Topology optimization II: Frames

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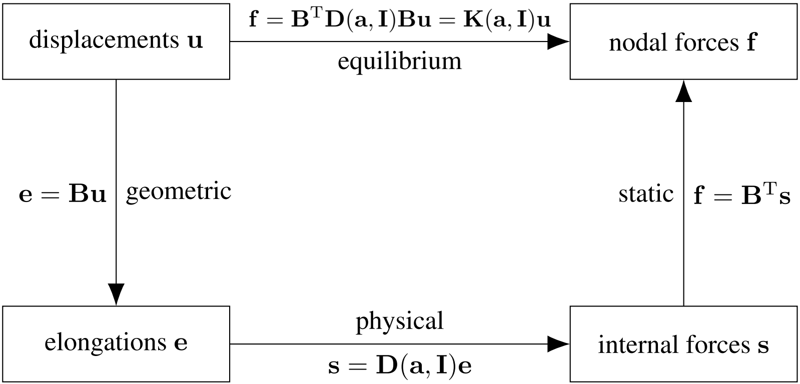
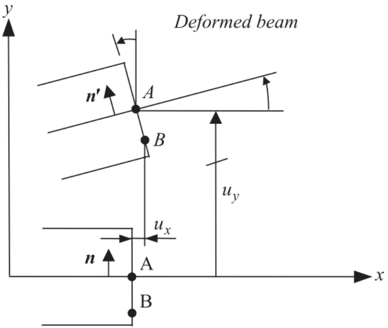
Finite elements overview


displacements $\mathbf{u}$
$\mathbf{f} = \mathbf{B}^{\mathrm{T}}\mathbf{D}(\mathbf{a},\mathbf{I})\mathbf{B}\mathbf{u} = \mathbf{K}(\mathbf{a},\mathbf{I})\mathbf{u}$
equilibrium
nodal forces $\mathbf{f}$
$\mathbf{e} = \mathbf{B}\mathbf{u}$
geometric
static
$\mathbf{f} = \mathbf{B}^{\mathrm{T}}\mathbf{s}$
elongations $\mathbf{e}$
physical
$\mathbf{s} = \mathbf{D}(\mathbf{a},\mathbf{I})\mathbf{e}$
internal forces $\mathbf{s}$


y
Deformed beam
n′
A
B
u_y
u_x
n
A
B
x

- Geometric matrix

$$\mathbf{B}_i = \begin{pmatrix} -\cos(\alpha_i) & -\sin(\alpha_i) & 0 & \cos(\alpha_i) & \sin(\alpha_i) & 0 \\ \frac{\sin(\alpha_i)}{\ell} & -\frac{\cos(\alpha_i)}{\ell} & -1 & -\frac{\sin(\alpha_i)}{\ell} & \frac{\cos(\alpha_i)}{\ell} & 0 \\ \frac{\sin(\alpha_i)}{\ell} & -\frac{\cos(\alpha_i)}{\ell} & 0 & -\frac{\sin(\alpha_i)}{\ell} & \frac{\cos(\alpha_i)}{\ell} & -1 \end{pmatrix}$$

- Material stiffness matrix (not diagonal!)

$$\mathbf{D}_i(a_i, I_i) = \begin{pmatrix} E_i a_i/\ell_i & 0 & 0 \\ 0 & \frac{4E_i I_i}{\ell_i} & \frac{2E_i I_i}{\ell_i} \\ 0 & \frac{2E_i I_i}{\ell_i} & \frac{4E_i I_i}{\ell_i} \end{pmatrix}$$

- Element stiffness matrix

$$\mathbf{K}_i(a_i, I_i) = \mathbf{B}_i^{\mathrm{T}} \mathbf{D}_i(a_i, I_i) \mathbf{B}_i$$

- Stiffness matrix in 2D has rank 3. Thus, it can also be written using a generalized geometric matrix

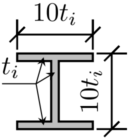
$$\hat{\mathbf{B}}_i = \begin{pmatrix} -\cos(\alpha_i) & -\sin(\alpha_i) & 0 & \cos(\alpha_i) & \sin(\alpha_i) & 0 \\ -\frac{2\sin(\alpha_i)}{\ell} & \frac{2\cos(\alpha_i)}{\ell} & 1 & \frac{2\sin(\alpha_i)}{\ell} & -\frac{2\cos(\alpha_i)}{\ell} & 1 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{pmatrix}$$

and diagonal generalized material stiffness matrix

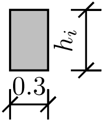
$$\hat{\mathbf{D}}_i(a_i, I_i) = \begin{pmatrix} \frac{E_i a_i}{\ell_i} & 0 & 0 \\ 0 & \frac{3E_i I_i}{\ell_i} & 0 \\ 0 & 0 & \frac{E_i I_i}{\ell_i} \end{pmatrix}$$

