2D LINEAR ELASTICITY

ASSUMPTIONS:
1) SMALL DEFORMATIONS
2) LINEAR MATERIAL BEHAVIOR
3) NEGLECTED DYNAMIC EFFECTS
4) NO GAPS / OVERLAPS IN DEFORMATION

KINEMATICS

- vektor with two displacements components

$$\underline{u} = \begin{bmatrix} u_x \\ u_y \end{bmatrix}$$

- extensional strains

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} \qquad \varepsilon_{yy} = \frac{\partial u_y}{\partial y} \qquad \gamma_{xy} = \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}$$

engineering shear strain

$$\varepsilon_{xy} = \frac{\gamma_{xy}}{2}$$

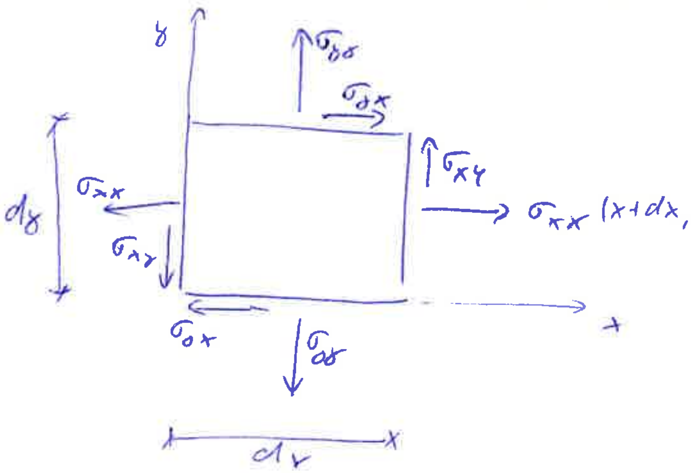
tensor shear strain

in vektor form:

$$\underline{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \underline{\nabla}_s \, \underline{u} = \underbrace{\begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}}_{\text{symmetric gradient matrix operator}} \begin{bmatrix} u_x \\ u_y \end{bmatrix}$$

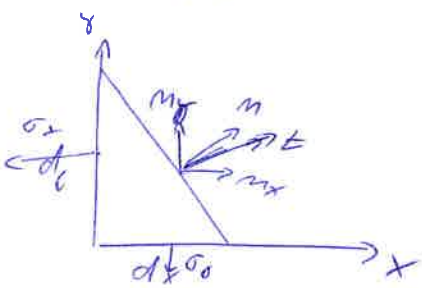
STRESS AND TRACTION

- stress = force / unit area that acts on the plane normal to ...



moment equilibrium: $\sigma_{xy} = \sigma_{yx}$

$$\underline{\sigma} = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix}$$



$$\begin{bmatrix} t_x \\ t_y \end{bmatrix} = \overbrace{\begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix}}^{\text{tensorial notation}} \begin{bmatrix} n_x \\ n_y \end{bmatrix}$$

$$\underline{t} = \underline{\tau} \, \underline{n}$$

EQUILIBRIUM

$$\rightarrow \quad \frac{\delta\sigma_{xx}}{\delta x} + \frac{\delta\sigma_{xy}}{\delta y} + b_x = 0$$

$$y\downarrow \quad \frac{\delta\sigma_{yx}}{\delta x} + \frac{\delta\sigma_{yy}}{\delta y} + b_y = 0$$

$$\underline{\nabla_S}^T \underline{\sigma} + \underline{b} = \underline{0}$$

↳ because of this operator appearing in both equilibrium and strain-displacement equation, the PDE is self-adjoint and the stiffness matrix symmetric

CONSTITUTIVE EQUATIONS

- linear elasticity, generalized Hookes law

$$\underline{\sigma} = \underline{D}\,\underline{\varepsilon}$$

- $\underline{D}$ - material stiffness matrix, 3x3, positive-definite

a) PLANE STRESS

$$\underline{D} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

object is thin (plate)
no stress on z surface

b) PLANE STRAIN

$$\underline{D} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

body is thick relative to xy plane
$\rightarrow \varepsilon_z = 0,\ \gamma_{xz} = \gamma_{yz} = 0$

