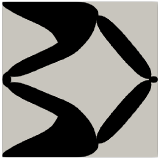
Topology optimization IV:
Continuum topology optimization II

Department of Mechanics
Faculty of Civil Engineering
Czech Technical University in Prague

Marek Tyburec

Design of compliant mechanisms

- Mobility gained from structural flexibility
- No need for body components such as hinges or sliders



- Basic formulation to minimize the (negative value of) output displacement

$$\begin{aligned} \min_{\boldsymbol{\rho}} \; & \mathbf{l}_{\text{out}}^{\mathrm{T}} \mathbf{u} \\ \text{subject to } & \mathbf{K}(\boldsymbol{\rho})\mathbf{u} = \mathbf{f} \\ & \mathbf{v}^{\mathrm{T}} \boldsymbol{\rho} \le \overline{V} \\ & \mathbf{0} \le \boldsymbol{\rho} \le \mathbf{1} \end{aligned}$$

- Is the formulation convex? (hint: write Hessian)

- Slight generalization of compliance minimization problems:

$$d(\boldsymbol{\rho}) = \mathbf{I}_{\text{out}}^{\text{T}} \mathbf{u}, \text{ where } \mathbf{K}(\boldsymbol{\rho})\mathbf{u} = \mathbf{f}$$

- What are the sensitivities of $d(\boldsymbol{\rho})$?
- Since the equilibrium equation must be satisfied, it must hold for any but fixed $\boldsymbol{\lambda} \in \mathbb{R}^{n_{\text{dof}}}$ that

$$d(\boldsymbol{\rho}) = \mathbf{I}_{\text{out}}^{\text{T}} \mathbf{u} + \boldsymbol{\lambda}^{\text{T}} \left(\mathbf{K}(\boldsymbol{\rho})\mathbf{u} - \mathbf{f}\right)$$

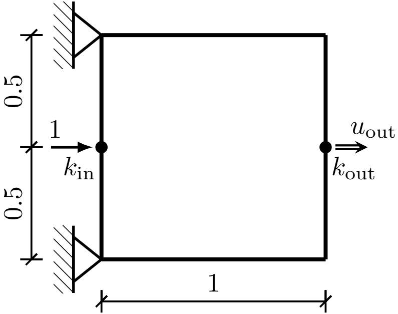
- Using the chain rule, we receive

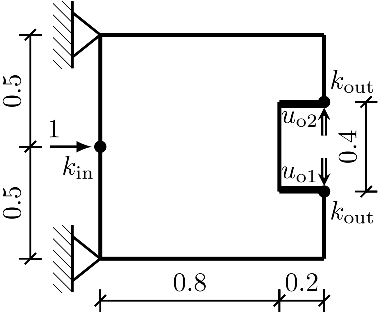
$$\begin{aligned} \frac{\partial d}{\partial \rho_i} &= \mathbf{I}_{\text{out}}^{\text{T}} \frac{\partial \mathbf{u}}{\partial \rho_i} + \boldsymbol{\lambda}^{\text{T}} \left(\frac{\partial \mathbf{K}(\boldsymbol{\rho})}{\partial \rho_i} \mathbf{u} + \mathbf{K}(\boldsymbol{\rho}) \frac{\partial \mathbf{u}}{\partial \rho_i} \right) \\ &= \left(\mathbf{I}_{\text{out}}^{\text{T}} + \boldsymbol{\lambda}^{\text{T}} \mathbf{K}(\boldsymbol{\rho}) \right) \frac{\partial \mathbf{u}}{\partial \rho_i} + \boldsymbol{\lambda}^{\text{T}} \frac{\partial \mathbf{K}(\boldsymbol{\rho})}{\partial \rho_i} \mathbf{u} \end{aligned}$$

- Select $\boldsymbol{\lambda}$ such that $\mathbf{K}(\boldsymbol{\rho})\boldsymbol{\lambda} = -\mathbf{I}_{\text{out}}$

$$\frac{\partial d}{\partial \rho_i} = \boldsymbol{\lambda}^{\text{T}} \frac{\partial \mathbf{K}(\boldsymbol{\rho})}{\partial \rho_i} \mathbf{u}$$