- Then, $\hat{\mathbf{s}}$ is a vector of generalized internal forces and $\hat{\mathbf{e}}$ a vector of generalized elongations
- $\hat{\mathbf{s}}$ and $\hat{\mathbf{e}}$ span the same spaces as $\mathbf{s}$ and $\mathbf{e}$ do but they have a different physical interpretation

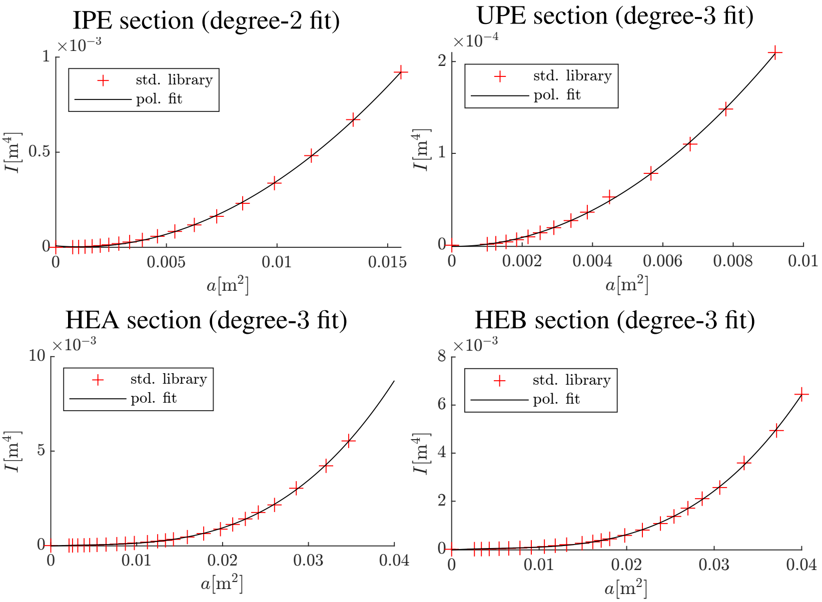
- If I_i and a_i are independent, we can build a convex optimization problem (analogously to SOCP and linear SDP formulations for trusses)
- However, $\overline{V}$ does not bound I_i from above. $I_i \to \infty$ is optimal
- Geometrically, this is equivalent to infinitely-large infinitely-thin hollow cross-section with the area a_i
- To avoid such situation, we restrict the optimization to a family of cross-sections with the moment of inertia being a polynomial function of a_i: $I_i(a_i) = c_1 a_i + c_2 a_i^2 + c_3 a_i^3$



- $a_i = 28t_i^2$
- $I_i = 2 \cdot \frac{1}{12} \cdot 10t_i \cdot t_i^3 + \frac{1}{12} \cdot t_i \cdot 8t_i^3 + 2 \cdot 10t_i^2 \cdot (4.5t_i)^2 = 449.\overline{3}t_i^4$
- $c_2 = \frac{I_i}{a_i^2} = \frac{449.\overline{3}}{784} \approx 0.573 \; (c_1 = c_3 = 0)$



- $a_i = 0.3h_i$
- $I_i = \frac{1}{12} \cdot 0.3 \cdot h_i^3 = 0.025h_i^3$
- $c_3 = \frac{I_i}{a_i^3} = \frac{0.025}{0.027} \approx 0.926 \; (c_1 = c_2 = 0)$


IPE section (degree-2 fit)
×10⁻³
std. library
pol. fit
I [m⁴]
a [m²]
0
0.5
1
0.005
0.01
0.015
UPE section (degree-3 fit)
×10⁻⁴
std. library
pol. fit
I [m⁴]
a [m²]
0
1
2
0.002
0.004
0.006
0.008
0.01
HEA section (degree-3 fit)
×10⁻³
std. library
pol. fit
I [m⁴]
a [m²]
0
5
10
0.01
0.02
0.03
0.04
HEB section (degree-3 fit)
×10⁻³
std. library
pol. fit
I [m⁴]
a [m²]
0
2
4
6
8
0.01
0.02
0.03
0.04

Compliance minimization: sizing via optimality criteria

- Ground structure design domain
- Compliance minimization is non-convex (in contrast to trusses)
- Impose $\mathbf{a} \geq \varepsilon \mathbf{1} > \mathbf{0}$ to secure $\mathbf{K}(\mathbf{a}) \in \mathbb{S}^{n_{\mathrm{dof}}}_{\succ 0}$
- Sizing problem formalized as

$$\min_{\mathbf{a}\in\mathbb{R}^{n_{\mathrm{e}}},\mathbf{u}\in\mathbb{R}^{n_{\mathrm{dof}}}} \quad \mathbf{f}^{\mathrm{T}}\mathbf{u} \tag{1a}$$

$$\text{subject to} \quad \mathbf{K}(\mathbf{a})\mathbf{u} = \mathbf{f} \tag{1b}$$

$$\boldsymbol{\ell}^{\mathrm{T}}\mathbf{a} \leq \overline{V} \tag{1c}$$

$$\mathbf{a} \geq \varepsilon \mathbf{1} \tag{1d}$$

- Simple and efficient heuristic optimization algorithm: optimality criteria (OC)
  - Usually converges to good-quality locally-optimal points
  - Convergence to a stationary point not guaranteed