- isotropic material: $G = \frac{E}{2\cdot(1+\nu)}$

STRONG FORM

a) $\underline{\nabla_S}^T \underline{\sigma} + \underline{b} = 0$

b) $\underline{\sigma} = \underline{D}\,\nabla_s u$

c) $\underline{\sigma}\, \underline{n} = \underline{\bar{t}}$ on Γ_t

d) $\underline{u} = \underline{\bar{u}}$ on Γ_u

WEAK FORM

a) $\int_\Omega w_x\, \vec{\nabla}\cdot\vec{\sigma}_x \, d\Omega + \int_\Omega w_x b_x \, d\Omega = 0 \qquad \forall w_x \in U_0$

b) $\int_\Omega w_y\, \vec{\nabla}\cdot\vec{\sigma}_y \, d\Omega + \int_\Omega w_y b_y \, d\Omega = 0 \qquad \forall w_y \in U_0$

c) $\int_{\Gamma_t} w_x (\bar{t}_x - \vec{\sigma}_x \vec{n})\, d\Gamma = 0 \qquad \forall w_x \in U_0$

d) $\int_{\Gamma_t} w_y (\bar{t}_y - \vec{\sigma}_y \vec{n})\, d\Gamma = 0 \qquad \forall w_y \in U_0$

applying the Green-Gauss theorem, we get

$$\int_\Omega (\nabla_s \underline{w})^T \underline{D}\, \nabla_s \underline{u}\, d\Omega = \int_{\Gamma_t} \underline{w}^T \underline{t}\, d\Gamma + \int_\Omega \underline{w}^T \underline{b}\, d\Omega \qquad \forall w \in U_0$$

$w = 0$ on Γ_u

FINITE ELEMENT DISCRETIZATION

- displacement fields u_x and u_y usually approximated by the same shape function
- 2 degrees of freedom per node (x and y displacements)

$$\underline{d} = \begin{bmatrix} u_{x1} \\ u_{y1} \\ \vdots \\ u_{xn} \\ u_{yn} \end{bmatrix} \qquad n \ldots \text{number of nodes}$$

$$\underline{u}(x,y) \approx \underline{u}^e(x,y) = \underline{N}^e(x,y)\, \underline{d}^e \qquad (x,y) \in \Omega^e$$

$$\underline{w}^T(x,y) \approx \underline{w}^{eT}(x,y) = \underline{w}^{eT} \underline{N}^e(x,y)^T \qquad (x,y) \in \Omega^e$$

$$\underline{N}^e = \begin{bmatrix} N_1^e & 0 & N_2^e & 0 & \ldots & N_n^e & 0 \\ 0 & N_1^e & 0 & N_2^e & \ldots & 0 & N_n^e \end{bmatrix}$$

$$\underline{w}^e = \left[w_{x1}^e \;\; w_{y1}^e \;\; \ldots \;\; w_{xn}^e \;\; w_{yn}^e \right]^T$$

INTEGRAL OVER THE DOMAIN Ω IS EQUAL TO THE SUM OF INTEGRALS OVER ALL ELEMENTS, Ω_e

$$\sum_{e=1}^{n_{el}} \left[\int_{\Omega^e} (\nabla_s \underline{w}^e)^T \underline{D}^e\, \nabla_s \underline{u}^e\, d\Omega - \int_{\Gamma_t^e} \underline{w}^{eT} \underline{t}\, d\Gamma - \int_{\Omega^e} \underline{w}^{eT} \underline{b}\, d\Omega \right] = 0$$

STRAIN

$$\underline{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} \approx \underline{\varepsilon}^e = \nabla_s \underline{u}^e = \nabla_s \underline{N}^e \underline{d}^e = \underline{B}^e \underline{d}^e$$

strain-displacement matrix

$$\underline{B}^e = \nabla_s \underline{N}^e = \begin{bmatrix} \frac{\partial N_1^e}{\partial x} & 0 & & \frac{\partial N_n^e}{\partial x} & 0 \\ 0 & \frac{\partial N_1^e}{\partial y} & \ldots & 0 & \frac{\partial N_n^e}{\partial y} \\ \frac{\partial N_1^e}{\partial y} & \frac{\partial N_1^e}{\partial x} & & \frac{\partial N_n^e}{\partial y} & \frac{\partial N_n^e}{\partial x} \end{bmatrix}$$

$$(\nabla_s \underline{w}^e)^T = \underline{w}^{eT} \underline{B}^{eT}$$

after substitution ...

$$\underline{w}^T\left[\sum_{e=1}^{n_{el}} \underline{L}^{eT}\left(\underbrace{\int_{\Omega^e} \underline{B}^{eT}\underline{D}^e\underline{B}^e\, d\Omega}_{\text{ELEMENT STIFFNESS MATRIX}}\ \underline{L}^e\underline{d} - \underbrace{\int_{\Gamma_t^e}\underline{N}^{eT}\bar{t}\,d\Gamma + \int_{\Omega^e}\underline{N}^{eT}\underline{b}\,d\Omega}_{\text{ELEMENT EXTERNAL FORCE MATRIX}}\right)\right]=0 \qquad \forall w \text{ vanishing on } E$$

discretized weak form:

