Topology optimization III: Continuum topology optimization I

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Finite elements overview

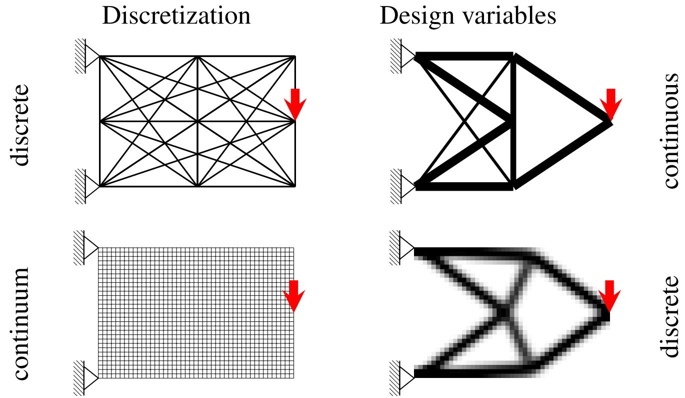
$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\text{displacements } \mathbf{u} \arrow[r, "\mathbf{f} = \sum_{i=1}^{n_\mathrm{e}} \left[\int_{\Omega_i} \mathbf{B}_i^\mathrm{T} \mathbf{D}_i(\rho_i) \mathbf{B}_i \mathrm{d}\Omega_i\right] \mathbf{u} = \mathbf{K}(\boldsymbol{\rho})\mathbf{u}", "\text{equilibrium}"'] \arrow[d, "\boldsymbol{\varepsilon} = \mathbf{B}\mathbf{u}"', "\text{geometric}"] & \text{body forces } \mathbf{b} \\
\text{strains } \boldsymbol{\varepsilon} \arrow[r, "\text{physical}", "\boldsymbol{\sigma} = \mathbf{D}(\boldsymbol{\rho})\boldsymbol{\varepsilon}"'] & \text{stresses } \boldsymbol{\sigma} \arrow[u, "\text{static}", "-\mathbf{b} = \mathbf{B}^\mathrm{T}\boldsymbol{\sigma}"']
\end{tikzcd}
$$

- Material stiffness matrix (plane stress)

$$
\mathbf{D}_i(\rho_i) = \frac{E(\rho_i)}{1-\nu^2} \begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{pmatrix}
$$

Continuum topology optimization

- Aim: the most efficient distribution of mass for a given volume
- Discretized continuum (FEM), optimizing (vanishing) properties of elements



- Find optimal placement of predefined (isotropic) material
- Representation similar to monochrome raster image: material (black), void (white)
- Thus, we search binary design field $\boldsymbol{\rho} \in \mathbb{B}^{n_\mathrm{e}}$. For compliance minimization, we have

$$\begin{aligned} \min_{\boldsymbol{\rho}\in\mathbb{B}^{n_\mathrm{e}},\mathbf{u}\in\mathbb{R}^{n_\mathrm{dof}}} \quad & \mathbf{f}^\mathrm{T}\mathbf{u} \\ \text{subject to} \quad & \mathbf{K}(\boldsymbol{\rho})\mathbf{u} = \mathbf{f} \\ & V(\boldsymbol{\rho}) \leq \overline{V} \end{aligned}$$

- Global solution to (large-scale) integer programs remains challenging[1]
- Commonly, the integrality requirement is alleviated by linear relaxation $\mathbf{0} \leq \boldsymbol{\rho} \leq \mathbf{1}$, but intermediate values are penalized at the same time $\Rightarrow$ loss of convexity of the relaxation

[1] Bilevel knapsack problem: D. Y. Gao, On topology optimization and canonical duality method, *Computer Methods in Applied Mechanics and Engineering*, 341:249–277, 2018, doi: 10.1016/j.cma.2018.06.027

- Proportional stiffness model Simple/Simplified/Solid Isotropic Material with Penalization (SIMP)

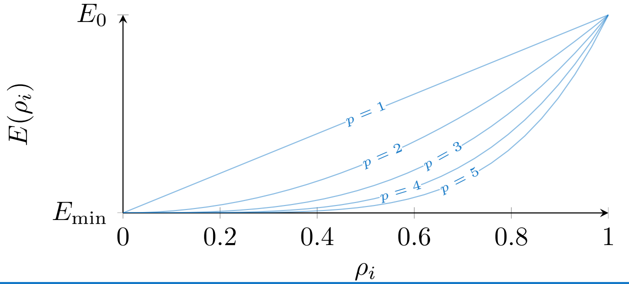
$$E(\rho_i) = \rho_i^p E_0, \text{ where } p \geq 1, \rho_i \in \left[\rho_{\min}, 1\right], \rho_{\min} \in \mathbb{R}_{>0} \Rightarrow E_{\min} = E_0 \rho_{\min}^p$$

