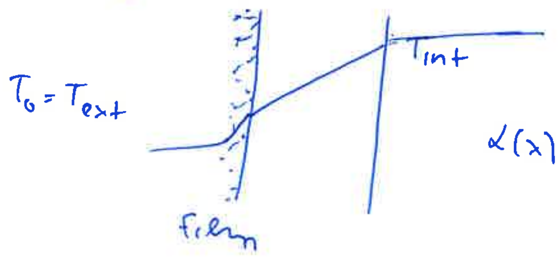
$$q_x(x) \cdot \overset{\{-1,1\}}{n(x)} = \bar{q}_n(x)$$

$\bar{q}$ – positive when energy (heat) flows out of bar

Ⅰ CONSTANT (Neumann b.c.) – perfectly insulated

$$Q|_{\Gamma_{\bar{q}}} = \bar{Q}$$

Ⅱ TEMPERATURE – DEPENDENT



$$Q|_{\Gamma_{qc}} = \alpha(x)\,(T_{\Gamma_{qc}} - T_0)$$

$\alpha(x)$ – film coefficient $[J/(m^2 K s)]$

III) FROM RADIATION

$$Q|_{\Gamma_{qr}} = \varepsilon(x)\,\sigma\left(T_{\Gamma_{qr}}^4 - T_\infty^4\right)$$

$\varepsilon \in (0,1)$ ... surface emissivity

4 types of B.C. $\begin{cases} \Gamma_q = \Gamma_{\bar{q}} + \Gamma_{q_c} + \Gamma_{q_r} \\ \Gamma_{\bar{T}} \end{cases}$

WEAK FORM

$$\int_\Omega \frac{\delta T}{dx} A \lambda \frac{dT}{dx}\, d\Omega = -(\delta T A \bar{q})\Big|_{\Gamma_q} + \int_\Omega \delta T \bar{Q}\, d\Omega$$

ELEMENT MATRICES

CONDUCT~~IVITY~~ANCE MATRIX:

$$\underline{K}^e = \int_{\Omega^e} (\underline{B}^e)^T A^e \lambda^e (\underline{B}^e)\, dx$$

BOUNDARY TEMPERATURE:

$$\underline{f}_{\Gamma_T^e}^e = \left[(\underline{N}^e)^T A^e \bar{T}\right]_{\Gamma_T^e}$$

INTERNAL HEAT MATRIX:

$$\underline{F}_{\Omega^e}^e = \int_\Omega (\underline{N}^e)^T \underline{N}^e\, dx\, \bar{\underline{Q}} = \frac{L^e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} \bar{Q}_1 \\ \bar{Q}_2 \end{bmatrix}$$

BOUNDARY FLUX MATRIX:

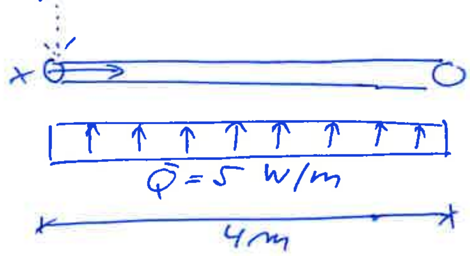
$$\underline{f}_{\Gamma_{\bar{q}}}^e = -\begin{bmatrix} \bar{q}_1 \\ \bar{q}_2 \end{bmatrix}$$

$$\underline{f}_{\Gamma_{q_c}}^e = \int_{\Gamma_{q_c}^e} (\underline{N}^e)^T \alpha(x)\, \underline{N}^e\, \bar{\underline{T}}_0^e\, d\Gamma_{q_c}^e =$$

$$= \underline{K}_\Gamma^e\, \bar{\underline{T}}_0 = \begin{bmatrix} \alpha_1^e & 0 \\ 0 & \alpha_2^e \end{bmatrix}\begin{bmatrix} \bar{T}_{0,1}^e \\ \bar{T}_{0,2}^e \end{bmatrix} = \begin{bmatrix} \alpha_1^e \bar{T}_{0,1}^e \\ \alpha_2^e \bar{T}_{0,2}^e \end{bmatrix}$$

$\boxed{\text{EXAMPLE 1}}$

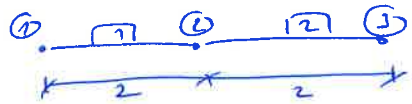
$T(x=0) = \bar{T} = 0\,K$ $\qquad$ $q(x=4)m = \bar{q} = 5\,\frac{W}{m^2}$