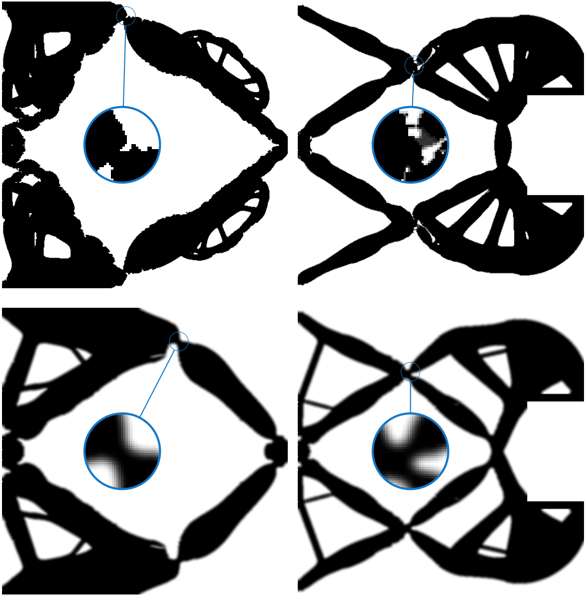
- Standard benchmark problems: inverter and gripper mechanisms
- Problems: length-scale control, de-facto hinges, concentration of stress, blurred interface



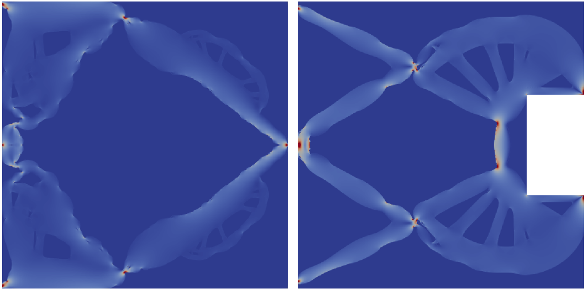


sensitivity filter

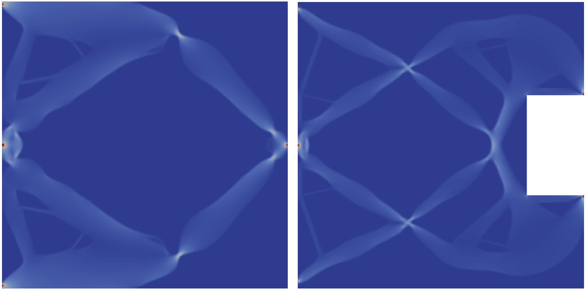
density filter



sensitivity filter



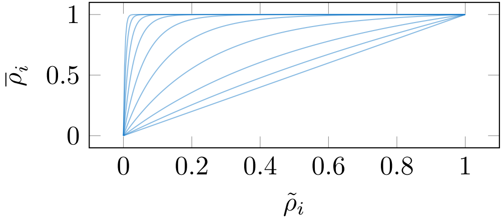
density filter



Projections

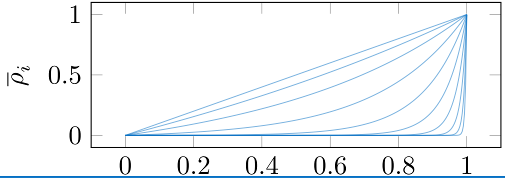
- Smooth Heaviside function (dilate)

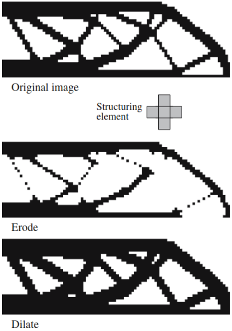
$$\overline{\rho}_i = 1 - e^{-\beta\tilde{\rho}_i} + \tilde{\rho}_i e^{-\beta}$$



- Smooth Heaviside function (erode)

$$\overline{\rho}_i = e^{-\beta(1-\tilde{\rho}_i)} - (1-\tilde{\rho}_i)e^{-\beta}$$



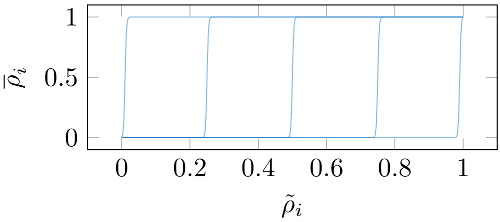


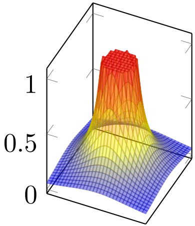
Courtesy of O. Sigmund, Morphology-based black and white filters for topology optimization, *Structural and Multidisciplinary Optimization*, 33(4-5): 401–424, 2007, doi: 10.1007/s00158-006-0087-x