- Modified SIMP: $\min_{\rho_i \in \mathbb{R}}(E(\rho_i))$ independent of p

$$E(\rho_i) = E_{\min} + (E_0 - E_{\min})\rho_i^p, \text{ where } \rho \in [0, 1]$$

- Rational Approximation of Material Properties (RAMP)

$$E(\rho_i) = E_{\min} + \frac{\rho_i}{1 + q(1 - \rho_i)}(E_0 - E_{\min}), \text{ where } \rho \in [0, 1]$$



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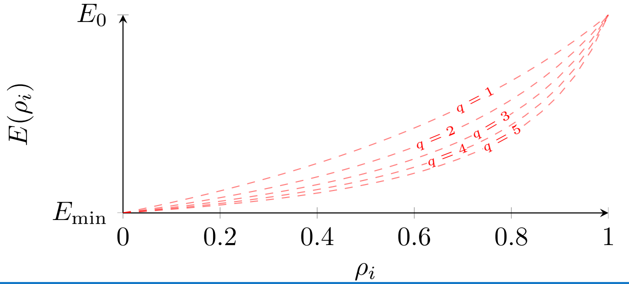

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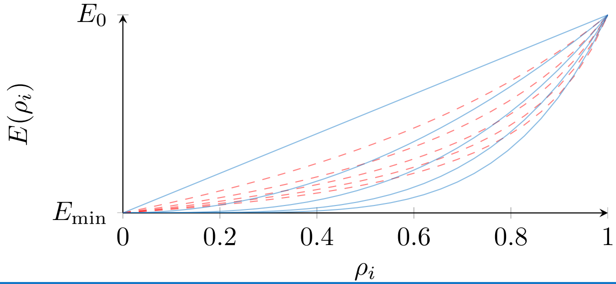
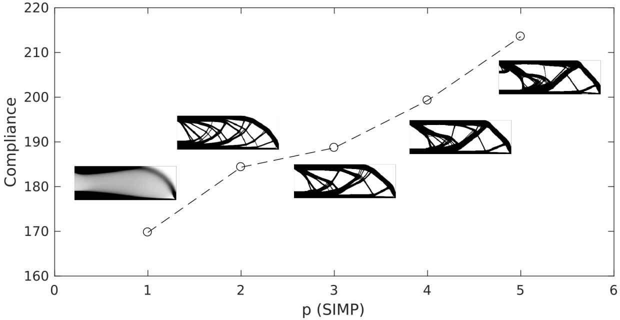


Material interpolation

- Using (modified) SIMP:
  - $p = 1$: the problem is convex (recall truss lecture) and equivalent to a variable-thickness-sheet (VTS) problem
  - p "low": grayscale regions
  - p "high": monochrome but increased number of local minimizers
  - $p = 3$: "magic number", realization via composites[2]



[2] M. P. Bendsøe and O. Sigmund, Material interpolation schemes in topology optimization, *Archive of Applied Mechanics (Ingenieur Archiv)*, 69(9-10):635–654, 1999, doi: 10.1007/s004190050248

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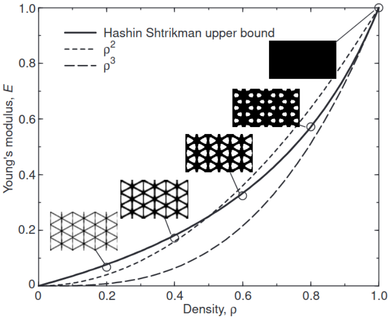


[2] M. P. Bendsøe and O. Sigmund, Material interpolation schemes in topology optimization, *Archive of Applied Mechanics (Ingenieur Archiv)*, 69(9-10):635–654, 1999, doi: 10.1007/s004190050248

Sensitivity of the compliance function

- Recall that

$$c(\boldsymbol{\rho}) = \mathbf{f}^{\mathrm{T}}\mathbf{u}, \quad \text{where } \mathbf{K}(\boldsymbol{\rho})\mathbf{u} = \mathbf{f}$$

- Since the equilibrium equation must be satisfied, it must hold for any but fixed $\boldsymbol{\lambda} \in \mathbb{R}^{n_{\mathrm{dof}}}$ that

$$c(\boldsymbol{\rho}) = \mathbf{f}^{\mathrm{T}}\mathbf{u} + \boldsymbol{\lambda}^{\mathrm{T}}\left(\mathbf{K}(\boldsymbol{\rho})\mathbf{u} - \mathbf{f}\right)$$