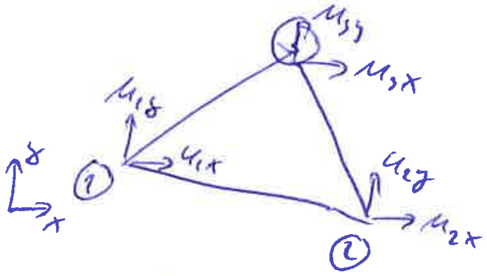
$$\underline{w}^T\left[\underbrace{\left(\sum_{e=1}^{n_{el}} \underline{L}^{eT}\underline{k}^e\underline{L}^e\right)\underline{d}} - \underbrace{\left(\sum_{e=1}^{n_{el}}\underline{L}^{eT}\underline{f}^e\right)}\right]=0$$

$$\underline{w}^T\left(\underline{K}\,\underline{d} - \underline{F}\right)=0$$

$$\boxed{\underline{K}\,\underline{d}=\underline{F}}$$

THREE-NODE TRIANGULAR ELEMENT

- linear displacements → constant strain
- nodes labeled counterclockwise



$$\underline{d}^e=\begin{bmatrix}u_{x1}^e\\ u_{y1}^e\\ u_{x2}^e\\ u_{y2}^e\\ u_{x3}^e\\ u_{y3}^e\end{bmatrix}$$

- displacement field in the element

a) $$\begin{bmatrix}u_x\\ u_y\end{bmatrix}^e=\begin{bmatrix}N_1^e & 0 & N_2^e & 0 & N_3^e & 0\\ 0 & N_1^e & 0 & N_2^e & 0 & N_3^e\end{bmatrix}\underline{d}^e$$

- strains

b) $$\begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{bmatrix}=\left\{\underbrace{\frac{1}{2A^e}\begin{bmatrix}y_{23}^e & 0 & y_{31}^e & 0 & y_{12}^e & 0\\ 0 & x_{32}^e & 0 & x_{13}^e & 0 & x_{21}^e\\ x_{32}^e & y_{23}^e & x_{13}^e & y_{31}^e & x_{21}^e & y_{12}^e\end{bmatrix}}_{\text{CONSTANT OVER ELEMENT}}\right.\begin{bmatrix}u_{x1}\\ u_{y1}\\ u_{x2}\\ u_{y2}\\ u_{x3}\\ u_{y3}\end{bmatrix} \qquad x_{ij}^e=x_i^e-x_j^e$$

- STIFFNESS MATRIX

for constant mat. prop. and unit thickness

$$5)\quad \underline{K}^e = \int_{\Omega^e} \underline{B}^{eT} \underline{D}^e \underline{B}^e \, d\Omega = \underbrace{A^e \cdot \underline{B}^{eT} \cdot \underline{D}^e \underline{B}^e}$$

6x6 MATRIX

EVALUATED BY FEM SOFTWARE

- ELEMENT BODY FORCE MATRIX

→ numerical integration
→ interpolation of b_x b_y a linear function

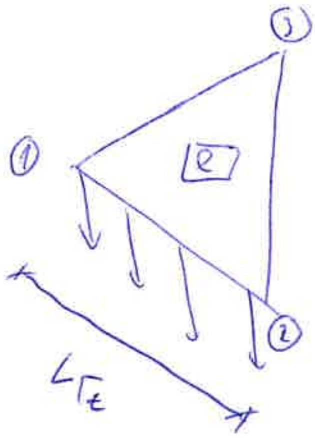
$$\underline{b} = \begin{bmatrix} b_x \\ b_y \end{bmatrix} = \sum_{\substack{i=1 \\ \text{node}}}^{3} (N_i)^T \begin{bmatrix} b_{xi} \\ b_{yi} \end{bmatrix}$$

$$\underline{F}_\Omega^e = \int_{\Omega^e} (\underline{N}^e)^T \underline{b} \, d\Omega =$$

$$= \int_{\Omega^e} \begin{bmatrix} N_1^{3T} & 0 \\ 0 & N_1^{3T} \\ N_2^{3T} & 0 \\ 0 & N_2^{3T} \\ N_3^{3T} & 0 \\ 0 & N_3^{3T} \end{bmatrix} \sum_{i=1}^{3} N_i^{3T} \begin{bmatrix} b_{xi} \\ b_{yi} \end{bmatrix} = \frac{A^e}{12} \begin{bmatrix} 2b_{x1} + b_{x2} + b_{x3} \\ 2b_{y1} + b_{y2} + b_{y3} \\ b_{x1} + 2b_{x2} + b_{x3} \\ b_{y1} + 2b_{y2} + b_{y3} \\ b_{x1} + b_{x2} + 2b_{x3} \\ b_{y1} + b_{y2} + 2b_{y3} \end{bmatrix}$$