Heaviside function with threshold

- Using a parameter η ($\eta = 0.0$ for dilate and $\eta = 1.0$ for erode)

$$\overline{\rho}_i = \frac{\tanh(\beta\eta) + \tanh(\beta(\tilde{\rho}_i - \eta))}{\tanh(\beta\eta) + \tanh(\beta(1-\eta))}$$





- Sensitivity

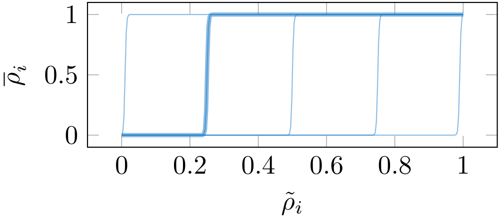
$$\frac{\partial \overline{\rho}_i}{\partial \tilde{\rho}_i} = \frac{1 - \tanh^2(\beta(\tilde{\rho}_i - \eta))}{\tanh(\beta\eta) + \tanh(\beta(1-\eta))}\beta$$

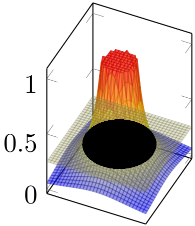
- Chain rule (density filtering)

$$\frac{\partial f}{\partial \rho_i} = \sum_{j=1}^{n_e} \frac{\partial f}{\partial \overline{\rho}_j} \frac{\overline{\rho}_j}{\partial \tilde{\rho}_j} \frac{\partial \tilde{\rho}_j}{\partial \rho_j}, \text{ where } \frac{\partial \tilde{\rho}_j}{\partial \rho_j} = \frac{w(\mathbf{x}_j)v_j}{\sum_{k=1}^{n_e} w(\mathbf{x}_k)v_k}$$

- Using a parameter η ($\eta = 0.0$ for dilate and $\eta = 1.0$ for erode)

$$\bar{\rho}_i = \frac{\tanh(\beta\eta) + \tanh(\beta(\tilde{\rho}_i - \eta))}{\tanh(\beta\eta) + \tanh(\beta(1 - \eta))}$$





- Sensitivity

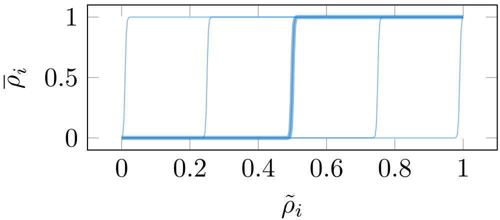
$$\frac{\partial \bar{\rho}_i}{\partial \tilde{\rho}_i} = \frac{1 - \tanh^2(\beta(\tilde{\rho}_i - \eta))}{\tanh(\beta\eta) + \tanh(\beta(1 - \eta))}\beta$$

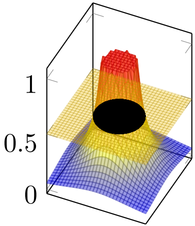
- Chain rule (density filtering)

$$\frac{\partial f}{\partial \rho_i} = \sum_{j=1}^{n_e} \frac{\partial f}{\partial \bar{\rho}_j} \frac{\bar{\rho}_j}{\partial \tilde{\rho}_j} \frac{\partial \tilde{\rho}_j}{\partial \rho_j}, \text{ where } \frac{\partial \tilde{\rho}_j}{\partial \rho_j} = \frac{w(\mathbf{x}_j) v_j}{\sum_{k=1}^{n_e} w(\mathbf{x}_k) v_k}$$

- Using a parameter η ($\eta = 0.0$ for dilate and $\eta = 1.0$ for erode)

$$\bar{\rho}_i = \frac{\tanh(\beta\eta) + \tanh(\beta(\tilde{\rho}_i - \eta))}{\tanh(\beta\eta) + \tanh(\beta(1-\eta))}$$





- Sensitivity

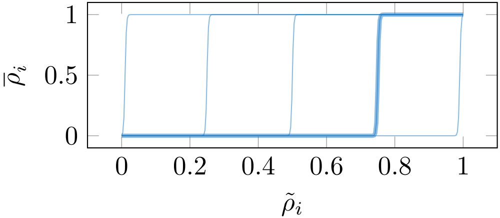
$$\frac{\partial \bar{\rho}_i}{\partial \tilde{\rho}_i} = \frac{1 - \tanh^2(\beta(\tilde{\rho}_i - \eta))}{\tanh(\beta\eta) + \tanh(\beta(1-\eta))}\beta$$