- Using the chain rule, we receive

$$\begin{aligned}
\frac{\partial c}{\partial \rho_i} &= \mathbf{f}^{\mathrm{T}}\frac{\partial \mathbf{u}}{\partial \rho_i} + \boldsymbol{\lambda}^{\mathrm{T}}\left(\frac{\partial \mathbf{K}(\boldsymbol{\rho})}{\partial \rho_i}\mathbf{u} + \mathbf{K}(\boldsymbol{\rho})\frac{\partial \mathbf{u}}{\partial \rho_i}\right) \\
&= \left(\mathbf{f}^{\mathrm{T}} + \boldsymbol{\lambda}^{\mathrm{T}}\mathbf{K}(\boldsymbol{\rho})\right)\frac{\partial \mathbf{u}}{\partial \rho_i} + \boldsymbol{\lambda}^{\mathrm{T}}\frac{\partial \mathbf{K}(\boldsymbol{\rho})}{\partial \rho_i}\mathbf{u}
\end{aligned}$$

- Select $\boldsymbol{\lambda} = -\mathbf{u}$

$$\frac{\partial c}{\partial \rho_i} = -\mathbf{u}^{\mathrm{T}}\frac{\partial \mathbf{K}(\boldsymbol{\rho})}{\partial \rho_i}\mathbf{u}$$

- **HW**: Derive sensitivities for problems with self-weight, $\mathbf{f}(\boldsymbol{\rho})$

- We have a sizing problem, so $\forall \boldsymbol{\rho} \in [0,1]^{n_\mathrm{e}} : \mathbf{K}(\boldsymbol{\rho}) \succ 0$. Therefore,

$$c(\boldsymbol{\rho}) = \mathbf{f}^\mathrm{T}\mathbf{K}(\boldsymbol{\rho})^{-1}\mathbf{f}$$

- Only $\mathbf{K}(\boldsymbol{\rho})^{-1}$ is a function of $\boldsymbol{\rho}$, thus we need $\frac{\partial \mathbf{K}(\boldsymbol{\rho})^{-1}}{\partial \rho_i}$

$$\mathbf{I} = \mathbf{K}(\boldsymbol{\rho})^{-1}\mathbf{K}(\boldsymbol{\rho})$$

- Using the chain rule,

$$\mathbf{0} = \frac{\partial \mathbf{K}(\boldsymbol{\rho})^{-1}}{\partial \rho_i}\mathbf{K}(\boldsymbol{\rho}) + \mathbf{K}(\boldsymbol{\rho})^{-1}\frac{\partial \mathbf{K}(\boldsymbol{\rho})}{\partial \rho_i}$$

$$\frac{\partial \mathbf{K}(\boldsymbol{\rho})^{-1}}{\partial \rho_i} = -\mathbf{K}(\boldsymbol{\rho})^{-1}\frac{\partial \mathbf{K}(\boldsymbol{\rho})}{\partial \rho_i}\mathbf{K}(\boldsymbol{\rho})^{-1}$$

- Inserting back, we receive

$$\frac{\partial c(\boldsymbol{\rho})}{\partial \rho_i} = \mathbf{f}^\mathrm{T}\frac{\partial \mathbf{K}(\boldsymbol{\rho})^{-1}}{\partial \rho_i}\mathbf{f} = -\mathbf{f}^\mathrm{T}\mathbf{K}(\boldsymbol{\rho})^{-1}\frac{\partial \mathbf{K}(\boldsymbol{\rho})}{\partial \rho_i}\mathbf{K}(\boldsymbol{\rho})^{-1}\mathbf{f} = -\mathbf{u}^\mathrm{T}\frac{\partial \mathbf{K}(\boldsymbol{\rho})}{\partial \rho_i}\mathbf{u}$$

- How does material interpolation enter the sensitivities?
- For modified SIMP, we have

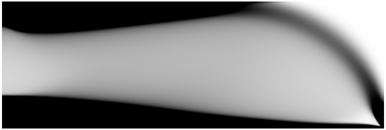
$$\mathbf{K}_i(\rho_i) = \left(E_{\min} + \left[E_0 - E_{\min}\right]\rho_i^p\right)\mathbf{K}_{0,i}$$
$$\frac{\partial \mathbf{K}(\rho_i)}{\partial \rho_i} = p\left(E_0 - E_{\min}\right)\rho_i^{p-1}\mathbf{K}_{0,i}$$

- For RAMP, we obtain

$$\mathbf{K}_i(\rho_i) = E_{\min} + \frac{\rho_i(E_0 - E_{\min})}{1 + q(1-\rho_i)}\mathbf{K}_{0,i}$$
$$\frac{\partial \mathbf{K}(\rho_i)}{\partial \rho_i} = \frac{\left(E_0 - E_{\min}\right)(1+q)}{\left[1 + q(1-\rho_i)\right]^2}\mathbf{K}_{0,i}$$

Filters

- Intermediate densities → reduced by penalization



- Checkerboard patterns: non-physical stiffness oscillation, relation to locking (also present in VTS problems!)