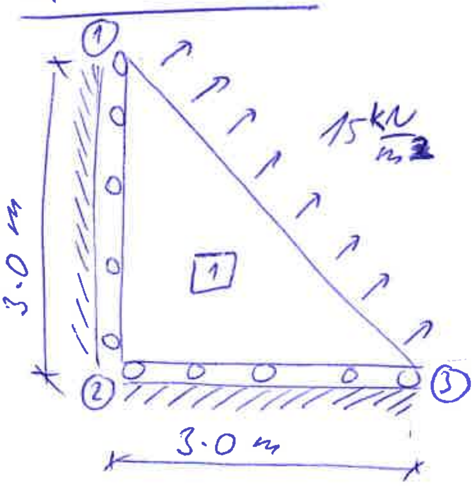
- BOUNDARY FORCE MATRIX

$$\underline{F}_\Gamma^e = \int_{\Gamma_t^e} \underline{N}^{eT} \bar{t} \, d\Gamma = \frac{L_{\Gamma_t}}{2} \begin{bmatrix} \bar{t}_x \\ \bar{t}_y \\ \bar{t}_x \\ \bar{t}_y \\ 0 \\ 0 \end{bmatrix}$$



$$\bar{t} = \begin{bmatrix} \bar{t}_x \\ \bar{t}_y \end{bmatrix}$$

EXAMPLE 1



$E = 300\ GPa$

$\nu = 0.3$

$t = 1\ m$

$u_{1x} = 0$

$u_{1y} = ?$

$u_{2x} = 0$

$u_{2y} = 0$

$u_{3x} = ?$

$u_{3y} = 0$

ELEMENT MATRICES

$A^{(1)} = \frac{3^2}{2} = 4.5\ m^2$

$x_1 = 0 \quad y_1 = 3$

$x_2 = 0 \quad y_2 = 0$

$x_3 = 3 \quad y_3 = 0$

$$\underline{B}^{(1)} = \frac{1}{9}\begin{bmatrix} 0 & 0 & -3 & 0 & 3 & 0 \\ 0 & 3 & 0 & -3 & 0 & 0 \\ 3 & 0 & -3 & -3 & 0 & 3 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & 0 & -\frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & 0 & -\frac{1}{3} & 0 & 0 \\ \frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix}$$

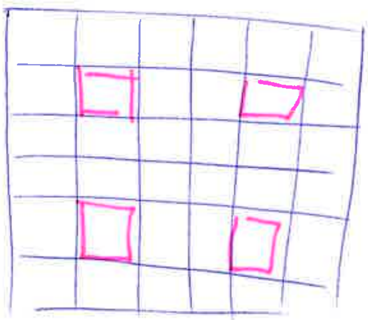
$$\underline{D}^{(1)} = \frac{300 \times 10^9}{1 - 0.09}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}$$

$$\underline{K}^{(1)} = A^{(1)}\, \underline{B}^{(1)T}\, \underline{D}^{(1)}\, \underline{B}^{(1)} =$$

$$= 10^9 \cdot \begin{bmatrix} 57.7 & 0 & -57.7 & -57.7 & 0 & 57.7 \\ & 164.8 & -49.5 & \begin{matrix} -164.8 \\ \cancel{-49.5} \end{matrix} & 49.5 & 0 \\ & & 222.5 & 107.1 & -164.8 & -57.7 \\ & \text{Sym.} & & 222.5 & -49.5 & -57.7 \\ & & & & 164.8 & 0 \\ & & & & & 57.7 \end{bmatrix}$$

$$L_t = \sqrt{3^2+3^2} = 4.2\ m$$

$$\underline{F}_{\Gamma_t}^{(1)} = \frac{4.2}{2}\begin{bmatrix} 15\cdot\cos(45) \\ 15\cdot\sin(45) \\ 0 \\ 0 \\ 15\cdot\cos(45) \\ 15\cdot\sin(45) \end{bmatrix}\times 10^3 = \begin{bmatrix} 22.5 \\ 22.5 \\ 0 \\ 0 \\ 22.5 \\ 22.5 \end{bmatrix}\times 10^3$$