- Lagrangian function:

$$\mathcal{L}(\mathbf{a}, \mathbf{u}, \boldsymbol{\lambda}, \mu, \boldsymbol{\nu}) = \mathbf{f}^{\mathrm{T}}\mathbf{u} + \boldsymbol{\lambda}^{\mathrm{T}}\left(\mathbf{K}(\mathbf{a})\mathbf{u} - \mathbf{f}\right) + \mu\left(\boldsymbol{\ell}^{\mathrm{T}}\mathbf{a} - \overline{V}\right) + \boldsymbol{\nu}^{\mathrm{T}}\left(\varepsilon\mathbf{1} - \mathbf{a}\right)$$

- Karush-Kuhn-Tucker conditions:

primal feasibility:
$$\mathbf{0} = \mathbf{K}(\mathbf{a}^*)\mathbf{u}^* - \mathbf{f}$$
$$\overline{V} \geq \boldsymbol{\ell}^{\mathrm{T}}\mathbf{a}$$
$$\mathbf{a} \geq \varepsilon\mathbf{1}$$

dual feasibility:
$$0 \leq \mu^*$$
$$\mathbf{0} \leq \boldsymbol{\nu}^*$$

complementary slackness:
$$0 = \mu^*\left(\boldsymbol{\ell}^{\mathrm{T}}\mathbf{a}^* - \overline{V}\right)$$
$$\mathbf{0} = \left(\boldsymbol{\nu}^*\right)^{\mathrm{T}}\left(\varepsilon\mathbf{1} - \mathbf{a}^*\right)$$

stationarity:
$$\frac{\partial\mathcal{L}}{\partial\mathbf{u}} = \mathbf{f}^{\mathrm{T}} + \left(\boldsymbol{\lambda}^*\right)^{\mathrm{T}}\mathbf{K}(\mathbf{a}^*) = \mathbf{0} \rightarrow \boldsymbol{\lambda}^* = -\mathbf{u}$$
$$\frac{\partial\mathcal{L}}{\partial a_i} = \left(\boldsymbol{\lambda}^*\right)^{\mathrm{T}}\frac{\partial\mathbf{K}(\mathbf{a}^*)}{\partial a_i}\mathbf{u}^* + \mu^*\ell_i - \nu_i^*$$
$$= -\left(\mathbf{u}^*\right)^{\mathrm{T}}\frac{\partial\mathbf{K}(\mathbf{a}^*)}{\partial a_i}\mathbf{u}^* + \mu^*\ell_i - \nu_i^* = 0$$

- Assume that $\mathbf{a}^* > \varepsilon\mathbf{1}$. Then, KKT conditions imply that $\boldsymbol{\nu}^* = \mathbf{0}$
- Consequently, the elements $\mathbf{a}^* > \varepsilon\mathbf{1}$ must have equal energy

$$\mu^* = \frac{1}{\ell_i} \left(\mathbf{u}^*\right)^{\mathrm{T}} \frac{\partial \mathbf{K}(\mathbf{a}^*)}{\partial a_i} \mathbf{u}^*$$

  and the volume constraint is active
- OC method: update scheme that balances μ^* for all the elements
  1. Bisection to find $\mu^{(k)}$ such that

$$\overline{V} = \sum_{i=1}^{n_{\mathrm{e}}} \left[\ell_i \max\left\{ a_i^{(k-1)} \frac{\frac{1}{\ell_i} \left(\mathbf{u}^{(k-1)}\right)^{\mathrm{T}} \frac{\partial \mathbf{K}(\mathbf{a}^{(k-1)})}{\partial a_i} \mathbf{u}^{(k-1)}}{\mu^{(k)}}, \varepsilon \right\} \right]$$

     $\mu^{(k)} = 1$ implies a local optimum
  2. Update step

$$a_i^{(k)} = \max\left\{ a_i^{(k-1)} \frac{\frac{1}{\ell_i} \left(\mathbf{u}^{(k-1)}\right)^{\mathrm{T}} \frac{\partial \mathbf{K}(\mathbf{a}^{(k-1)})}{\partial a_i} \mathbf{u}^{(k-1)}}{\mu^{(k)}}, \varepsilon \right\}$$

Compliance minimization: complementary-strain-energy-based formulation

- Generalized formulation for trusses by exploiting the rank-3 stiffness matrix decomposition
- Complementary strain energy function (1/2 removed to stay consistent)

$$\mathit{\Pi}_2(\mathbf{a}, \mathbf{I}, \hat{\mathbf{s}}) = \sum_{i=1}^{n_\mathrm{e}} \left(\frac{\ell_i \hat{s}_{1,i}^2}{E_i a_i} + \frac{\ell_i \hat{s}_{2,i}^2}{3 E_i I_i} + \frac{\ell_i \hat{s}_{3,i}^2}{E_i I_i} \right) \quad \text{such that } \hat{\mathbf{B}}^\mathrm{T} \hat{\mathbf{s}} = \mathbf{f}$$