- Mesh-dependency: finer discretization provides refined structural details (micro-perforated material), no length-scale control

- Existence of solutions? → We need regularization

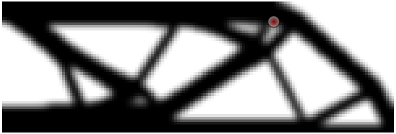
Density filter

- Regularization of the density field by a filter with radius r

$$\tilde{\rho}_i = \frac{\sum_{j=1}^{n_\mathrm{e}} w(\mathbf{x}_j) v_j \rho_j}{\sum_{j=1}^{n_\mathrm{e}} w(\mathbf{x}_j) v_j}, \text{ where } w_j(\mathbf{x}_j) = \max\left\{r - \|\mathbf{x}_j - \mathbf{x}_i\|_2, 0\right\}$$

$$\frac{\partial f}{\partial \rho_i} = \sum_{j=1}^{n_\mathrm{e}} \frac{\partial f}{\partial \tilde{\rho}_j} \frac{\partial \tilde{\rho}_j}{\partial \rho_j}, \text{ where } \frac{\partial \tilde{\rho}_j}{\partial \rho_j} = \frac{w(\mathbf{x}_j) v_j}{\sum_{k=1}^{n_\mathrm{e}} w(\mathbf{x}_k) v_k}$$

- Existence of solutions[3]
- Drawback: blurred design (we will deal with this in the next lecture)



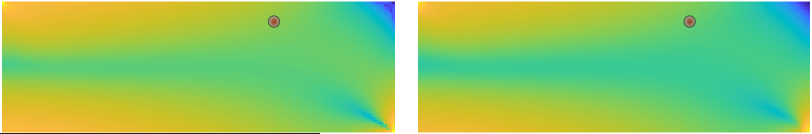
[3]B. Bourdin, Filters in topology optimization, *International Journal for Numerical Methods in Engineering*, 50(9):2143–2158, 2001, doi: 10.1002/nme.116

Sensitivity filter

- Modify the sensitivity field only $\rightarrow$ which objective function do we minimize? Is the gradient direction descent?

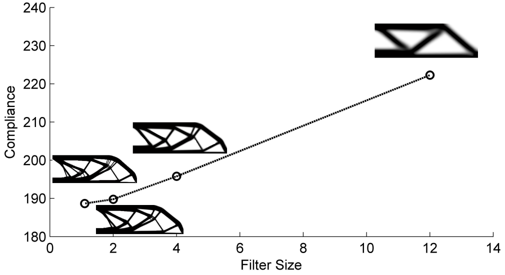
$$\frac{\partial \tilde{f}}{\partial \rho_i} = \frac{\sum_{i=1}^{n_\mathrm{e}} w(\mathbf{x}_i) \rho_i \frac{\partial f}{\partial \rho_i}}{\max\{\rho_i, \varepsilon\} \sum_{i=1}^{n_\mathrm{e}} w(\mathbf{x}_i)}$$

- Sensitivity filter can be seen in the perspective of non-local elasticity problems in continuum mechanics[4]
- No direct length-scale control
- Compliance sensitivities (absolute value in logarithmic scale)

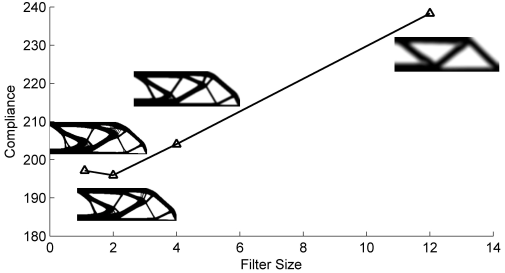


[4] O. Sigmund and K. Maute, Sensitivity filtering from a continuum mechanics perspective, *Structural and Multidisciplinary Optimization*, 46(4):471–475, 2012, doi: 10.1007/s00158-012-0814-4

Sensitivity filter



Density filter



Optimization algorithms

1. Initialize problem
   - FEM problem: discretization, boundary conditions, material properties
   - Optimization problem: objective, constraints
   - Prepare filter
2. Starting point $\boldsymbol{\rho}$ (e.g., $\boldsymbol{\rho} = \mathbf{1}\frac{\overline{V}}{\sum_{i=1}^{n_\mathrm{e}} v_i}$)
3. While not converged (usually $\|\boldsymbol{\rho}\|_\infty < 0.01$) or exceeded iteration limit
   - Solve the state problem ✓
   - Evaluate the sensitivities ✓
   - Filtering ✓
   - (Projections—next lecture)
   - Update the design variables