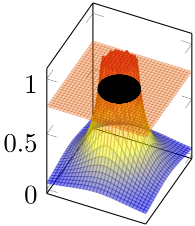
- Chain rule (density filtering)

$$\frac{\partial f}{\partial \rho_i} = \sum_{j=1}^{n_e} \frac{\partial f}{\partial \bar{\rho}_j} \frac{\bar{\rho}_j}{\partial \tilde{\rho}_j} \frac{\partial \tilde{\rho}_j}{\partial \rho_j}, \text{ where } \frac{\partial \tilde{\rho}_j}{\partial \rho_j} = \frac{w(\mathbf{x}_j) v_j}{\sum_{k=1}^{n_e} w(\mathbf{x}_k) v_k}$$

- Using a parameter η ($\eta = 0.0$ for dilate and $\eta = 1.0$ for erode)

$$\overline{\rho}_i = \frac{\tanh(\beta\eta) + \tanh(\beta(\tilde{\rho}_i - \eta))}{\tanh(\beta\eta) + \tanh(\beta(1-\eta))}$$





- Sensitivity

$$\frac{\partial \overline{\rho}_i}{\partial \tilde{\rho}_i} = \frac{1 - \tanh^2(\beta(\tilde{\rho}_i - \eta))}{\tanh(\beta\eta) + \tanh(\beta(1-\eta))}\beta$$

- Chain rule (density filtering)

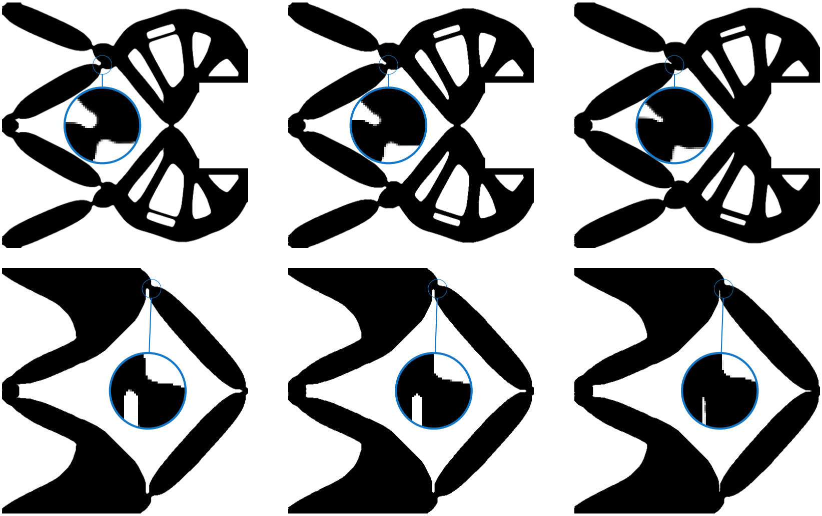
$$\frac{\partial f}{\partial \rho_i} = \sum_{j=1}^{n_\mathrm{e}} \frac{\partial f}{\partial \overline{\rho}_j} \frac{\overline{\rho}_j}{\partial \tilde{\rho}_j} \frac{\partial \tilde{\rho}_j}{\partial \rho_j}, \text{ where } \frac{\partial \tilde{\rho}_j}{\partial \rho_j} = \frac{w(\mathbf{x}_j)v_j}{\sum_{k=1}^{n_\mathrm{e}} w(\mathbf{x}_k)v_k}$$

- Manufacturing-tolerant optimization: concurrent optimization of three fields $\boldsymbol{\rho}_{\mathrm{d}}$, $\boldsymbol{\rho}_{\mathrm{e}}$, $\boldsymbol{\rho}$

$$
\begin{aligned}
\min_{\boldsymbol{\rho}_{\mathrm{d}},\boldsymbol{\rho}_{\mathrm{e}},\boldsymbol{\rho},s} \quad & s \\
\text{subject to} \quad & \mathbf{I}_{\mathrm{out}}^{\mathrm{T}}\mathbf{u}_{\mathrm{d}} \leq s \\
& \mathbf{I}_{\mathrm{out}}^{\mathrm{T}}\mathbf{u}_{\mathrm{e}} \leq s \\
& \mathbf{I}_{\mathrm{out}}^{\mathrm{T}}\mathbf{u} \leq s \\
& \mathbf{K}(\boldsymbol{\rho}_{\mathrm{d}})\mathbf{u}_{\mathrm{d}} = \mathbf{f} \\
& \mathbf{K}(\boldsymbol{\rho}_{\mathrm{e}})\mathbf{u}_{\mathrm{e}} = \mathbf{f} \\
& \mathbf{K}(\boldsymbol{\rho})\mathbf{u} = \mathbf{f} \\
& \mathbf{0} \leq \boldsymbol{\rho}_{\mathrm{d}} \leq \mathbf{1} \\
& \mathbf{0} \leq \boldsymbol{\rho}_{\mathrm{e}} \leq \mathbf{1} \\
& \mathbf{0} \leq \boldsymbol{\rho} \leq \mathbf{1} \\
& \mathbf{v}^{\mathrm{T}}\boldsymbol{\rho} \leq \overline{V}
\end{aligned}
$$