- The same procedure as in truss SOCP formulation

$$w_{1,i} \geq \frac{\ell_i \hat{s}_{1,i}^2}{E_i a_i} \quad \rightarrow w_{1,i} + a_i \geq \left\| \left(2\sqrt{\frac{\ell_i}{E_i}} \hat{s}_{1,i} \quad w_{1,i} - a_i \right) \right\|_2$$

$$w_{2,i} \geq \frac{\ell_i \hat{s}_{2,i}^2}{3 E_i I_i} \quad \rightarrow w_{2,i} + I_i \geq \left\| \left(2\sqrt{\frac{\ell_i}{3 E_i}} \hat{s}_{2,i} \quad w_{2,i} - I_i \right) \right\|_2$$

$$w_{3,i} \geq \frac{\ell_i \hat{s}_{3,i}^2}{E_i I_i} \quad \rightarrow w_{3,i} + I_i \geq \left\| \left(2\sqrt{\frac{\ell_i}{E_i}} \hat{s}_{3,i} \quad w_{3,i} - I_i \right) \right\|_2$$

- Introduce linearized monomials $a_i^{(1)}$, $a_i^{(2)}$, and $a_i^{(3)}$

$$\min_{\mathbf{a},\mathbf{a}^{(2)},\mathbf{a}^{(3)},\hat{\mathbf{s}},\mathbf{w}} \sum_{j=1}^{3} \mathbf{1}^{\mathrm{T}}\mathbf{w}_j \tag{2a}$$

$$\text{subject to } \hat{\mathbf{B}}^{\mathrm{T}}\hat{\mathbf{s}} = \mathbf{f} \tag{2b}$$

$$w_{1,i} + a_i \geq \left\| \begin{pmatrix} 2\sqrt{\frac{\ell_i}{E_i}}\hat{s}_{1,i} & w_{1,i} - a_i \end{pmatrix} \right\|_2 \tag{2c}$$

$$w_{2,i} + \left(c_1 a_i + c_2 a_i^{(2)} + c_3 a_i^{(3)}\right) \geq \left\| \begin{pmatrix} 2\sqrt{\frac{\ell_i}{3E_i}}\hat{s}_{2,i} & w_{2,i} - \left[c_1 a_i + c_2 a_i^{(2)} + c_3 a_i^{(3)}\right] \end{pmatrix} \right\|_2 \tag{2d}$$

$$w_{3,i} + \left(c_1 a_i + c_2 a_i^{(2)} + c_3 a_i^{(3)}\right) \geq \left\| \begin{pmatrix} 2\sqrt{\frac{\ell_i}{E_i}}\hat{s}_{3,i} & w_{3,i} - \left[c_1 a_i + c_2 a_i^{(2)} + c_3 a_i^{(3)}\right] \end{pmatrix} \right\|_2 \tag{2e}$$

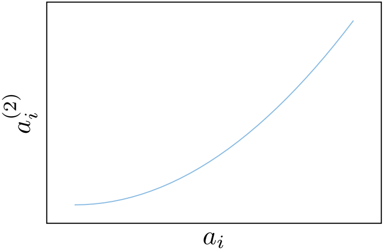
$$a_i^{(2)} = a_i^2 \tag{2f}$$

$$a_i^{(3)} = a_i^{(2)} a_i \tag{2g}$$

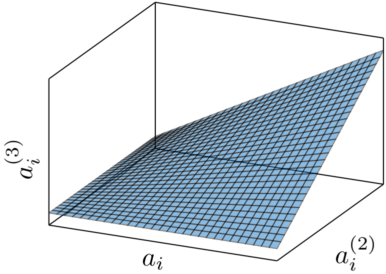
$$\boldsymbol{\ell}^{\mathrm{T}}\mathbf{a} \leq \overline{V} \tag{2h}$$

$$\mathbf{a} \geq \mathbf{0} \tag{2i}$$

- Solution via branch-and-bound-type method, Gurobi optimizer
  - Spatial branching of non-convex constraints
  - Very nice explanation at https://www.gurobi.com/resource/non-convex-quadratic-optimization/



$a_i^{(2)} = a_i^2$



$a_i^{(3)} = a_i a_i^{(2)}$

Compliance minimization: potential-energy-based SDP

- Recall (truss lecture) that the compliance optimization problem is equivalent to the ~~linear~~ SDP formulation:

$$\min_{\mathbf{a}\in\mathbb{R}^{n_\mathrm{e}}, c\in\mathbb{R}} \quad c \tag{3a}$$

$$\text{subject to} \quad \begin{pmatrix} c & -\mathbf{f}^\mathrm{T} \\ -\mathbf{f} & \mathbf{K}(\mathbf{a}) \end{pmatrix} \succeq 0, \tag{3b}$$

$$\boldsymbol{\ell}^\mathrm{T}\mathbf{a} \leq \overline{V}, \tag{3c}$$

$$\mathbf{a} \geq \mathbf{0} \tag{3d}$$

- Such reformulation also works here, there were no assumptions on linearity of $\mathbf{K}(\mathbf{a})$
- For frames, the polynomial matrix inequality makes the problem non-convex
- Rewriting (3b) via characteristic polynomial results in polynomial inequality $\rightarrow$ basic semi-algebraic feasible set