Optimality criteria method

- Formulation

$$\min_{\boldsymbol{\rho}\in\mathbb{R}^{n_\mathrm{e}},\mathbf{u}\in\mathbb{R}^{n_\mathrm{dof}}} \mathbf{f}^\mathrm{T}\mathbf{u}$$

$$\text{subject to } \mathbf{K}(\boldsymbol{\rho})\mathbf{u} = \mathbf{f}$$

$$\sum_{i=1}^{n_\mathrm{e}} v_i\rho_i \leq \overline{V}$$

$$\mathbf{0} \leq \boldsymbol{\rho} \leq \mathbf{1}$$

- Extension of the idea for trusses/frames
- Lagrangian function:

$$\mathcal{L}(\boldsymbol{\rho},\mathbf{u},\boldsymbol{\lambda},\mu,\boldsymbol{\nu},\boldsymbol{\psi}) = \mathbf{f}^\mathrm{T}\mathbf{u} + \boldsymbol{\lambda}^\mathrm{T}\left(\mathbf{K}(\boldsymbol{\rho})\mathbf{u} - \mathbf{f}\right) + \mu\left(\mathbf{v}^\mathrm{T}\boldsymbol{\rho} - \overline{V}\right) + \boldsymbol{\nu}^\mathrm{T}\left(-\boldsymbol{\rho}\right) + \boldsymbol{\psi}^\mathrm{T}\left(\boldsymbol{\rho} - \mathbf{1}\right)$$

- Karush-Kuhn-Tucker conditions:

$$\begin{aligned}
\text{dual feas.} \quad & 0 \leq \mu^*, \mathbf{0} \leq \boldsymbol{\nu}^*, \mathbf{0} \leq \boldsymbol{\psi}^* \\
\text{compl. slack.} \quad & 0 = \mu^* \left(\mathbf{v}^{\mathrm{T}} \boldsymbol{\rho}^* - \overline{V} \right) \\
& \mathbf{0} = \left(\boldsymbol{\nu}^* \right)^{\mathrm{T}} \left(-\boldsymbol{\rho}^* \right) \\
& \mathbf{0} = \left(\boldsymbol{\psi}^* \right)^{\mathrm{T}} \left(\boldsymbol{\rho}^* - \mathbf{1} \right) \\
\text{primal feas.} \quad & \mathbf{f} = \mathbf{K}(\boldsymbol{\rho}^*)\mathbf{u}^*, \sum_{i=1}^{n_\mathrm{e}} v_i \rho_i \leq \overline{V}, \mathbf{0} \leq \boldsymbol{\rho} \leq \mathbf{1} \\
\text{stationarity} \quad & \frac{\partial \mathcal{L}}{\partial \mathbf{u}} = \mathbf{f}^{\mathrm{T}} + \left(\boldsymbol{\lambda}^* \right)^{\mathrm{T}} \mathbf{K}(\boldsymbol{\rho}^*) = \mathbf{0} \rightarrow \boldsymbol{\lambda}^* = -\mathbf{u} \\
& \frac{\partial \mathcal{L}}{\partial \rho_i} = \left(\boldsymbol{\lambda}^* \right)^{\mathrm{T}} \frac{\partial \mathbf{K}(\boldsymbol{\rho}^*)}{\partial \rho_i} \mathbf{u}^* + \mu^* v_i - \nu_i^* + \psi_i^* \\
& \qquad = -\left(\mathbf{u}^* \right)^{\mathrm{T}} \frac{\partial \mathbf{K}(\boldsymbol{\rho}^*)}{\partial \rho_i} \mathbf{u}^* + \mu^* v_i - \nu_i^* + \psi_i^* = 0
\end{aligned}$$

- For $\mathbf{0} < \boldsymbol{\rho}^* < \mathbf{1}$, KKT conditions imply $\boldsymbol{\nu}^* = \boldsymbol{\psi}^* = \mathbf{0}$
- Consequently, the elements $\mathbf{0} < \boldsymbol{\rho}^* < \mathbf{1}$ have equal energy

$$\mu^* = \frac{1}{v_i} \left(\mathbf{u}^* \right)^{\mathrm{T}} \frac{\partial \mathbf{K}(\boldsymbol{\rho}^*)}{\partial \rho_i} \mathbf{u}^*$$

- Let

$$b_i = \frac{(\mathbf{u})^{\mathrm{T}} \frac{\partial \mathbf{K}(\boldsymbol{\rho})}{\rho_i} \mathbf{u}}{v_i \mu} = \frac{-\frac{\partial c}{\partial \rho_i}}{\mu \frac{\partial V}{\partial \rho_i}}$$

- OC method: update scheme that balances μ^* for all the elements
- Extension from trusses/frames: move limit m, upper bounds $\mathbf{1}$, damping coefficient η
  1. Bisection to find $\mu^{(k)}$ such that

$$\overline{V} = \sum_{i=1}^{n_\mathrm{e}} v_i \max\left\{0, \max\left\{\rho_i^{(k-1)} - m, \min\left\{1, \min\left\{\rho_i^{(k-1)} + m, \rho_i^{(k-1)} \left(b_i^{(k)}\right)^{\eta}\right\}\right\}\right\}\right\}$$