- Volume constraint must be set through $\boldsymbol{\rho}_{\mathrm{d}}$

Three-field robust optimization



$\eta = 0.75$ $\eta = 0.50$ $\eta = 0.25$

Stress constraints

Stress constraints

- Two main challenges:
  - Local nature: large number of stress constraints
  - Singularity: elements with low stiffness usually possess large strain levels. Thus, stress constrains prevent introduction of holes
- Main state-of-the-art approaches:
  - Stress aggregation (“traditional”)
    - Global description $\rightarrow$ single adjoint problem
    - Relaxation procedure to avoid singularity
    - Slow continuation procedure needed
  - Augmented Lagrangian (recent result)
    - Augmented Lagrangian objective function
    - Single adjoint equation yet treating the constraints locally

- Stress

$$\boldsymbol{\sigma}_i(\rho_i) = \mathbf{D}_i(\rho_i)\mathbf{B}_i\mathbf{u}_i$$

- Equivalent von Mises stress

$$\begin{aligned}
\sigma_{\mathrm{eq}}(\rho_i)^2 &= \boldsymbol{\sigma}_i(\rho_i)^{\mathrm{T}}\mathbf{V}\boldsymbol{\sigma}_i(\rho_i) \\
&= \mathbf{u}_i^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}}\mathbf{D}_i(\rho_i)\mathbf{V}\mathbf{D}_i(\rho_i)\mathbf{B}_i\mathbf{u}_i \\
&= f_\sigma(\rho_i)^2\mathbf{u}_i^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}}\mathbf{D}_{0,i}\mathbf{V}\mathbf{D}_{0,i}\mathbf{B}_i\mathbf{u}_i \\
&= f_\sigma(\rho_i)^2\mathbf{u}_i^{\mathrm{T}}\mathbf{M}_{0,i}\mathbf{u}_i \\
\sigma_{\mathrm{eq}}(\rho_i) &= f_\sigma(\rho_i)\sqrt{\mathbf{u}_i^{\mathrm{T}}\mathbf{M}_{0,i}\mathbf{u}_i + \sigma_{\min}^2}
\end{aligned}$$

- In plane stress, we have

$$\mathbf{V} = \begin{pmatrix} 1 & -1/2 & 0 \\ -1/2 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

- $\sigma_{\min}$ to circumvent numerical issues in sensitivities for $\mathbf{u}_i^{\mathrm{T}}\mathbf{M}_{0,i}\mathbf{u}_i \to 0$
- Stress interpolation function, e.g., $f_\sigma(\rho_i) = \frac{\rho_i}{\varepsilon(1-\rho_i)+\rho_i}$, where $\varepsilon = 0.2$

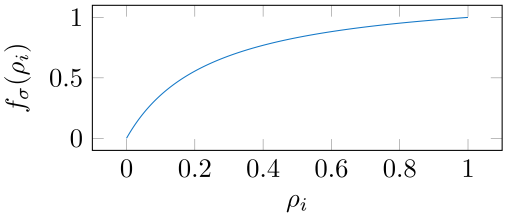
- Stress

$$\boldsymbol{\sigma}_i(\rho_i) = \mathbf{D}_i(\rho_i)\mathbf{B}_i\mathbf{u}_i$$

- Equivalent von Mises stress

$$\sigma_{\mathrm{eq}}(\rho_i) = f_\sigma(\rho_i)\sqrt{\mathbf{u}_i^{\mathrm{T}}\mathbf{M}_{0,i}\mathbf{u}_i + \sigma_{\min}^2}$$

- $\sigma_{\min}$ to circumvent numerical issues in sensitivities for $\mathbf{u}_i^{\mathrm{T}}\mathbf{M}_{0,i}\mathbf{u}_i \to 0$
- Stress interpolation function, e.g., $f_\sigma(\rho_i) = \frac{\rho_i}{\varepsilon(1-\rho_i)+\rho_i}$, where $\varepsilon = 0.2$