- Let $f(c) = c$ be the objective function
- The constraints can be incorporated into a single matrix inequality

$$\mathbf{G}(\mathbf{a}, c) = \begin{pmatrix} c & -\mathbf{f}^{\mathrm{T}} & 0 & 0 & \dots & 0 \\ -\mathbf{f} & \mathbf{K}(\mathbf{a}) & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ 0 & \mathbf{0} & \overline{V} - \boldsymbol{\ell}^{\mathrm{T}}\mathbf{a} & 0 & \dots & 0 \\ 0 & \mathbf{0} & 0 & a_i & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \mathbf{0} & 0 & 0 & \dots & a_{n_{\mathrm{e}}} \end{pmatrix}$$

- Therefore, the problem can also be formalized as

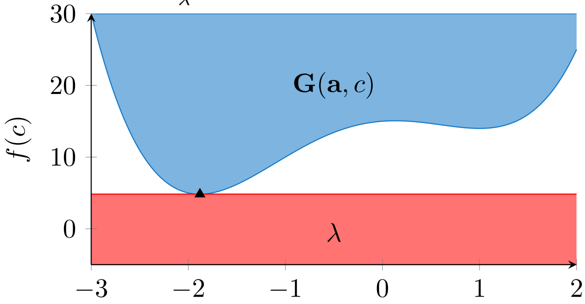
$$f^* = \min f(c), \quad \text{subject to } \mathbf{G}(\mathbf{a}, c) \succeq 0$$

- This is equivalent to an infinite-dimensional **convex** problem: we maximize λ such that we "touch" the feasible domain $\mathcal{K}(\mathbf{G})$

$$f^* = \sup_{\lambda} \lambda, \quad \text{s.t. } \forall \mathbf{a}, c \in \mathcal{K}(\mathbf{G}) : \lambda \leq f(c)$$

- Since $f(c)$ and λ are polynomial functions, this is equivalent to $(f - \lambda)$ being a non-negative polynomial on $\mathcal{K}(\mathbf{G})$

$$f^* = \sup_{\lambda} \lambda, \quad \text{s.t. } (f - \lambda) \in C_k(\mathcal{K}(\mathbf{G}))$$



- Assume there exists p_0 and $\mathbf{R}$ that are Sum-Of-Squares such that $p_0 + \langle \mathbf{R}, \mathbf{G} \rangle \geq 0$ is compact
- Then, Putinar's Positivestellensatz[1] extended to the PMI case[2] enables maximization of λ with certified feasible set non-negativity

$$\sup_{\lambda, p_0, \mathbf{R}} \lambda \tag{4a}$$

$$\text{subject to } f - \lambda = p_0 + \langle \mathbf{R}, \mathbf{G} \rangle , \quad p_0, \mathbf{R} \text{ are SOS} \tag{4b}$$

- For the maximum degree in the polynomials p_0 and $\mathbf{R}$ fixed to $2r$, non-negatitivity of $f - \lambda$ on $\mathcal{K}(\mathbf{G})^{(r)}$ can be checked via a finite-dimensional linear SDP
- Problems: degree is not known! Recovery of global minimizers?

[1] M. Putinar, Positive polynomials on compact semi-algebraic sets, *Indiana University Mathematics Journal*, 42(3):969–984, 1993, doi: 10.2307/24897130

[2] D. Henrion and J.-B. Lasserre, Convergent relaxations of polynomial matrix inequalities and static output feedback, *IEEE Transactions on Automatic Control*, 51(2):192–202, 2006, doi: 10.1109/tac.2005.863494

- Is $p(x)$ non-negative on $\mathcal{K}(x)$?

$$p(x) = 2x^3 - 10x^2 + 6x + 18 \text{ with } \mathcal{K}(x) = \left\{1 - x^2 \geq 0\right\}$$

- Based on Putinar's Positivestellensatz

$$p(x) = \begin{pmatrix} 1 & x & x^2 \end{pmatrix} \begin{pmatrix} A_0 & A_1 & A_2 \\ A_1 & A_3 & A_4 \\ A_2 & A_4 & A_5 \end{pmatrix} \begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix} + \begin{pmatrix} 1 & x \end{pmatrix} \begin{pmatrix} B_0 & B_1 \\ B_1 & B_2 \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix} \left[1 - x^2\right]$$