$A = 0.1\,m^2$

$\lambda = 2\,W/(m\cdot K)$

2 LINEAR 2-NODE ELEMENTS:



$\boxed{1}$

CONDUCTANCE MATRIX:

see elasticity

$$\underline{K}^{(1)} = \int_{\Omega^{(1)}} (B^{(1)})^T A^{(1)} \lambda^{(1)} B^{(1)}\, dx = \frac{A^{(1)}\lambda^{(1)}}{L^{(1)}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} =$$

$$= \frac{0.1 \times 2}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{array}{c c} \begin{array}{cc} 1 & 2 \end{array} & \\ \begin{bmatrix} 0.1 & -0.1 \\ -0.1 & 0.1 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \end{array} \end{array} \quad \frac{W}{K}$$

$\boxed{2}$

$$\underline{K}^{(2)} = \frac{0.1 \times 2}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{array}{c c} \begin{array}{cc} 2 & 3 \end{array} & \\ \begin{bmatrix} 0.1 & -0.1 \\ -0.1 & 0.1 \end{bmatrix} & \begin{array}{c} 2 \\ 3 \end{array} \end{array} \quad \frac{W}{K}$$

$$\underline{K} = \underline{K}^{(1)} + \underline{K}^{(2)} = \begin{array}{c c} \begin{bmatrix} 0.1 & -0.1 & 0 \\ -0.1 & 0.2 & -0.1 \\ 0 & -0.1 & 0.1 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \\ \begin{array}{ccc} 1 & 2 & 3 \end{array} & \end{array} \quad \frac{W}{K}$$

BOUNDARY FLUX MATRIX

1 at boundary

$$\underline{F}^e_{\Gamma_{\bar{q}}} = -\left[(\underline{N}^e)^T A^e \bar{q} \right]_{\Gamma^e_{\bar{q}}}$$

$$\underline{F}^e_{\Gamma_{\bar{q}}} = \begin{bmatrix} 0 \\ 0 \\ -(5 \times 0.1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -0.5 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \quad \frac{W}{m^2}$$

ELEMENT SOURCE FLUX

$$\underline{f}^e_{\Omega} = \frac{L_e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \bar{Q}_1 \\ \bar{Q}_2 \end{bmatrix} \Rightarrow \underline{f}^{(1)}_{\Omega} = \frac{2}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 15 \\ 15 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

$$\underline{F}^{(2)}_{\Omega} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

$$\underline{f}_{\Omega} = \underline{F}^{(1)}_{\Omega} + \underline{F}^{(2)}_{\Omega} = \begin{bmatrix} 5 \\ 10 \\ 5 \end{bmatrix} \quad W/m^2$$

SOLUTION: $\underline{K}\,\underline{T} = \underline{F}$

$$\underbrace{\begin{bmatrix} 0.1 & -0.1 & 0 \\ -0.1 & 0.2 & -0.1 \\ 0 & -0.1 & 0.1 \end{bmatrix}}_{\underline{K}} \cdot \underbrace{\begin{bmatrix} 0 \\ T_2 \\ T_3 \end{bmatrix}}_{\underline{T}} = \underbrace{\begin{bmatrix} 5 \\ 10 \\ 5 \end{bmatrix}}_{f_\Omega} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ -0.5 \end{bmatrix}}_{f_{\bar{q}}} + \begin{bmatrix} R_1 \\ 0 \\ 0 \end{bmatrix}$$

FREE NODES ... 2, 3

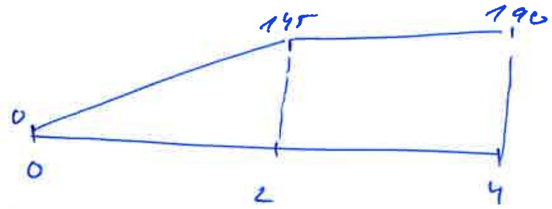
ESSENTIAL B.C. ... 1

$$\underline{K}_{FF} \cdot \underline{T}_F = \underline{f}_F$$

$$\begin{bmatrix} 0.2 & -0.1 \\ -0.1 & 0.1 \end{bmatrix} \begin{bmatrix} T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 4.5 \end{bmatrix}$$