- Adjoint method:

$$\sigma_{\mathrm{eq},i}(\boldsymbol{\rho}) = f_\sigma(\rho_i)\sqrt{\mathbf{u}^{\mathrm{T}}\mathbf{M}_{0,i}\mathbf{u} + \sigma_{\min}^2} + \boldsymbol{\lambda}^{\mathrm{T}}(\mathbf{K}(\boldsymbol{\rho})\mathbf{u} - \mathbf{f})$$

- After differentiation, we receive

$$\frac{\partial \sigma_{\mathrm{eq},i}(\boldsymbol{\rho})}{\partial \rho_j} = \frac{\partial f_\sigma(\rho_i)}{\partial \rho_j}\sqrt{\mathbf{u}^{\mathrm{T}}\mathbf{M}_{0,i}\mathbf{u} + \sigma_{\min}^2} + \frac{f_\sigma(\rho_i)\mathbf{u}^{\mathrm{T}}\mathbf{M}_{0,i}\frac{\partial \mathbf{u}}{\partial \rho_j}}{\sqrt{\mathbf{u}^{\mathrm{T}}\mathbf{M}_{0,i}\mathbf{u} + \sigma_{\min}^2}} + \boldsymbol{\lambda}^{\mathrm{T}}\left(\frac{\partial \mathbf{K}(\boldsymbol{\rho})}{\partial \rho_j}\mathbf{u} + \mathbf{K}(\boldsymbol{\rho})\frac{\partial \mathbf{u}}{\partial \rho_j}\right)$$

- After rearranging the terms

$$\frac{\partial \sigma_{\mathrm{eq},i}(\boldsymbol{\rho})}{\partial \rho_j} = \frac{\partial f_\sigma(\rho_i)}{\partial \rho_j}\sqrt{\mathbf{u}^{\mathrm{T}}\mathbf{M}_{0,i}\mathbf{u} + \sigma_{\min}^2} + \boldsymbol{\lambda}^{\mathrm{T}}\frac{\partial \mathbf{K}(\boldsymbol{\rho})}{\partial \rho_j}\mathbf{u} + \left(\frac{f_\sigma(\rho_i)\mathbf{u}^{\mathrm{T}}\mathbf{M}_{0,i}}{\sqrt{\mathbf{u}^{\mathrm{T}}\mathbf{M}_{0,i}\mathbf{u} + \sigma_{\min}^2}} + \boldsymbol{\lambda}^{\mathrm{T}}\mathbf{K}(\boldsymbol{\rho})\right)\frac{\partial \mathbf{u}}{\partial \rho_j}$$

- To eliminate $\frac{\partial \mathbf{u}}{\partial \rho_j}$, we set

$$\boldsymbol{\lambda} = -\mathbf{K}(\boldsymbol{\rho})^{-1} \frac{f_\sigma(\rho_i)\mathbf{M}_{0,i}\mathbf{u}}{\sqrt{\mathbf{u}^{\mathrm{T}}\mathbf{M}_{0,i}\mathbf{u} + \sigma_{\min}^2}}$$

- Hence,

$$\frac{\partial \sigma_{\mathrm{eq},i}(\boldsymbol{\rho})}{\partial \rho_j} = \frac{\partial f_\sigma(\rho_i)}{\partial \rho_j}\sqrt{\mathbf{u}^{\mathrm{T}}\mathbf{M}_{0,i}\mathbf{u} + \sigma_{\min}^2} - \mathbf{u}^{\mathrm{T}}\frac{\partial \mathbf{K}(\boldsymbol{\rho})}{\partial \rho_j}\mathbf{K}(\boldsymbol{\rho})^{-1}\frac{f_\sigma(\rho_i)\mathbf{M}_{0,i}\mathbf{u}}{\sqrt{\mathbf{u}^{\mathrm{T}}\mathbf{M}_{0,i}\mathbf{u} + \sigma_{\min}^2}}$$

- Maximum stress level should be below the yield function $f_\alpha(\sigma_y)$

$$\frac{\max_{i=1}^{n_\mathrm{e}}\{\sigma_{\mathrm{eq},i}(\boldsymbol{\rho})\}}{f_\alpha(\sigma_y)} \leq 1$$