  2. Update step

$$\rho_i^{(k+1)} = \begin{cases} \max\left\{0, \rho_i^{(k)} - m\right\} & \text{if } \rho_i^{(k)} \left[b_i^{(k)}\right]^{\eta} \leq \max\left\{0, \rho_i^{(k)} - m\right\} \\ \min\left\{1, \rho_i^{(k)} + m\right\} & \text{if } \rho_i^{(k)} \left[b_i^{(k)}\right]^{\eta} \geq \min\left\{1, \rho_i^{(k)} + m\right\} \\ \rho_i^{(k)} \left[b_i^{(k)}\right]^{\eta} & \text{otherwise} \end{cases}$$

Method of moving asymptotes

- Simple and versatile method used regularly in topology optimization
- Idea: convex separable approximation at current iteration, efficient solution to a dual subproblem (size depends on the number of constraints)
- Optimization problem

$$P: \min_{\mathbf{x}\in\mathbb{R}^n} \; f_0(\mathbf{x})$$
$$\text{subject to } f_i(\mathbf{x}) \leq \hat{f}_i, \; \forall i \in \{1, \ldots, m\}$$

- General procedure
  0. Starting point $\mathbf{x}^{(0)}$, iteration $k = 0$
  1. Compute $f_i(\mathbf{x}^{(k)})$ and $\nabla f_i(\mathbf{x}^{(k)})$, $\forall i \in \{0, \ldots, m\}$
  2. Generate subproblem $P^{(k)}$ that approximates P at $\mathbf{x}^{(k)}$
  3. Solve $P^{(k)}$. Set $\mathbf{x}^{(k+1)}$ and $k = k + 1$
  4. Go to 1. if not converged

- Approximate the functions $f_i(\mathbf{x})$ via vertical asymptotes $L_j^{(k)} < x_j < U_j^{(k)}$ as

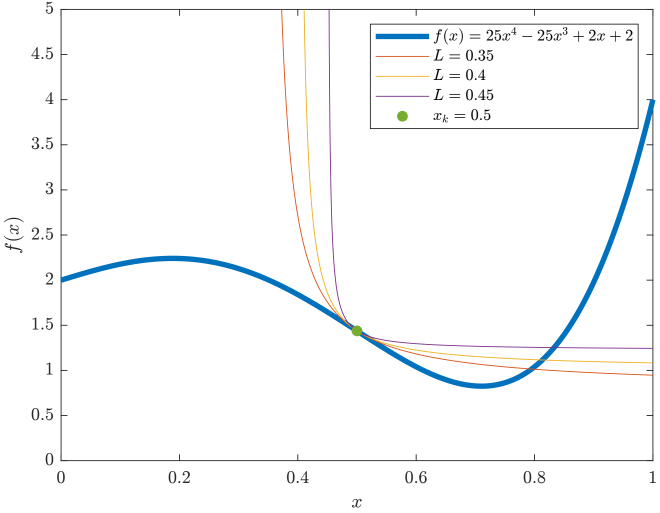
$$f_i^{(k)}(\mathbf{x}) = r_i^{(k)} + \sum_{j=1}^{n} \left(\frac{p_{ij}^{(k)}}{U_j^{(k)} - x_j} + \frac{q_{ij}^{(k)}}{x_j - L_j^{(k)}} \right)$$

where

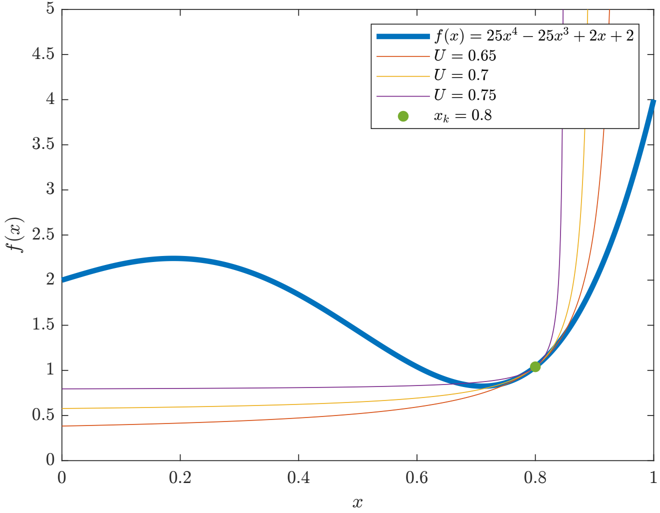
$$p_{ij}^{(k)} = \begin{cases} \left(U_j^{(k)} - x_j^{(k)}\right)^2 \frac{\partial f_i}{\partial x_j}(\mathbf{x}^{(k)}) & \text{if } \frac{\partial f_i}{\partial x_j} > 0 \\ 0 & \text{if } \frac{\partial f_i}{\partial x_j} \leq 0 \end{cases}$$

$$q_{ij}^{(k)} = \begin{cases} 0 & \text{if } \frac{\partial f_i}{\partial x_j} \geq 0 \\ -\left(x_j(k) - L_j^{(k)}\right)^2 \frac{\partial f_i}{\partial x_j}(\mathbf{x}^{(k)}) & \text{if } \frac{\partial f_i}{\partial x_j} < 0 \end{cases}$$

$$r_i^{(k)} = f_i(\mathbf{x}^{(k)}) - \sum_{j=1}^{n} \left(\frac{p_{ij}^{(k)}}{U_j^{(k)} - x_j^{(k)}} + \frac{q_{ij}^{(k)}}{x_j^{(k)} - L_j^{(k)}} \right)$$