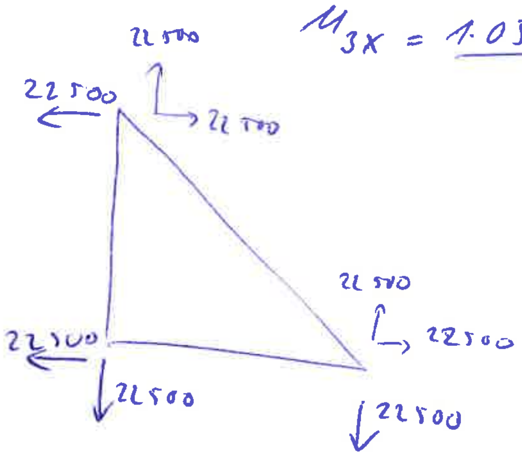
$$\underline{K}\ \underline{d}: \quad \begin{bmatrix} 0 \\ \boxed{u_{1y}} \\ 0 \\ 0 \\ \boxed{u_{3x}} \\ 0 \end{bmatrix} = \begin{bmatrix} 22.5 \\ \boxed{22.5} \\ 0 \\ 0 \\ \boxed{22.5} \\ 22.5 \end{bmatrix} + \begin{bmatrix} r_{1x} \\ 0 \\ r_{2x} \\ r_{2y} \\ 0 \\ r_{3x} \end{bmatrix}$$

$\underline{K}$ $\quad\underline{d}\quad$ $\underline{F}_{\Gamma_t}$

$$\rightarrow u_{1y} = \underline{1.05\cdot 10^{-7}}\ m$$

$$u_{3x} = \underline{1.05\cdot 10^{-7}}\ m$$

$$r_{1x} = -22\,500\ N$$
$$r_{2x} = -22\,500\ N$$
$$r_{2y} = -22\,500\ N$$
$$r_{3x} = -22\,500\ N$$



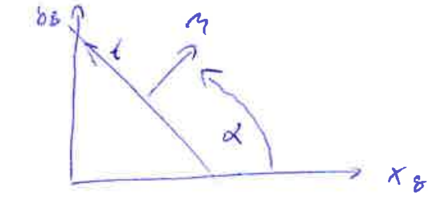
STRAIN

$$\underline{\varepsilon}^{(1)} = \underline{B}^{(1)}\underline{d}^{(1)} = \begin{bmatrix} 0 & 0 & -\frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & 0 & -\frac{1}{3} & 0 & 0 \\ \frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 0 \\ 1.05\times 10^{-7} \\ 0 \\ 0 \\ 1.05\times 10^{-7} \\ 0 \end{bmatrix} = \begin{bmatrix} 3.5\times 10^{-8} \\ 3.5\times 10^{-8} \\ 0 \end{bmatrix}$$

STRESS

$$\underline{\sigma}^{(1)} = \underline{D}^{(1)}\underline{\varepsilon}^{(1)} = \frac{300\times 10^9}{1-0.09}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix} * \begin{bmatrix} 3.5\times 10^{-8} \\ 3.5\times 10^{-8} \\ 0 \end{bmatrix} = \begin{bmatrix} 15\,000 \\ 15\,000 \\ 0 \end{bmatrix} Pa$$

STRESS ON EDGE ①-⑤



$$\overset{V_R}{\begin{bmatrix} V_n \\ V_t \end{bmatrix}} = \overset{R}{\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}} \overset{V_G}{\begin{bmatrix} V_x \\ V_y \end{bmatrix}}$$

stress - second order tensor

tensor product $V_R \times V_R \Leftrightarrow V_R V_R^T = (R \cdot V_G)(R V_G)^T = R \overset{\text{second order tensor}}{V_G V_G^T} R^T$

$$\begin{bmatrix} \sigma_{nn} & \sigma_{nt} \\ \sigma_{tn} & \sigma_{tt} \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \cdot \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} =$$

$$= \begin{bmatrix} 0.71 & 0.71 \\ -0.71 & 0.71 \end{bmatrix} \begin{bmatrix} 15000 & 0 \\ 0 & 15000 \end{bmatrix} \begin{bmatrix} 0.71 & -0.71 \\ 0.71 & 0.71 \end{bmatrix} = \begin{bmatrix} \overset{\sigma_{nn}}{15000} & 0 \\ 0 & \underset{\sigma_{tt}}{15000} \end{bmatrix}$$