- Stress aggregation: approximate the $\max$ operator by a smooth function
- Usually, we use p-norm or p-mean functions, where p should be high enough to capture the operator but small enough due to numerical stability
- For any p, we have:

$$\left[\frac{1}{n_\mathrm{e}}\sum_{i=1}^{n_\mathrm{e}}\left(\frac{\sigma_{\mathrm{eq},i}(\boldsymbol{\rho})}{f_\alpha(\sigma_y)}\right)^p\right]^{1/p} \leq \frac{\max_{i=1}^{n_\mathrm{e}}\{\sigma_{\mathrm{eq},i}(\boldsymbol{\rho})\}}{f_\alpha(\sigma_y)} \leq \left[\sum_{i=1}^{n_\mathrm{e}}\left(\frac{\sigma_{\mathrm{eq},i}(\boldsymbol{\rho})}{f_\alpha(\sigma_y)}\right)^p\right]^{1/p}$$

- While p-mean is thus an under-estimator, p-norm an over-estimator
- Continuation on p or a discontinuous corrector

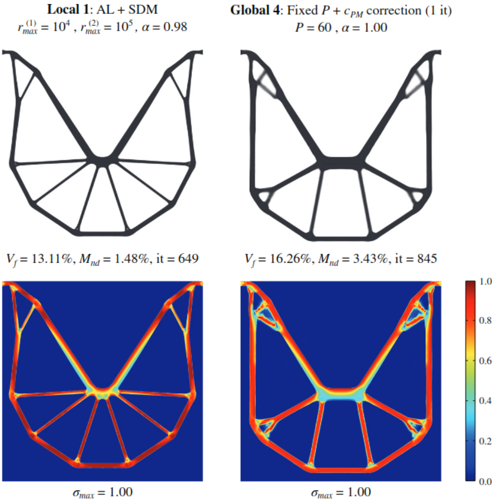
- The idea is to replace original stress-constrained problem by a sequence of stress-unconstrained problems and an augmented objective function

$$\mathcal{L}(\boldsymbol{\rho}, \boldsymbol{\mu}, r) = f(\boldsymbol{\rho}) + \frac{r}{2} \sum_{i=1}^{n_\mathrm{e}} \max \left\{ \frac{\mu_i}{r} + \frac{\sigma_{\mathrm{eq},i}(\boldsymbol{\rho})}{f_\alpha(\sigma_y)} - 1, 0 \right\}^2$$

- Here, r is a penalization parameter and $\boldsymbol{\mu}$ Lagrange multipliers
- Lagrange multipliers are kept constant within subproblem solution, and updated as

$$\mu_k = \max \left\{ r \left(\frac{\sigma_{\mathrm{eq},i}(\boldsymbol{\rho})}{f_\alpha(\sigma_y)} - 1 \right) + \mu_i, 0 \right\}$$

$$r = \min \left\{ \gamma_r r, \frac{r_{\max}}{n_\mathrm{e}} \right\}$$



1

[1]Courtesy of G. A. da Silva, N. Aage, A. T. Beck, and O. Sigmund, Local versus global stress constraint strategies in topology optimization: A comparative study, *International Journal for Numerical Methods in Engineering*, 122(21):6003–6036, 2021, doi: 10.1002/nme.6781

Inverse homogenization

- Homogenization method averages the microstructure by a representative periodic unit cell
- Inverse approach – inverse homogenization – optimize a PUC with prescribed/maximal effective macroscopic material properties

$$\mathsf{E}^{\mathrm{H}}_{ijkl} = \frac{1}{|Y|} \sum_{e=1}^{n_{\mathrm{e}}} \left(\mathbf{u}_e^{A(kl)} \right)^{\mathrm{T}} \mathbf{k}_e \mathbf{u}_e^{A(ij)},$$

where

$$\mathbf{u}_e^{A(kl)} = \mathbf{u}_e^{0(kl)} - \mathbf{u}_e^{*(kl)}$$

- $\mathbf{u}_e^{0(kl)}$ corresponds to displacements from three unit prestrains
- $\mathbf{u}_e^{*(kl)}$ is a periodic solution to the prestress

- Optimization problem formulation:

$$\min_{\boldsymbol{\rho}} \frac{1}{|Y|}\sum_{e=1}^{N} v_e\rho_e$$

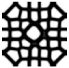
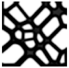
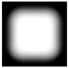
$$\text{subject to } \mathbf{K}\mathbf{u}^{A(kl)} = \mathbf{f}^{kl}, \quad \forall k,l \in \{1,2\},$$

$$E^*_{ijkl} - E^H_{ijkl}(\boldsymbol{\rho}) = 0, \quad \forall i,j,k,l \in \{1,2\},$$

$$0 \le \rho_e \le 1, \quad \forall e \in \{1, \ \dots \ , n_\mathrm{e}\},$$

- Periodic boundary conditions
- Sensitivity:

$$\frac{\partial \mathbf{E}^{\mathrm{H}}_{ijkl}}{\partial \rho_e} = \frac{1}{|Y|}\sum_{e=1}^{n_\mathrm{e}} \left(\mathbf{u}^{A(kl)}\right)^{\mathrm{T}} \frac{\partial \mathbf{K}(\boldsymbol{\rho})}{\partial \rho_e}\mathbf{u}^{A(ij)}$$