$f(x) = 25x^4 - 25x^3 + 2x + 2$
$L = 0.35$
$L = 0.4$
$L = 0.45$
$x_k = 0.5$
$f(x)$
x
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
0.2
0.4
0.6
0.8



$f(x) = 25x^4 - 25x^3 + 2x + 2$
$U = 0.65$
$U = 0.7$
$U = 0.75$
$x_k = 0.8$
$f(x)$
x
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
0
0.2
0.4
0.6
0.8
1

- Clearly, $\forall i \in \{0, \dots, m\} : f_i^{(k)}(\mathbf{x}^{(k)}) = f_i(\mathbf{x}^{(k)})$ and $\frac{\partial f_i^{(k)}}{\partial x_j} = \frac{\partial f_i}{\partial x_j}$ at $\mathbf{x}^{(k)} \rightarrow$ first-order approximation
- Positive (semi)definite Hessian matrix $\rightarrow$ (strictly) convex function

$$\frac{\partial^2 f_i^{(k)}}{\partial x_j^2} = \frac{2p_{ij}^{(k)}}{\left(U_j^{(k)} - x_j\right)^3} + \frac{2q_{ij}^{(k)}}{\left(x_j - L_j^{(k)}\right)^3}$$

$$\frac{\partial^2 f_i^{(k)}}{\partial x_j x_\ell} = 0, \text{ where } j \neq \ell$$

- Subproblem

$$\min \sum_{j=1}^{n} \left(\frac{p_{0j}^{(k)}}{U_j^{(k)} - x_j} + \frac{q_{0j}^{(k)}}{x_j - L_j^{(k)}} \right) + r_0^{(k)}$$

$$\text{subject to} \sum_{j=1}^{n} \left(\frac{p_{ij}^{(k)}}{U_j^{(k)} - x_j} + \frac{q_{ij}^{(k)}}{x_j - L_j^{(k)}} \right) \leq \hat{f}_i - r_i^{(k)}, \quad \forall i \in \{1, \dots, m\}$$

$$L_j^{(k)} < \max\{\underline{x}_j, \alpha_j^{(k)}\} \leq x_j \leq \min\{\overline{x}_j, \beta_j^{(k)}\} < U_j^{(k)}, \forall j \in \{1, \dots, n\}$$

- How to set the asymptotes?
  - Oscillating problem → stabilizing by tightening the asymptotes
  - Slow convergence → relax by weakening asymptotes
- Define $b_i = \hat{f}_i - r_i$ and $\mathbf{b} = (b_1, \ldots, b_m)^{\mathrm{T}}$, $\mathbf{p}_j = (p_{1j}, \ldots, p_{mj})^{\mathrm{T}}$, $\mathbf{q}_j = (q_{1j}, \ldots, q_{mj})^{\mathrm{T}}$
- Dual problem

$$\max_{\mathbf{y}} r_0 - \mathbf{y}^{\mathrm{T}}\mathbf{b} + \sum_{j=1}^{n} \left(\frac{p_{0j} + \mathbf{y}^{\mathrm{T}}\mathbf{p}_j}{U_j - x_j(\mathbf{y})} + \frac{q_{0j} + \mathbf{y}^{\mathrm{T}}\mathbf{q}_j}{x_j(\mathbf{y}) - L_j} \right)$$

subject to $\mathbf{y} \geq \mathbf{0}$

where

$$x_j(\mathbf{y}) = \frac{\sqrt{p_{0j} + \mathbf{y}^{\mathrm{T}}\mathbf{p}_j}\, L_j + \sqrt{q_{0j} + \mathbf{y}^{\mathrm{T}}\mathbf{q}_j}\, U_j}{\sqrt{p_{0j} + \mathbf{y}^{\mathrm{T}}\mathbf{p}_j} + \sqrt{q_{0j} + \mathbf{y}^{\mathrm{T}}\mathbf{q}_j}}$$