- Thus, we search for

$$\begin{aligned} \text{find } & \mathbf{A}, \mathbf{B} \\ \text{subject to } & 18 = A_0 + B_0 \\ & 6 = 2A_1 + 2B_1 \\ & -10 = 2A_2 + A_3 + B_2 - B_0 \\ & 2 = 2A_4 - 2B_1 \\ & 0 = A_5 - B_2 \\ & \mathbf{A}, \mathbf{B} \succeq 0 \quad b(x)^{\mathrm{T}} \mathbf{A} b(x) = b(x)^{\mathrm{T}} \mathbf{L} \mathbf{L}^{\mathrm{T}} b(x) \end{aligned}$$

- Non-negativity certificate

$$p(x) = (x^2 - 2x - 3)^2 + (x - 3)^2 (1 - x^2)$$

- The dual problem is equivalent to an infinite-dimensional ($r \to \infty$) linear SDP

$$f^{(r)} = \min_{\mathbf{y}} \mathbf{q}^{\mathrm{T}}\mathbf{y} \quad \text{subject to } \mathbf{M}_{2r}(\mathbf{y}) \succeq 0, \quad \mathbf{M}_{2r-d}(\mathbf{G}\mathbf{y}) \succeq 0$$

in the moment variables

$$\mathbf{y} = \begin{pmatrix} y_{c^1} & y_{a_1^1} & \cdots & y_{a_{n_e}^1} & y_{c^2} & \cdots & y_{a_{n_e}^{2r}} \end{pmatrix}$$

that are associated with the polynomial space basis $\mathbf{b}(c, \mathbf{a})$

- No free lunch: we cannot solve infinite dimensional convex SDP
- However, we may consider a hierarchy of finite-dimensional convex **truncations** of increasing size
- Why MSOS instead of the SOS hierarchy? We can recognize global optimality

$$\min_{a,c} c \tag{5a}$$

$$\text{subject to } \begin{pmatrix} c & \overline{f} \\ \overline{f} & a^2 \end{pmatrix} \succeq 0, \tag{5b}$$

$$\overline{V} - a \geq 0, \tag{5c}$$

$$a \geq 0. \tag{5d}$$

- First relaxation (linearization): $\mathbf{b}_1(a,c) = \begin{pmatrix} 1 & c & a & c^2 & ca & a^2 \end{pmatrix}^\mathrm{T}$

$$\min_{\mathbf{y}} y_{c^1} \tag{6a}$$

$$\text{subject to } \begin{pmatrix} y_{c^1} & \overline{f} \\ \overline{f} & y_{a^2} \end{pmatrix} \succeq 0, \tag{6b}$$

$$\overline{V} - y_{a^1} \geq 0, \tag{6c}$$

$$y_{a^1} \geq 0, \tag{6d}$$

$$\begin{pmatrix} 1 & y_{c^1} & y_{a^1} \\ y_{c^1} & y_{c^2} & y_{c^1a^1} \\ y_{a^1} & y_{c^1a^1} & y_{a^2} \end{pmatrix} \succeq 0 \tag{6e}$$

- Second relaxation: $\mathbf{b}_2(a,c) = (1\;\; c\;\; a\;\; c^2\;\; ca\;\; a^2\;\; c^3\;\; c^2a\;\; ca^2\;\; a^3\;\; c^4\;\; c^3a\;\; c^2a^2\;\; ca^3\;\; a^4)^{\mathrm{T}}$

$$\min_{\mathbf{y}}\; y_{c1} \tag{7a}$$

subject to

$$\begin{pmatrix} y_{c1} & \overline{f} & y_{c2} & \overline{f}y_{c1} & y_{c1a1} & \overline{f}y_{a1} \\ \overline{f} & y_{a2} & \overline{f}y_{c1} & y_{c1a2} & \overline{f}y_{a1} & y_{a3} \\ y_{c2} & \overline{f}y_{c1} & y_{c3} & \overline{f}y_{c2} & y_{c2a1} & \overline{f}y_{c1a1} \\ \overline{f}y_{c1} & y_{c1a2} & \overline{f}y_{c2} & y_{c2a2} & \overline{f}y_{c1a1} & y_{c1a3} \\ y_{c1a1} & \overline{f}y_{a1} & y_{c2a1} & \overline{f}y_{c1a1} & y_{c1a2} & \overline{f}y_{a2} \\ \overline{f}y_{a1} & y_{a3} & \overline{f}y_{c1a1} & y_{c1a3} & \overline{f}y_{a2} & y_{a4} \end{pmatrix} \succeq 0, \tag{7b}$$

$$\begin{pmatrix} \overline{V} - y_{a1} & \overline{V}y_{c1} - y_{c1a1} & \overline{V}y_{a1} - y_{a2} \\ \overline{V}y_{c1} - y_{c1a1} & \overline{V}y_{c2} - y_{c2a1} & \overline{V}y_{c1a1} - y_{c1a2} \\ \overline{V}y_{a1} - y_{a2} & \overline{V}y_{c1a1} - y_{c1a2} & \overline{V}y_{a2} - y_{a3} \end{pmatrix} \succeq 0, \tag{7c}$$