$\mathbf{E}^{\mathrm{H}}$	PUC	$\mathbf{E}^{\mathrm{H}}$	PUC
$\begin{pmatrix}0.3360 & 0.0632 & 0.0712\\ 0.0632 & 0.0726 & 0.0699\\ 0.0712 & 0.0699 & 0.0776\end{pmatrix}$		$\begin{pmatrix}0.2503 & 0.0767 & 0.1365\\ 0.0767 & 0.0283 & 0.0456\\ 0.1365 & 0.0456 & 0.0774\end{pmatrix}$	
$\begin{pmatrix}0.1390 & 0.1300 & 0.0897\\ 0.1300 & 0.1258 & 0.0946\\ 0.0897 & 0.0946 & 0.1208\end{pmatrix}$		$\begin{pmatrix}0.2412 & 0.0746 & 0.1143\\ 0.0746 & 0.1645 & 0.0295\\ 0.1143 & 0.0295 & 0.0638\end{pmatrix}$	
$\begin{pmatrix}0.1991 & 0.1001 & 0.1089\\ 0.1001 & 0.1300 & 0.0497\\ 0.1089 & 0.0497 & 0.0906\end{pmatrix}$		$\begin{pmatrix}0.1683 & 0.0791 & 0.000\\ 0.0791 & 0.1673 & 0.000\\ 0.000 & 0.000 & 0.0307\end{pmatrix}$	
$\begin{pmatrix}0.1642 & 0.1309 & 0.0000\\ 0.1309 & 0.1642 & 0.0000\\ 0.0000 & 0.0000 & 0.1174\end{pmatrix}$		$\begin{pmatrix}0.1052 & 0.0169 & 0.0000\\ 0.0169 & 0.4087 & 0.0000\\ 0.0000 & 0.0000 & 0.0022\end{pmatrix}$	
$\begin{pmatrix}0.2767 & 0.0383 & 0.0000\\ 0.0383 & 0.2767 & 0.0000\\ 0.0000 & 0.0000 & 0.0169\end{pmatrix}$		$\begin{pmatrix}0.1495 & 0.1229 & 0.0000\\ 0.1229 & 0.1437 & 0.0000\\ 0.0000 & 0.0000 & 0.0521\end{pmatrix}$	
$\begin{pmatrix}0.0861 & 0.0936 & 0.0000\\ 0.0936 & 0.3183 & 0.0000\\ 0.0000 & 0.0000 & 0.0374\end{pmatrix}$		$\begin{pmatrix}0.2893 & 0.1934 & 0.0000\\ 0.1934 & 0.2891 & 0.0000\\ 0.0000 & 0.0000 & 0.1459\end{pmatrix}$	
$\begin{pmatrix}0.1973 & 0.0188 & 0.0000\\ 0.0188 & 0.1973 & 0.0000\\ 0.0000 & 0.0000 & 0.0093\end{pmatrix}$		$\begin{pmatrix}0.0000 & 0.0000 & 0.0000\\ 0.0000 & 0.4732 & 0.0000\\ 0.0000 & 0.0000 & 0.0000\end{pmatrix}$	

Summary

- Extension of TO to different problems is straightforward
- Heaviside projection can be used to suppress gray regions when used with density filtering
- Morphology operators (dilate/erode) provide simple manufacturing-tolerant designs and eliminate de-facto hinges
- Best procedure for stress constraints seems to be the Augmented Lagrangian method

References

- B. S. Lazarov, F. Wang, and O. Sigmund, Length scale and manufacturability in density-based topology optimization, *Archive of Applied Mechanics*, 86(1-2):189–218, 2016, doi: 10.1007/s00419-015-1106-4
- G. A. da Silva, N. Aage, A. T. Beck, and O. Sigmund, Local versus global stress constraint strategies in topology optimization: A comparative study, *International Journal for Numerical Methods in Engineering*, 122(21):6003–6036, 2021, doi: 10.1002/nme.6781