- Small smooth concave problem solvable by gradient methods

- For compliance minimization, we have the subproblems ($\boldsymbol{\nabla} c \leq \mathbf{0}$)

$$\min_{\boldsymbol{\rho}\in\mathbb{R}^{n_\mathrm{e}}} \; c(\boldsymbol{\rho})^{(k)} - \sum_{i=1}^{n_\mathrm{e}} \frac{(\rho_i^{(k)} - L_i)^2}{\rho_i - L_i} \frac{\partial c}{\partial \rho_i}(\boldsymbol{\rho}^{(k)}) \tag{2a}$$

$$\text{s.t. } \mathbf{v}^\mathrm{T}\boldsymbol{\rho} \leq \overline{V} \tag{2b}$$

$$\rho_{\min}\mathbf{1} \leq \boldsymbol{\rho} \leq \mathbf{1} \tag{2c}$$

- Lagrangian function for $\rho_{\min}\mathbf{1} \leq \boldsymbol{\rho} \leq \mathbf{1}$

$$\mathcal{L}(\boldsymbol{\rho}, \mu) = c(\boldsymbol{\rho})^{(k)} - \sum_{i=1}^{n_\mathrm{e}} \frac{(\rho_i^{(k)} - L_i)^2}{\rho_i - L_i} \frac{\partial c}{\partial \rho_i}(\boldsymbol{\rho}^{(k)}) + \mu(\mathbf{v}^\mathrm{T}\boldsymbol{\rho} - \overline{V})$$

- For compliance minimization, we have the subproblems ($\boldsymbol{\nabla} c \leq \mathbf{0}$)

$$\min_{\boldsymbol{\rho}\in\mathbb{R}^{n_\mathrm{e}}} c(\boldsymbol{\rho})^{(k)} - \sum_{i=1}^{n_\mathrm{e}} \frac{(\rho_i^{(k)} - L_i)^2}{\rho_i - L_i} \frac{\partial c}{\partial \rho_i}(\boldsymbol{\rho}^{(k)}) \tag{2a}$$

$$\text{s.t. } \mathbf{v}^\mathrm{T}\boldsymbol{\rho} \leq \overline{V} \tag{2b}$$

$$\rho_{\min}\mathbf{1} \leq \boldsymbol{\rho} \leq \mathbf{1} \tag{2c}$$

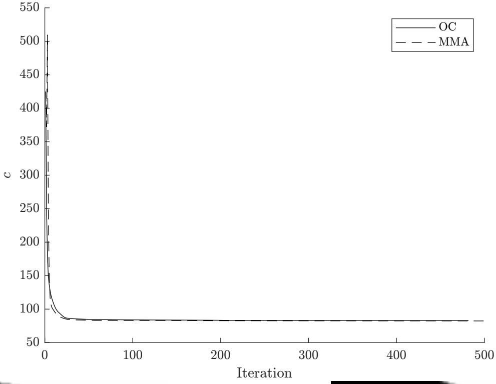
- Lagrangian function for $\rho_{\min}\mathbf{1} \leq \boldsymbol{\rho} \leq \mathbf{1}$ and $L_i = 0 \Rightarrow$ OC with $\eta = 0.5$ and $m = \infty$

$$\mathcal{L}(\boldsymbol{\rho}, \mu) = c(\boldsymbol{\rho})^{(k)} - \sum_{i=1}^{n_\mathrm{e}} \frac{\left(\rho_i^{(k)}\right)^2}{\rho_i} \frac{\partial c}{\partial \rho_i}(\boldsymbol{\rho}^{(k)}) + \mu(\mathbf{v}^\mathrm{T}\boldsymbol{\rho} - \overline{V})$$

- Stationarity of $\mathcal{L}(\boldsymbol{\rho}^*, \mu^*)$:

$$\frac{\mathcal{L}(\boldsymbol{\rho}^*, \mu^*)}{\rho_i} = \mu^* v_i + \frac{\partial c}{\partial \rho_i}(\boldsymbol{\rho}^{(k)}) \frac{\left(\rho_i^{(k)}\right)^2}{\rho_i^2} = 0 \Leftrightarrow \rho_i^* = \rho_i^{(k)} \overbrace{\sqrt{\frac{-\frac{\partial c}{\partial \rho_i}}{\mu^* v_i}}}^{\left(b_i^{(k)}\right)^\eta}$$

- Dual problem: $\max_{\mu\in\mathbb{R}_{\geq 0}} \mathcal{L}(\mu)$ such that $\boldsymbol{\rho}^{(k+1)}$ satisfies the volume constraint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Iteration
OC
MMA

Summary

- Material interpolation schemes for penalizing grayscale designs
- Regularization for mesh independence and solution existence
- Computation of sensitivities by adjoint/direct method
- Steps of topology optimization
  - Solution of a state problem
  - Evaluation of sensitivities
  - Filtering
  - Update step (OC, MMA)
- More in the next lecture

References

- M. P. Bendsøe and O. Sigmund, *Topology Optimization*. Springer Berlin Heidelberg, 2004, doi: 10.1007/978-3-662-05086-6