$$\begin{pmatrix} y_{a1} & y_{c1a1} & y_{a2} \\ y_{c1a1} & y_{c2a1} & y_{c1a2} \\ y_{a2} & y_{c1a2} & y_{a3} \end{pmatrix} \succeq 0, \tag{7d}$$

$$\begin{pmatrix} 1 & y_{c1} & y_{a1} & y_{c2} & y_{c1a1} & y_{a2} \\ y_{c1} & y_{c2} & y_{c1a1} & y_{c3} & y_{c2a1} & y_{c1a2} \\ y_{a1} & y_{c1a1} & y_{a2} & y_{c2a1} & y_{c1a2} & y_{a3} \\ y_{c2} & y_{c3} & y_{c2a1} & y_{c4} & y_{c3a1} & y_{c2a2} \\ y_{c1a1} & y_{c2a1} & y_{c1a2} & y_{c3a1} & y_{c2a2} & y_{c1a3} \\ y_{a2} & y_{c1a2} & y_{a3} & y_{c2a2} & y_{c1a3} & y_{a4} \end{pmatrix} \succeq 0 \tag{7e}$$

- Finite-dimensional relaxations: $\forall r : f^{(r)} \leq f^*$
- **Monotonic convergence** (larger portion of the infinite-dimensional SDP is considered with increasing r): $f^{(r)} \uparrow f^*$ as $r \to \infty$
- **Sufficient condition of global optimality** from the theory of moments — rank flatness of the moment matrices
- Numerical procedure for **extraction** of (the) globally-optimal solutions
- For our problem
  - $\mathbf{a} > \mathbf{0}$, $\mathbf{c} \to \boldsymbol{\infty}$ is a feasible solution, hence not compact
  - $0 \leq a_i \leq \overline{a}_i$, where $\overline{a}_i = \overline{V}/\ell_i$
  - $0 \leq c \leq \overline{c}$, where $\overline{c} = \mathbf{f}^\mathrm{T}\mathbf{K}(\tilde{\mathbf{a}})^{-1}\mathbf{f}$
  - Scaling to the $[-1, 1]$ domains
  - Bound constraints $a_{\mathrm{s},i}^2 \leq 1$ and $c_\mathrm{s}^2 \leq 1$
- Then, the feasible set is algebraically compact

$$
\begin{aligned}
\min_{\mathbf{a}_\mathrm{s}, c_\mathrm{s}} \quad & 0.5\,(c_\mathrm{s}+1)\,\overline{c} \\
\text{subject to} \quad & \begin{pmatrix} \frac{1}{2}\,(c_\mathrm{s}+1)\,\overline{c} & -\mathbf{f}^\mathrm{T} \\ -\mathbf{f} & \mathbf{K}(\mathbf{a}_\mathrm{s}) \end{pmatrix} \succeq 0, \\
& 2 - n_\mathrm{e} - \mathbf{1}^\mathrm{T}\mathbf{a}_\mathrm{s} \geq 0, \\
& \mathbf{a}_\mathrm{sc}^2 \leq \mathbf{1}, \\
& c_\mathrm{sc}^2 \leq 1
\end{aligned}
$$

- Based on $\mathbf{y}$ associated with (unscaled) 1-degree monomials $\hat{\mathbf{a}}$, $\hat{c}$ we can compute feasible upper bounds
  - $\tilde{\mathbf{a}}$ satisfies convex constraints
  - The pair $\hat{\mathbf{a}}$ and $\hat{c}$ generally fails to satisfy the equilibrium PMI
  - Correct the compliance

$$
\tilde{c} = \mathbf{f}^\mathrm{T}\mathbf{K}(\hat{\mathbf{a}})^\dagger\mathbf{f}
$$

  - This can be done because $\mathbf{f} \in \mathrm{Im}(\mathbf{K}(\hat{\mathbf{a}}))$ is guaranteed

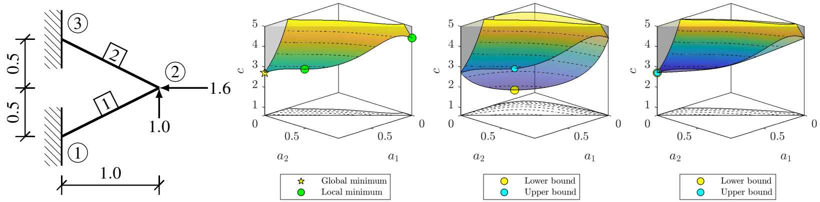
Lemma

$\tilde{c} - f^{(r)} \leq \varepsilon$ is a sufficient condition of global ε-optimality.

- Because hierarchy convergence is independent of the objective function, $\mathcal{K}(\mathbf{G})^{(r)} \uparrow \mathrm{conv}(\mathcal{K}(\mathbf{G}))$
- For $r \to \infty$ optimization of $f(\mathbf{a}, c)$ over $\mathcal{K}(\mathbf{G})$ is equivalent to optimization of $f(\mathbf{x})$ over $\mathrm{conv}(\mathcal{K}(\mathbf{G}))$

Theorem

For optimization problems with global minimizers forming a convex set, it holds that $\tilde{c} - f^{(r)} = 0$ as $r \to \infty$.