$T_2 = 145\ K$

$T_3 = 190\ K$



TEMPERATURE GRADIENT:

$T \approx \underline{N}(x)\,\underline{T}$

$$\frac{dT}{dx} \approx \underline{B}(x)\,\underline{T}$$

$$\frac{dT^{(1)}}{dx} = \begin{bmatrix} -\frac{1}{L^e} & \frac{1}{L^e} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 145 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 145 \end{bmatrix} = 72.5\ \frac{K}{m}$$

$$\frac{dT^{(2)}}{dx} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 145 \\ 190 \end{bmatrix} = 22.5\ \frac{K}{m}$$

EXACT SOLUTION:

→ can be done only for some simple problems, primarily 1D

$$\frac{d}{dx}\left(-\lambda \frac{dT}{dx}\right) = Q$$

$$\frac{d}{dx}\left(0.2\,\frac{dT}{dx}\right) + 5 = 0$$

$$0.2\,\frac{d^2T}{dx^2} = -5$$

$$\frac{d^2T}{dx^2} = -25$$

BOUNDARY CONDITIONS:

a) $T(0) = 0$

b) $\bar{q}(4) = -\overset{2}{\lambda}\dfrac{dT}{dx}\cdot\hat{m}\Big|_{x=4} = 5 \quad \Longleftrightarrow \quad \dfrac{dT}{dx}(4) = \dfrac{5}{-2} = -2.5$

INTEGRATION OF GOVERNING EQUATION:

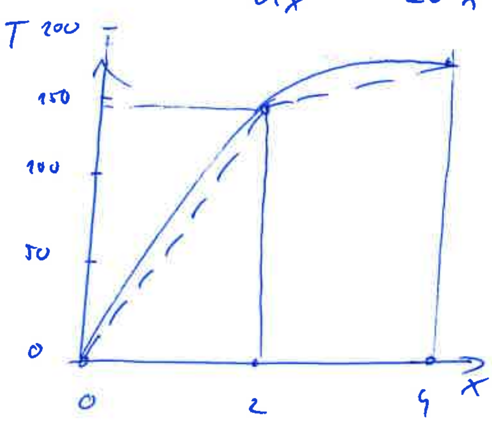
$$\frac{d^2T}{dx^2} = -25$$

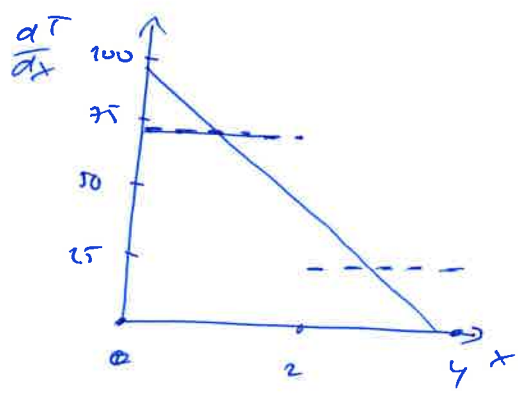
$$\frac{dT}{dx} = -25x + C_1 \qquad b)\ -2.5 = -25\cdot 4 + C_1$$

$$\underline{C_1} = 100 - 2.5 = \underline{97.5}$$

$$\underbrace{T = -12.5x^2 + 97.5x + C_2 \qquad a)\ 0 = C_2}_{T = -12.5x^2 + 97.5x}$$

$$\frac{dT}{dx} = -25x + 97.5$$

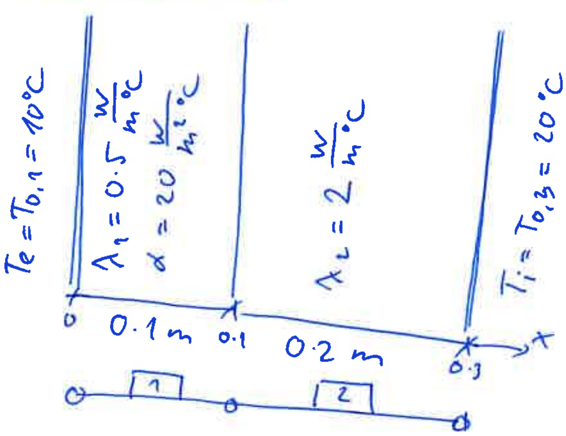




--- FEM

—— EXACT

EXAMPLE 2



ESSENTIAL B.C ... $\varnothing$

CONDUCTANCE MATRIX

$$\underline{K}_{\Omega}^{(1)} = \int_0^{0.1} (\underline{B}^{(1)})^T \lambda^{(1)} A^{(1)} \underline{B}^{(1)} \, dx = \frac{\lambda^{(1)}}{L^{(1)}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{0.5}{0.1} \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix} \left[\frac{W}{°C\, m^2} \right]$$