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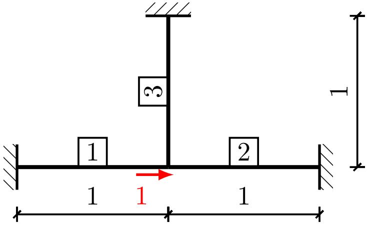
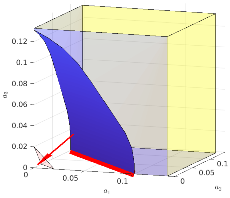
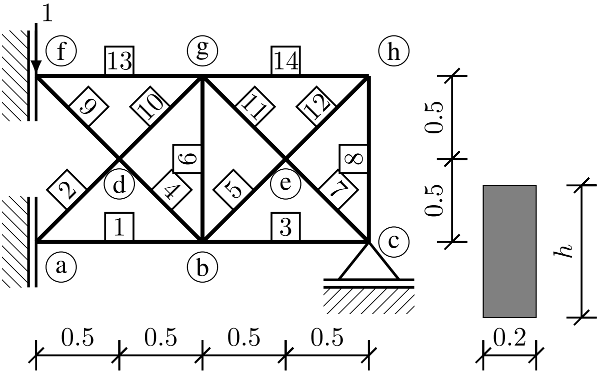




Examples


1
f
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b
0.5
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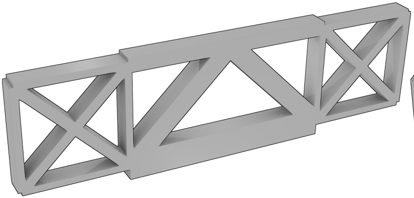
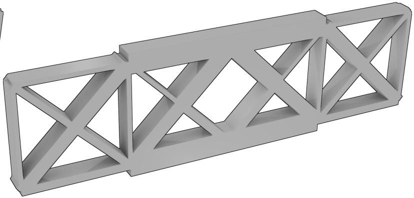
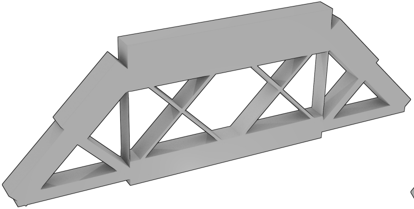
h



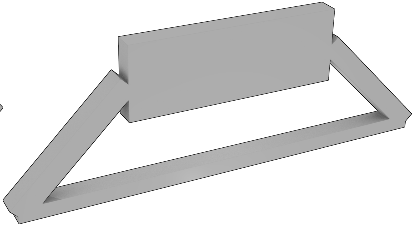
(a) $c_{\mathrm{truss}} = 324.0$ ($c_{\mathrm{asframe}} = 314.5$)



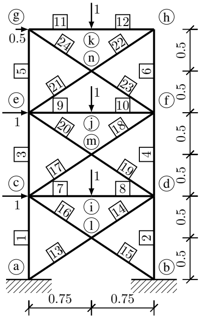
(b) $c_{\mathrm{OC}} = 327.4$

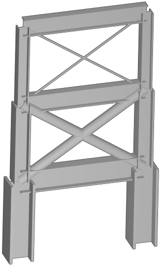


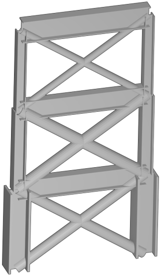
(c) $c_{\mathrm{PO2}} = 280.1$

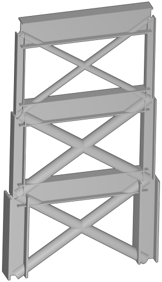


(d) $c^{*}_{\mathrm{PO3}} = 210.3$




(a) $c_{\text{PO1}} = 6584.0$, LB=2248.9


(b) $c_{\text{PO2}} = 5255.9$, LB=4629.8


(c) $c^*_{\text{PO3}} = 4996.5$, LB=4996.5

- Optimization of frame structures generalizes truss TO
- Resulting optimization problems are non-convex even in the case of compliance minimization
- Solution techniques:
  - Non-linear programming/OC method: local optimum is usually achieved
  - Global approaches: non-convex quadratic programming (Gurobi) and moment-sum-of-squares hierarchy (e.g., Mosek)
  - No free lunch: global optimality is usually achieved for small-scale problems only

- Y. Kanno, Mixed-integer second-order cone programming for global optimization of compliance of frame structure with discrete design variables, *Structural and Multidisciplinary Optimization*, 54(2): 301–316, 2016, doi: 10.1007/s00158-016-1406-5
- M. Tyburec, J. Zeman, M. Kružík, and D. Henrion, Global optimality in minimum compliance topology optimization of frames and shells by moment-sum-of-squares hierarchy, *Structural and Multidisciplinary Optimization*, 64(4):1963–1981, 2021, doi: 10.1007/s00158-021-02957-5