$$1$$

$$\underline{K}_{\Omega}^{(2)} = \int_{0.1}^{0.3} (\underline{B}^{(2)})^T \lambda^{(2)} A^{(2)} \underline{B}^{(2)} \, dx = \frac{\lambda^{(2)}}{L^{(2)}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{2}{0.2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 10 & -10 \\ -10 & 10 \end{bmatrix} \frac{W}{°C\, m^2}$$

$$\underline{K}_{\Gamma}^{(1)} = (\underline{N}^{(1)})^T \alpha^{(1)} \underline{N}^{(1)} \Big|_{\Gamma} = N_{(x_1^{(1)})}^{(1)\,T} \alpha^{(1)}_{(x_1^{(1)})} \underline{N}^{(1)}_{(x_1^{(1)})} + N_{(x_2^{(1)})}^{(1)\,T} \alpha^{(1)}_{(x_2^{(1)})} \underline{N}^{(1)}_{(x_2^{(1)})} =$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \times 20 \times [1 \quad 0] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \times \times [0 \quad 1] =$$

$$= \begin{bmatrix} 20 & 0 \\ 0 & 0 \end{bmatrix} \quad \frac{W}{°C\, m^2}$$

$$\underline{K}_{\Gamma}^{(2)} = \begin{bmatrix} 0 & 0 \\ 0 & 20 \end{bmatrix} \quad \frac{W}{°C\, m^2}$$

$$\underline{K} = \underline{K}_{\Omega}^{(1)} + \underline{K}_{\Omega}^{(2)} + \underline{K}_{\Gamma}^{(1)} + \underline{K}_{\Gamma}^{(2)} = \begin{bmatrix} 5 \cancel{+20} +20 & -5 & \\ -5 & 5+10 & -10 \\ & -10 & 10+20 \end{bmatrix} =$$

$$= \begin{bmatrix} 25 & -5 & \\ -5 & 15 & -10 \\ & -10 & 30 \end{bmatrix} \quad \frac{W}{°C\, m^2}$$

CONVECTIVE FLUX MATRIX

$$\underline{f}_{\Gamma_c}^{e} = \underline{K}_{\Gamma}^{e} \begin{bmatrix} T_{0,1}^{e} \\ T_{0,2}^{e} \end{bmatrix} = \begin{bmatrix} \alpha_1^e & 0 \\ 0 & \alpha_2^e \end{bmatrix} \begin{bmatrix} T_{0,1}^{e} \\ T_{0,2}^{e} \end{bmatrix} = \begin{bmatrix} \alpha_1^e \, T_{0,1}^{e} \\ \alpha_2^e \, T_{0,2}^{e} \end{bmatrix}$$

$$\underline{f}_{\Gamma_c}^{(1)} = \begin{bmatrix} 20 \times 10 \\ 0 \end{bmatrix} = \begin{bmatrix} 200 \\ 0 \end{bmatrix} \quad W/m^2$$

$$\underline{F}_{\Gamma_c}^{(2)} = \begin{bmatrix} 0 \\ 20 \times 20 \end{bmatrix} = \begin{bmatrix} 0 \\ 400 \end{bmatrix} \quad W/m^2$$

$$\underline{F} = \begin{bmatrix} 200 \\ 0 \\ 400 \end{bmatrix}$$

SOLUTION

$\underline{K} \cdot \underline{T} = \underline{F}$

$$\begin{bmatrix} 25 & -5 & \\ -5 & 15 & -10 \\ & -10 & 30 \end{bmatrix} \cdot \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} 200 \\ 0 \\ 400 \end{bmatrix} \Rightarrow \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} 11.25 \\ 16.25 \\ 18.75 \end{bmatrix} \quad °C$$